



HAESE MATHEMATICS

WORKED SOLUTIONS

Mathematics

**Applications and
Interpretation SL**

2



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for use with

IB Diploma Programme

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MATHEMATICS:

APPLICATIONS AND INTERPRETATION SL WORKED SOLUTIONS

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FOREWORD

This book gives you fully worked solutions for every question in Exercises, Review Sets, Activities, and Investigations (which do not involve student experimentation) in each chapter of our textbook *Mathematics: Applications and Interpretation SL*.

Correct answers can sometimes be obtained by different methods. In this book, where applicable, each worked solution is modelled on the worked example in the textbook.

Be aware of the limitations of calculators and computer modelling packages. Understand that when your calculator gives an answer that is different from the answer you find in the book, you have not necessarily made a mistake, but the book may not be wrong either.

Please contact us if you notice any errors in this book.

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Chapter 1

APPROXIMATIONS AND ERROR

EXERCISE 1A.1

- 1**
 - a** $86 \approx 90$ {round up, as 6 is greater than 5}
 - b** $81 \approx 80$ {round down, as 1 is less than 5}
 - c** $85 \approx 90$ {5 is rounded up}
 - d** $128 \approx 130$ {round up, as 8 is greater than 5}
 - e** $162 \approx 160$ {round down, as 2 is less than 5}
 - f** $104 \approx 100$ {round down, as 4 is less than 5}
 - g** $635 \approx 640$ {5 is rounded up}
 - h** $1822 \approx 1820$ {round down, as 2 is less than 5}
 - i** $699 \approx 700$ {round up, as 9 is greater than 5}
 - j** $3045 \approx 3050$ {5 is rounded up}

- 2**
 - a** $215 \approx 200$ {round down, as 1 is less than 5}
 - b** $264 \approx 300$ {round up, as 6 is greater than 5}
 - c** $3750 \approx 3800$ {5 is rounded up}
 - d** $3950 \approx 4000$ {5 is rounded up}
 - e** $26\,341 \approx 26\,300$ {round down, as 4 is less than 5}

- 3**
 - a** $8365 \approx 8000$ {round down, as 3 is less than 5}
 - b** $3500 \approx 4000$ {5 is rounded up}
 - c** $19\,210 \approx 19\,000$ {round down, as 2 is less than 5}
 - d** $19\,650 \approx 20\,000$ {round up, as 6 is greater than 5}
 - e** $114\,823 \approx 115\,000$ {round up, as 8 is greater than 5}

- 4**
 - a** $8848\text{ m} \approx 8850\text{ m}$ {round up, as 8 is greater than 5}
 - b** $31\,722\text{ km}^2 \approx 32\,000\text{ km}^2$ {round up, as 7 is greater than 5}
 - c** $4\,749\,598\text{ people} \approx 4\,750\,000\text{ people}$ {5 is rounded up}
 - d** $85\,512\text{ people} \approx 85\,500\text{ people}$ {round down, as 1 is less than 5}
 - e** $7\,692\,000\text{ km}^2 \approx 7\,700\,000\text{ km}^2$ {round up, as 9 is greater than 5}
 - f** $5443\text{ kg} \approx 5400\text{ kg}$ {round down, as 4 is less than 5}
 - g** $16\,950\text{ km} \approx 17\,000\text{ km}$ {5 is rounded up}
 - h** $384\,400\text{ km} \approx 400\,000\text{ km}$ {round up, as 8 is greater than 5}
 - i** $428\,240\,515\text{ people} \approx 428\,000\,000\text{ people}$ {round down, as 2 is less than 5}

EXERCISE 1A.2

- 1
 - a $6.181 \approx 6.2$ {round up, as $8 > 5$ }
 - b $6.181 \approx 6.18$ {round down, as $1 < 5$ }
 - c $3.25 \approx 3.3$ {5 is rounded up}
 - d $17.403 \approx 17.40$ {round down, as $3 < 5$ }
 - e $2.131\,58 \approx 2.132$ {5 is rounded up}
 - f $0.1940 \approx 0.2$ {round up, as $9 > 5$ }
 - g $0.0972 \approx 0.10$ {round up, as $7 > 5$ }
 - h $102.382 \approx 102.38$ {round down, as $2 < 5$ }
- 2 $9.58 \text{ seconds} \approx 9.6 \text{ seconds}$ {round up, as $8 > 5$ }
- 3 $1.435 \text{ m} \approx 1.44 \text{ m}$ {5 is rounded up}
- 4 $0.012 \text{ cm} \approx 0.01 \text{ cm}$ {round down, as $2 < 5$ }
- 5
 - a $3.141\,59 \approx 3.1$ {round down, as $4 < 5$ }
 - b $3.141\,59 \approx 3.142$ {5 is rounded up}
 - c $3.141\,59 \approx 3.1416$ {round up, as $9 > 5$ }
- 6
 - a $0.263\,157\,895 \approx 0.3$ {round up, as $6 > 5$ }
 - b $0.263\,157\,895 \approx 0.26$ {round down, as $3 < 5$ }
 - c $0.263\,157\,895 \approx 0.263\,158$ {round up, as $8 > 5$ }
- 7

<ol style="list-style-type: none"> a $\sqrt{2} \approx 1.414\,213 \dots$ ≈ 1.414 {round down, as $2 < 5$} c $\sqrt{23} \approx 4.795\,831 \dots$ ≈ 4.796 {round up, as $8 > 5$} e $\sqrt[3]{-15} \approx -2.466\,212 \dots$ ≈ -2.466 {round down, as $2 < 5$} 	<ol style="list-style-type: none"> b $\sqrt{5} \approx 2.236\,067 \dots$ ≈ 2.236 {round down, as $0 < 5$} d $\sqrt[3]{4} \approx 1.587\,401 \dots$ ≈ 1.587 {round down, as $4 < 5$} f $\sqrt[3]{450} \approx 7.663\,094 \dots$ ≈ 7.663 {round down, as $0 < 5$}
---	---
- 8

<ol style="list-style-type: none"> a $(16.8 + 12.4) \times 17.1 = 29.2 \times 17.1$ $= 499.32$ c $127 \div 9 - 5 = \frac{127}{9} - 5$ ≈ 9.11 e $37.4 - 16.1 \div (4.2 - 2.7)$ $= 37.4 - 16.1 \div 1.5$ ≈ 26.67 g $\frac{27.4}{3.2} - \frac{18.6}{16.1} \approx 7.41$ 	<ol style="list-style-type: none"> b $16.8 + 12.4 \times 17.1 = 16.8 + 212.04$ $= 228.84$ d $127 \div (9 - 5) = \frac{127}{4}$ $= 31.75$ f $\frac{16.84}{7.9 + 11.2} = \frac{16.84}{19.1}$ ≈ 0.88 h $\frac{27.9 - 17.3}{8.6} + 4.7 = \frac{10.6}{8.6} + 4.7$ ≈ 5.93
--	---

$$\text{i} \quad \frac{0.0768 + 7.1}{18.69 - 3.824} = \frac{7.1768}{14.866} \approx 0.48$$

$$\text{9} \quad \frac{40 \text{ goals}}{23 \text{ games}} \approx 1.74 \text{ goals per game}$$

\therefore Kerry scored an average of 1.74 goals per game.

- 10** While $2.45 \text{ m} \approx 2.5 \text{ m}$ is correct, Wang should have used the original value of 2.45 m to round to the nearest integer, so $2.45 \text{ m} \approx 2 \text{ m}$.

EXERCISE 1A.3

1 a $128 \approx 130$ {2 significant figures}



This is the second significant figure, so we look at the next digit which is 8. The 8 tells us to round the 2 up to a 3, so we convert the 8 into 0.

b $8342 \approx 8300$ {2 significant figures}



This is the second significant figure, so we look at the next digit which is 4. The 4 tells us to keep the original 3, so we convert the 42 into 00.

c $2.568 \approx 2.6$ {2 significant figures}



This is the second significant figure, so we look at the next digit which is 6. The 6 tells us to round the 5 up to a 6 and delete the remaining digits.

d $0.0134 \approx 0.013$ {2 significant figures}



These zeros at the front are place holders and so must stay. The first significant figure is 1, and the second significant figure is 3. The 4 tells us to leave the 3 as it is and delete the remaining digits.

e $163\,870 \approx 160\,000$ {2 significant figures}



This is the second significant figure, so we look at the next digit which is 3. The 3 tells us to keep the original 6, so we convert the 3870 into 0000.

f $1.086 \approx 1.1$ {2 significant figures}



This is the second significant figure, as it lies between two non-zero digits. The 8 tells us to round the 0 up to a 1 and delete the remaining digits.

g $3958 \approx 4000$ {2 significant figures}




This is the second significant figure, so we look at the next digit which is 5. The 5 tells us to round the 39 to 40, so we convert the 58 into 00.

h $6.611 \approx 6.6$ {2 significant figures}




This is the second significant figure, so we look at the next digit which is 1. The 1 tells us to leave the 6 as it is and delete the remaining digits.


2 a $83\,064 \approx 83\,100$ {3 significant figures}

 This is the third significant figure, so we look at the next digit which is 6. The 6 tells us to round the 0 to 1, so we convert the 64 into 00.


b $10\,044 \approx 10\,000$ {3 significant figures}

 This is the third significant figure, so we look at the next digit which is 4. The 4 tells us to keep the original 0, so we convert the 44 into 00.


c $0.105\,26 \approx 0.105$ {3 significant figures}

 This is the third significant figure, so we look at the next digit which is 2. The 2 tells us to leave the 5 as it is and delete the remaining digits.


d $31.695 \approx 31.7$ {3 significant figures}

 This is the third significant figure, so we look at the next digit which is 9. The 9 tells us to round the 6 up to a 7, and delete the remaining digits.


e $70.707 \approx 70.7$ {3 significant figures}

 This is the third significant figure, so we look at the next digit which is 0. The 0 tells us to leave the 7 as it is and delete the remaining digits.


f $4.0007 \approx 4.00$ {3 significant figures}

 This is the third significant figure, so we look at the next digit which is 0. The 0 tells us to leave the original 0 as it is and delete the remaining digits.


g $0.036\,71 \approx 0.0367$ {3 significant figures}

 This is the third significant figure, so we look at the next digit which is 1. The 1 tells us to leave the 7 as it is and delete the remaining digits.


h $19.989 \approx 20.0$ {3 significant figures}

 This is the third significant figure, so we look at the next digit which is 8. The 8 tells us to round the 19.9 to 20.0 and delete the remaining digits.


3 a $16.382 \approx 16.38$ {4 significant figures}

 This is the fourth significant figure, so we look at the next digit which is 2. The 2 tells us to leave the 8 as it is and delete the remaining digits.

b $438.207 \approx 438.2$ {4 significant figures}

 This is the fourth significant figure, so we look at the next digit which is 0. The 0 tells us to leave the 2 as it is and delete the remaining digits.

c $6\,873\,681 \approx 6\,874\,000$ {4 significant figures}

 This is the fourth significant figure, so we look at the next digit which is 6. The 6 tells us to round the 3 up to a 4, so we convert the 681 into 000.

d $0.028\,885 \approx 0.028\,89$ {4 significant figures}

This is the fourth significant figure, so we look at the next digit which is 5.
The 5 tells us to round the 8 up to a 9 and delete the remaining digits.

4 a $96\,257 \text{ people} \approx 100\,000 \text{ people}$ {1 significant figure}

first significant figure, next digit is 6, so we round up

b $96\,257 \text{ people} \approx 96\,000 \text{ people}$ {2 significant figures}

second significant figure, next digit is 2, so we round down

c $96\,257 \text{ people} \approx 96\,300 \text{ people}$ {3 significant figures}

third significant figure, next digit is 5, so we round up

5 a $\sqrt{7} \approx 2.645\,751 \dots$

≈ 2.65 {3 significant figures}

b $2\pi \approx 6.283\,185 \dots$

≈ 6.28 {3 significant figures}

c $36 \div 17 \approx 2.117\,647 \dots$

≈ 2.12 {3 significant figures}

d $517 \times 3802 = 1\,965\,634$

$\approx 1\,970\,000$ {3 significant figures}

e $(0.986)^5 \approx 0.931\,932 \dots$

≈ 0.932 {3 significant figures}

f $\frac{16.3 - 2.68}{3.1} = \frac{13.62}{3.1}$

$\approx 4.393\,548 \dots$

≈ 4.39 {3 significant figures}

g $\sqrt{5.4 - 2.18} = \sqrt{3.22}$

$\approx 1.794\,435 \dots$

≈ 1.79 {3 significant figures}

h $\frac{9.58}{\sqrt{2.8}} \approx 5.725\,145 \dots$

≈ 5.73 {3 significant figures}

6 $32 \text{ rows} \times 28 \text{ seats} = 896 \text{ seats}$

$\approx 900 \text{ seats}$ {2 significant figures}

\therefore there are about 900 seats in the theatre.

7 $30.1 \text{ m} \times 8.5 \text{ m} = 255.85 \text{ m}^2$

$\approx 256 \text{ m}^2$ {3 significant figures}

\therefore the area of the ballroom is about 256 m^2 .

8 $\frac{\$752.25}{4} = \188.0625

\therefore each person receives \$188.0625

a $\$188.0625 \approx \188 {3 significant figures}

b $\$188.0625 \approx \188.06 {5 significant figures}

9 1 hour = 60 min
 $= 60 \times 60 = 3600 \text{ s}$

\therefore in one hour, sound will travel $343 \text{ m s}^{-1} \times 3600 \text{ s} = 1\,234\,800 \text{ m}$
 $\approx 1\,200\,000 \text{ m}$ {2 significant figures}

10 a Eric has rounded each answer to 3 significant figures before using it in the next calculation, rather than using exact values.

b Diameter of circle = $\sqrt{6^2 + 6^2} = \sqrt{72} \text{ cm}$

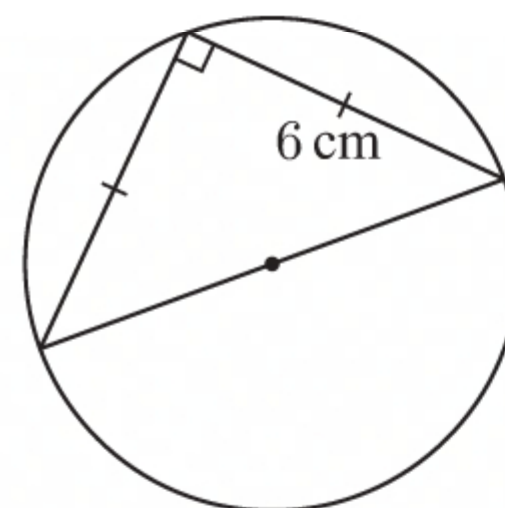
\therefore radius of circle = $\frac{\sqrt{72}}{2} \text{ cm}$

\therefore area of circle = $\pi \times \left(\frac{\sqrt{72}}{2}\right)^2 \text{ cm}^2$

$= \frac{72\pi}{4} \text{ cm}^2$

$\approx 56.548\,667 \dots \text{ cm}^2$

$\approx 56.5 \text{ cm}^2$ {3 significant figures}



EXERCISE 1B

1 a 32×6
 $\approx 30 \times 6$
 ≈ 180

b 58×7
 $\approx 60 \times 7$
 ≈ 420

c 81×30
 $\approx 80 \times 30$
 ≈ 2400

d 207×3
 $\approx 200 \times 3$
 ≈ 600

e 487×50
 $\approx 500 \times 50$
 $\approx 25\,000$

f 6117×4
 $\approx 6000 \times 4$
 $\approx 24\,000$

g 48×23
 $\approx 50 \times 20$
 ≈ 1000

h 61×42
 $\approx 60 \times 40$
 ≈ 2400

i 103×47
 $\approx 100 \times 50$
 ≈ 5000

j 3125×18
 $\approx 3000 \times 20$
 $\approx 60\,000$

k 422×307
 $\approx 400 \times 300$
 $\approx 120\,000$

l 3818×27
 $\approx 4000 \times 30$
 $\approx 120\,000$

m 2.7×1.15
 $\approx 3 \times 1$
 ≈ 3

n 5.36×0.68
 $\approx 5 \times 0.7$
 ≈ 3.5

o 28.37×6.13
 $\approx 30 \times 6$
 ≈ 180

2 a $86 \div 3$
 $\approx 90 \div 3$
 ≈ 30

b $64 \div 5$
 $\approx 60 \div 5$
 ≈ 12

c $512 \div 21$
 $\approx 500 \div 20$
 ≈ 25

d $610 \div 43$
 $\approx 600 \div 40$
 ≈ 15

e $4182 \div 19$
 $\approx 4000 \div 20$
 ≈ 200

f $78\,638 \div 82$
 $\approx 80\,000 \div 80$
 ≈ 1000

g $318 \div 62$
 $\approx 300 \div 60$
 ≈ 5

h $47\,320 \div 193$
 $\approx 50\,000 \div 200$
 ≈ 250

i $0.628 \div 3$
 $\approx 0.6 \div 3$
 ≈ 0.2

j $46.1 \div 5.2$
 $\approx 50 \div 5$
 ≈ 10

k $631.7 \div 0.29$
 $\approx 600 \div 0.3$
 ≈ 2000

l $18.7 \div 3.86$
 $\approx 20 \div 4$
 ≈ 5

- 3 a** $8 \text{ kg} \times \text{€}2.80/\text{kg}$
 $\approx 8 \text{ kg} \times \text{€}3/\text{kg}$
 $\approx \text{€}24$
- c** $7 \text{ tickets} \times \87.30 per ticket
 $\approx 7 \text{ tickets} \times \90 per ticket
 $\approx \$630$
- 4 a** $4.2 \text{ h} \times 63 \text{ km h}^{-1}$
 $\approx 4 \text{ h} \times 60 \text{ km h}^{-1}$
 $\approx 240 \text{ km}$
 \therefore Brodie will travel about 240 km.
- c** $\frac{423 \text{ tonnes}}{18 \text{ trucks}} \approx \frac{400 \text{ tonnes}}{20 \text{ trucks}}$
 $\approx 20 \text{ tonnes per truck}$
 \therefore each truck will carry about 20 tonnes.
- b** $3 \text{ tickets} \times \213 per ticket
 $\approx 3 \text{ tickets} \times \200 per ticket
 $\approx \$600$
- d** $55 \text{ L} \times \text{£}1.49/\text{L}$
 $\approx 60 \text{ L} \times \text{£}1/\text{L}$
 $\approx \text{£}60$
- b** $14 \text{ years} \times 365.25 \text{ days/year}$
 $\approx 10 \text{ years} \times 400 \text{ days/year}$
 $\approx 4000 \text{ days}$
 \therefore there are about 4000 days in 14 years.
- d** $18.2 \text{ hours per week} \times 12 \text{ weeks}$
 $\approx 20 \text{ hours per week} \times 10 \text{ weeks}$
 $\approx 200 \text{ hours}$
 $200 \text{ hours} \times \text{€}21.50 \text{ per hour}$
 $\approx 200 \text{ hours} \times \text{€}20 \text{ per hour}$
 $\approx \text{€}4000$
 \therefore the part-time worker will earn about €4000 over the 12 weeks.

EXERCISE 1C

- 1 a** The tape measure is accurate to $\pm \frac{1}{2}$ cm.
- b** The measuring cylinder is accurate to $\pm \frac{1}{2}$ mL.
- c** The beaker is accurate to $\pm \frac{1}{2} \times 100 \text{ mL} = \pm 50 \text{ mL}$.
- d** The set of scales is accurate to $\pm \frac{1}{2} \times 500 \text{ g} = \pm 250 \text{ g}$.
- e** The thermometer is accurate to $\pm \frac{1}{2} \times 0.1^\circ\text{C} = \pm 0.05^\circ\text{C}$.
- 2** The scales have accuracy $\pm \frac{1}{2} \text{ kg}$ \therefore the range of values is $68 \pm \frac{1}{2} \text{ kg}$.
 So Roni's actual weight is in the range 67.5 kg to 68.5 kg.
 $\therefore 67.5 \text{ kg} < w < 68.5 \text{ kg}$
- 3 a** The range of values is $27 \pm \frac{1}{2} \text{ mm}$.
 So, the actual value is in the range 26.5 mm to 27.5 mm.
- b** $38.3 \text{ cm} = 383 \text{ mm}$
 The range of values is $383 \pm \frac{1}{2} \text{ mm}$.
 So, the actual value is in the range 382.5 mm to 383.5 mm
 or 38.25 cm to 38.35 cm.

c $4.8 \text{ m} = 480 \text{ cm}$

The range of values is $480 \pm \frac{1}{2} \times 10 \text{ cm}$
 $= 480 \pm 5 \text{ cm}.$

So, the actual value is in the range
 $475 \text{ cm to } 485 \text{ cm}$
or $4.75 \text{ m to } 4.85 \text{ m}.$

e The range of values is $25 \pm \frac{1}{2} \text{ g}.$

So, the actual value is in the range
 $24.5 \text{ g to } 25.5 \text{ g}.$

d $1.5 \text{ kg} = 1500 \text{ g}$

The range of values is $1500 \pm \frac{1}{2} \times 100 \text{ g}$
 $= 1500 \pm 50 \text{ g}.$

So, the actual value is in the range
 $1450 \text{ g to } 1550 \text{ g}$
or $1.45 \text{ kg to } 1.55 \text{ kg}.$

f $3.75 \text{ kg} = 3750 \text{ g}$

The range of values is $3750 \pm \frac{1}{2} \times 10 \text{ g}$
 $= 3750 \pm 5 \text{ g}.$

So, the actual value is in the range
 $3745 \text{ g to } 3755 \text{ g}$
or $3.745 \text{ kg to } 3.755 \text{ kg}.$

4 The thermometer reads 36.4°C , so it must be accurate to $\pm \frac{1}{2} \times 0.1^\circ\text{C} = \pm 0.05^\circ\text{C}.$

\therefore the range of values is $36.4 \pm 0.05^\circ\text{C}.$

Tom's actual temperature lies between 36.35°C and $36.45^\circ\text{C}.$

$\therefore 36.35^\circ\text{C} < T < 36.45^\circ\text{C}$

5 For distances less than 10 km, the exercise watch is accurate to $\pm \frac{0.01}{2} \text{ km} = \pm 0.005 \text{ km}.$

For distances between 10 km and 100 km, the exercise watch is accurate to $\pm \frac{0.1}{2} \text{ km} = \pm 0.05 \text{ km}.$

a The range of values is $1.06 \pm 0.005 \text{ km},$ or $1.055 \text{ km to } 1.065 \text{ km}.$

\therefore if the watch displays 1.06 km, the least distance Joanne could have run is 1.055 km.

b The range of values is $9.72 \pm 0.005 \text{ km},$ or $9.715 \text{ km to } 9.725 \text{ km}.$

\therefore if the watch displays 9.72 km, the least distance Joanne could have run is 9.715 km.

c The range of values is $10.1 \pm 0.05 \text{ km},$ or $10.05 \text{ km to } 10.15 \text{ km}.$

\therefore if the watch displays 10.1 km, the least distance Joanne could have run is 10.05 km.

6 a The tape measure is accurate to $\pm 0.05 \text{ m}$

\therefore for a measurement of 6.1 m, the range is $6.05 \text{ m to } 6.15 \text{ m}$

for a measurement of 6.4 m, the range is $6.35 \text{ m to } 6.45 \text{ m}$

for a measurement of 6.0 m, the range is $5.95 \text{ m to } 6.05 \text{ m}$

for a measurement of 6.1 m, the range is $6.05 \text{ m to } 6.15 \text{ m}$

\therefore 6.4 m is likely to be the incorrect measurement, as its range of values is furthest from any of the other measurements.

b 6.05 m, as it is in the range of values for a measurement of 6.0 m or 6.1 m.

c 10 cm graduations, as all of the measurements were given to the nearest 0.1 m, or 10 cm.

7 a $2.4 \text{ m} = 240 \text{ cm}$

The range of values is $240 \pm \frac{1}{2} \times 10 \text{ cm}$
 $= 240 \pm 5 \text{ cm}.$

So, the actual length of a rope is in the range $235 \text{ cm to } 245 \text{ cm}.$

$\therefore 2.35 \text{ m} < l < 2.45 \text{ m}$

- b** If each rope is 2.35 m, then n of these ropes will have total length $2.35n$ m.
 If each rope is 2.45 m, then n of these ropes will have total length $2.45n$ m.
 $\therefore 2.35n \text{ m} < L < 2.45n \text{ m}$

- 8** The times are accurate to $\pm \frac{1}{2}$ s.

The range of Jiao's times is $2 \text{ min } 8 \pm \frac{1}{2} \text{ s}$.

So, Jiao's actual time is in the range $2 \text{ min } 7.5 \text{ s}$ to $2 \text{ min } 8.5 \text{ s}$.

The range of Liang's times is $2 \text{ min } 13 \pm \frac{1}{2} \text{ s}$.

So, Liang's actual time is in the range $2 \text{ min } 12.5 \text{ s}$ to $2 \text{ min } 13.5 \text{ s}$.

\therefore the least time t by which Jiao beat Liang is $12.5 - 8.5 = 4 \text{ s}$

and the greatest time t by which Jiao beat Liang is $13.5 - 7.5 = 6 \text{ s}$

$\therefore 4 \text{ s} < t < 6 \text{ s}$

- 9** The length of the bath mat could be from $85\frac{1}{2} \text{ cm}$ to $86\frac{1}{2} \text{ cm}$.

The width of the bath mat could be from $37\frac{1}{2} \text{ cm}$ to $38\frac{1}{2} \text{ cm}$.

\therefore the lower boundary of the perimeter is $2 \times 85\frac{1}{2} + 2 \times 37\frac{1}{2} = 246 \text{ cm}$

and the upper boundary of the perimeter is $2 \times 86\frac{1}{2} + 2 \times 38\frac{1}{2} = 250 \text{ cm}$

The perimeter is between 246 cm and 250 cm, which is $248 \pm 2 \text{ cm}$.

- 10** The length of the garden bed could be from $251\frac{1}{2} \text{ cm}$ to $252\frac{1}{2} \text{ cm}$.

The width of the garden bed could be from $142\frac{1}{2} \text{ cm}$ to $143\frac{1}{2} \text{ cm}$.

\therefore the lower boundary for the length of edging l is $2 \times 251\frac{1}{2} + 2 \times 142\frac{1}{2} = 788 \text{ cm}$

and the upper boundary for the length of edging l is $2 \times 252\frac{1}{2} + 2 \times 143\frac{1}{2} = 792 \text{ cm}$

$\therefore 788 \text{ cm} < l < 792 \text{ cm}$

- 11** The length of the rectangle could be from 5.5 cm to 6.5 cm.

The width of the rectangle could be from 7.5 cm to 8.5 cm.

$$\begin{aligned} \text{a Largest area} &= 6.5 \times 8.5 \text{ cm}^2 \\ &= 55.25 \text{ cm}^2 \end{aligned}$$

$$\begin{aligned} \text{b Smallest area} &= 5.5 \times 7.5 \text{ cm}^2 \\ &= 41.25 \text{ cm}^2 \end{aligned}$$

- 12** The length of the glass window could be from $41\frac{1}{2} \text{ cm}$ to $42\frac{1}{2} \text{ cm}$.

The width of the glass window could be from $25\frac{1}{2} \text{ cm}$ to $26\frac{1}{2} \text{ cm}$.

\therefore the lower boundary of the area is $41\frac{1}{2} \times 25\frac{1}{2} = 1058.25 \text{ cm}^2$

and the upper boundary of the area is $42\frac{1}{2} \times 26\frac{1}{2} = 1126.25 \text{ cm}^2$

The area is between 1058.25 cm^2 and 1126.25 cm^2 , which is $1092.25 \pm 34 \text{ cm}^2$.

- 13** The base of the triangle could be from 8.5 cm to 9.5 cm.

The height of the triangle could be from 7.5 cm to 8.5 cm.

\therefore the lower boundary of the area A is $\frac{1}{2} \times 8.5 \times 7.5 = 31.875 \text{ cm}^2$

and the upper boundary of the area A is $\frac{1}{2} \times 9.5 \times 8.5 = 40.375 \text{ cm}^2$

$\therefore 31.875 \text{ cm}^2 < A < 40.375 \text{ cm}^2$

14 The length of the box could be from 3.5 cm to 4.5 cm.

The width of the box could be from 7.5 cm to 8.5 cm.

The depth of the box could be from 5.5 cm to 6.5 cm.

\therefore the lower boundary of the volume is $3.5 \times 7.5 \times 5.5 = 144.375 \text{ cm}^3$

and the upper boundary of the volume is $4.5 \times 8.5 \times 6.5 = 248.625 \text{ cm}^3$

The volume is between 144.375 cm^3 and 248.625 cm^3 , which is $196.5 \pm 52.125 \text{ cm}^3$.

15 The length of the house brick could be from 21.25 cm to 21.35 cm.

The width of the house brick could be from 9.75 cm to 9.85 cm.

The depth of the house brick could be from 7.25 cm to 7.35 cm.

\therefore the lower boundary of the volume is $21.25 \times 9.75 \times 7.25 \approx 1502.11 \text{ cm}^3$

and the upper boundary of the volume is $21.35 \times 9.85 \times 7.35 \approx 1545.69 \text{ cm}^3$

$\therefore 1502.11 \text{ cm}^3 < V < 1545.69 \text{ cm}^3$

16 The radius of the cylinder could be from 4.5 cm to 5.5 cm.

The height of the cylinder could be from 14.5 cm to 15.5 cm.

\therefore the lower boundary of the volume is $\pi \times 4.5^2 \times 14.5 \approx 922.45 \text{ cm}^3$

and the upper boundary of the volume is $\pi \times 5.5^2 \times 15.5 \approx 1473.01 \text{ cm}^3$

The volume is between about 922.45 cm^3 and 1473.01 cm^3 , which is about $1197.73 \pm 275.28 \text{ cm}^3$.

17 The radius of the cone could be from 8.35 cm to 8.45 cm.

The height of the cone could be from 4.55 cm to 4.65 cm.

\therefore the lower boundary of the volume is $\frac{1}{3} \times \pi \times 8.35^2 \times 4.55 \approx 332.21 \text{ cm}^3$

and the upper boundary of the volume is $\frac{1}{3} \times \pi \times 8.45^2 \times 4.65 \approx 347.69 \text{ cm}^3$

The volume is between about 332.21 cm^3 and 347.69 cm^3 , which is about $339.95 \pm 7.74 \text{ cm}^3$.

18 Volume of a sphere = $\frac{4}{3}\pi r^3$, surface area of a sphere = $4\pi r^2$

The rounding will have more effect on the volume, as the error is multiplied through 3 times rather than twice.

19 The base side length of the square-based pyramid could be from 4.55 cm to 4.65 cm.

The height of the square-based pyramid could be from 5.15 cm to 5.25 cm.

a The lower boundary of the volume is $\frac{1}{3} \times 4.55^2 \times 5.15 \approx 35.54 \text{ cm}^3$

and the upper boundary of the volume is $\frac{1}{3} \times 4.65^2 \times 5.25 \approx 37.84 \text{ cm}^3$

The volume is between about 35.54 cm^3 and 37.84 cm^3 , which is about $36.69 \pm 1.15 \text{ cm}^3$.

b The lower boundary of the surface area is found when the base side length is 4.55 cm and the height is 5.15 cm.

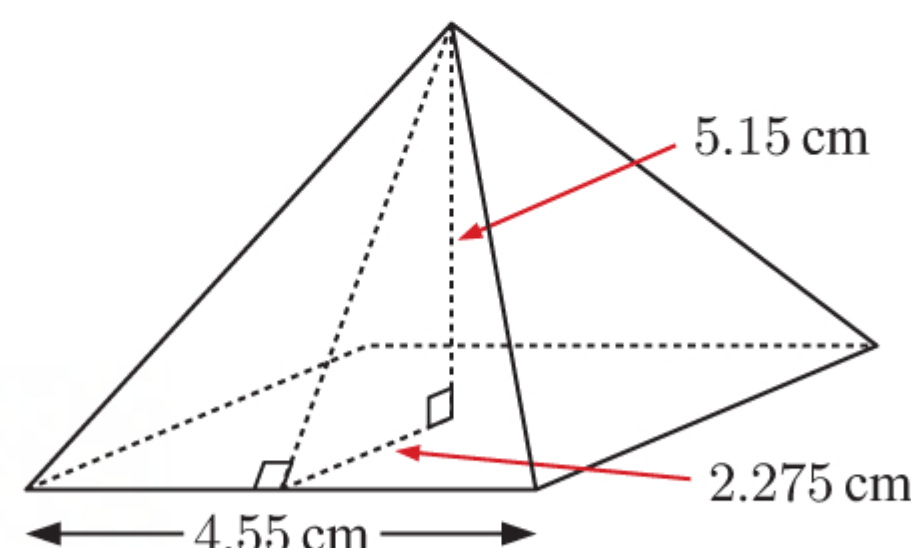
The height of a triangular face

$$= \sqrt{2.275^2 + 5.15^2} \text{ cm} \quad \{\text{Pythagoras}\}$$

The lower boundary of the surface area

$$= 4 \times \left(\frac{1}{2} \times 4.55 \times \sqrt{2.275^2 + 5.15^2} \right) + 4.55^2$$

$$\approx 71.94 \text{ cm}^2$$



The upper boundary of the surface area is found when the base side length is 4.65 cm and the height is 5.25 cm.

The height of a triangular face

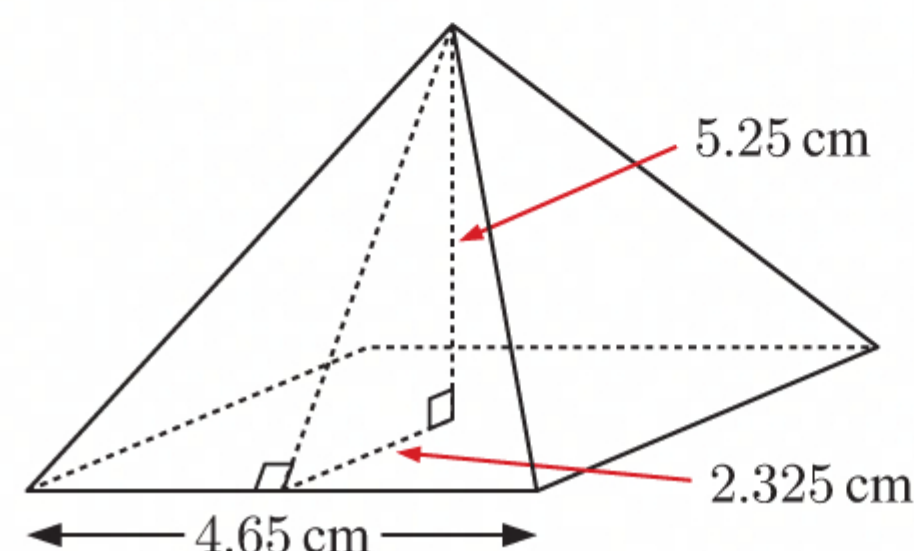
$$= \sqrt{2.325^2 + 5.25^2} \text{ cm} \quad \{\text{Pythagoras}\}$$

The upper boundary of the surface area

$$= 4 \times \left(\frac{1}{2} \times 4.65 \times \sqrt{2.325^2 + 5.25^2} \right) + 4.65^2$$

$$\approx 75.02 \text{ cm}^2$$

The surface area is between about 71.94 cm^2 and 75.02 cm^2 , which is about $73.48 \pm 1.54 \text{ cm}^2$.



EXERCISE 1D

1 a Absolute error $= |V_A - V_E|$
 $= \text{€} |1\,370\,000 - 1\,367\,540|$
 $= \text{€}2460$

$$\text{percentage error} = \frac{|V_A - V_E|}{V_E} \times 100\%$$

$$= \frac{2460}{1\,367\,540} \times 100\%$$

$$\approx 0.180\%$$

b Absolute error $= |V_A - V_E|$
 $= |31\,000 - 31\,467| \text{ people}$
 $= |-467| \text{ people}$
 $= 467 \text{ people}$

$$\text{percentage error} = \frac{|V_A - V_E|}{V_E} \times 100\%$$

$$= \frac{467}{31\,467} \times 100\%$$

$$\approx 1.48\%$$

c Absolute error $= |V_A - V_E|$
 $= \$ |460\,000 - 458\,110|$
 $= \$1890$

$$\text{percentage error} = \frac{|V_A - V_E|}{V_E} \times 100\%$$

$$= \frac{1890}{458\,110} \times 100\%$$

$$\approx 0.413\%$$

d Absolute error $= |V_A - V_E|$
 $= |3000 - 2811| \text{ cars}$
 $= 189 \text{ cars}$

$$\text{percentage error} = \frac{|V_A - V_E|}{V_E} \times 100\%$$

$$= \frac{189}{2811} \times 100\%$$

$$\approx 6.72\%$$

2 a Absolute error $= |V_A - V_E|$
 $= |5 - 6.238| \text{ kg}$
 $= |-1.238| \text{ kg}$
 $= 1.238 \text{ kg}$

$$\text{percentage error} = \frac{|V_A - V_E|}{V_E} \times 100\%$$

$$= \frac{1.238}{6.238} \times 100\%$$

$$\approx 19.8\%$$

$$\begin{aligned}
 \text{b Absolute error} &= |V_A - V_E| \\
 &= |100 - 97.6| \text{ m} \\
 &= 2.4 \text{ m}
 \end{aligned}$$

$$\begin{aligned}
 \text{percentage error} &= \frac{|V_A - V_E|}{V_E} \times 100\% \\
 &= \frac{2.4}{97.6} \times 100\% \\
 &\approx 2.46\%
 \end{aligned}$$

$$\begin{aligned}
 \text{c Absolute error} &= |V_A - V_E| \\
 &= |20 - 23.8| \text{ L} \\
 &= |-3.8| \text{ L} \\
 &= 3.8 \text{ L}
 \end{aligned}$$

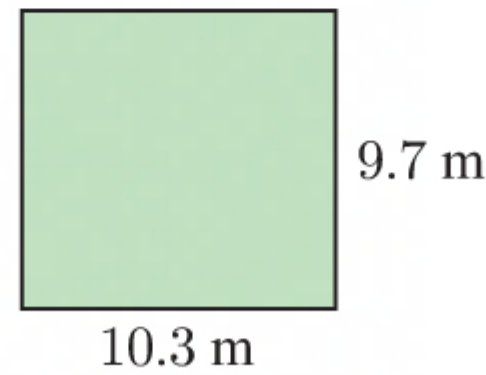
$$\begin{aligned}
 \text{percentage error} &= \frac{|V_A - V_E|}{V_E} \times 100\% \\
 &= \frac{3.8}{23.8} \times 100\% \\
 &\approx 16.0\%
 \end{aligned}$$

$$\begin{aligned}
 \text{d Absolute error} &= |V_A - V_E| \\
 &= |50 - 72| \text{ hours} \\
 &= |-22| \text{ hours} \\
 &= 22 \text{ hours}
 \end{aligned}$$

$$\begin{aligned}
 \text{percentage error} &= \frac{|V_A - V_E|}{V_E} \times 100\% \\
 &= \frac{22}{72} \times 100\% \\
 &\approx 30.6\%
 \end{aligned}$$

$$\begin{aligned}
 \text{3 a Actual area} &= 10.3 \text{ m} \times 9.7 \text{ m} \\
 &= 99.91 \text{ m}^2
 \end{aligned}$$

$$\begin{aligned}
 \text{b Area} &\approx 10 \text{ m} \times 10 \text{ m} \\
 &\approx 100 \text{ m}^2
 \end{aligned}$$



$$\begin{aligned}
 \text{c Absolute error} &= |V_A - V_E| \\
 &= |100 - 99.91| \text{ m}^2 \\
 &= 0.09 \text{ m}^2
 \end{aligned}$$

$$\begin{aligned}
 \text{percentage error} &= \frac{|V_A - V_E|}{V_E} \times 100\% \\
 &= \frac{0.09}{99.91} \times 100\% \\
 &\approx 0.0901\%
 \end{aligned}$$

$$\begin{aligned}
 \text{4 a Actual volume} &= 23.9 \text{ cm} \times 14.8 \text{ cm} \times 9.2 \text{ cm} \\
 &= 3254.224 \text{ cm}^3
 \end{aligned}$$

$$\begin{aligned}
 \text{b Volume} &\approx 24 \text{ cm} \times 15 \text{ cm} \times 9 \text{ cm} \\
 &\approx 3240 \text{ cm}^3
 \end{aligned}$$

$$\begin{aligned}
 \text{c Absolute error} &= |V_A - V_E| \\
 &= |3240 - 3254.224| \text{ cm}^3 \\
 &= |-14.224| \text{ cm}^3 \\
 &= 14.224 \text{ cm}^3
 \end{aligned}$$

$$\begin{aligned}
 \text{percentage error} &= \frac{|V_A - V_E|}{V_E} \times 100\% \\
 &= \frac{14.224}{3254.224} \times 100\% \\
 &\approx 0.437\%
 \end{aligned}$$

$$\begin{aligned}
 \text{5 a Area} &\approx 8 \text{ m} \times 9 \text{ m} \\
 &\approx 72 \text{ m}^2
 \end{aligned}$$

$$\begin{aligned}
 \text{b Cost} &= \$85 \times 72 \\
 &= \$6120
 \end{aligned}$$

$$\begin{aligned}
 \text{c Actual area} &= 8.2 \text{ m} \times 9.4 \text{ m} \\
 &= 77.08 \text{ m}^2
 \end{aligned}$$

$$\begin{aligned}
 \text{d Percentage error} &= \frac{|V_A - V_E|}{V_E} \times 100\% \\
 &= \frac{|72 - 77.08|}{77.08} \times 100\% \\
 &= \frac{|-5.08|}{77.08} \times 100\% \\
 &\approx 6.59\%
 \end{aligned}$$

- e Exact area $77.08 \text{ m}^2 >$ approximate area 72 m^2
 \therefore there will not be enough grass to cover the courtyard.

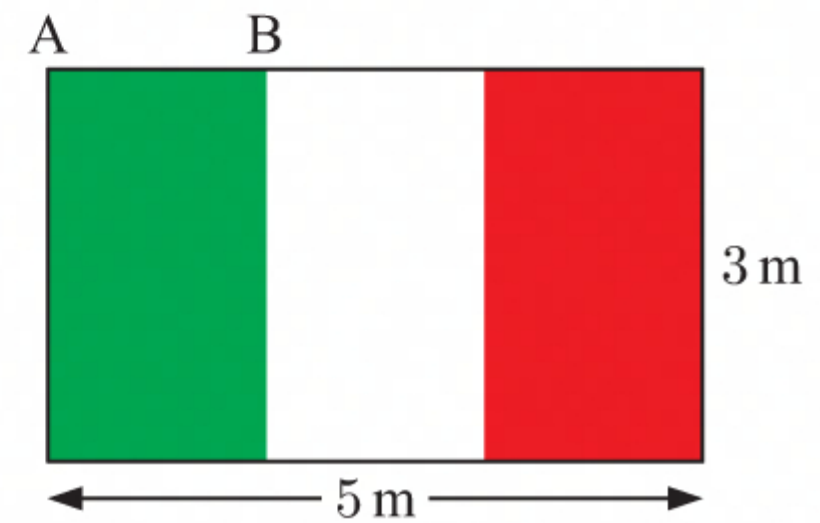
f Area $\approx 9 \text{ m} \times 10 \text{ m}$
 $\approx 90 \text{ m}^2$
 Cost $= \$85 \times 90$
 $= \$7650$

- 6 a Area of flag $= 5 \text{ m} \times 3 \text{ m} = 15 \text{ m}^2$
 \therefore area of green section $= \frac{1}{3} \times 15 \text{ m}^2 = 5 \text{ m}^2$

b $AB = \frac{5}{3} \approx 1.666 \dots \text{ cm}$
 $\approx 1.7 \text{ cm}$

c Area of green section $\approx 1.7 \text{ m} \times 3 \text{ m}$
 $\approx 5.1 \text{ m}^2$

d Percentage error $= \frac{|V_A - V_E|}{V_E} \times 100\%$
 $= \frac{|5.1 - 5|}{5} \times 100\%$
 $= \frac{|-0.1|}{5} \times 100\%$
 $= 2\%$



7 a Average speed $= \frac{\text{distance}}{\text{time}}$
 $= \frac{87 \text{ km}}{\frac{4}{3} \text{ hours}}$
 $= 65.25 \text{ km h}^{-1}$

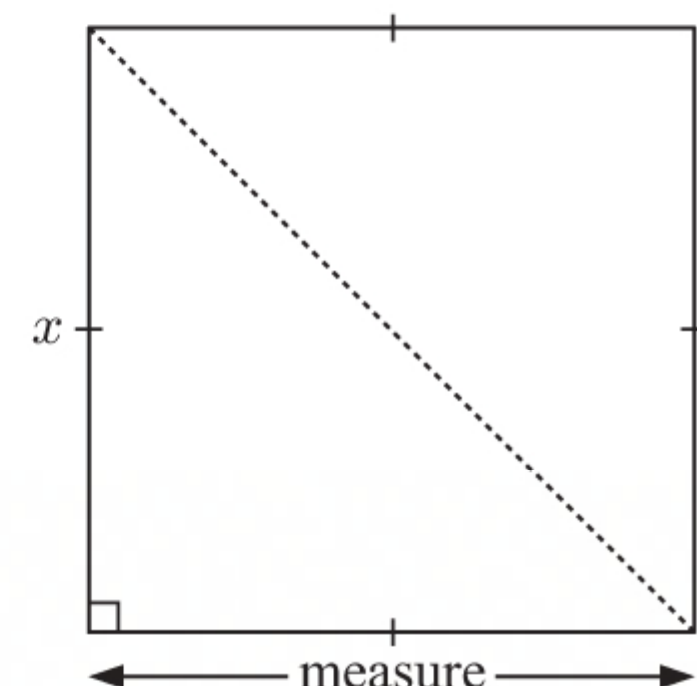
b Absolute error $= |V_A - V_E|$
 $= |70 - 65.25| \text{ km h}^{-1}$
 $= 4.75 \text{ km h}^{-1}$

percentage error $= \frac{|V_A - V_E|}{V_E} \times 100\%$
 $= \frac{4.75}{65.25} \times 100\%$
 $\approx 7.28\%$

- 8 a Let the square have side length x .

Length of diagonal $= \sqrt{x^2 + x^2}$ {Pythagoras}
 $= \sqrt{2x^2}$
 $= \sqrt{2} \times \sqrt{x^2}$
 $= \sqrt{2}x$ {as $x > 0$ }

\therefore the diagonal of a square is $\sqrt{2}$ times the measure of its side.



- b** We let the measure be 1, so that the length of the diagonal is $\sqrt{2}$.

$$\begin{aligned} \therefore \sqrt{2} &\approx 1 + \underbrace{\frac{1}{3}}_{\substack{\text{measure} \\ \text{increased} \\ \text{by its third}}} + \underbrace{\frac{1}{3 \times 4}}_{\substack{\text{increased} \\ \text{again by a} \\ \text{fourth of} \\ \text{this third}}} - \underbrace{\frac{1}{3 \times 4 \times 34}}_{\substack{\text{less the} \\ \text{thirtyfourth} \\ \text{part of that} \\ \text{fourth}}} \\ &\approx 1 + \frac{1}{3} + \frac{1}{12} - \frac{1}{408} \\ &\approx \frac{408}{408} + \frac{136}{408} + \frac{34}{408} - \frac{1}{408} \\ &\approx \frac{577}{408} \end{aligned}$$

c Percentage error = $\frac{|V_A - V_E|}{V_E} \times 100\%$

$$\begin{aligned} &\approx \frac{\left| \frac{577}{408} - \sqrt{2} \right|}{\sqrt{2}} \times 100\% \\ &\approx 1.50 \times 10^{-4} \% \end{aligned}$$

9 a Area $\approx 2.3 \text{ m} \times 1.4 \text{ m}$
 $\approx 3.22 \text{ m}^2$

- b** The length of the rectangle could be from 2.25 m to 2.35 m.
 The width of the rectangle could be from 1.35 m to 1.45 m.

\therefore the lower boundary of the area is $2.25 \times 1.35 = 3.0375 \text{ m}^2$
 and the upper boundary of the area is $2.35 \times 1.45 = 3.4075 \text{ m}^2$

c If the exact area V_E was 3.0375 m^2 , the

$$\begin{aligned} \text{percentage error} &= \frac{|V_A - V_E|}{V_E} \times 100\% \\ &= \frac{|3.22 - 3.0375|}{3.0375} \times 100\% \\ &\approx 6.01\% \end{aligned}$$

If the exact area V_E was 3.4075 m^2 , the

$$\begin{aligned} \text{percentage error} &= \frac{|V_A - V_E|}{V_E} \times 100\% \\ &= \frac{|3.22 - 3.4075|}{3.4075} \times 100\% \\ &\approx 5.50\% \end{aligned}$$

\therefore the maximum percentage error in the estimate is about 6.01%.

10 a Volume $\approx (\pi \times 4^2) \text{ cm}^2 \times 5 \text{ cm}$
 $\approx 80\pi \text{ cm}^3$
 $\approx 251 \text{ cm}^3$

- b** The radius of the can could be from 3.5 cm to 4.5 cm.
 The height of the can could be from 4.5 cm to 5.5 cm.

\therefore the lower boundary of the volume of the can is $\pi \times 3.5^2 \times 4.5 = 55.125\pi \text{ cm}^3$
 $\approx 173 \text{ cm}^3$

and the upper boundary of the volume of the can is $\pi \times 4.5^2 \times 5.5 = 111.375\pi \text{ cm}^3$
 $\approx 350 \text{ cm}^3$

$\therefore 173 \text{ cm}^3 < V < 350 \text{ cm}^3$

- c If the exact volume V_E was $55.125\pi \text{ cm}^3$,
the percentage error

$$\begin{aligned} &= \frac{|V_A - V_E|}{V_E} \times 100\% \\ &= \frac{|80\pi - 55.125\pi|}{55.125\pi} \times 100\% \\ &\approx 45.1\% \end{aligned}$$

\therefore the maximum percentage error in the estimate is about 45.1%.

- If the exact volume V_E was $111.375\pi \text{ cm}^3$,
the percentage error

$$\begin{aligned} &= \frac{|V_A - V_E|}{V_E} \times 100\% \\ &= \frac{|80\pi - 111.375\pi|}{111.375\pi} \times 100\% \\ &\approx 28.2\% \end{aligned}$$

11 a Time taken = $\frac{\text{distance}}{\text{speed}}$

$$\begin{aligned} &\approx \frac{250 \text{ km}}{56.8 \text{ km h}^{-1}} \\ &\approx \frac{625}{142} \text{ hours} \\ &\approx 4.40 \text{ hours } (\approx 4 \text{ h } 24 \text{ min } 5 \text{ s}) \end{aligned}$$

- b** The length of the trip could be from 249.5 km to 250.5 km.

The average speed for the trip could be from 56.75 km h^{-1} to 56.85 km h^{-1} .

\therefore the lower boundary of the time taken is $\frac{249.5}{56.85} = \frac{4990}{1137} \approx 4.39$ hours

and the upper boundary of the time taken is $\frac{250.5}{56.75} = \frac{1002}{227} \approx 4.41$ hours

- i** If the exact time V_E was $\frac{4990}{1137}$ hours,

the absolute error = $|V_A - V_E|$

$$\begin{aligned} &= \left| \frac{625}{142} - \frac{4990}{1137} \right| \\ &\approx 0.0127 \text{ hours} \end{aligned}$$

- If the exact time V_E was $\frac{1002}{227}$ hours,

the absolute error = $|V_A - V_E|$

$$\begin{aligned} &= \left| \frac{625}{142} - \frac{1002}{227} \right| \\ &\approx 0.0127 \text{ hours} \end{aligned}$$

\therefore the maximum absolute error is about 0.0127 hours, which is approximately 45.7 seconds.

- ii** If the exact time V_E was $\frac{4990}{1137}$ hours, the

percentage error = $\frac{|V_A - V_E|}{V_E} \times 100\%$

$$\begin{aligned} &= \frac{\left| \frac{625}{142} - \frac{4990}{1137} \right|}{\frac{4990}{1137}} \times 100\% \\ &\approx 0.289\% \end{aligned}$$

- If the exact time V_E was $\frac{1002}{227}$ hours, the

percentage error = $\frac{|V_A - V_E|}{V_E} \times 100\%$

$$\begin{aligned} &= \frac{\left| \frac{625}{142} - \frac{1002}{227} \right|}{\frac{1002}{227}} \times 100\% \\ &\approx 0.287\% \end{aligned}$$

\therefore the maximum percentage error in the estimate is about 0.289%.

REVIEW SET 1A

- 1
 - a $7423 \approx 7400$ {round down, as 2 is less than 5}
 - b $32\,191 \approx 32\,200$ {round up, as 9 is greater than 5}
 - c $10\,543 \approx 10\,500$ {round down, as 4 is less than 5}
 - d $408\,961 \approx 409\,000$ {round up, as 6 is greater than 5}

- 2 $e \approx 2.718\,281\,828\,459 \dots$
 - a $e \approx 2.72$ {round up, as 8 is greater than 5}
 - b $e \approx 2.718\,28$ {round down, as 1 is less than 5}
 - c $e \approx 2.718\,281\,83$ {round up, as 8 is greater than 5}

- 3
 - a $\sqrt{27} \approx 5.196\,152 \dots$
 ≈ 5.20 {round up, as 6 is greater than 5}
 - b $\frac{2.3 \times 9.4}{1.3} \approx 16.630\,769 \dots$
 ≈ 16.6 {round down, as 3 is less than 5}
 - c $0.307^3 \approx 0.028\,934 \dots$
 ≈ 0.0289 {round down, as 3 is less than 5}

- 4

<ol style="list-style-type: none"> a 47×7 $\approx 50 \times 7$ ≈ 350 	<ol style="list-style-type: none"> b 89×16 $\approx 90 \times 20$ ≈ 1800 	<ol style="list-style-type: none"> c $267 \div 48$ $\approx 300 \div 50$ ≈ 6
--	---	--

- 5
 - a $5877 \div 32$
 $\approx 6000 \div 30$
 ≈ 200
 - b The exact value of $5877 \div 32$ is *less* than our estimate, as in our one figure approximation we increased the numerator and decreased the denominator.

- 6 The first side length could be from 7.5 m to 8.5 m.
 The second side length could be from 11.5 m to 12.5 m.
 The third side length could be from 13.5 m to 14.5 m.
 \therefore the lower boundary of the perimeter P of the garden is $7.5 + 11.5 + 13.5 = 32.5$ m
 and the upper boundary of the perimeter P of the garden is $8.5 + 12.5 + 14.5 = 35.5$ m
 $\therefore 32.5 \text{ m} < P < 35.5 \text{ m}$

- 7
 - a The tape measure is accurate to $\pm \frac{1}{2}$ cm.
 - b The range of values is $36 \pm \frac{1}{2}$ cm.
 So, the possible values are in the range 35.5 cm to 36.5 cm.
 - c If the side length is 35.5 cm, then the area A is $35.5 \times 35.5 = 1260.25 \text{ cm}^2$
 If the side length is 36.5 cm, then the area A is $36.5 \times 36.5 = 1332.25 \text{ cm}^2$
 $\therefore 1260.25 \text{ cm}^2 < A < 1332.25 \text{ cm}^2$

$$\begin{aligned}
 \text{8 a Absolute error} &= |V_A - V_E| \\
 &= \$|2000 - 2590| \\
 &= \$|-590| \\
 &= \$590
 \end{aligned}
 \qquad
 \begin{aligned}
 \text{percentage error} &= \frac{|V_A - V_E|}{V_E} \times 100\% \\
 &= \frac{590}{2590} \times 100\% \\
 &\approx 22.8\%
 \end{aligned}$$

$$\begin{aligned}
 \text{b Absolute error} &= |V_A - V_E| \\
 &= |26 - 26.109| \text{ cm} \\
 &= |-0.109| \text{ cm} \\
 &= 0.109 \text{ cm}
 \end{aligned}
 \qquad
 \begin{aligned}
 \text{percentage error} &= \frac{|V_A - V_E|}{V_E} \times 100\% \\
 &= \frac{0.109}{26.109} \times 100\% \\
 &\approx 0.417\%
 \end{aligned}$$

$$\begin{aligned}
 \text{c Absolute error} &= |V_A - V_E| \\
 &= |4000 - 4386| \text{ people} \\
 &= |-386| \text{ people} \\
 &= 386 \text{ people}
 \end{aligned}
 \qquad
 \begin{aligned}
 \text{percentage error} &= \frac{|V_A - V_E|}{V_E} \times 100\% \\
 &= \frac{386}{4386} \times 100\% \\
 &\approx 8.80\%
 \end{aligned}$$

$$9 \quad V_A = 5.29 \text{ runs}, \quad V_E = \frac{37}{7} \text{ runs}$$

$$\begin{aligned}
 \text{Percentage error} &= \frac{|V_A - V_E|}{V_E} \times 100\% \\
 &= \frac{\left|5.29 - \frac{37}{7}\right|}{\frac{37}{7}} \times 100\% \\
 &\approx 0.0811\%
 \end{aligned}$$

$$10 \quad \text{a The radius of the gazebo is } \frac{2.8}{2} = 1.4 \text{ m}$$

$$\therefore \text{ the exact area of the gazebo is } \pi \times 1.4^2 = 1.96\pi \approx 6.16 \text{ m}^2$$

$$\begin{aligned}
 \text{b If the diameter 2.8 m is rounded up to 3 m, then the radius is estimated to be } \frac{3}{2} = 1.5 \text{ m.} \\
 \therefore \text{ the approximate area of the gazebo is } \pi \times 1.5^2 = 2.25\pi \approx 7.07 \text{ m}^2
 \end{aligned}$$

$$\begin{aligned}
 \text{c Absolute error} &= |V_A - V_E| \\
 &= |2.25\pi - 1.96\pi| \text{ m}^2 \\
 &= 0.29\pi \text{ m}^2 \\
 &\approx 0.911 \text{ m}^2
 \end{aligned}
 \qquad
 \begin{aligned}
 \text{percentage error} &= \frac{|V_A - V_E|}{V_E} \times 100\% \\
 &= \frac{0.29\pi}{1.96\pi} \times 100\% \\
 &\approx 14.8\%
 \end{aligned}$$

REVIEW SET 1B

- 1
 - a $74815 \approx 74820$ {5 is rounded up}
 - b $74815 \approx 74800$ {round down, as 1 is less than 5}
 - c $74815 \approx 75000$ {round up, as 8 is greater than 5}

$$2 \quad \frac{248 \text{ customers}}{13 \text{ days}} \approx 19.1 \text{ customers per day}$$

3 a $\sqrt{5.4 \times 7.6} \approx 6.406\,246\dots$
 ≈ 6.41 {round up, as 6 is greater than 5}

b $\sqrt{5.4 \times 7.6} \approx 6.406\,246\dots$
 ≈ 6.406 {round down, as 2 is less than 5}

4 a 28×74
 $\approx 30 \times 70$
 ≈ 2100

b 5.84×8.09
 $\approx 6 \times 8$
 ≈ 48

c $57.9 \div 23.5$
 $\approx 60 \div 20$
 ≈ 3

5 a Exact total cost = $88 \times \text{£}4.90$
 $= \text{£}431.20$

b Total cost $\approx 90 \times \text{£}5$
 $\approx \text{£}450$

c Percentage error = $\frac{|V_A - V_E|}{V_E} \times 100\%$
 $= \frac{|450 - 431.20|}{431.20} \times 100\%$
 $\approx 4.36\%$

6 a Dafne's time could be from 14.85 s to 14.95 s.
 $\therefore 14.85 \text{ s} < t < 14.95 \text{ s}$

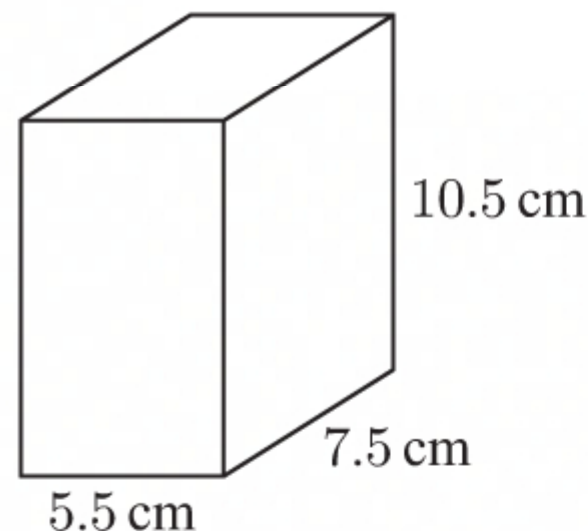
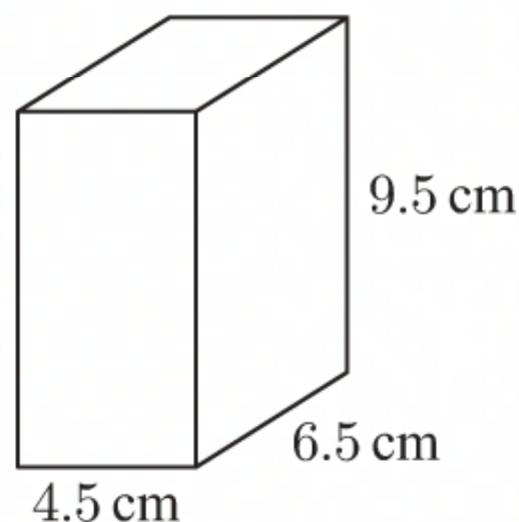
b Dafne's time could be from 14.85 s to 14.95 s.

\therefore the lower boundary of Dafne's speed s is $\frac{100}{14.95} \approx 6.69 \text{ m s}^{-1}$

and the upper boundary of Dafne's speed s is $\frac{100}{14.85} \approx 6.73 \text{ m s}^{-1}$

$\therefore 6.69 \text{ m s}^{-1} < s < 6.73 \text{ m s}^{-1}$

- 7** The length of the box could be from 4.5 cm to 5.5 cm.
 The width of the box could be from 6.5 cm to 7.5 cm.
 The height of the box could be from 9.5 cm to 10.5 cm.

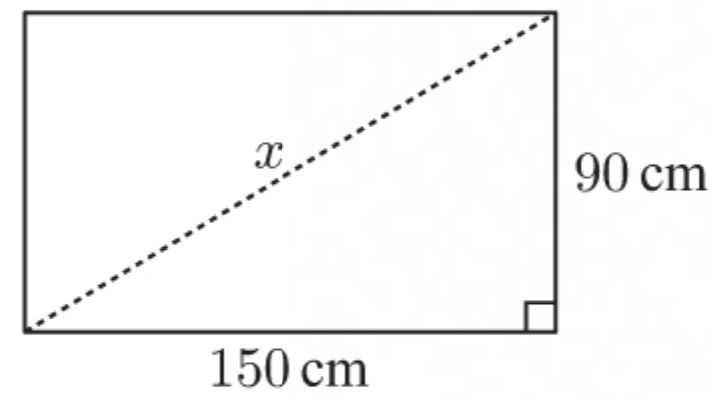


lower boundary for the surface area
 $= 2 \times (4.5 \times 6.5) + 2 \times (6.5 \times 9.5)$
 $+ 2 \times (4.5 \times 9.5)$
 $= 267.5 \text{ cm}^2$

$\therefore 267.5 \text{ cm}^2 < A < 355.5 \text{ cm}^2$

upper boundary for the surface area
 $= 2 \times (5.5 \times 7.5) + 2 \times (7.5 \times 10.5)$
 $+ 2 \times (5.5 \times 10.5)$
 $= 355.5 \text{ cm}^2$

$$\begin{aligned}
 8 \quad a \quad x^2 &= 150^2 + 90^2 \quad \{\text{Pythagoras}\} \\
 \therefore x &= \sqrt{150^2 + 90^2} \quad \{\text{as } x > 0\} \\
 &= \sqrt{30\,600} \\
 &= 30\sqrt{34} \\
 &\approx 175
 \end{aligned}$$



\therefore the length of the diagonal of the screen is approximately 175 cm.

$$\begin{aligned}
 b \quad \text{Absolute error} &= |V_A - V_E| \\
 &= |30\sqrt{34} - 177.8| \text{ cm} \\
 &\approx 2.87 \text{ cm} \\
 \text{percentage error} &= \frac{|V_A - V_E|}{V_E} \times 100\% \\
 &= \frac{|30\sqrt{34} - 177.8|}{177.8} \times 100\% \\
 &\approx 1.61\%
 \end{aligned}$$

$$9 \quad a \quad i \quad \sqrt{5} \text{ m} \approx 2 \text{ m} \qquad ii \quad \sqrt{5} \text{ m} \approx 2.24 \text{ m} \approx 224 \text{ cm}$$

$$iii \quad \sqrt{5} \text{ m} \approx 2.236 \text{ m} \approx 2236 \text{ mm}$$

$$b \quad i \quad 1 \text{ significant figure} \qquad ii \quad 3 \text{ significant figures} \qquad iii \quad 4 \text{ significant figures}$$

$$c \quad \text{Percentage error} = \frac{|V_A - V_E|}{V_E} \times 100\%$$

$$\begin{aligned}
 \text{percentage error for nearest metre} &= \frac{|2 - \sqrt{5}|}{\sqrt{5}} \times 100\% \\
 &\approx 10.6\%
 \end{aligned}$$

$$\begin{aligned}
 \text{percentage error for nearest centimetre} &= \frac{|2.24 - \sqrt{5}|}{\sqrt{5}} \times 100\% \\
 &\approx 0.176\%
 \end{aligned}$$

$$\begin{aligned}
 \text{percentage error for nearest millimetre} &= \frac{|2.236 - \sqrt{5}|}{\sqrt{5}} \times 100\% \\
 &\approx 0.003\,04\%
 \end{aligned}$$

The nearest centimetre length, 2.24 m, and the nearest millimetre length, 2.236 m, satisfy the architect's requirements.

$$\begin{aligned}
 10 \quad a \quad \text{Percentage error} &= \frac{|V_A - V_E|}{V_E} \times 100\% \\
 &= \frac{|3 - \pi|}{\pi} \times 100\% \\
 &\approx 4.51\%
 \end{aligned}$$

$$\begin{aligned}
 b \quad \text{Percentage error} &= \frac{|V_A - V_E|}{V_E} \times 100\% \\
 &= \frac{|3.1 - \pi|}{\pi} \times 100\% \\
 &\approx 1.32\%
 \end{aligned}$$

$$\begin{aligned}
 c \quad \text{Percentage error} &= \frac{|V_A - V_E|}{V_E} \times 100\% \\
 &= \frac{|3.14 - \pi|}{\pi} \times 100\% \\
 &\approx 0.0507\%
 \end{aligned}$$

$$\begin{aligned}
 d \quad \text{Percentage error} &= \frac{|V_A - V_E|}{V_E} \times 100\% \\
 &= \frac{\left| \frac{22}{7} - \pi \right|}{\pi} \times 100\% \\
 &\approx 0.0402\%
 \end{aligned}$$

$$\begin{aligned}
 \text{Percentage error} &= \frac{|V_A - V_E|}{V_E} \times 100\% \\
 &= \frac{\left| \frac{355}{113} - \pi \right|}{\pi} \times 100\% \\
 &\approx 8.49 \times 10^{-6} \%
 \end{aligned}$$

11 a Area $\approx \pi \times 3.5^2 \approx 12.25\pi \approx 38.5 \text{ cm}^2$

b The radius of the circle could be from 3.45 cm to 3.55 cm.

\therefore the lower bound of the area A is $\pi \times 3.45^2 = 11.9025\pi \approx 37.4 \text{ cm}^2$

and the upper bound of the area A is $\pi \times 3.55^2 = 12.6025\pi \approx 39.6 \text{ cm}^2$

$\therefore 37.4 \text{ cm}^2 < A < 39.6 \text{ cm}^2$

c If the exact area V_E was $11.9025\pi \text{ cm}^2$,
the percentage error

$$\begin{aligned}
 &= \frac{|V_A - V_E|}{V_E} \times 100\% \\
 &= \frac{|12.25\pi - 11.9025\pi|}{11.9025\pi} \times 100\% \\
 &\approx 2.92\%
 \end{aligned}$$

If the exact time V_E was $12.6025\pi \text{ cm}^2$,
the percentage error

$$\begin{aligned}
 &= \frac{|V_A - V_E|}{V_E} \times 100\% \\
 &= \frac{|12.25\pi - 12.6025\pi|}{12.6025\pi} \times 100\% \\
 &\approx 2.80\%
 \end{aligned}$$

\therefore the maximum percentage error in the estimate is about 2.92%.

Chapter 2

LOANS AND ANNUITIES

EXERCISE 2A

- 1 a $N = 5 \times 12 = 60$, $I\% = 6$, $PV = 12\,000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$

A financial calculator screen with the following inputs:
n = 60
I% = 6
PV = 12000
PMT = -231.9936184 (highlighted)
FV = 0
P/Y = 12
The screen also shows buttons for n, I%, PV, PMT, FV, and AMORTZN at the bottom.

$$\therefore PMT \approx -232.00$$

The monthly repayment is \$232.

- b Total repayment = monthly repayment \times number of months
 $= \$232 \times 60$
 $= \$13\,920$
- c Interest = total repayment – amount borrowed
 $= \$13\,920 - \$12\,000$
 $= \$1\,920$

- 2 a $N = 3 \times 12 = 36$, $I\% = 4.5$, $PV = 9500$, $FV = 0$, $P/Y = 12$, $C/Y = 12$

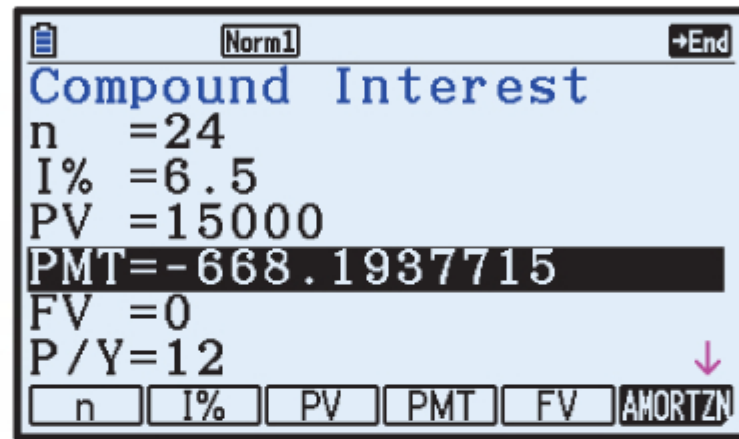
A financial calculator screen with the following inputs:
n = 36
I% = 4.5
PV = 9500
PMT = -282.5957825 (highlighted)
FV = 0
P/Y = 12
The screen also shows buttons for n, I%, PV, PMT, FV, and AMORTZN at the bottom.

$$\therefore PMT \approx -282.60$$

The monthly repayment is £282.60.

- b Total repayment = monthly repayment \times number of months
 $= £282.60 \times 36$
 $= £10\,173.60$
- c Interest = total repayment – amount borrowed
 $= £10\,173.60 - £9500$
 $= £673.60$

- 3 a** $N = 2 \times 12 = 24$, $I\% = 6.5$, $PV = 15\,000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$



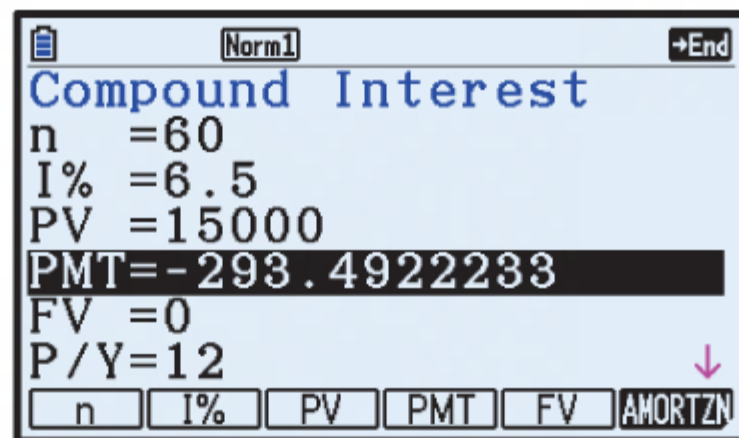
$$\therefore PMT \approx -668.20$$

The monthly repayment is \$668.20.

$$\begin{aligned} \text{Total repayment} &= \text{monthly repayment} \times \text{number of months} \\ &= \$668.20 \times 24 \\ &= \$16\,036.80 \end{aligned}$$

$$\begin{aligned} \text{Interest} &= \text{total repayment} - \text{amount borrowed} \\ &= \$16\,036.80 - \$15\,000 \\ &= \$1\,036.80 \end{aligned}$$

- b** $N = 5 \times 12 = 60$, $I\% = 6.5$, $PV = 15\,000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$



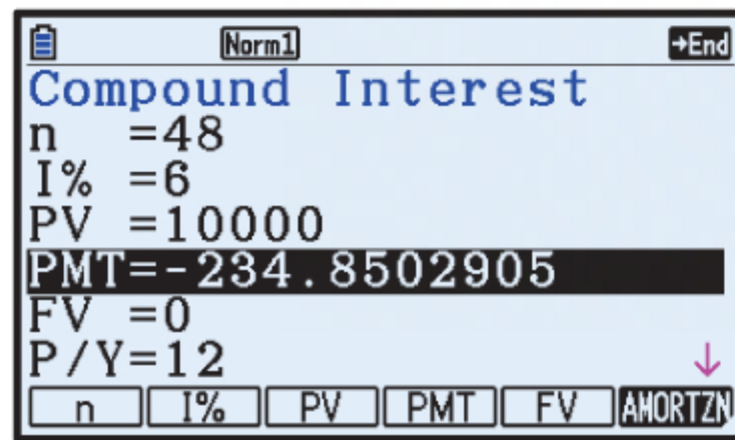
$$\therefore PMT \approx -293.50$$

The monthly repayment is \$293.50.

$$\begin{aligned} \text{Total repayment} &= \text{monthly repayment} \times \text{number of months} \\ &= \$293.50 \times 60 \\ &= \$17\,610 \end{aligned}$$

$$\begin{aligned} \text{Interest} &= \text{total repayment} - \text{amount borrowed} \\ &= \$17\,610 - \$15\,000 \\ &= \$2\,610 \end{aligned}$$

4 $N = 4 \times 12 = 48$, $I\% = 6$, $PV = 10\,000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$



$\therefore PMT \approx -234.86$

The monthly repayment is €234.86.

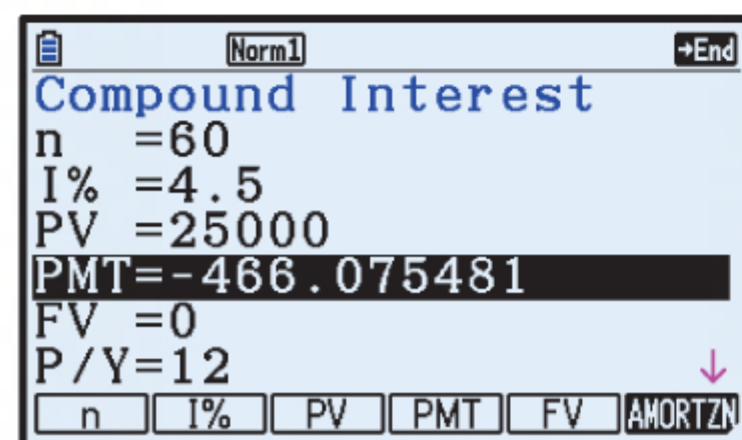
$$\begin{aligned} \text{Total repayment} &= \text{monthly repayment} \times \text{number of months} \\ &= €234.86 \times 48 \\ &= €11\,273.28 \end{aligned}$$

$$\begin{aligned} \text{Interest} &= \text{total repayment} - \text{amount borrowed} \\ &= €11\,273.28 - €10\,000 \\ &= €1273.28 \end{aligned}$$

$$\begin{aligned} \text{Total cost of taking out the loan} &= \text{interest} + \text{application fee} + \text{monthly service fee} \times \text{number of months} \\ &= €1273.28 + €150 + €10 \times 48 \\ &= €1903.28 \end{aligned}$$

5 a *Balance Bank:*

$N = 5 \times 12 = 60$, $I\% = 4.5$, $PV = 25\,000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$

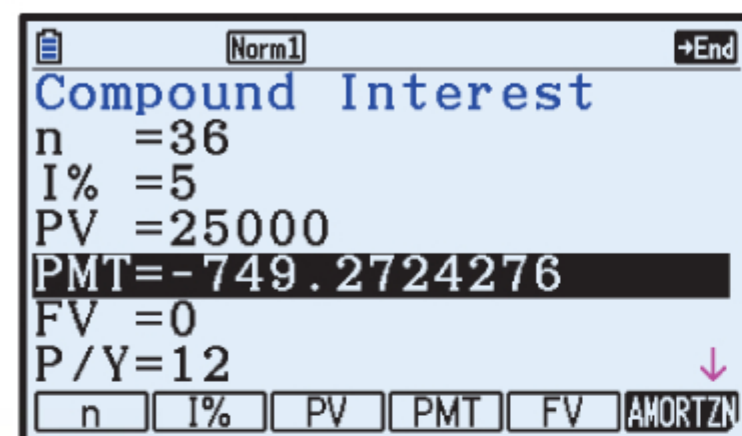


$\therefore PMT \approx -466.08$

The monthly repayment is \$466.08.

Cash Credit Union:

$N = 3 \times 12 = 36$, $I\% = 5$, $PV = 25\,000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$



$\therefore PMT \approx -749.28$

The monthly repayment is \$749.28.

Balance Bank has the smaller monthly repayments.

b *Balance Bank:*

$$\begin{aligned}
 \text{Total repayment} &= \text{monthly repayment} \times \text{number of months} \\
 &= \$466.08 \times 60 \\
 &= \$27\,964.80
 \end{aligned}$$

$$\begin{aligned}
 \text{Interest} &= \text{total repayment} - \text{amount borrowed} \\
 &= \$27\,964.80 - \$25\,000 \\
 &= \$2\,964.80
 \end{aligned}$$

Cash Credit Union:

$$\begin{aligned}
 \text{Total repayment} &= \text{monthly repayment} \times \text{number of months} \\
 &= \$749.28 \times 36 \\
 &= \$26\,974.08
 \end{aligned}$$

$$\begin{aligned}
 \text{Interest} &= \text{total repayment} - \text{amount borrowed} \\
 &= \$26\,974.08 - \$25\,000 \\
 &= \$1\,974.08
 \end{aligned}$$

The Cash Credit Union loan charges less interest in total.

- c** Provided Becky is able to afford the larger repayments, she should choose the Cash Credit Union loan as she will pay \$990.72 less interest overall.

6

	A	B	C	D	E
1	LOAN SPREADSHEET				
2					
3	Loan amount	\$30,000.00			
4	Number of years	5			
5	Rate p.a.	8.00%			
6	Periods p.a.	12			
7	Rate per period	0.667%			
8	Repayment	\$608.30			
9					
10	Month	Amount	Interest	Repayment	Balance
11	1	\$30,000.00	\$200.00	\$608.30	\$29,591.70
12	2	\$29,591.70	\$197.28	\$608.30	\$29,180.68
13	3	\$29,180.68	\$194.54	\$608.30	\$28,766.92
14	4	\$28,766.92	\$191.78	\$608.30	\$28,350.40
15	5	\$28,350.40	\$189.00	\$608.30	\$27,931.10
16	6	\$27,931.10	\$186.21	\$608.30	\$27,509.01

a $N = 5 \times 12 = 60$, $I\% = 8$, $PV = 30\,000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$

Norm1	+End
Compound Interest	
n	=60
I%	=8
PV	=30000
PMT	=-608.2918287
FV	=0
P/Y	=12
n	I% PV PMT FV AMORTZN

$$\therefore PMT \approx -608.30$$

The monthly repayment is \$608.30 which is the same as that given in the spreadsheet.

- b** From the spreadsheet, the account balance after six months is \$27 509.01.

- c i From the spreadsheet, \$200 is paid in interest in month 1.
 - ii From the spreadsheet, \$186.21 is paid in interest in month 6.
- The balance of the loan is less in month 6 which means the interest paid will also be less.
- d i Using the spreadsheet given, \$4.02 is paid in interest in the 60th month.
 - ii Using the spreadsheet given, \$6497.40 is paid in interest in total (or \$6498 using technology).
- e The monthly repayment was rounded up so every month the payments have reduced the balance by a little extra which means that the final payment is slightly less.

7 a $N = 2 \times 26 = 52$, $I\% = 9.9$, $PV = 7000$, $FV = 0$, $P/Y = 26$, $C/Y = 26$

Norm1 +End
Compound Interest
n =52
I% =9.9
PV =7000
PMT=-148.6371002
FV =0
P/Y=26
↓
n I% PV PMT FV AMORTZN

$\therefore PMT \approx -148.64$

The fortnightly repayment is \$148.64.

- b Total interest = total repayment – starting principal
- $$= \$148.64 \times 52 - \$7000$$
- $$= \$7729.28 - \$7000$$
- $$= \$729.28$$

8 a $N = 4 \times 12 = 48$, $I\% = 8.25$, $PV = 20\,000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$

Norm1 +End
Compound Interest
n =48
I% =8.25
PV =20000
PMT=-490.6088587
FV =0
P/Y=12
↓
n I% PV PMT FV AMORTZN

$\therefore PMT \approx -490.61$

The monthly repayment is £490.61.

b $N = 1 \times 12 = 12$, $I\% = 8.25$, $PV = 20\,000$, $PMT = -490.61$, $P/Y = 12$, $C/Y = 12$

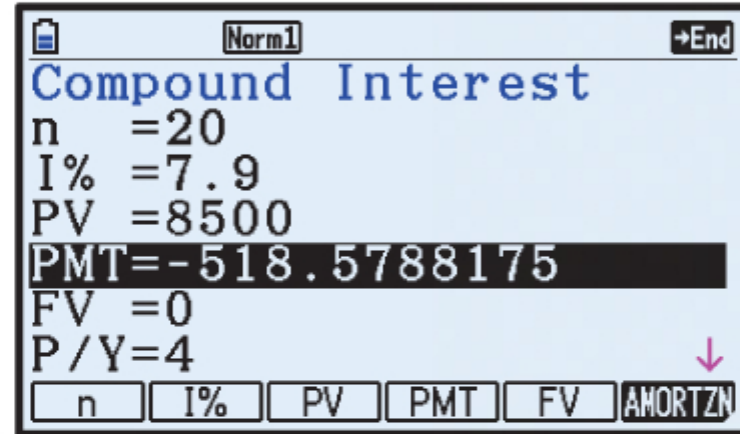
Norm1 +End
Compound Interest
n =12
I% =8.25
PV =20000
PMT=-490.61
FV=-15598.72712
P/Y=12
↓
n I% PV PMT FV AMORTZN

$\therefore FV \approx -15\,598.73$

The outstanding balance on the loan after 1 year is £15 598.73.

9 a Jacob will borrow $\$12\,000 - \$3\,500 = \$8\,500$.

b i $N = 5 \times 4 = 20$, $I\% = 7.9$, $PV = 8500$, $FV = 0$, $P/Y = 4$, $C/Y = 4$

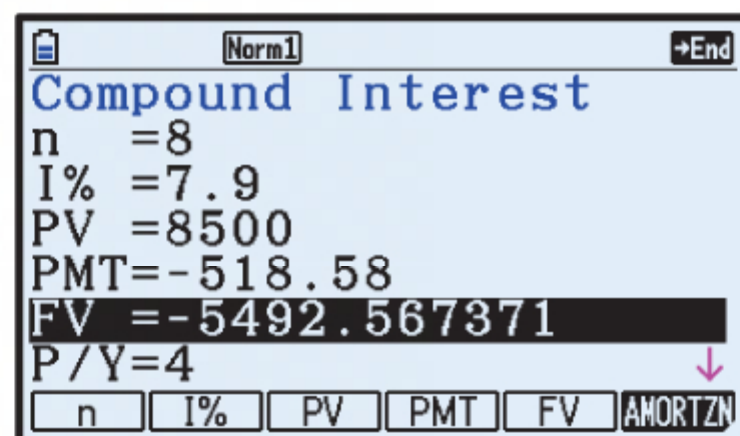


$$\therefore PMT \approx -518.58$$

The quarterly repayment is \$518.58.

ii Total interest = total repayment – starting principal
 $= \$518.58 \times 20 - \$8\,500$
 $= \$10\,371.60 - \$8\,500$
 $= \$1\,871.60$

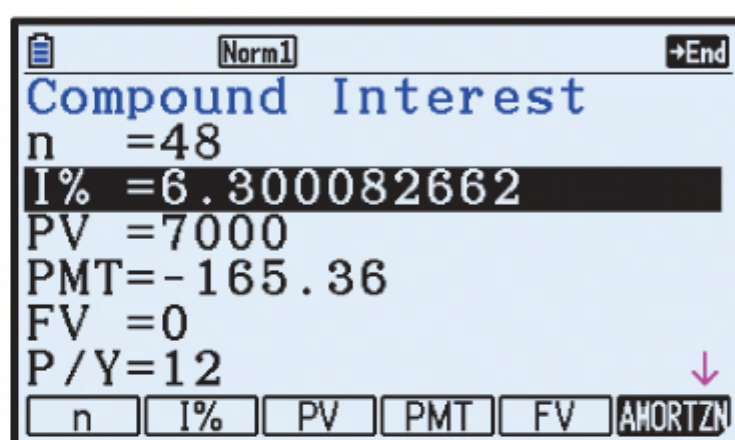
iii $N = 2 \times 4 = 8$, $I\% = 7.9$, $PV = 8500$, $PMT = -518.58$, $P/Y = 4$, $C/Y = 4$



$$\therefore FV \approx -5492.57$$

The outstanding balance on the loan after 2 years is \$5492.57.

10 a $N = 4 \times 12 = 48$, $PV = 7000$, $FV = 0$, $PMT = -165.36$, $P/Y = 12$, $C/Y = 12$



$$\therefore I\% \approx 6.30$$

So, the annual interest rate is 6.30% p.a.

- b** $N = 1 \times 12 = 12$, $I\% = 6.30$, $PV = 7000$, $PMT = -165.36$, $P/Y = 12$, $C/Y = 12$

Norm1 +End
Compound Interest
n =12
I% =6.3
PV =7000
PMT=-165.36
FV=-5411.327486
P/Y=12
n I% PV PMT FV AMORTZ

$$\therefore FV \approx -5411.33$$

So, the outstanding balance on the loan after 1 year is \$5411.33.

\therefore Simon has paid off $\$7000 - \$5411.33 = \$1588.67$ of the loan after 1 year.

$$\begin{aligned} \text{Total interest paid in first year} &= \text{total repayment} - \text{amount paid off in first year} \\ &= \$165.36 \times 12 - \$1588.67 \\ &= \$1984.32 - \$1588.67 \\ &= \$395.65 \end{aligned}$$

- 11 a i** $N = 3 \times 12 = 36$, $I\% = 8.5$, $PV = 25\,000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$

Norm1 +End
Compound Interest
n =36
I% =8.5
PV =25000
PMT=-789.1884356
FV =0
P/Y=12
n I% PV PMT FV AMORTZ

$$\therefore PMT \approx -789.19$$

So, the monthly repayment is \$789.19.

- ii** $N = 5 \times 12 = 60$, $I\% = 8.5$, $PV = 25\,000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$

Norm1 +End
Compound Interest
n =60
I% =8.5
PV =25000
PMT=-512.9132832
FV =0
P/Y=12
n I% PV PMT FV AMORTZ

$$\therefore PMT \approx -512.92$$

So, the monthly repayment is \$512.92.

- iii** $N = 7 \times 12 = 84$, $I\% = 8.5$, $PV = 25\,000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$

Norm1 +End
Compound Interest
n =84
I% =8.5
PV =25000
PMT=-395.9121359
FV =0
P/Y=12
n I% PV PMT FV AMORTZ

$$\therefore PMT \approx -395.92$$

So, the monthly repayment is \$395.92.

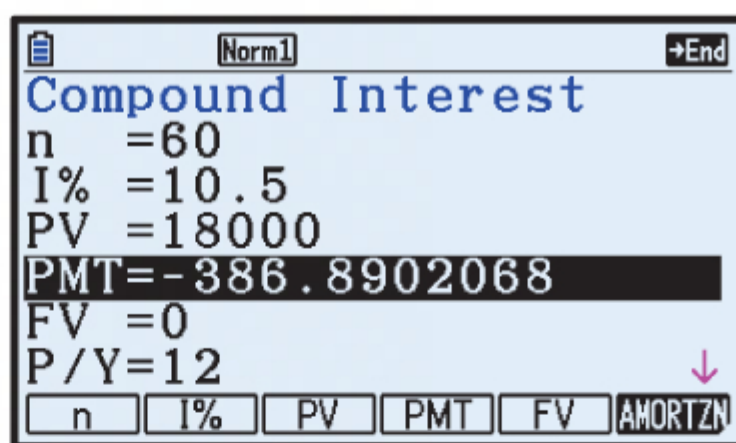
$$\begin{aligned}
 \text{b Total interest charged for 3 year loan} &= \text{total repayment} - \text{starting principal} \\
 &= \$789.19 \times 36 - \$25\,000 \\
 &= \$28\,410.84 - \$25\,000 \\
 &= \$3410.84
 \end{aligned}$$

$$\begin{aligned}
 \text{Total interest charged for 5 year loan} &= \text{total repayment} - \text{starting principal} \\
 &= \$512.92 \times 60 - \$25\,000 \\
 &= \$30\,775.20 - \$25\,000 \\
 &= \$5775.20
 \end{aligned}$$

$$\begin{aligned}
 \text{Total interest charged for 7 year loan} &= \text{total repayment} - \text{starting principal} \\
 &= \$395.92 \times 84 - \$25\,000 \\
 &= \$33\,257.28 - \$25\,000 \\
 &= \$8257.28
 \end{aligned}$$

The 3 year loan charges the least interest of \$3410.84 as more is paid off the balance each month and therefore less interest is charged overall.

12 a $N = 5 \times 12 = 60$, $I\% = 10.5$, $PV = 18\,000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$

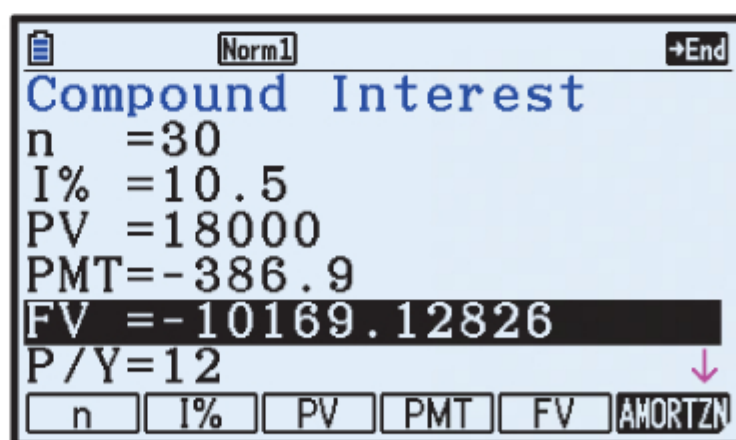


$$\therefore PMT \approx -386.90$$

So, the monthly repayment is €386.90.

$$\begin{aligned}
 \text{b Total interest} &= \text{total repayment} - \text{starting principal} \\
 &= €386.90 \times 60 - €18\,000 \\
 &= €23\,214 - €18\,000 \\
 &= €5214
 \end{aligned}$$

c $N = 2.5 \times 12 = 30$, $I\% = 10.5$, $PV = 18\,000$, $PMT = -386.90$, $P/Y = 12$, $C/Y = 12$



$$\therefore FV \approx -10\,169.13$$

So, the outstanding balance on the loan after $2\frac{1}{2}$ years is €10 169.13.

- d** Ally pays more interest in the first $2\frac{1}{2}$ years than in the second $2\frac{1}{2}$ years, so she still has more than half the loan to pay off when half the loan period has passed.

- 13 a** $N = 20 \times 12 = 240$, $I\% = 6.25$, $PV = 250\,000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$

Norm1 End
Compound Interest
n = 240
I% = 6.25
PV = 250000
PMT = -1827.320506
FV = 0
P/Y = 12
↓
n I% PV PMT FV AMORTZN

$$\therefore PMT \approx -1827.33$$

So, the monthly repayment is \$1827.33.

- b** Total interest = total repayment – principal
 $= \$1827.33 \times 240 - \$250\,000$
 $= \$438\,559.20 - \$250\,000$
 $= \$188\,559.20$

- c** $N = 10 \times 12 = 120$, $I\% = 6.25$, $PV = 250\,000$, $PMT = -1827.33$, $P/Y = 12$, $C/Y = 12$

Norm1 End
Compound Interest
n = 120
I% = 6.25
PV = 250000
PMT = -1827.33
FV = -162745.0322
P/Y = 12
↓
n I% PV PMT FV AMORTZN

$$\therefore FV \approx -162\,745.03$$

The outstanding balance on the loan after 10 years is \$162 745.03.

- d i** $N = 5 \times 12 = 60$, $I\% = 6.25$, $PV = 162\,745.03$, $FV = 0$, $P/Y = 12$, $C/Y = 12$

Norm1 End
Compound Interest
n = 60
I% = 6.25
PV = 162745.03
PMT = -3165.270676
FV = 0
P/Y = 12
↓
n I% PV PMT FV AMORTZN

$$\therefore PMT \approx -3165.28$$

The monthly repayment for the next 5 years is \$3165.28.

- ii** Total interest
 $= (\text{total repayment in first 10 years} - \text{principal paid in first 10 years})$
 $+ (\text{total repayment in last 5 years} - \text{principal paid in last 5 years})$
 $= (\$1827.33 \times 120 - (\$250\,000 - \$162\,745.03)) + (\$3165.28 \times 60 - \$162\,745.03)$
 $= \$132\,024.63 + \$27\,171.77$
 $= \$159\,196.40$
- iii** Total interest saved = $\$188\,559.20 - \$159\,196.40$
 $= \$29\,362.80$

ACTIVITY 1

At the time of writing, the interest rates $\approx 2.5\%$.

1 Monthly repayments

	4%	6%	8%	10%	12%	2.5%
5 years	\$7366.61	\$7733.13	\$8110.56	\$8498.82	\$8897.78	\$7098.95
10 years	\$4049.81	\$4440.83	\$4853.11	\$5286.03	\$5738.84	\$3770.80
15 years	\$2958.76	\$3375.43	\$3822.61	\$4298.43	\$4800.68	\$2667.16
20 years	\$2423.93	\$2865.73	\$3345.77	\$3860.09	\$4404.35	\$2119.62

Total interest paid

	4%	6%	8%	10%	12%	2.5%
5 years	\$41 966.60	\$63 987.80	\$86 633.60	\$109 929.20	\$133 866.80	\$25 937.00
10 years	\$85 977.20	\$132 899.60	\$182 373.20	\$234 323.60	\$288 660.80	\$52 496.00
15 years	\$132 576.80	\$207 577.40	\$288 069.80	\$373 717.40	\$464 122.40	\$80 088.80
20 years	\$181 743.20	\$287 775.20	\$402 984.80	\$526 421.60	\$657 044.00	\$108 708.80

- 2** A higher interest rate will increase the total amount of interest paid over the course of a loan. Paying off a loan over a shorter time period will decrease the total interest paid, sometimes even when the interest rate is higher. For example, in the table above we can see that less interest is paid at 10% over 5 years than at 6% over 10 years. However, we have to consider whether or not a family can meet their regular payments since higher interest rates and shorter loan periods will increase the necessary regular repayment.

ACTIVITY 2**THE LOAN REPAYMENTS FORMULA**

- 1 a** The balance after 2 repayment periods $= PV(1+i)^2 - p(1+i) - p$
 The balance after 3 repayment periods $= [PV(1+i)^2 - p(1+i) - p](1+i) - p$
 $= PV(1+i)^3 - p(1+i)^2 - p(1+i) - p$
- b** The balance after 4 repayment periods $= [PV(1+i)^3 - p(1+i)^2 - p(1+i) - p](1+i) - p$
 $= PV(1+i)^4 - p(1+i)^3 - p(1+i)^2 - p(1+i) - p$

2 $PV(1+i)^n - p \frac{(1+i)^n - 1}{i} = 0$

$$\therefore p \frac{(1+i)^n - 1}{i} = PV(1+i)^n$$

$$\therefore p = PV(1+i)^n \times \frac{i}{(1+i)^n - 1}$$

$$\therefore p = \frac{PV \times i \times (1+i)^n}{(1+i)^n - 1}$$

$$3 \quad PV = 16\,500, \quad i = \frac{0.055}{12}, \quad n = 4 \times 12 = 48$$

$$\therefore p = \frac{16\,500 \times \frac{0.055}{12} \times \left(1 + \frac{0.055}{12}\right)^{48}}{\left(1 + \frac{0.055}{12}\right)^{48} - 1} \\ \approx 383.74 \quad \{\text{rounded up}\}$$

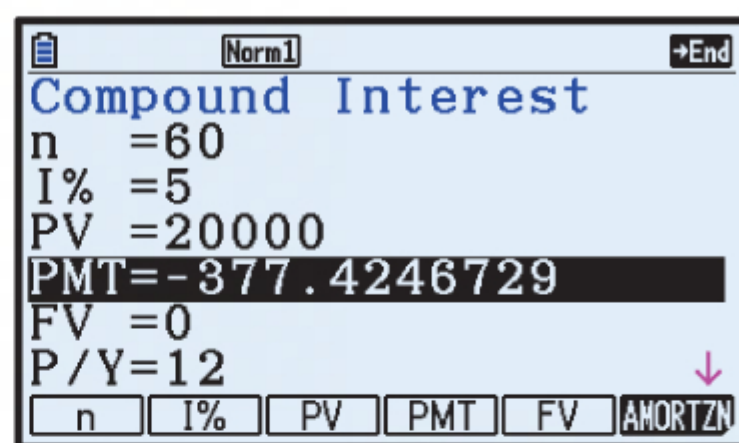
This agrees with our answer to **Example 1**.

$$5 \quad PV = 20\,000, \quad i = \frac{0.05}{12}, \quad n = 5 \times 12 = 60$$

$$\therefore p = \frac{20\,000 \times \frac{0.05}{12} \times \left(1 + \frac{0.05}{12}\right)^{60}}{\left(1 + \frac{0.05}{12}\right)^{60} - 1} \\ \approx 377.43 \quad \{\text{rounded up}\}$$

Using technology,

$$N = 5 \times 12 = 60, \quad I\% = 5, \quad PV = 20\,000, \quad FV = 0, \quad P/Y = 12, \quad C/Y = 12$$



$$\therefore PMT \approx -377.43$$

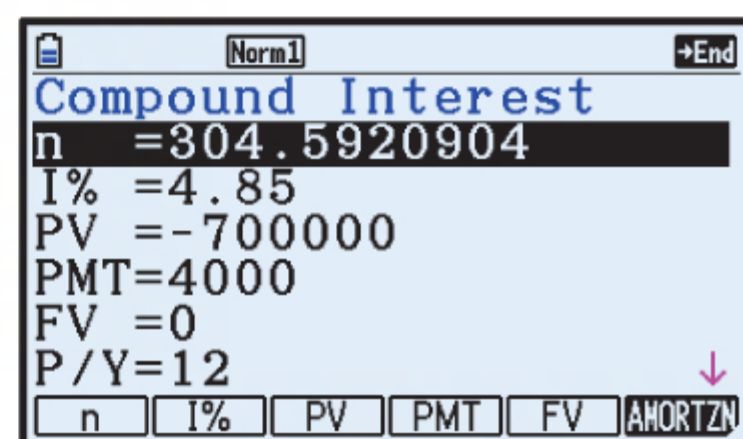
The monthly repayment is \$377.43.

- 7 When we know the formula and can implement it in a spreadsheet, we have greater flexibility in our calculations, and we can perform calculations very quickly.

EXERCISE 2B

- 1 a We need to find how long it will take for the future value to fall to \$0.

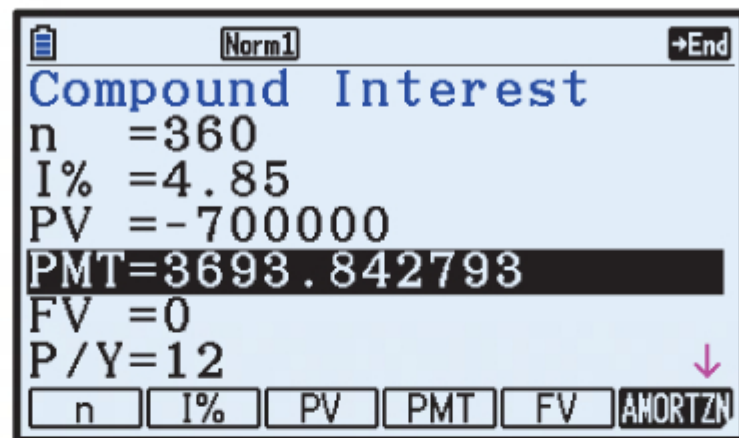
$$I\% = 4.85, \quad PV = -700\,000, \quad PMT = 4000, \quad FV = 0, \quad P/Y = 12, \quad C/Y = 12$$



$$\therefore N \approx 305$$

Sue will be able to withdraw \$4000 per month for 304 months, and then less in the 305th month (after 25 years 5 months).

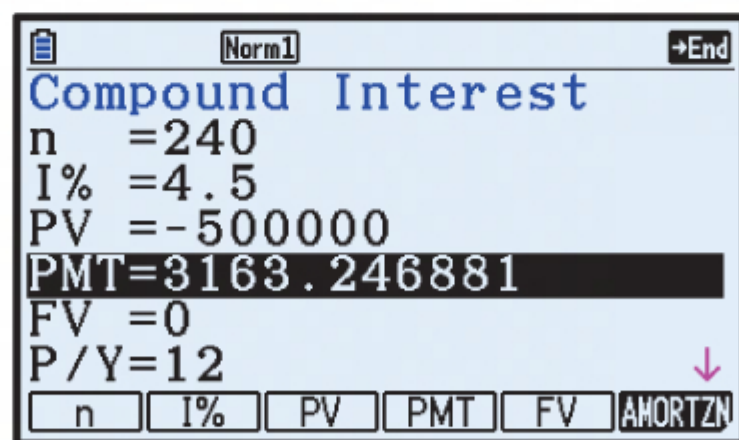
- b** $N = 30 \times 12 = 360$, $I\% = 4.85$, $PV = -700\,000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$



$$\therefore PMT \approx 3693.84$$

Sue can afford to withdraw \$3693.84 each month.

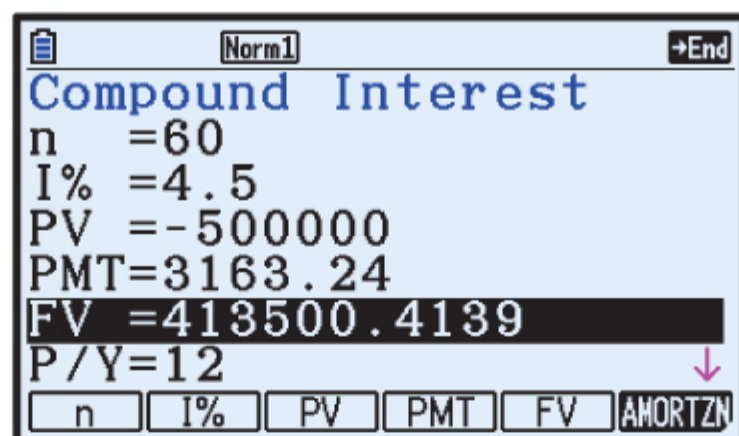
- 2 a** $N = 20 \times 12 = 240$, $I\% = 4.5$, $PV = -500\,000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$



$$\therefore PMT \approx 3163.24$$

Celia can afford to withdraw €3163.24 each month.

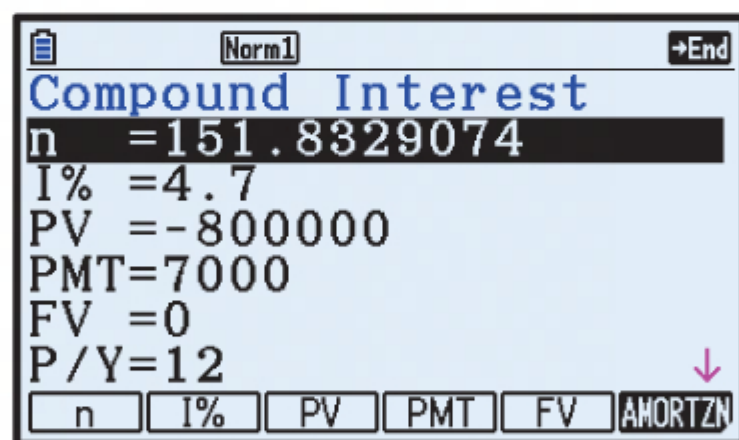
- b** $N = 5 \times 12 = 60$, $I\% = 4.5$, $PV = -500\,000$, $PMT = 3163.24$, $P/Y = 12$, $C/Y = 12$



$$\therefore FV \approx 413\,500.41$$

After 5 years, the outstanding balance of Celia's fund will be €413 500.41.

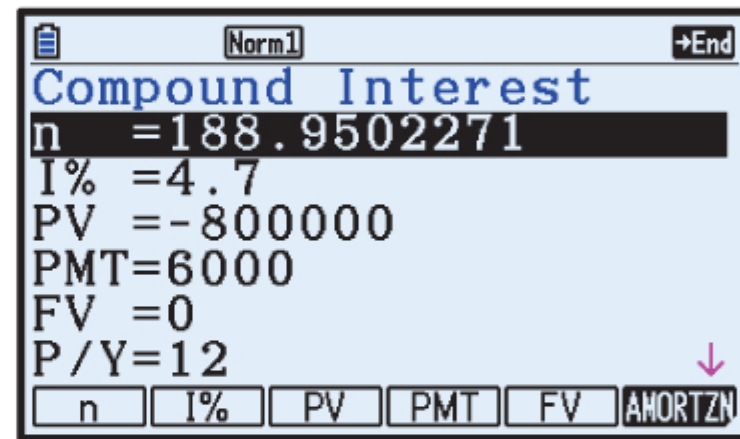
- 3 a** We need to find how long it will take for the future value to fall to \$0.
 $I\% = 4.7$, $PV = -800\,000$, $PMT = 7000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$



$$\therefore N \approx 152$$

Terence will be able to withdraw \$7000 per month for 151 months, and then less in the 152nd month (after 12 years 8 months).

- b** $I\% = 4.7$, $PV = -800\,000$, $PMT = 6000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$

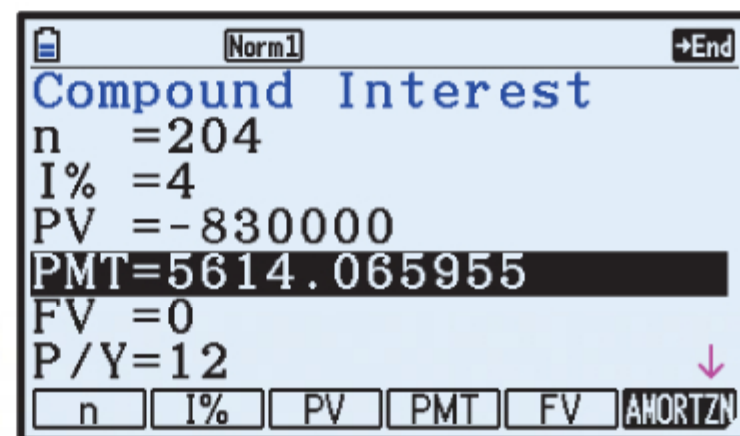


$\therefore N \approx 189$ {we round up since the money will last until the 189th month}

Terence's money will last $189 - 152 = 37$ months (or 3 years 1 month) longer if he only withdraws \$6000 each month.

- 4 a** Henry wants the money to last $85 - 68 = 17$ years.

$$N = 17 \times 12 = 204, \quad I\% = 4, \quad PV = -830\,000, \quad FV = 0, \quad P/Y = 12, \quad C/Y = 12$$



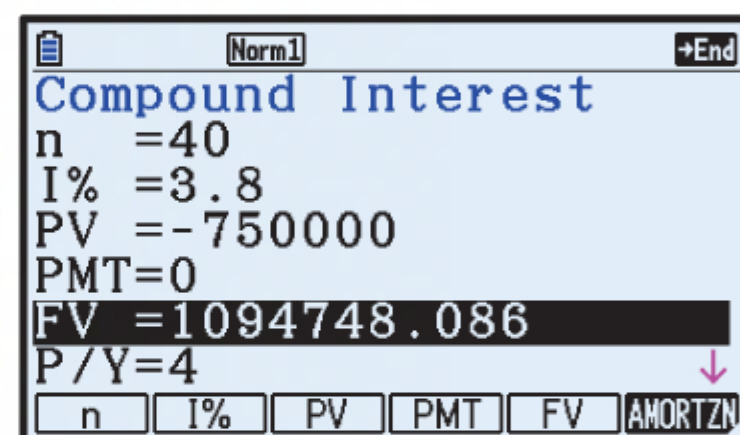
$$\therefore PMT \approx 5614.06$$

Henry can afford to withdraw £5614.06 each month.

- b** No, he can only afford to spend £5614.06 per month if he wants the money to last until he is 85, which is not enough to maintain his current lifestyle.

If he spends £6000 per month then his money will run out before he turns 84.

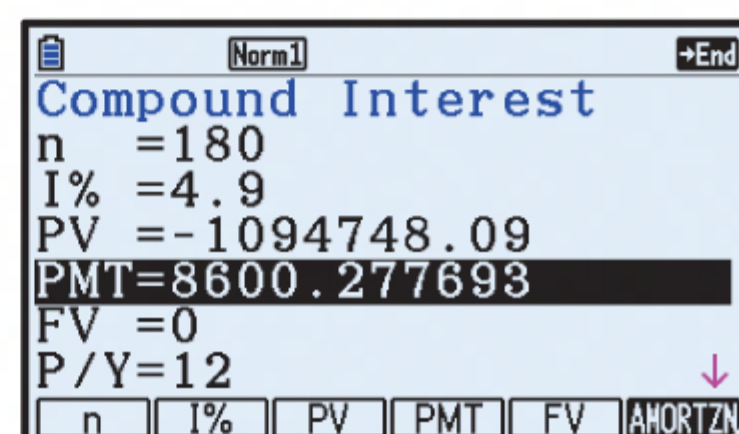
- 5 a** $N = 10 \times 4 = 40$, $I\% = 3.8$, $PV = -750\,000$, $PMT = 0$, $P/Y = 4$, $C/Y = 4$



$$\therefore FV \approx 1\,094\,748.09$$

There will be \$1 094 748.09 in the account after 10 years.

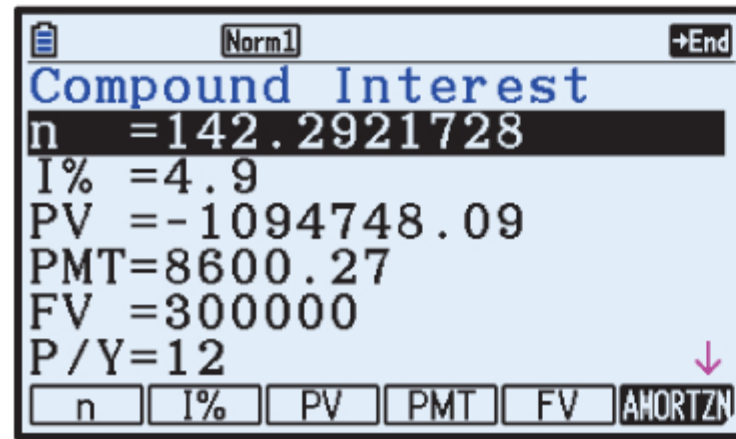
- b i** $N = 15 \times 12 = 180$, $I\% = 4.9$, $PV = -1\,094\,748.09$, $FV = 0$, $P/Y = 12$, $C/Y = 12$



$$\therefore PMT \approx 8600.27$$

Tamsyn can afford to withdraw \$8600.27 each month.

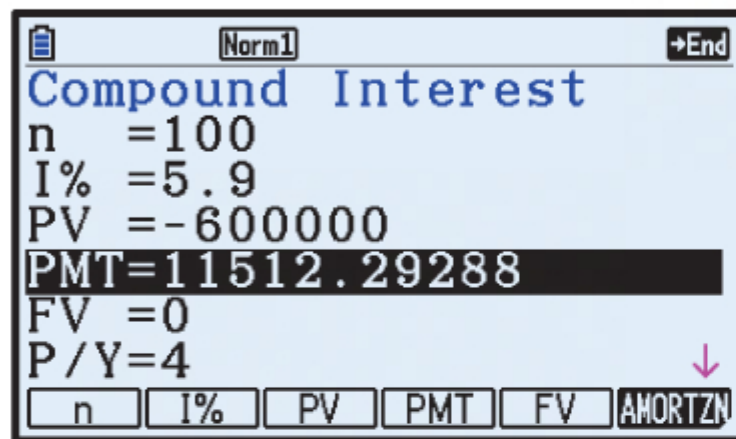
- ii $I\% = 4.9$, $PV = -1\,094\,748.09$, $PMT = 8600.27$, $FV = 300\,000$, $P/Y = 12$,
 $C/Y = 12$



$\therefore N \approx 143$ {we round up since after 142 months there will be more than \$300 000 in the account}

It will take 143 months (or 11 years 11 months) for the balance of the fund to fall below \$300 000.

- 6 a $N = 25 \times 4 = 100$, $I\% = 5.9$, $PV = -600\,000$, $FV = 0$, $P/Y = 4$, $C/Y = 4$

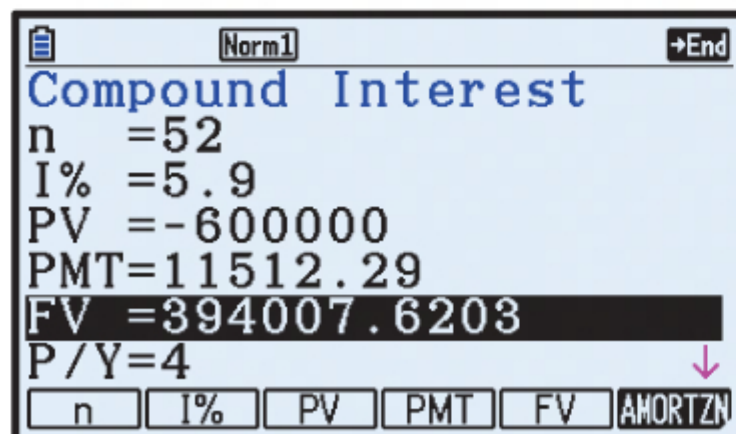


$\therefore PMT \approx 11\,512.29$

Danny can afford to withdraw £11 512.29 each quarter.

- b Danny is 55, so it will be 13 years until he turns 68.

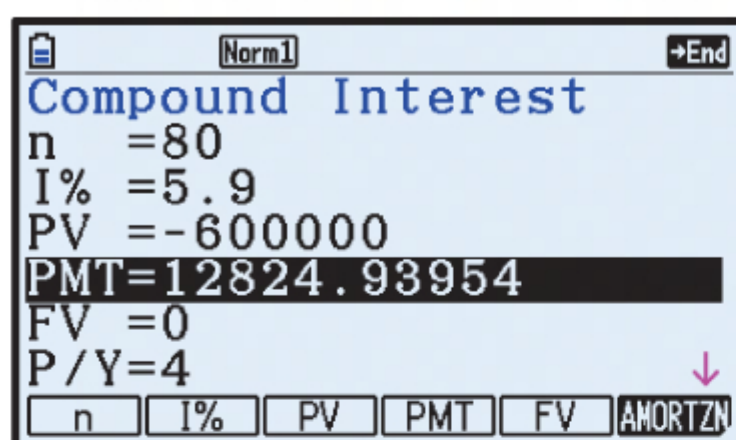
$N = 13 \times 4 = 52$, $I\% = 5.9$, $PV = -600\,000$, $PMT = 11\,512.29$, $P/Y = 4$,
 $C/Y = 4$



$\therefore FV \approx 394\,007.62$

There will be £394 007.62 in the account when Danny is 68.

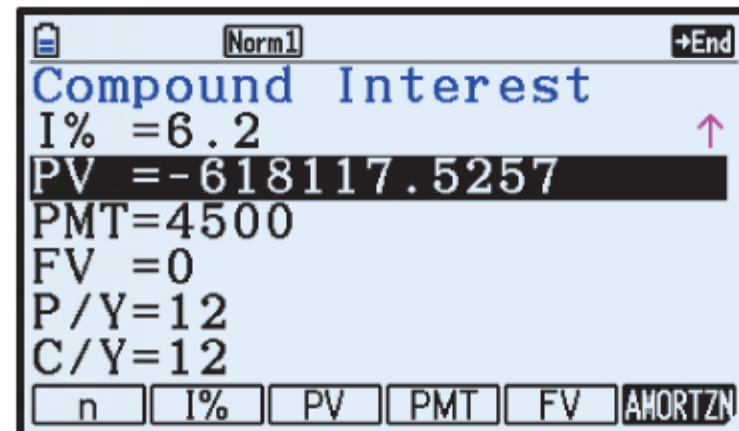
- c $N = 20 \times 4 = 80$, $I\% = 5.9$, $PV = -600\,000$, $FV = 0$, $P/Y = 4$, $C/Y = 4$



$\therefore PMT \approx 12\,824.93$

If Danny's money only needed to last 20 years, he could withdraw
 $\pounds 12\,824.93 - \pounds 11\,512.29 = \pounds 1\,312.64$ more each quarter.

- 7 a** Maggie needs the money to last $90 - 70 = 20$ years.
 $20 \text{ years} = 20 \times 12 = 240$ months.
 If Maggie withdraws \$4500 each month, she will withdraw a total of
 $\$4500 \times 240 = \$1\,080\,000$ from her annuity account.
- b** Maggie will earn interest on the money in the annuity account as she makes her regular withdrawals.
- c** $N = 20 \times 12 = 240$, $I\% = 6.2$, $PMT = 4500$, $FV = 0$, $P/Y = 12$, $C/Y = 12$

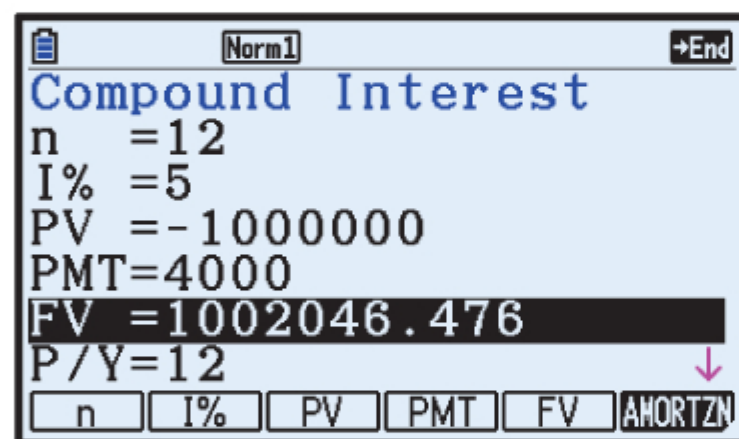


$$\therefore PV \approx -618\,117.53$$

Maggie will need \$618 117.53 in savings when she retires.

- 8** We find the balance of the fund after 1 year.

$$N = 1 \times 12 = 12, \quad I\% = 5, \quad PV = -1\,000\,000, \quad PMT = 4000, \quad P/Y = 12, \quad C/Y = 12$$



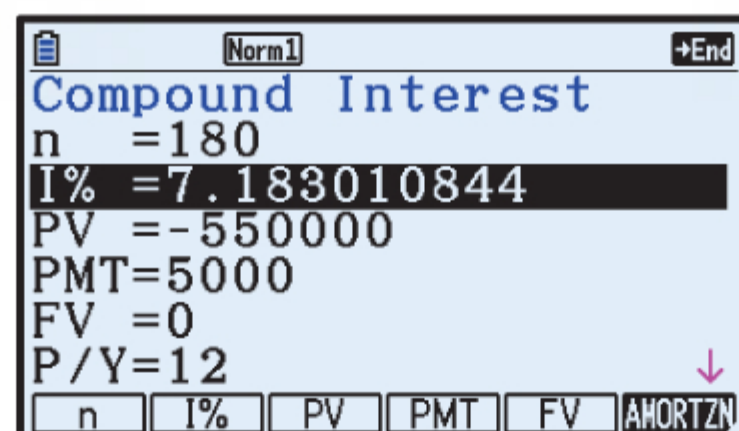
$$\therefore FV \approx 1\,002\,046.48$$

So, after 1 year, the amount in the fund has increased.

\therefore the amount in the fund will continue to increase.

\therefore the money will last forever.

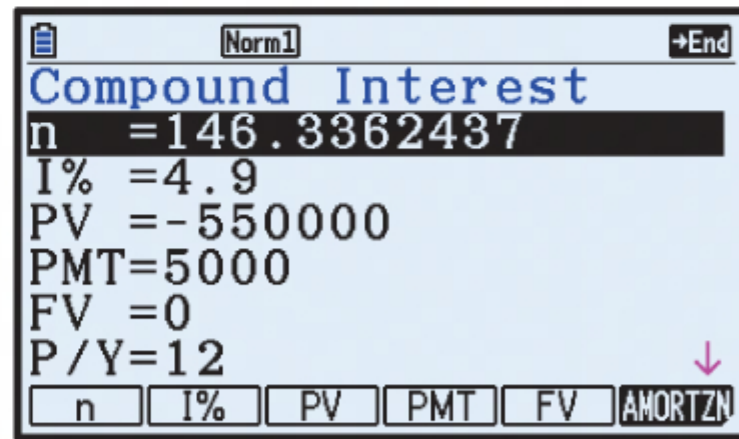
- 9 a** $N = 15 \times 12 = 180$, $PV = -550\,000$, $PMT = 5000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$



$$\therefore I\% \approx 7.19$$

Igor will require an interest rate of 7.19% p.a. compounded monthly.

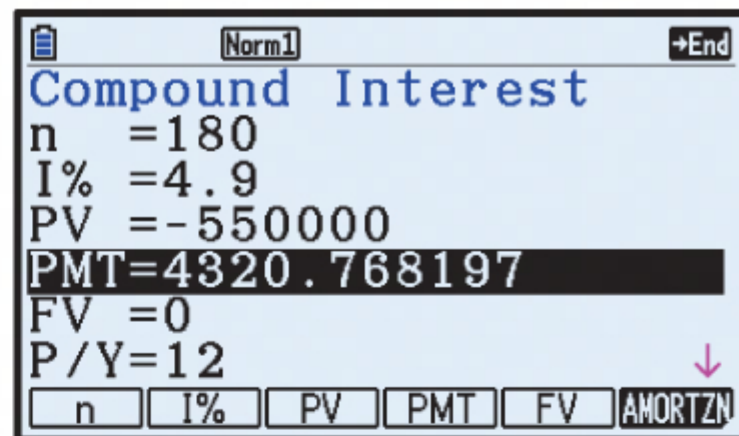
- b i** $I\% = 4.9$, $PV = -550\,000$, $PMT = 5000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$



$\therefore N \approx 147$ {we round up since the money will last until the 147th month}

The money will last $180 - 147 = 33$ months (or 2 years 9 months) less time.

- ii** $N = 15 \times 12 = 180$, $I\% = 4.9$, $PV = -550\,000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$

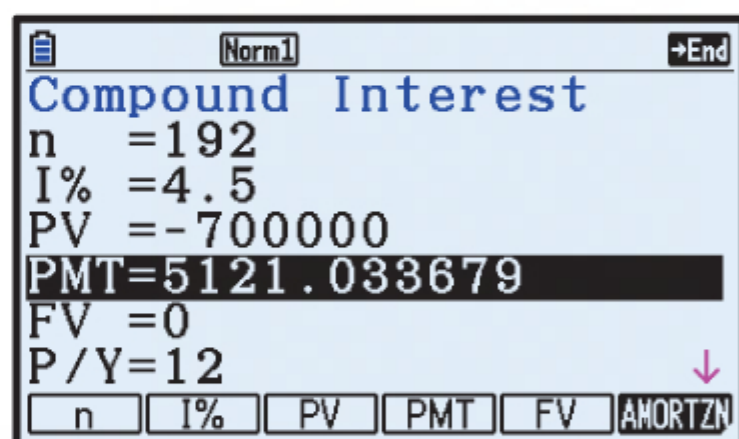


$\therefore PMT \approx 4320.76$

If Igor wants the money to last 15 years, he needs to withdraw

$\text{€}5000 - \text{€}4320.76 = \text{€}679.24$ less each month.

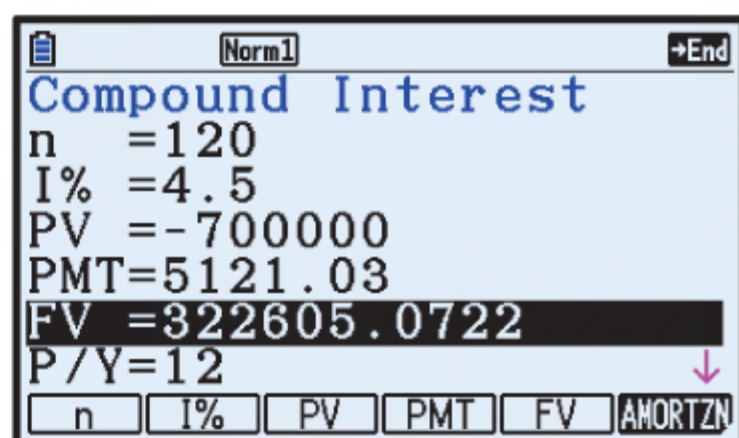
- 10 a** $N = 16 \times 12 = 192$, $I\% = 4.5$, $PV = -700\,000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$



$\therefore PMT \approx 5121.03$

Luke can withdraw \$5121.03 each month.

- b** $N = 10 \times 12 = 120$, $I\% = 4.5$, $PV = -700\,000$, $PMT = 5121.03$, $P/Y = 12$, $C/Y = 12$



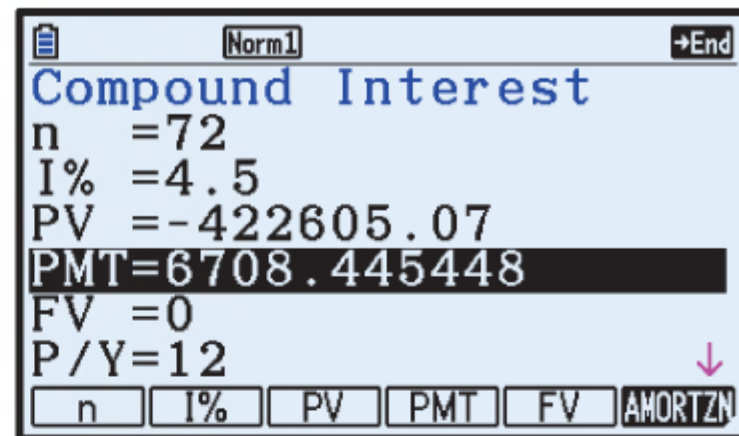
$\therefore FV \approx 322\,605.07$

The balance of the fund after 10 years is \$322 605.07.

- c Luke wants the money to last another $16 - 10 = 6$ years.

After receiving the inheritance, there is $\$322\,605.07 + \$100\,000 = \$422\,605.07$ in the fund.

$$N = 6 \times 12 = 72, \quad I\% = 4.5, \quad PV = -422\,605.07, \quad FV = 0, \quad P/Y = 12, \quad C/Y = 12$$



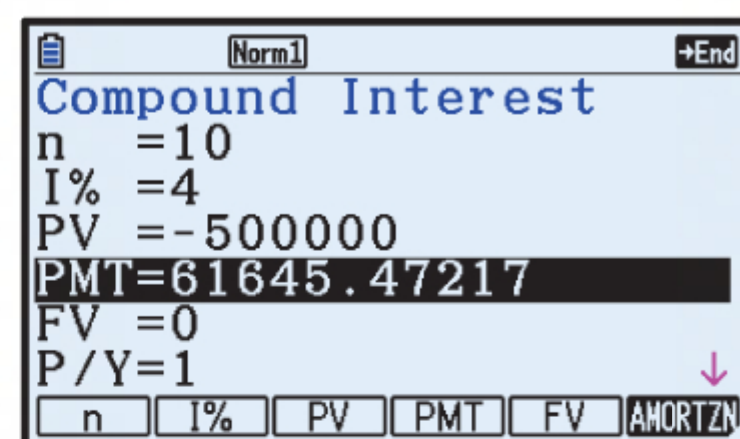
$$\therefore PMT \approx 6708.44$$

Luke is able to withdraw \$6708.44 each month for the remaining 6 years.

ACTIVITY 3

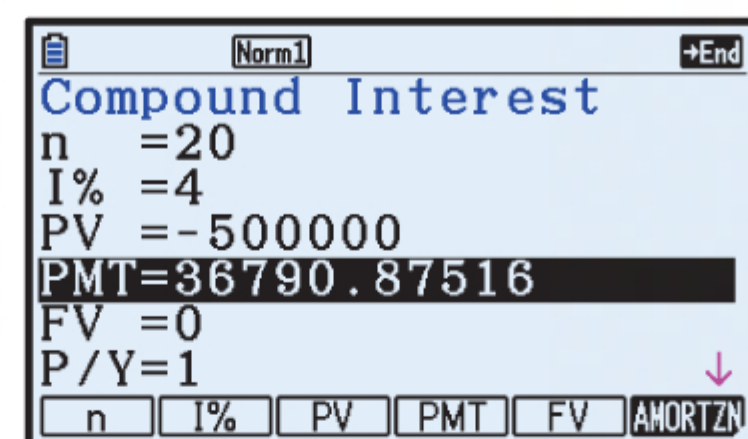
PERPETUITIES

- 1
 - a We expect to pay $PV = \frac{PMT}{r} = \frac{\$300}{0.05} = \$6000$.
 - b We expect to pay $PV = \frac{PMT}{r} = \frac{\$4000}{0.032} = \$125\,000$.
 - c We expect to pay $PV = \frac{PMT}{r} = \frac{\$18\,000}{0.045} = \$400\,000$.
- 2
 - a We expect to receive $PMT = PV \times r = €1000 \times 0.035 = €35$ per year.
 - b We expect to receive $PMT = PV \times r = €25\,000 \times 0.048 = €1200$ per year.
- 3
 - a We expect to receive $PMT = PV \times r = \$500\,000 \times 0.04 = \$20\,000$ per year.
 - b
 - i $N = 10, I\% = 4, PV = -500\,000, FV = 0, P/Y = 1, C/Y = 1$
 - ii $N = 20, I\% = 4, PV = -500\,000, FV = 0, P/Y = 1, C/Y = 1$



$$\therefore PMT \approx 61\,645.47$$

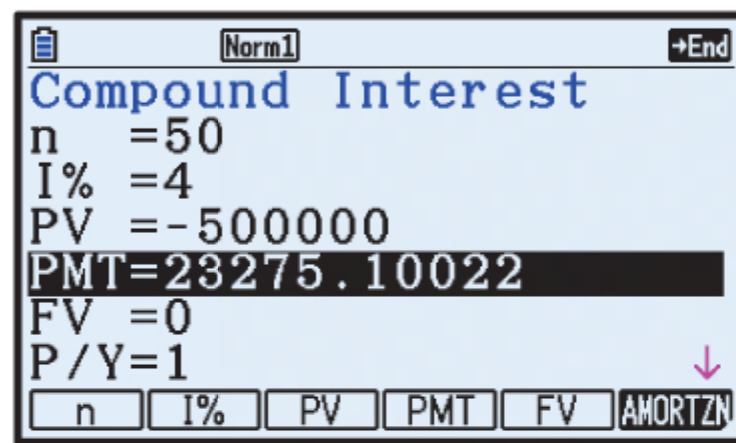
The money will last for 10 years if \$61 645.47 is withdrawn each year.



$$\therefore PMT \approx 36\,790.87$$

The money will last for 20 years if \$36 790.87 is withdrawn each year.

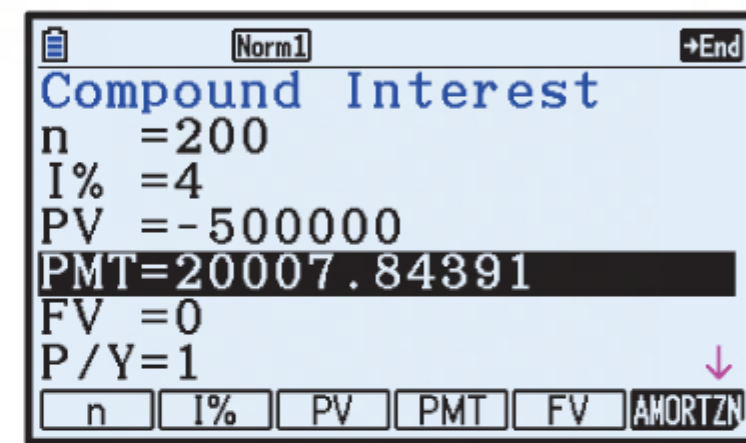
iii $N = 50$, $I\% = 4$, $PV = -500\,000$,
 $FV = 0$, $P/Y = 1$, $C/Y = 1$



$\therefore PMT \approx 23\,275.10$

The money will last for 50 years if \$23 275.10 is withdrawn each year.

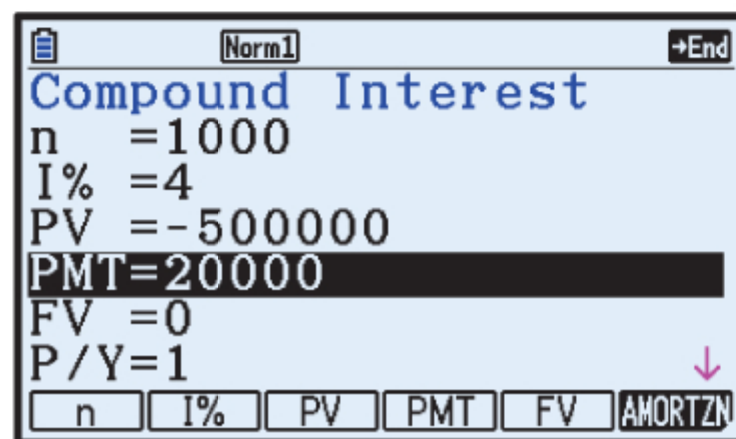
iv $N = 200$, $I\% = 4$, $PV = -500\,000$,
 $FV = 0$, $P/Y = 1$, $C/Y = 1$



$\therefore PMT \approx 20\,007.84$

The money will last for 200 years if \$20 007.84 is withdrawn each year.

v $N = 1000$, $I\% = 4$, $PV = -500\,000$,
 $FV = 0$, $P/Y = 1$, $C/Y = 1$



$\therefore PMT \approx 20\,000$

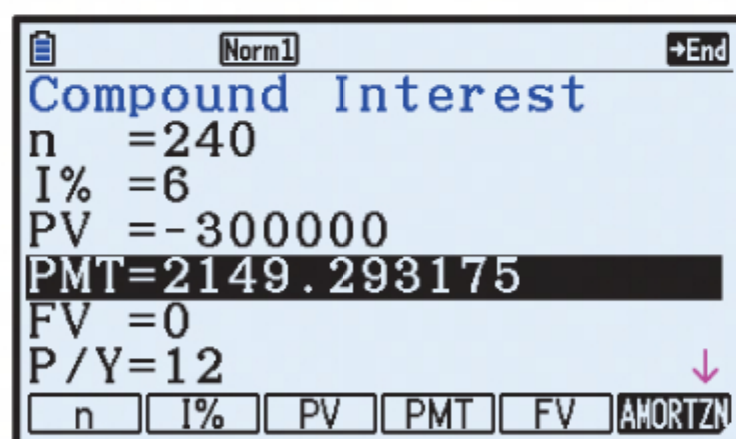
The money will last for 1000 years if \$20 000 is withdrawn each year.

- c As the number of years required increases, the amount that can be withdrawn approaches \$20 000. This is the same value as the payment received from the perpetuity, which is meant to continue indefinitely.

ACTIVITY 4

GROWING ANNUITIES

- 1 b The withdrawals are increasing over time.
 c \$2709.58 is withdrawn in the final time period.
 d $N = 20 \times 12 = 240$, $I\% = 6$, $PV = -300\,000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$



$\therefore PMT \approx 2149.29$

The regular withdrawal for this investment would be \$2149.29, which lies between the initial and final withdrawals in the growing annuity.

- e We would set the growth rate to 0%.

- 2 a i** Using the spreadsheet, the initial monthly withdrawal is \$1888.51.
- ii** We enter an expression such as “= SUM(D14 : D313)” to add up all of the values in the withdrawal column.

The total amount withdrawn over 25 years is \$842 289.06.

$$\begin{aligned}\text{b Total interest earned} &= \text{total withdrawn} - \text{starting principal} \\ &= \$842\,289.06 - \$500\,000 \\ &= \$342\,289.06\end{aligned}$$

This agrees with the value obtained by adding up all of the values in the interest column.

- 3 a** Increasing the amount originally deposited will *increase* the initial withdrawal.
- b** Increasing the duration of the annuity will *decrease* the initial withdrawal.
- c** Increasing the interest rate will *increase* the initial withdrawal.
- d** Increasing the growth rate will *decrease* the initial withdrawal.

4 a
$$w = \frac{PV \times (i - g) \times (1 + i)^n}{(1 + i)^n - (1 + g)^n}$$

In question **1** we have $PV = 300\,000$, $i = \frac{0.06}{12}$, $g = \frac{0.02}{12}$, and $n = 20 \times 12 = 240$.

$$\begin{aligned}\therefore w &= \frac{300\,000 \times \left(\frac{0.06}{12} - \frac{0.02}{12}\right) \times \left(1 + \frac{0.06}{12}\right)^{240}}{\left(1 + \frac{0.06}{12}\right)^{240} - \left(1 + \frac{0.02}{12}\right)^{240}} \\ &\approx 1819.92 \quad \checkmark\end{aligned}$$

In question **2** we have $PV = 500\,000$, $i = \frac{0.04}{12}$, $g = \frac{0.03}{12}$, and $n = 25 \times 12 = 300$.

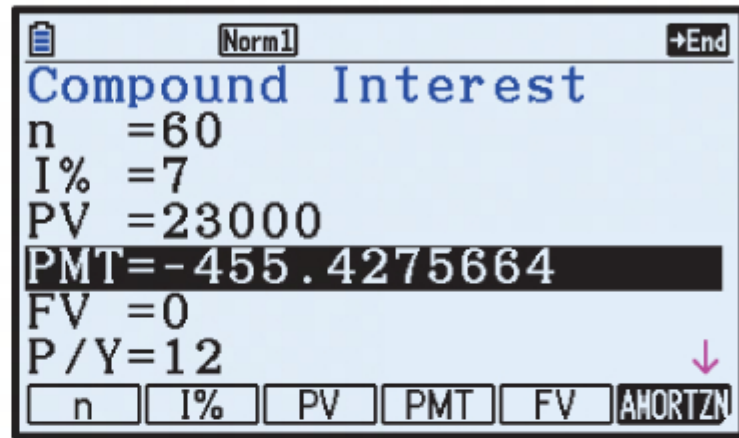
$$\begin{aligned}\therefore w &= \frac{500\,000 \times \left(\frac{0.04}{12} - \frac{0.03}{12}\right) \times \left(1 + \frac{0.04}{12}\right)^{300}}{\left(1 + \frac{0.04}{12}\right)^{300} - \left(1 + \frac{0.03}{12}\right)^{300}} \\ &\approx 1888.51 \quad \checkmark\end{aligned}$$

b If $g = 0$, then
$$w = \frac{PV \times i \times (1 + i)^n}{(1 + i)^n - 1}$$

which is the formula of the regular withdrawal amount from **Activity 2**.

REVIEW SET 2A

- 1 a $N = 5 \times 12 = 60$, $I\% = 7$, $PV = 23\,000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$

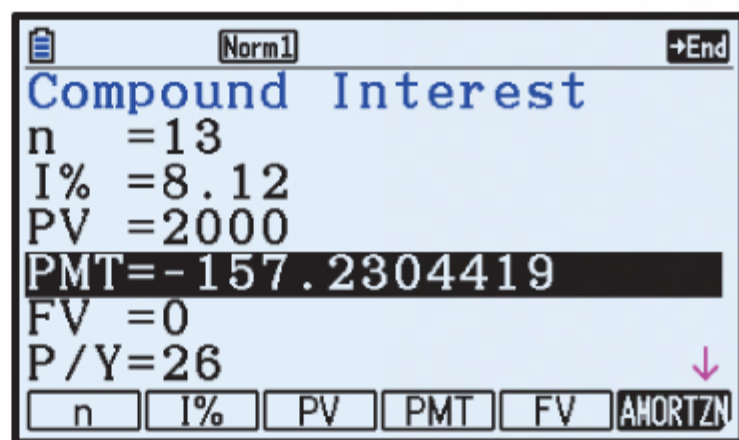


$$\therefore PMT \approx -455.43$$

The monthly repayment is \$455.43.

- b Total repayment = monthly repayment \times number of months
 $= \$455.43 \times 60$
 $= \$27\,325.80$
- c Interest = total repayment – amount borrowed
 $= \$27\,325.80 - \$23\,000$
 $= \$4\,325.80$

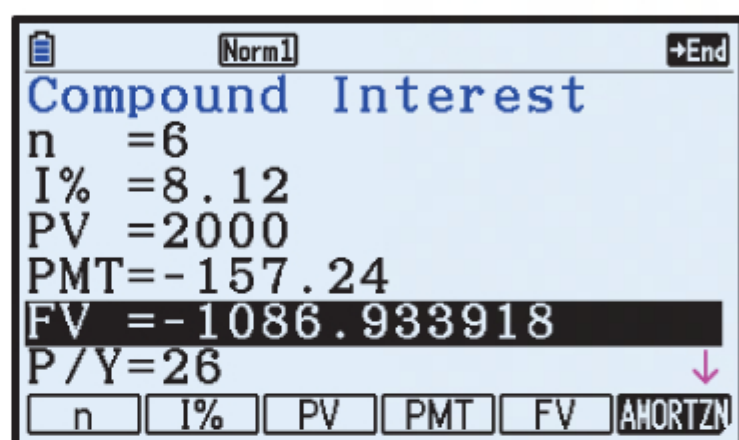
- 2 a $N = \frac{1}{2} \times 26 = 13$, $I\% = 8.12$, $PV = 2000$, $FV = 0$, $P/Y = 26$, $C/Y = 26$



$$\therefore PMT \approx -157.24$$

The fortnightly repayment is €157.24.

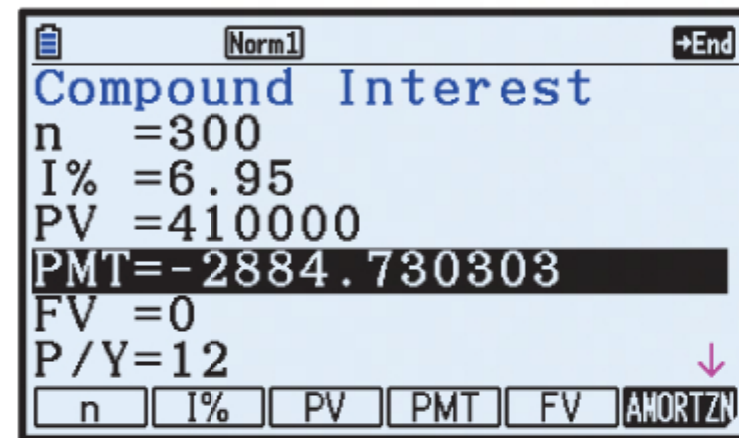
- b $N = 6$, $I\% = 8.12$, $PV = 2000$, $PMT = -157.24$, $P/Y = 26$, $C/Y = 26$



$$\therefore FV \approx -1086.93$$

The outstanding balance on the loan after 6 fortnights is €1086.93.

- 3 a $N = 25 \times 12 = 300$, $I\% = 6.95$, $PV = 410\,000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$



$$\therefore PMT \approx -2884.74$$

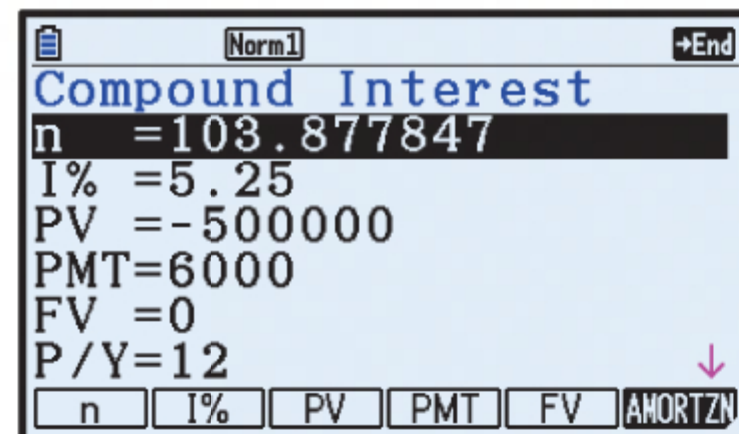
The minimum monthly repayment is \$2884.74.

- b Total interest = total repayment – principal
 $= \$2884.74 \times 300 - \$410\,000$
 $= \$865\,422 - \$410\,000$
 $= \$455\,422$

Simone will therefore pay more in interest than the amount she originally borrowed.

- 4 a We need to find how long it will take for the future value to fall to \$0.

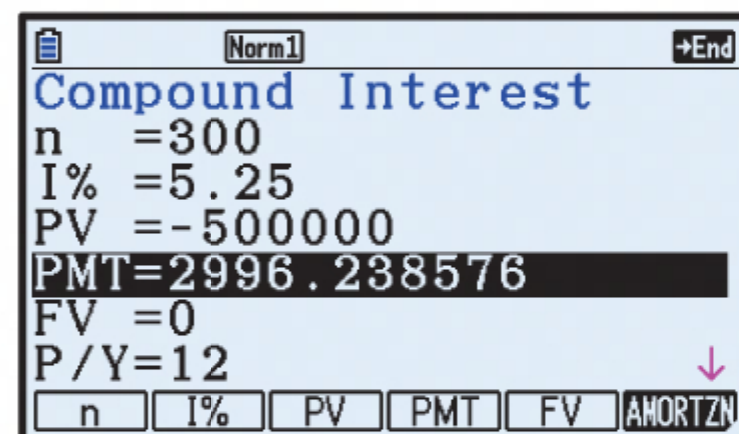
$$I\% = 5.25, \quad PV = -500\,000, \quad PMT = 6000, \quad FV = 0, \quad P/Y = 12, \quad C/Y = 12$$



$$\therefore N \approx 104$$

Yasmin will be able to withdraw \$6000 per month for 103 months, and then less in the 104th month (after 8 years 8 months).

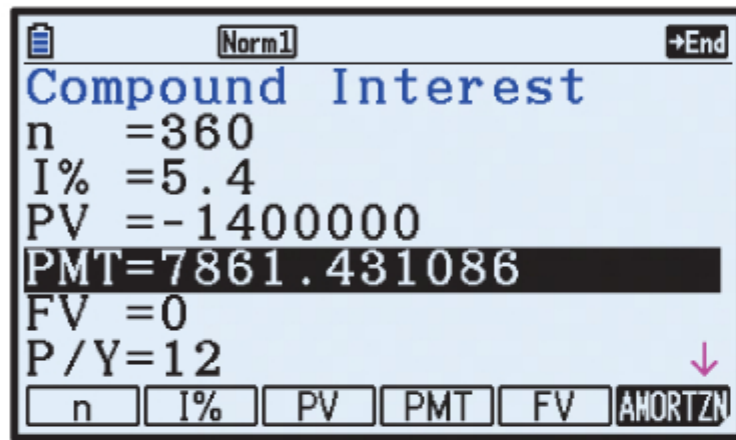
- b $N = 25 \times 12 = 300$, $I\% = 5.25$, $PV = -500\,000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$



$$\therefore PMT \approx 2996.23$$

Yasmin can afford to withdraw \$2996.23 per month.

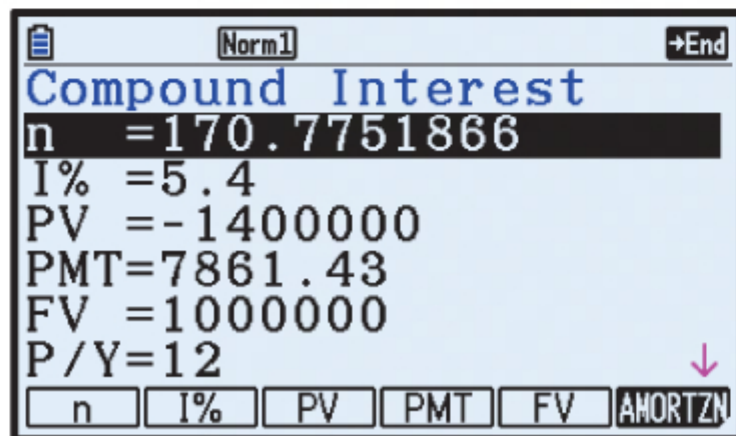
- 5 a $N = 30 \times 12 = 360$, $I\% = 5.4$, $PV = -1\,400\,000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$



$$\therefore PMT \approx 7861.43$$

Vasili can withdraw €7861.43 each month.

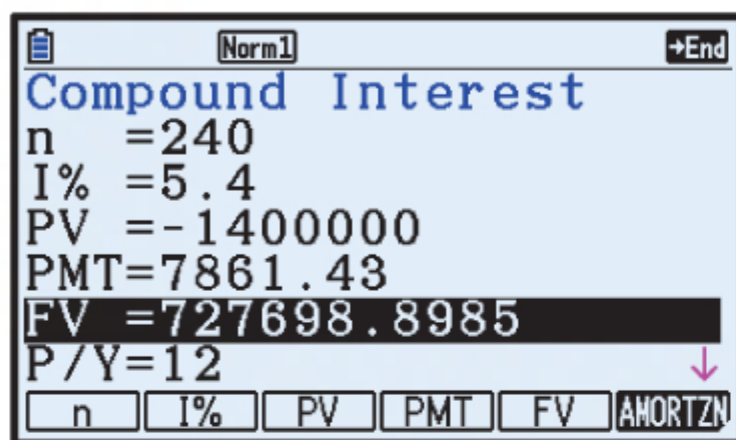
- b $I\% = 5.4$, $PV = -1\,400\,000$, $PMT = 7861.43$, $FV = 1\,000\,000$, $P/Y = 12$, $C/Y = 12$



$$\therefore N \approx 171 \quad \{\text{we round up since after 170 months the balance is still greater than €1\,000\,000}\}$$

It will take 171 months (or 14 years 3 months) for the balance of the fund to fall below €1 000 000.

- c $N = 20 \times 12 = 240$, $I\% = 5.4$, $PV = -1\,400\,000$, $PMT = 7861.43$, $P/Y = 12$, $C/Y = 12$

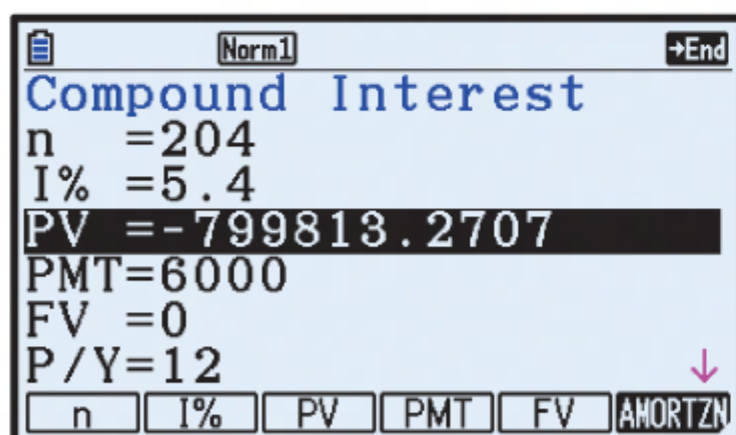


$$\therefore FV \approx 727\,698.90$$

After 20 years, there is €727 698.90 left in the fund.

- 6 a Scott's money needs to last for $85 - 68 = 17$ years.

$$N = 17 \times 12 = 204, \quad I\% = 5.4, \quad PMT = 6000, \quad FV = 0, \quad P/Y = 12, \quad C/Y = 12$$



$$\therefore PV \approx -799\,813.28$$

Scott will need to have \$799 813.28 in savings when he retires.

- b** Scott is 68 now, so he will be 80 in 12 years' time.

$$N = 12 \times 12 = 144, \quad I\% = 5.4, \quad PV = -799\,813.28, \quad PMT = 6000, \quad P/Y = 12, \\ C/Y = 12$$

Norm1 End
Compound Interest
n =144
I% =5.4
PV =-799813.28
PMT=6000
FV =314877.35
P/Y=12
n I% PV PMT FV AMORTZ

$$\therefore FV \approx 314\,877.35$$

When Scott is 80 there will be \$314 877.35 left in his account.

REVIEW SET 2B

- 1 a** $N = 4 \times 12 = 48, \quad I\% = 5.5, \quad PV = 12\,000, \quad FV = 0, \quad P/Y = 12, \quad C/Y = 12$

Norm1 End
Compound Interest
n =48
I% =5.5
PV =12000
PMT=-279.0777027
FV =0
P/Y=12
n I% PV PMT FV AMORTZ

$$\therefore PMT \approx -279.08$$

The monthly repayment is \$279.08.

- b** Total interest = total repayment – amount borrowed
 $= \$279.08 \times 48 - \$12\,000$
 $= \$1395.84$

- 2 a** Peter will borrow $\$22\,000 - \$4500 = \$17\,500$.

- b** $N = 4 \times 4 = 16, \quad I\% = 6.9, \quad PV = 17\,500, \quad FV = 0, \quad P/Y = 4, \quad C/Y = 4$

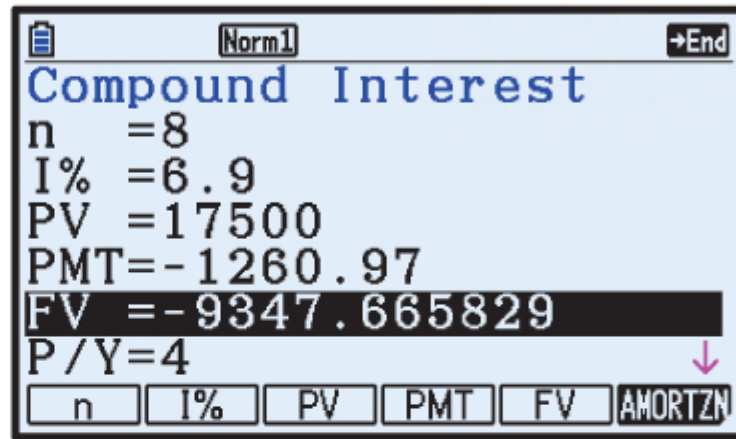
Norm1 End
Compound Interest
n =16
I% =6.9
PV =17500
PMT=-1260.969548
FV =0
P/Y=4
n I% PV PMT FV AMORTZ

$$\therefore PMT \approx -1260.97$$

The quarterly repayment is \$1260.97.

- c** Total interest = total repayment – amount borrowed
 $= \$1260.97 \times 16 - \$17\,500$
 $= \$20\,175.52 - \$17\,500$
 $= \$2675.52$

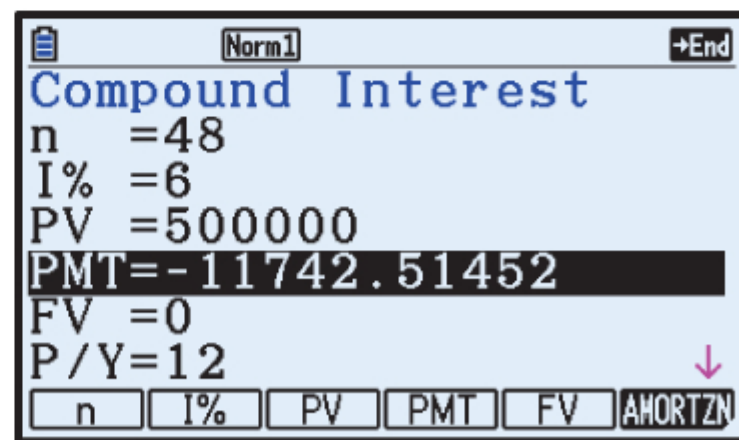
d $N = 2 \times 4 = 8$, $I\% = 6.9$, $PV = 17\,500$, $PMT = -1260.97$, $P/Y = 4$, $C/Y = 4$



$\therefore FV \approx -9347.67$

The outstanding balance on the loan after 2 years is \$9347.67.

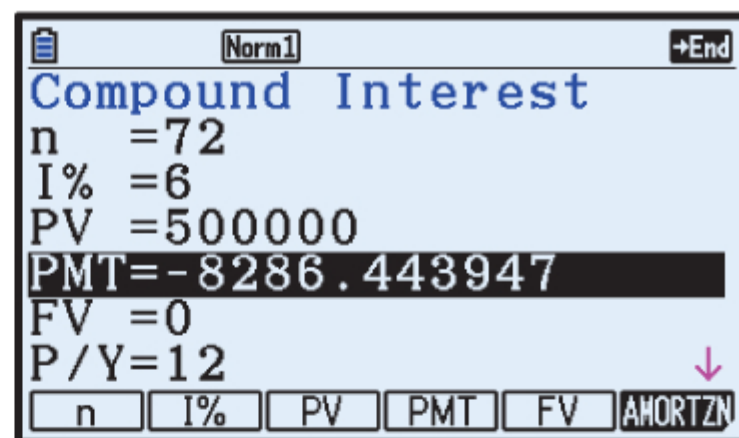
3 a i $N = 4 \times 12 = 48$, $I\% = 6$, $PV = 500\,000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$



$\therefore PMT \approx -11\,742.52$

The monthly loan repayments over 4 years would be 11 742.52 pesos.

ii $N = 6 \times 12 = 72$, $I\% = 6$, $PV = 500\,000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$



$\therefore PMT \approx -8286.45$

The monthly loan repayments over 6 years would be 8286.45 pesos.

b 4 year loan total interest = total repayment – amount borrowed
 $= 11\,742.52 \times 48 - 500\,000$
 $= 563\,640.96 - 500\,000$
 $= 63\,640.96$ pesos

6 year loan total interest = total repayment – amount borrowed
 $= 8286.45 \times 72 - 500\,000$
 $= 596\,624.40 - 500\,000$
 $= 96\,624.40$

The 4 year loan charges the least interest of 63 640.96 pesos as more is paid off each month and therefore less interest is charged.

4 a An annuity fund is an investment where an individual makes a lump-sum deposit, and then makes regular *withdrawals* from the account. We have previously considered compound interest investments that make regular *deposits* into an account.

b Diane is technically correct, but she will be able to withdraw more than £2000 per month since the money in the fund will earn interest.

- c $N = 25 \times 12 = 300$, $I\% = 4$, $PV = -600\,000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$

Norm1 End
Compound Interest
n = 300
I% = 4
PV = -600000
PMT = 3167.021042
FV = 0
P/Y = 12
n I% PV PMT FV AMORTZN

$$\therefore PMT \approx 3167.02$$

Diane will be able to withdraw £3167.02 each month.

- 5 a $N = 20 \times 12 = 240$, $I\% = 5.8$, $PV = -350\,000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$

Norm1 End
Compound Interest
n = 240
I% = 5.8
PV = -350000
PMT = 2467.293379
FV = 0
P/Y = 12
n I% PV PMT FV AMORTZN

$$\therefore PMT \approx 2467.29$$

Pia will be able to withdraw €2467.29 each month.

- b $N = 15 \times 12 = 180$, $I\% = 5.8$, $PV = -350\,000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$

Norm1 End
Compound Interest
n = 180
I% = 5.8
PV = -350000
PMT = 2915.814482
FV = 0
P/Y = 12
n I% PV PMT FV AMORTZN

$$\therefore PMT \approx 2915.81$$

Pia would be able to withdraw $€2915.81 - €2467.29 = €448.52$ more each month if the money only needed to last 15 years.

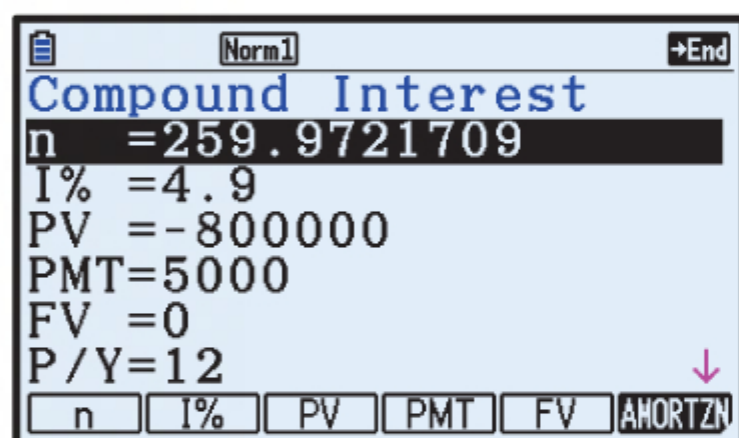
- 6 a $N = 15 \times 12 = 180$, $PV = -800\,000$, $PMT = 6284.75$, $FV = 0$, $P/Y = 12$, $C/Y = 12$

Norm1 End
Compound Interest
n = 180
I% = 4.899990988
PV = -800000
PMT = 6284.75
FV = 0
P/Y = 12
n I% PV PMT FV AMORTZN

$$\therefore I\% \approx 4.90$$

The account pays 4.9% p.a. compounded monthly.

- b** $I\% = 4.90$, $PV = -800\,000$, $PMT = 5000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$



$\therefore N \approx 260$ {we round up since the money will last until the 260th month}

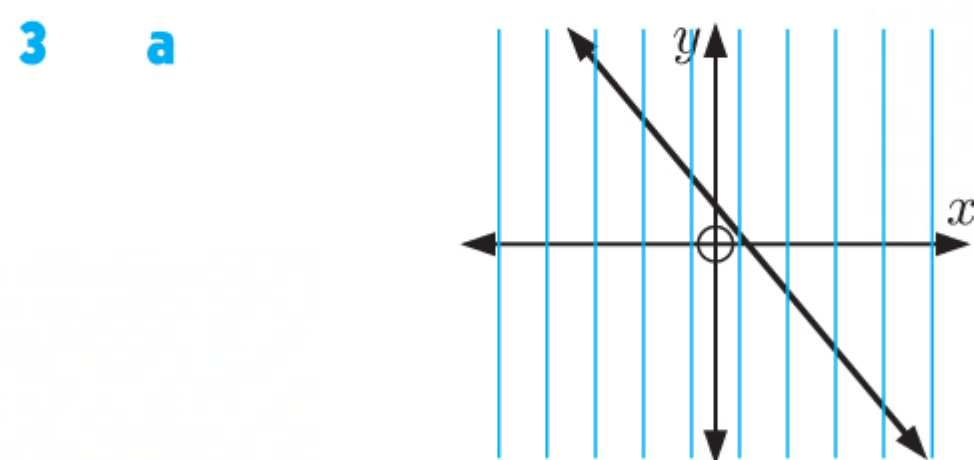
Harold's money will last $260 - 180 = 80$ months (or 6 years 8 months) longer if he withdraws £5000 each month.

Chapter 3

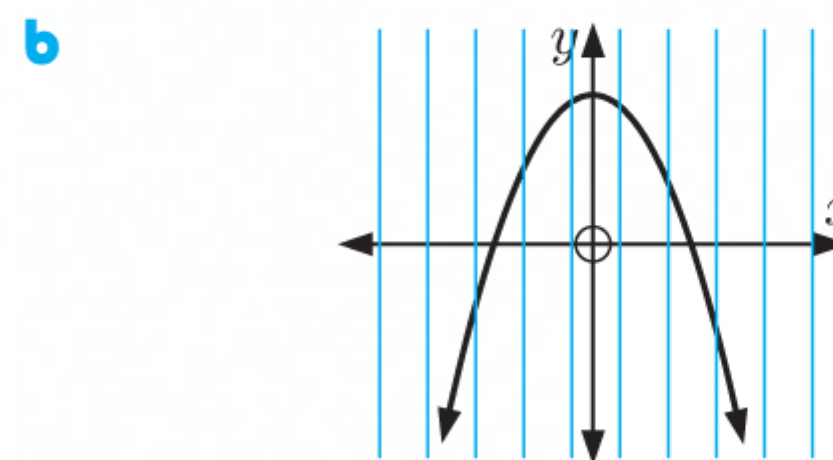
FUNCTIONS

EXERCISE 3A

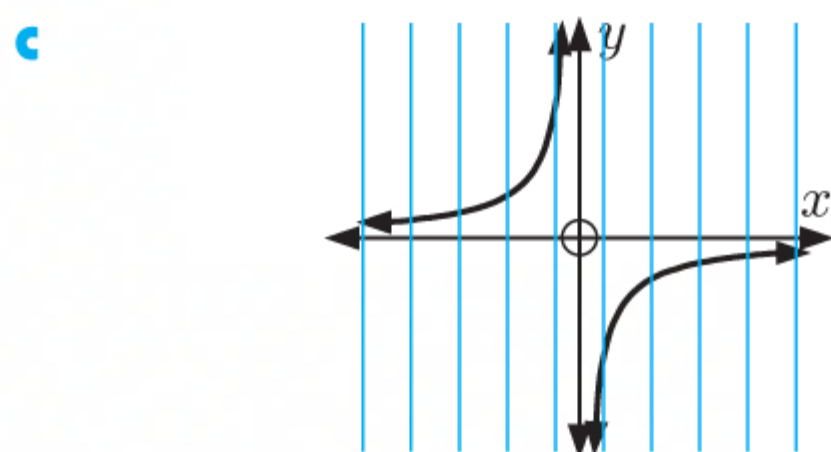
- 1
 - a $\{(1, 3), (2, 4), (3, 5), (4, 6)\}$ is a function since no two ordered pairs have the same x -coordinate.
 - b $\{(1, 3), (3, 2), (1, 7), (-1, 4)\}$ is not a function since two of the ordered pairs, $(1, 3)$ and $(1, 7)$, have the same x -coordinate 1.
 - c $\{(2, -1), (2, 0), (2, 3), (2, 11)\}$ is not a function since each ordered pair has the same x -coordinate 2.
 - d $\{(7, 6), (5, 6), (3, 6), (-4, 6)\}$ is a function since no two ordered pairs have the same x -coordinate.
 - e $\{(0, 0), (1, 0), (3, 0), (5, 0)\}$ is a function since no two ordered pairs have the same x -coordinate.
 - f $\{(0, 0), (0, -2), (0, 2), (0, 4)\}$ is not a function since each ordered pair has the same x -coordinate 0.
- 2
 - a $y = x^2 - 9$ is a function, since for any value of x there is at most one value of y .
 - b $x + y = 9$ is a function, since for any value of x there is at most one value of y .
 - c $x^2 + y^2 = 9$ is not a function. If $x^2 + y^2 = 9$, then $y = \pm\sqrt{9 - x^2}$. So, for example, for $x = 2$, $y = \pm\sqrt{5}$.



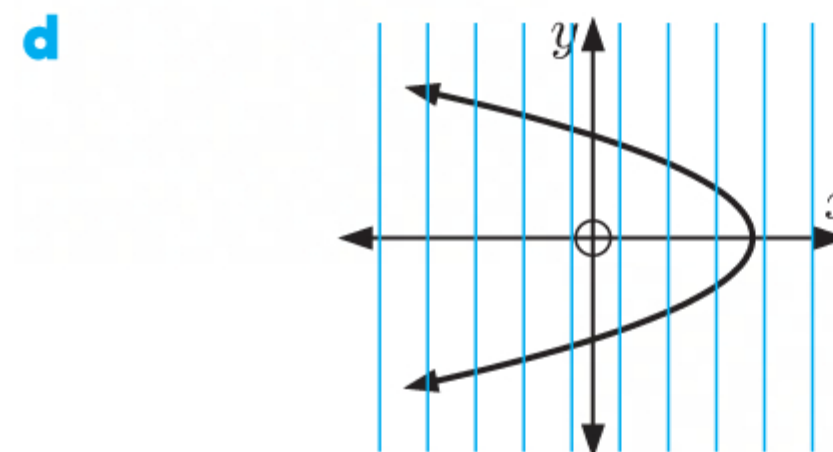
Each vertical line cuts the graph no more than once, so this relation is a function.



Each vertical line cuts the graph no more than once, so this relation is a function.

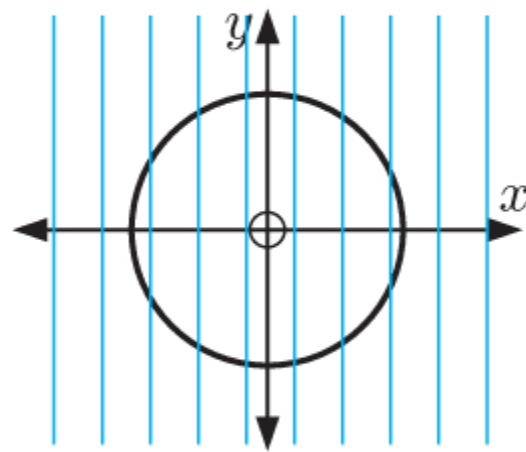


Each vertical line cuts the graph no more than once, so this relation is a function.



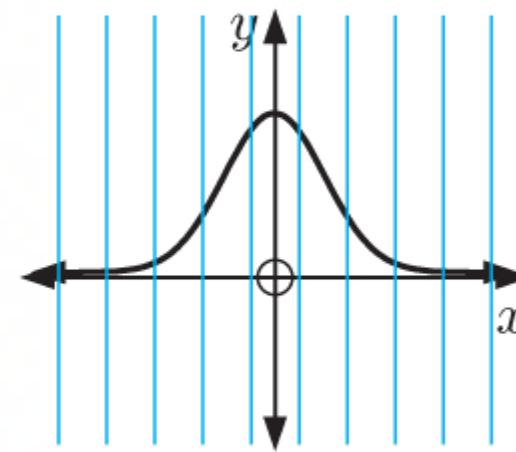
Some vertical lines cut the graph more than once, so this relation is not a function.

e



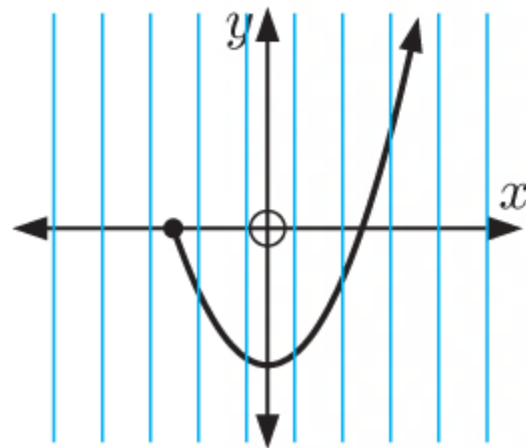
Some vertical lines cut the graph more than once, so this relation is not a function.

f



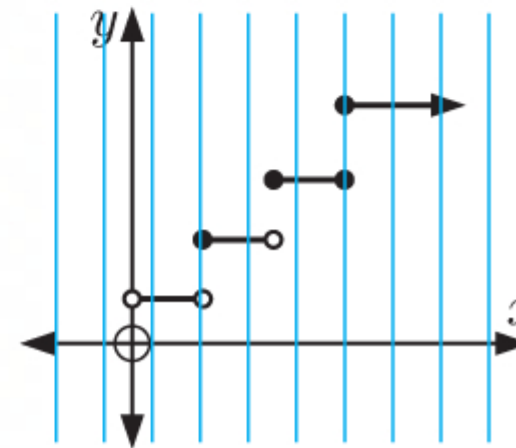
Each vertical line cuts the graph no more than once, so this relation is a function.

g



Each vertical line cuts the graph no more than once, so this relation is a function.

h



One vertical line cuts the graph more than once, so this relation is not a function.

- 4 The relation between *age* and *cost* is not a function as there are two corresponding values of *cost* for a 2 year old child.

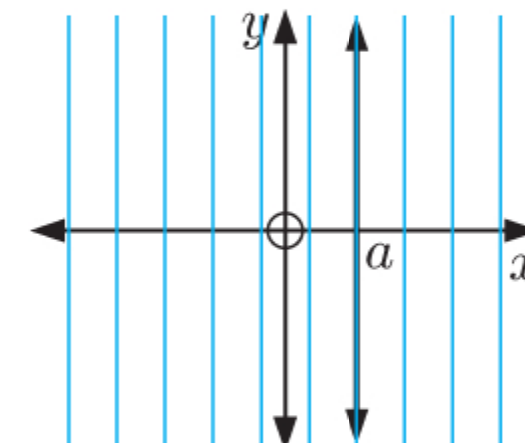
The categories 0 - 2 years and 2 - 16 years both include the age 2 years.

So, a ticket for a 2 year old child could cost either \$0 or \$20 according to the schedule.

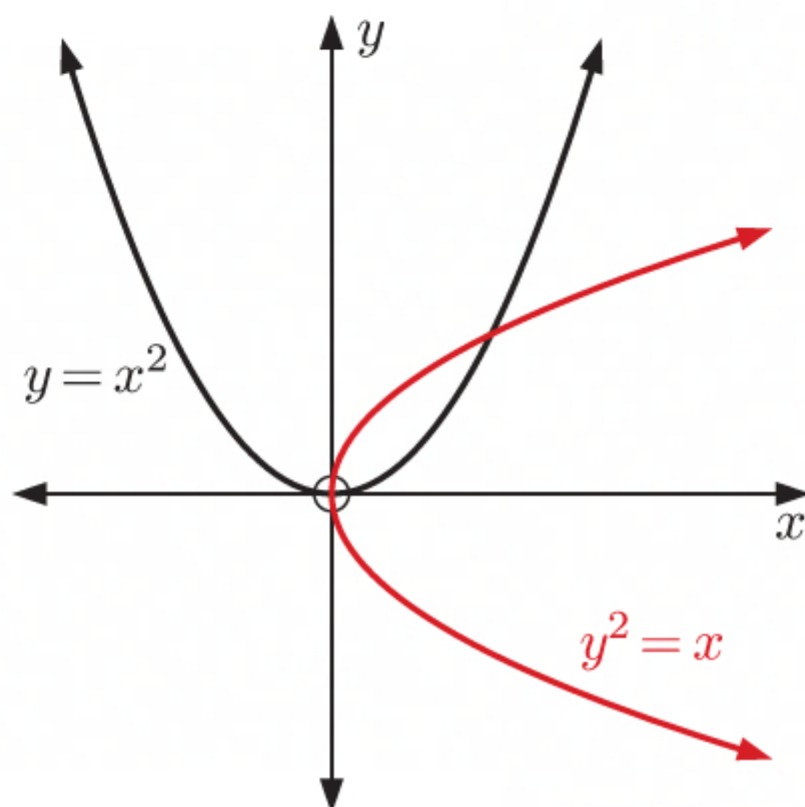
- 5 It is not possible for a function to have more than one *y*-intercept. The *y*-axis is a vertical line, so if a relation has more than one *y*-intercept, then a vertical line cuts the relation more than once. So, such a relation cannot be a function.

- 6 No, the graph of a straight line will not be a function if it is a vertical line which has the form $x = a$ for some constant a .

The vertical line through $x = a$ cuts the graph at every point, so the straight line $x = a$ does not pass the vertical line test and hence is not a function.



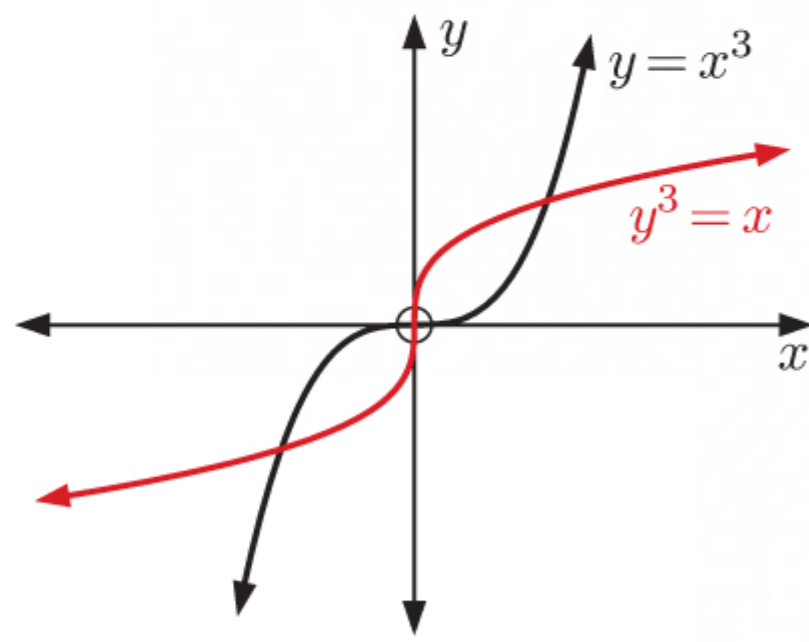
7



- a $y^2 = x$ is a relation but not a function.
 $y = x^2$ is a function (and a relation).
 $y^2 = x$ has a horizontal axis of symmetry (the x -axis).
 $y = x^2$ has a vertical axis of symmetry (the y -axis).
Both $y^2 = x$ and $y = x^2$ pass through $(0, 0)$ and $(1, 1)$.
 $y^2 = x$ is a rotation of $y = x^2$ clockwise through 90° about the origin or $y^2 = x$ is a reflection of $y = x^2$ in the line $y = x$.

- b i The part of the graph of $y^2 = x$ in the first quadrant corresponds to $y = \sqrt{x}$.
ii $y = \sqrt{x}$ is a function as any vertical line cuts the graph at most once.

8



a Both curves are functions since any vertical line will cut each curve at most once.

b $y^3 = x$
 $\therefore y = x^{\frac{1}{3}}$
 $\therefore y = \sqrt[3]{x}$

EXERCISE 3B

1 $f(x) = 3x + 2$

a $f(0) = 3(0) + 2$ {replacing x with (0) }
 $= 0 + 2$
 $= 2$

b $f(2) = 3(2) + 2$ {replacing x with (2) }
 $= 6 + 2$
 $= 8$

c $f(-1) = 3(-1) + 2$ {replacing x with (-1) }
 $= -3 + 2$
 $= -1$

d $f(-5) = 3(-5) + 2$ {replacing x with (-5) }
 $= -15 + 2$
 $= -13$

e $f(-\frac{1}{3}) = 3(-\frac{1}{3}) + 2$ {replacing x with $(-\frac{1}{3})$ }
 $= -1 + 2$
 $= 1$

2 $f(x) = 3x - x^2 + 2$

a $f(0) = 3(0) - 0^2 + 2$
 $= 0 - 0 + 2$
 $= 2$

b $f(3) = 3(3) - 3^2 + 2$
 $= 9 - 9 + 2$
 $= 2$

c $f(-3) = 3(-3) - (-3)^2 + 2$
 $= -9 - 9 + 2$
 $= -16$

d $f(-7) = 3(-7) - (-7)^2 + 2$
 $= -21 - 49 + 2$
 $= -68$

e $f(\frac{3}{2}) = 3(\frac{3}{2}) - (\frac{3}{2})^2 + 2$
 $= \frac{9}{2} - \frac{9}{4} + 2$
 $= \frac{17}{4}$

3 $g(x) = x - \frac{4}{x}$

a $g(1) = 1 - \frac{4}{1} = -3$

b $g(4) = 4 - \frac{4}{4} = 3$

c $g(-1) = -1 - \frac{4}{(-1)} = 3$

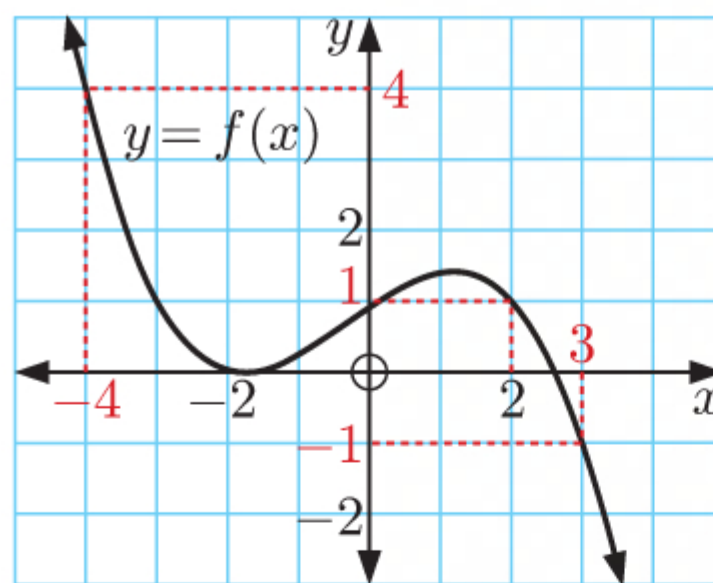
d $g(-4) = -4 - \frac{4}{(-4)} = -3$

e $g(-\frac{1}{2}) = -\frac{1}{2} - \frac{4}{(-\frac{1}{2})} = -\frac{1}{2} + 8 = \frac{15}{2}$

4 a i $f(2) = 1$

ii $f(3) = -1$

b When $y = f(x) = 4$, $x = -4$.



5 $G(x) = \frac{2x+3}{x-4}$

a i $G(2) = \frac{2(2)+3}{2-4}$
 $= \frac{7}{-2}$
 $= -\frac{7}{2}$

ii $G(0) = \frac{2(0)+3}{0-4}$
 $= \frac{3}{-4}$
 $= -\frac{3}{4}$

iii $G(-\frac{1}{2}) = \frac{2(-\frac{1}{2})+3}{-\frac{1}{2}-4}$
 $= \frac{-1+3}{(-\frac{9}{2})}$
 $= \frac{2}{(-\frac{9}{2})}$
 $= -\frac{4}{9}$

b $G(x) = \frac{2x+3}{x-4}$ is undefined when $x-4=0$
 $\therefore x=4$

So, when $x=4$, $G(x)$ does not exist.

c $G(x) = -3$, so $\frac{2x+3}{x-4} = -3$
 $\therefore 2x+3 = -3(x-4)$
 $\therefore 2x+3 = -3x+12$
 $\therefore 5x = 9$
 $\therefore x = \frac{9}{5}$

6 $f(x) = 1-3x$, $g(x) = \sqrt{x+5}$

a $f(-1) = 1-3(-1)$
 $= 1+3$
 $= 4$
 $g(11) = \sqrt{11+5}$
 $= \sqrt{16}$
 $= 4$

$\therefore f(-1) = g(11)$

b $f(x) = g(4)$
 $\therefore 1-3x = \sqrt{4+5}$
 $\therefore 1-3x = \sqrt{9}$
 $\therefore 1-3x = 3$
 $\therefore -3x = 2$
 $\therefore x = -\frac{2}{3}$

7 $f(x) = 7-3x$

a $f(a) = 7-3a$

b $f(-a) = 7-3(-a)$
 $= 7+3a$

c $f(a+3) = 7-3(a+3)$
 $= 7-3a-9$
 $= -3a-2$

d $f(2a) = 7-3(2a)$
 $= 7-6a$

e $f(x+2) = 7-3(x+2)$
 $= 7-3x-6$
 $= 1-3x$

f $f(x+h) = 7-3(x+h)$
 $= 7-3x-3h$

8 $F(x) = 2x^2 + 3x - 1$

a $F(x+4)$
 $= 2(x+4)^2 + 3(x+4) - 1$
 $= 2(x^2 + 8x + 16) + 3x + 12 - 1$
 $= 2x^2 + 16x + 32 + 3x + 11$
 $= 2x^2 + 19x + 43$

c $F(-x)$
 $= 2(-x)^2 + 3(-x) - 1$
 $= 2x^2 - 3x - 1$

e $F(3x)$
 $= 2(3x)^2 + 3(3x) - 1$
 $= 2(9x^2) + 9x - 1$
 $= 18x^2 + 9x - 1$

b $F(2-x)$
 $= 2(2-x)^2 + 3(2-x) - 1$
 $= 2(4 - 4x + x^2) + 6 - 3x - 1$
 $= 8 - 8x + 2x^2 + 5 - 3x$
 $= 2x^2 - 11x + 13$

d $F(x^2)$
 $= 2(x^2)^2 + 3(x^2) - 1$
 $= 2x^4 + 3x^2 - 1$

f $F(x+h)$
 $= 2(x+h)^2 + 3(x+h) - 1$
 $= 2(x^2 + 2xh + h^2) + 3x + 3h - 1$
 $= 2x^2 + 4xh + 2h^2 + 3x + 3h - 1$
 $= 2x^2 + (4h+3)x + 2h^2 + 3h - 1$

9 $f(x) = x^2$

a $f(3x) = (3x)^2$
 $= 9x^2$

c $3f(x) = 3x^2$

b $f\left(\frac{x}{2}\right) = \left(\frac{x}{2}\right)^2$
 $= \frac{x^2}{4}$

d $2f(x-1) + 5 = 2(x-1)^2 + 5$
 $= 2(x^2 - 2x + 1) + 5$
 $= 2x^2 - 4x + 2 + 5$
 $= 2x^2 - 4x + 7$

10 $f(x) = \frac{1}{x}$

a $f(-x) = \frac{1}{(-x)}$
 $= -\frac{1}{x}$

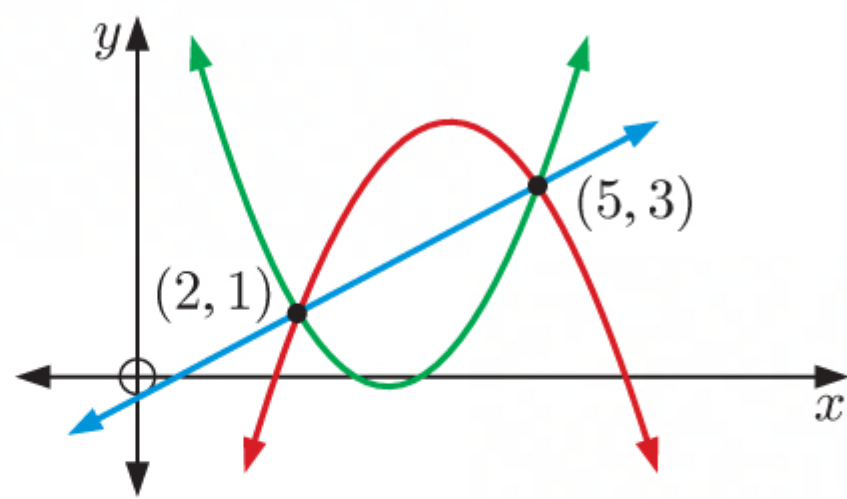
b $f\left(\frac{1}{2}x\right) = \frac{1}{\frac{1}{2}x}$
 $= \frac{1}{\left(\frac{x}{2}\right)}$
 $= \frac{2}{x}$

c $2f(x) + 3 = 2 \times \frac{1}{x} + 3$
 $= \frac{2}{x} + \frac{3x}{x}$
 $= \frac{2+3x}{x}$

d $3f(x-1) + 2 = 3 \times \frac{1}{x-1} + 2$
 $= \frac{3}{x-1} + \frac{2(x-1)}{x-1}$
 $= \frac{3+2x-2}{x-1}$
 $= \frac{2x+1}{x-1}$

11 f is the function which converts x into $f(x)$ whereas $f(x)$ is the value of the function at any value of x .

12

**Note:** Other answers are possible.First sketch the straight line which passes through the points $(2, 1)$ and $(5, 3)$.

Then sketch two quadratic functions which also pass through the two points.

13 $f(x) = ax + b$ where $f(2) = 1$ and $f(-3) = 11$

So, $a(2) + b = 1$

and $a(-3) + b = 11$

$\therefore 2a + b = 1$

$\therefore -3a + b = 11$

$\therefore b = 1 - 2a \quad \dots (*)$

$\therefore -3a + (1 - 2a) = 11 \quad \{\text{using } (*)\}$

$\therefore -5a = 10$

$\therefore a = -2$

Substituting $a = -2$ into $(*)$ gives $b = 1 - 2(-2) = 5$.So, $a = -2$, $b = 5$, and hence $f(x) = -2x + 5$.14 a $P(t) = 5 + 10t$

$$\begin{aligned} \therefore P(3) &= 5 + 10(3) \\ &= 35 \end{aligned}$$

There are 35 L of petrol in the tank after 3 minutes.

b When $P(t) = 50$, then $5 + 10t = 50$

$\therefore 10t = 45$

$\therefore t = 4.5$

After $4\frac{1}{2}$ minutes, there are 50 L of petrol in the tank.c When Samantha started to fill the tank, time $t = 0$.

$$\begin{aligned} \text{Now, when } t = 0, \quad P(0) &= 5 + 10(0) \\ &= 5 \end{aligned}$$

There were 5 L of petrol in the tank when Samantha started to fill it.

15 a $H(30) = 800$

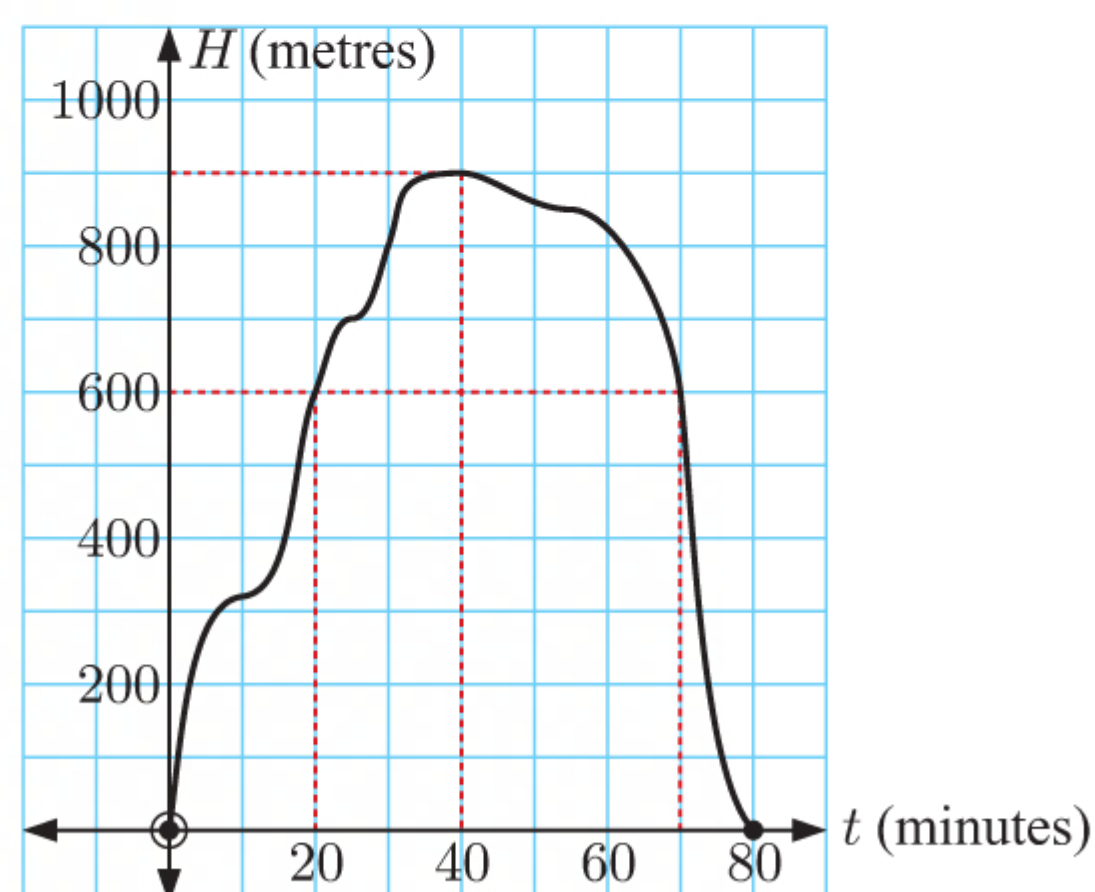
After 30 minutes, the balloon is 800 m high.

b $H(t) = 600$ when $t = 20$ or 70 .

After 20 minutes and after 70 minutes the balloon is 600 m high.

c The height of the balloon was recorded for $0 \leq t \leq 80$ minutes.

d The range of heights recorded was 0 m to 900 m.



16 $f(x) = ax + \frac{b}{x}$ where $f(1) = 1$ and $f(2) = 5$

$$\begin{aligned} \text{So, } a(1) + \frac{b}{(1)} &= 1 & \text{and } a(2) + \frac{b}{(2)} &= 5 \\ \therefore a + b &= 1 & \therefore 2a + \frac{b}{2} &= 5 \\ \therefore b &= 1 - a \quad \dots (*) & \therefore 4a + b &= 10 \\ & & \therefore 4a + (1 - a) &= 10 \quad \{\text{using } (*)\} \\ & & \therefore 3a &= 9 \\ & & \therefore a &= 3 \end{aligned}$$

Substituting $a = 3$ into $(*)$ gives $b = 1 - 3 = -2$.

So, $a = 3$, $b = -2$.

17 $T(x) = ax^2 + bx + c$ where $T(0) = -4$, $T(1) = -2$, and $T(2) = 6$

$$\begin{aligned} \text{So, } a(0)^2 + b(0) + c &= -4 \\ \therefore c &= -4 \end{aligned}$$

$$\begin{aligned} \text{Now } a(1)^2 + b(1) - 4 &= -2 & \text{and } a(2)^2 + b(2) - 4 &= 6 \\ \therefore a + b &= 2 & \therefore 4a + 2b &= 10 \\ \therefore b &= 2 - a \quad \dots (*) & \therefore 2a + b &= 5 \\ & & \therefore 2a + (2 - a) &= 5 \quad \{\text{using } (*)\} \\ & & \therefore a &= 3 \end{aligned}$$

Substituting $a = 3$ into $(*)$ gives $b = 2 - 3 = -1$.

So, $a = 3$, $b = -1$, and $c = -4$.

18 $V(t) = 9000 - 900t$

a
$$\begin{aligned} V(4) &= 9000 - 900(4) \\ &= 9000 - 3600 \\ &= 5400 \end{aligned}$$

$V(4)$ is the value of the photocopier in pounds after 4 years.

\therefore the value of the photocopier 4 years after purchase is £5400.

b
$$\begin{aligned} V(t) &= 3600, \text{ so } 9000 - 900t = 3600 \\ \therefore 900t &= 5400 \\ \therefore t &= 6 \end{aligned}$$

After 6 years, the value of the photocopier is £3600.

c The original purchase price is when $t = 0$.

$$\begin{aligned} \text{Now, } V(0) &= 9000 - 900(0) \\ &= 9000 \end{aligned}$$

The original purchase price was £9000.

d We have $t \geq 0$ and $V \geq 0$ since time and value are always positive.

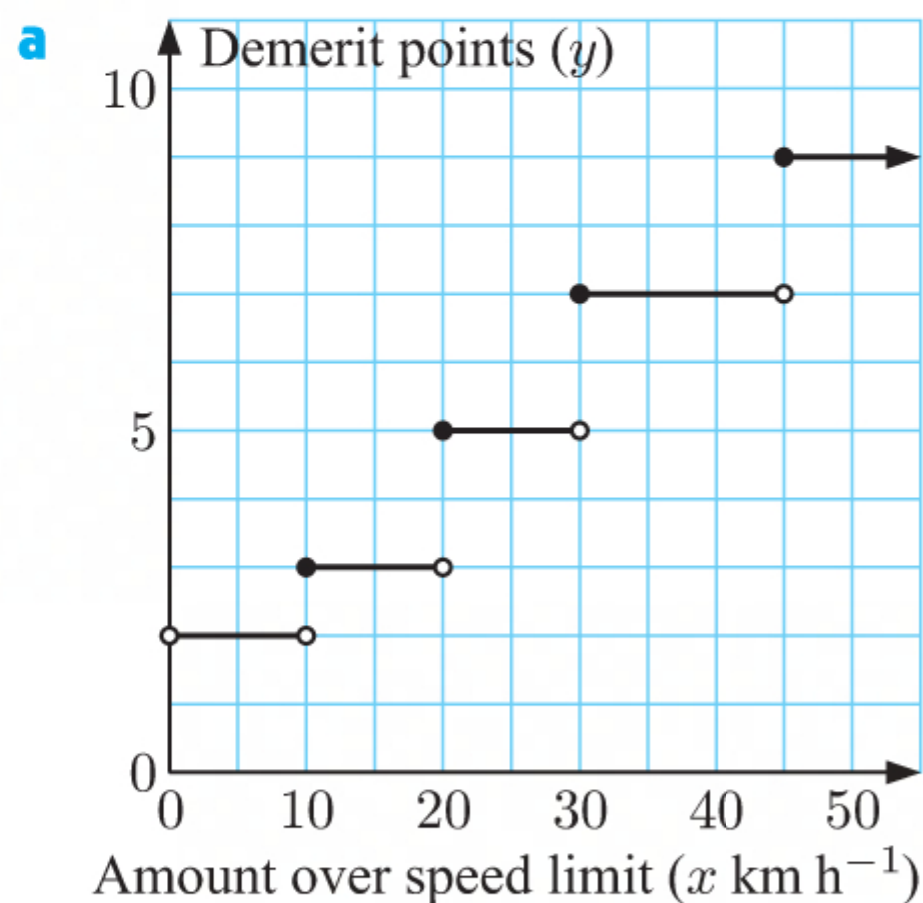
$$\begin{aligned} \text{So, } 9000 - 900t &\geq 0 \\ \therefore 9000 &\geq 900t \\ \therefore t &\leq 10 \end{aligned}$$

So, $0 \leq t \leq 10$ years.

EXERCISE 3C

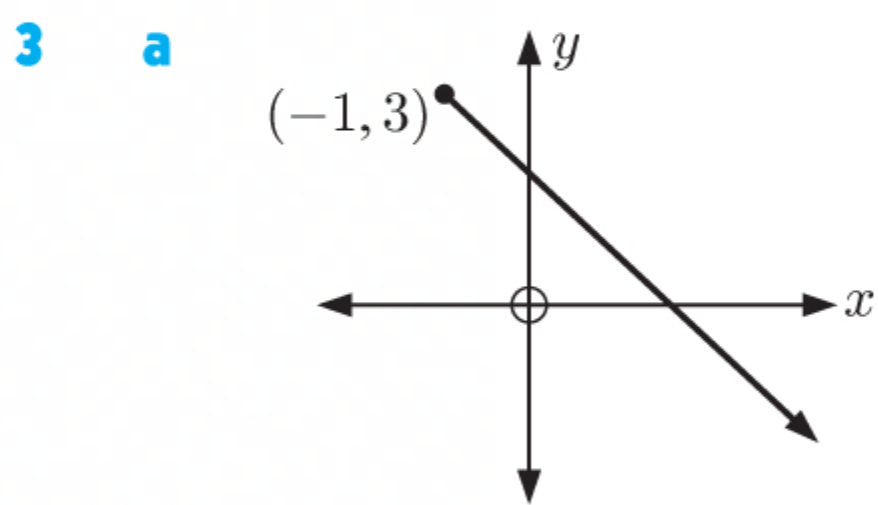
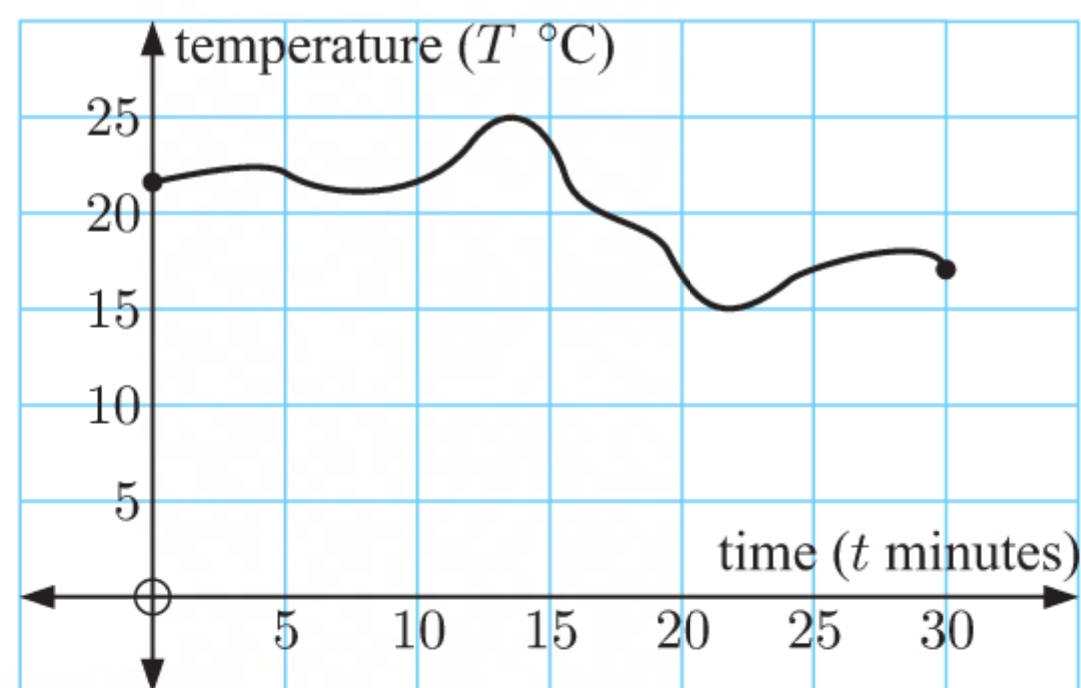
1

Amount over speed limit ($x \text{ km h}^{-1}$)	Demerit points (y)
$0 < x < 10$	2
$10 \leq x < 20$	3
$20 \leq x < 30$	5
$30 \leq x < 45$	7
$x \geq 45$	9

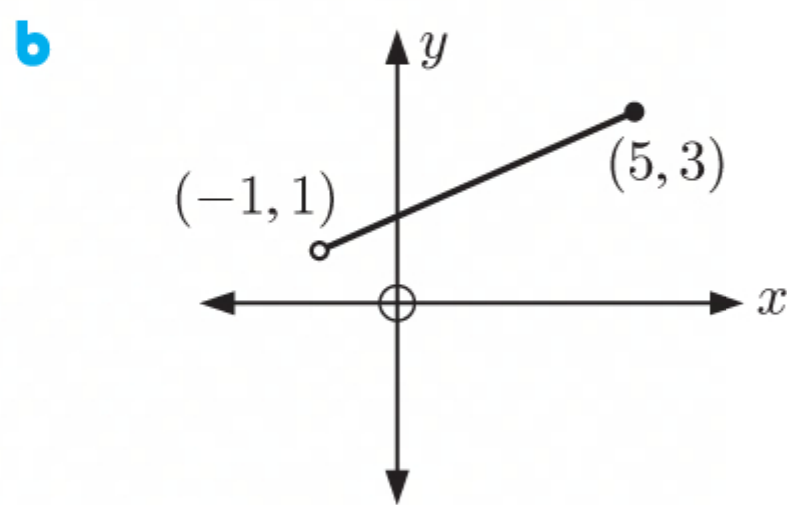


- b** The function is defined for x such that $x > 0$.
 \therefore the domain is $\{x \mid x > 0\}$.
 The possible demerit points are 2, 3, 5, 7, and 9.
 \therefore the range is $\{y \mid y = 2, 3, 5, 7, \text{ or } 9\}$.

- 2 a** At any moment in time there can be only one temperature, so the graph is a function.
- b** The temperature function is defined for all time t such that $0 \leq t \leq 30$ minutes.
 \therefore the domain is $\{t \mid 0 \leq t \leq 30\}$.
 The recorded temperatures lie between $T = 15^\circ\text{C}$ and $T = 25^\circ\text{C}$.
 \therefore the range is $\{T \mid 15 \leq T \leq 25\}$.

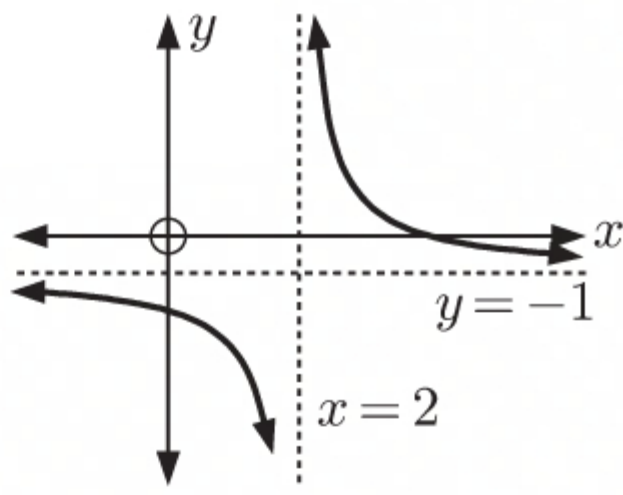


Domain is $\{x \mid x \geq -1\}$
 Range is $\{y \mid y \leq 3\}$

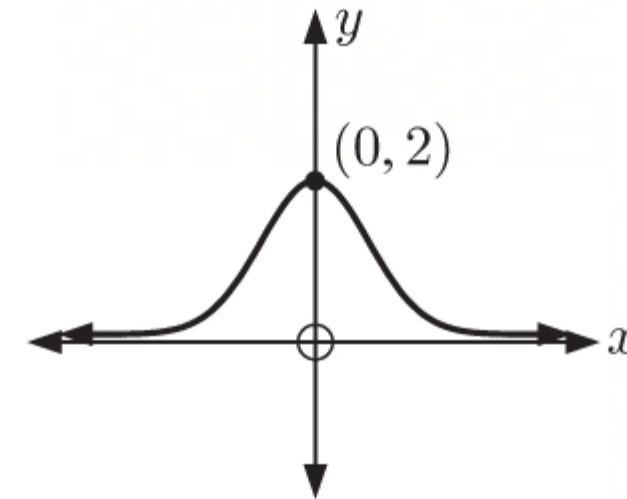


Domain is $\{x \mid -1 < x \leq 5\}$
 Range is $\{y \mid 1 < y \leq 3\}$

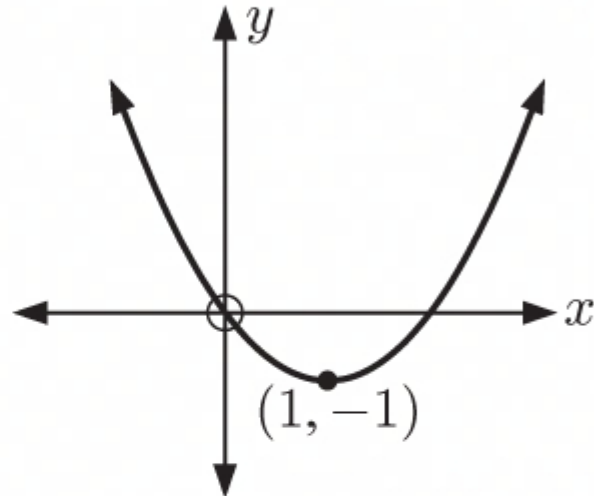
c

Domain is $\{x \mid x \neq 2\}$ Range is $\{y \mid y \neq -1\}$

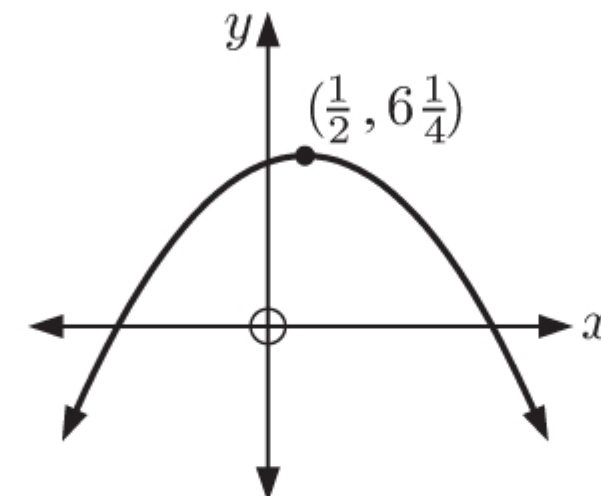
d

Domain is $\{x \mid x \in \mathbb{R}\}$ Range is $\{y \mid 0 < y \leq 2\}$

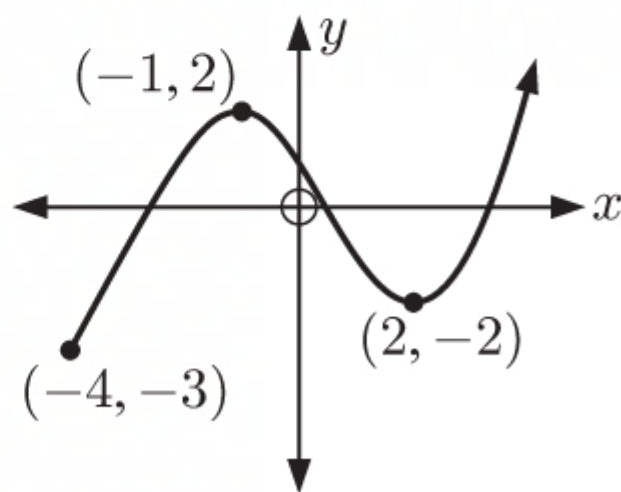
e

Domain is $\{x \mid x \in \mathbb{R}\}$ Range is $\{y \mid y \geq -1\}$

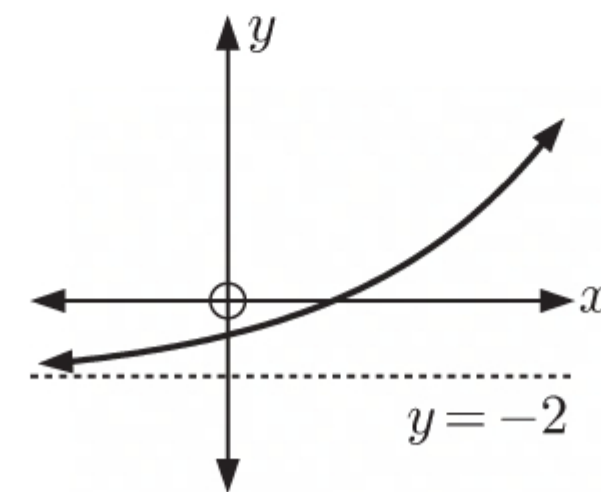
f

Domain is $\{x \mid x \in \mathbb{R}\}$ Range is $\{y \mid y \leq \frac{25}{4}\}$

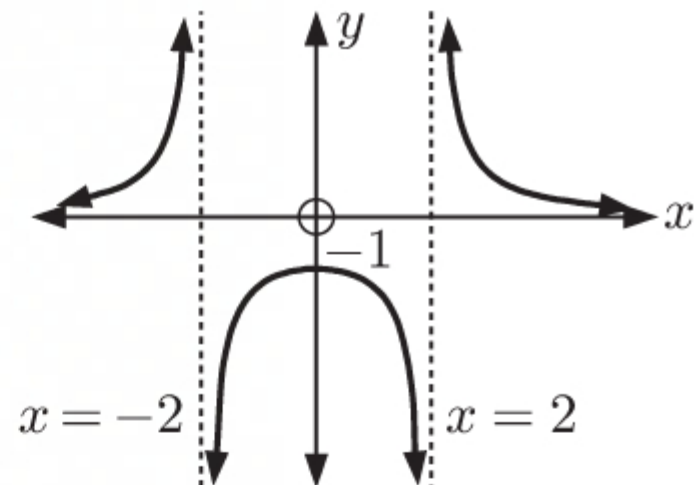
g

Domain is $\{x \mid x \geq -4\}$ Range is $\{y \mid y \geq -3\}$

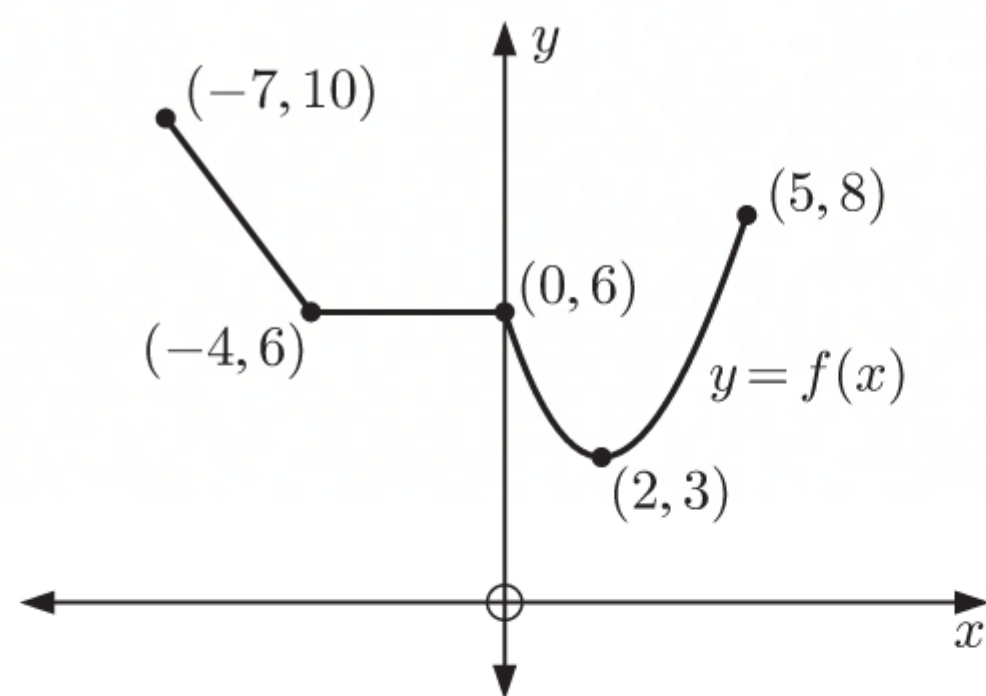
h

Domain is $\{x \mid x \in \mathbb{R}\}$ Range is $\{y \mid y > -2\}$

i

Domain is $\{x \mid x \neq \pm 2\}$ Range is $\{y \mid y \leq -1 \text{ or } y > 0\}$

4



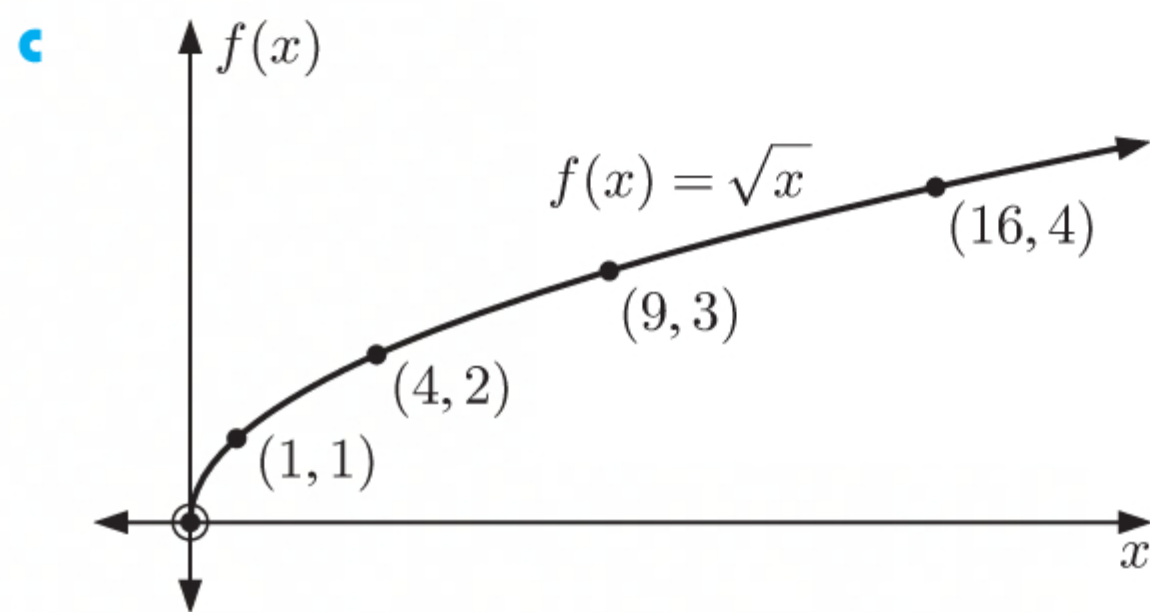
From the graph:

Domain is $\{x \mid -7 \leq x \leq 5\}$ Range is $\{y \mid 3 \leq y \leq 10\}$

- a $x = -5$ satisfies $-7 \leq x \leq 5$.
 \therefore “ -5 is in the domain of f ” is true.
- b $y = 2$ does not satisfy $3 \leq y \leq 10$.
 \therefore “ 2 is in the range of f ” is false.
- c $y = 9$ satisfies $3 \leq y \leq 10$.
 \therefore “ 9 is in the range of f ” is true.
- d $x = \sqrt{2} \approx 1.41$ satisfies $-7 \leq x \leq 5$.
 \therefore “ $\sqrt{2}$ is in the domain of f ” is true.

5 $f(x) = \sqrt{x}$

- a \sqrt{x} is defined when $x \geq 0$.
 \therefore the domain is $\{x \mid x \geq 0\}$.



x	0	1	4	9	16
$f(x)$	0	1	2	3	4

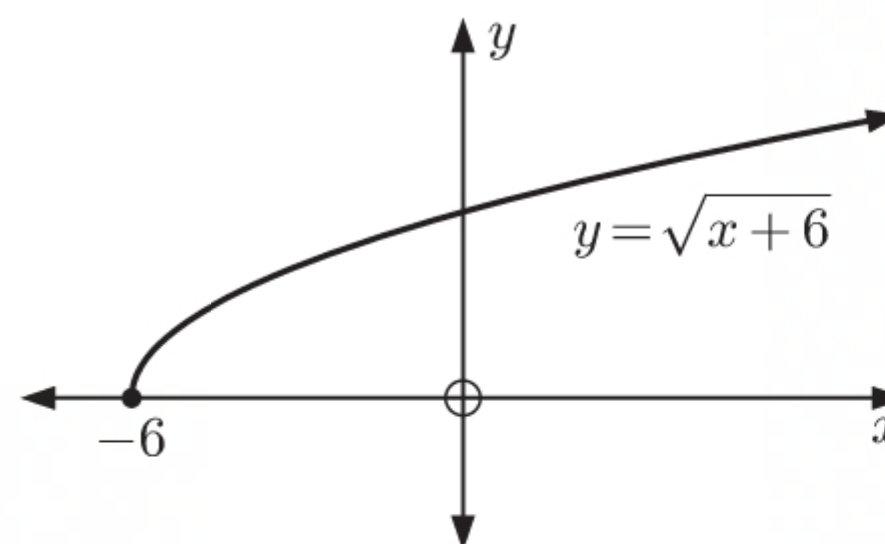
- d A square root cannot be negative.
 \therefore the range is $\{y \mid y \geq 0\}$.

6 a $\sqrt{x+6}$ is defined when $x+6 \geq 0$
 $\therefore x \geq -6$

\therefore the domain is $\{x \mid x \geq -6\}$.

A square root cannot be negative.

\therefore the range is $\{y \mid y \geq 0\}$.

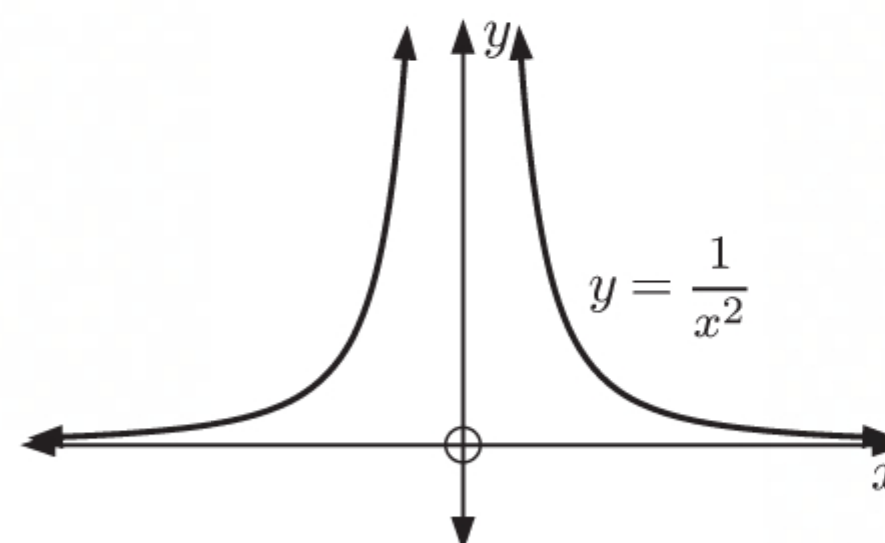


b $\frac{1}{x^2}$ is defined when $x^2 \neq 0$
 $\therefore x \neq 0$

\therefore the domain is $\{x \mid x \neq 0\}$.

$y = f(x)$ is always positive and never zero.

\therefore the range is $\{y \mid y > 0\}$.

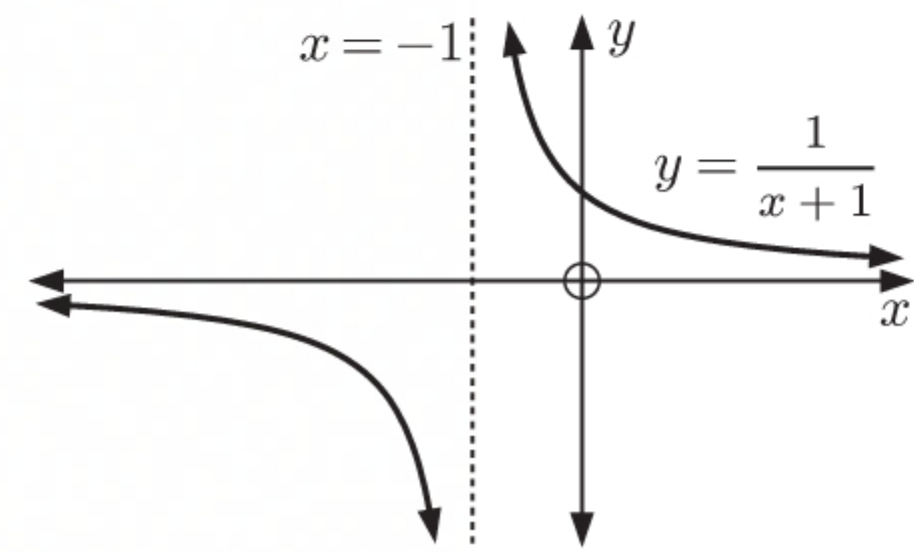


c $\frac{1}{x+1}$ is defined when $x+1 \neq 0$
 $\therefore x \neq -1$

\therefore the domain is $\{x \mid x \neq -1\}$.

No matter how large or small x is,
 $y = f(x)$ is never zero.

\therefore the range is $\{y \mid y \neq 0\}$.

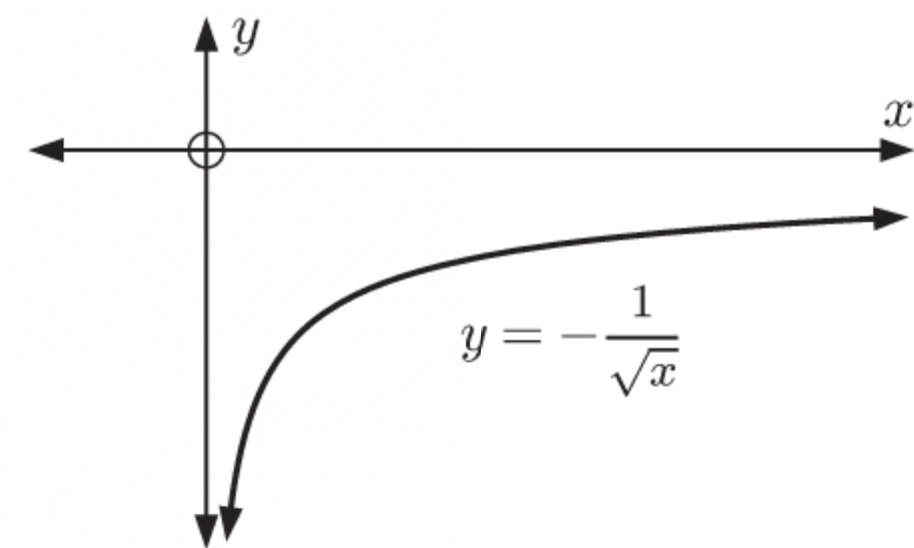


d $-\frac{1}{\sqrt{x}}$ is defined when $x > 0$

\therefore the domain is $\{x \mid x > 0\}$.

$y = f(x)$ is always negative and never zero.

\therefore the range is $\{y \mid y < 0\}$.

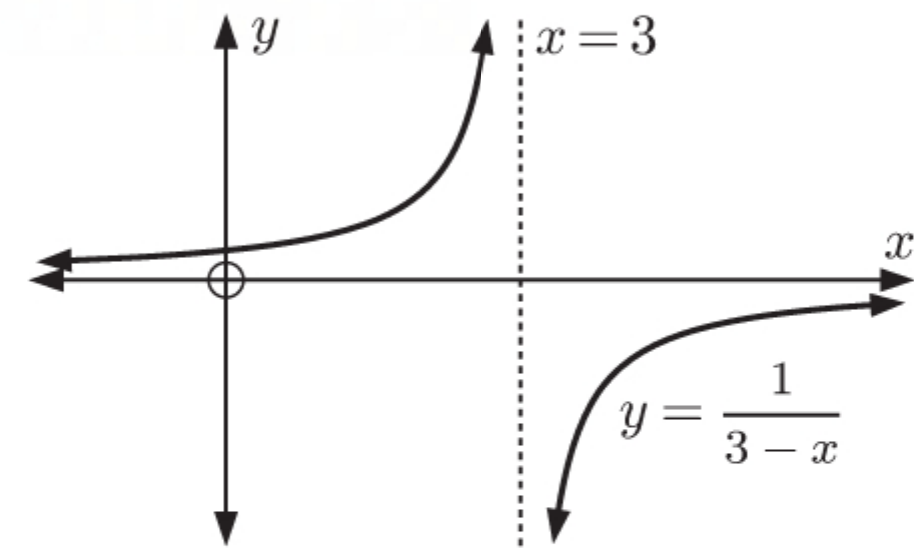


e $\frac{1}{3-x}$ is defined when $3-x \neq 0$
 $\therefore x \neq 3$

\therefore the domain is $\{x \mid x \neq 3\}$.

No matter how large or small x is,
 $y = f(x)$ is never zero.

\therefore the range is $\{y \mid y \neq 0\}$.

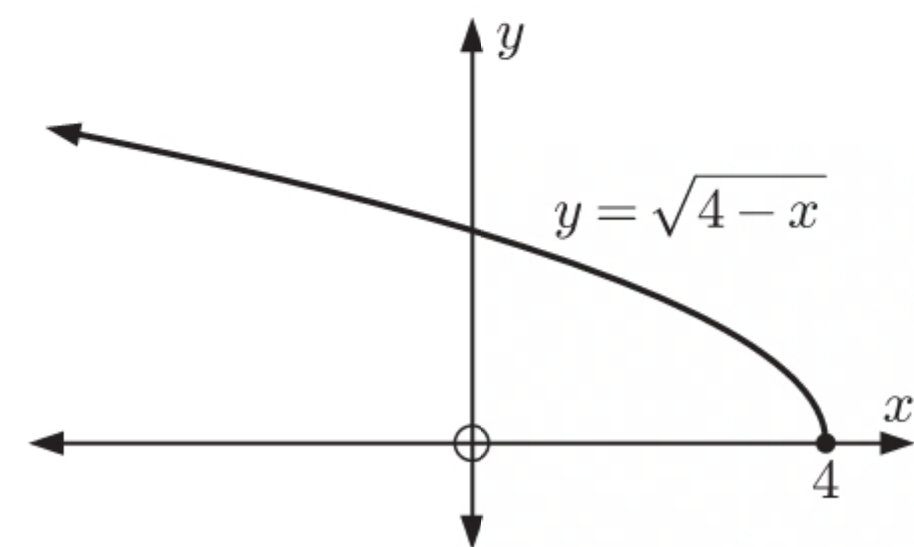


f $\sqrt{4-x}$ is defined when $4-x \geq 0$
 $\therefore x \leq 4$

\therefore the domain is $\{x \mid x \leq 4\}$.

A square root cannot be negative.

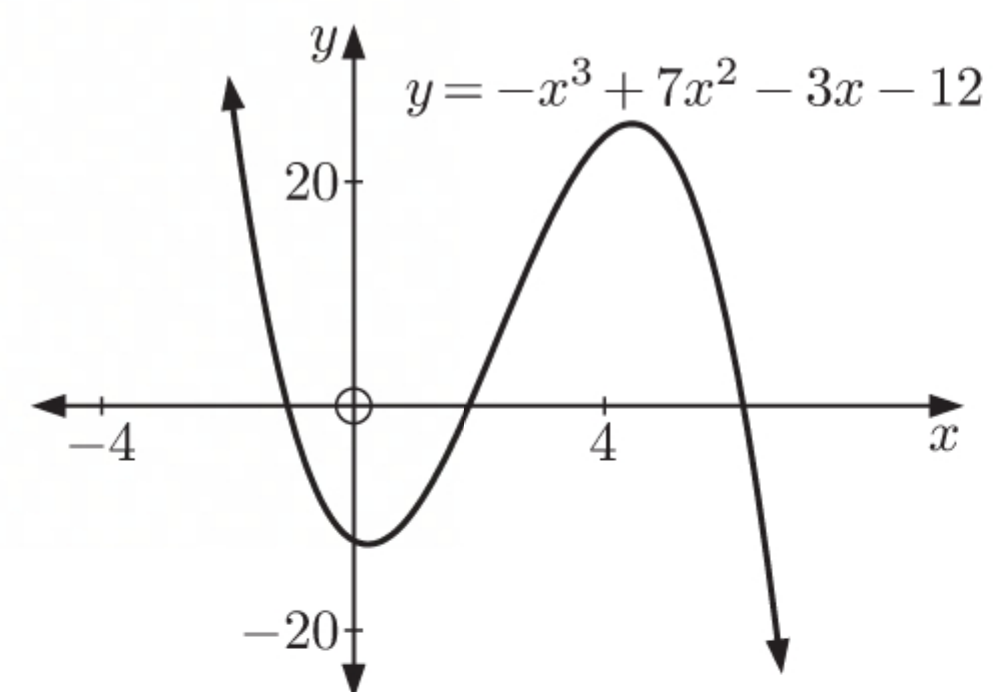
\therefore the range is $\{y \mid y \geq 0\}$.

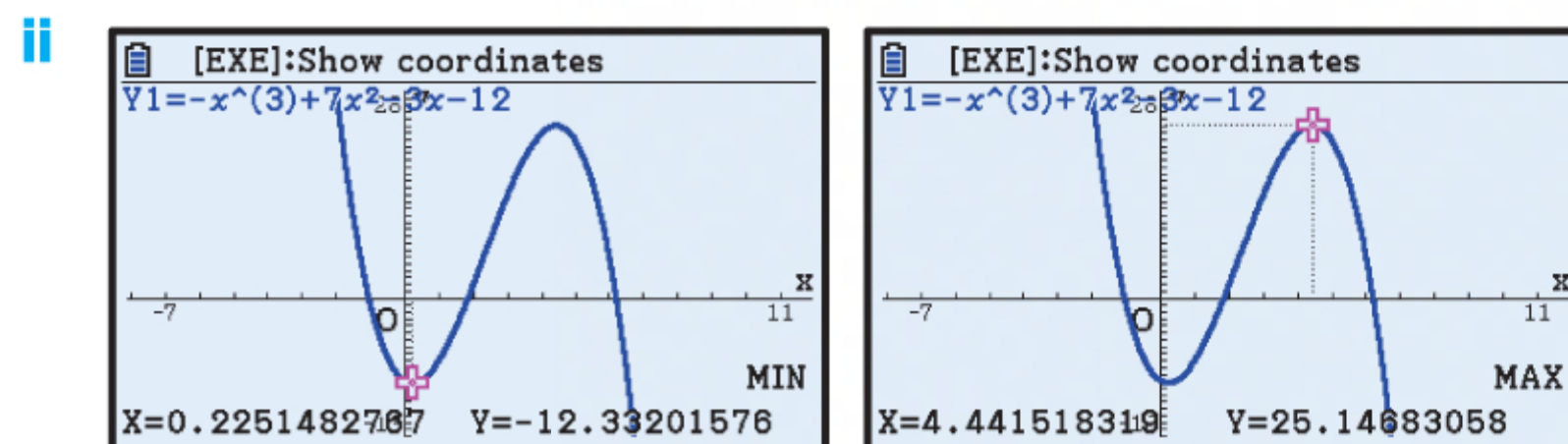
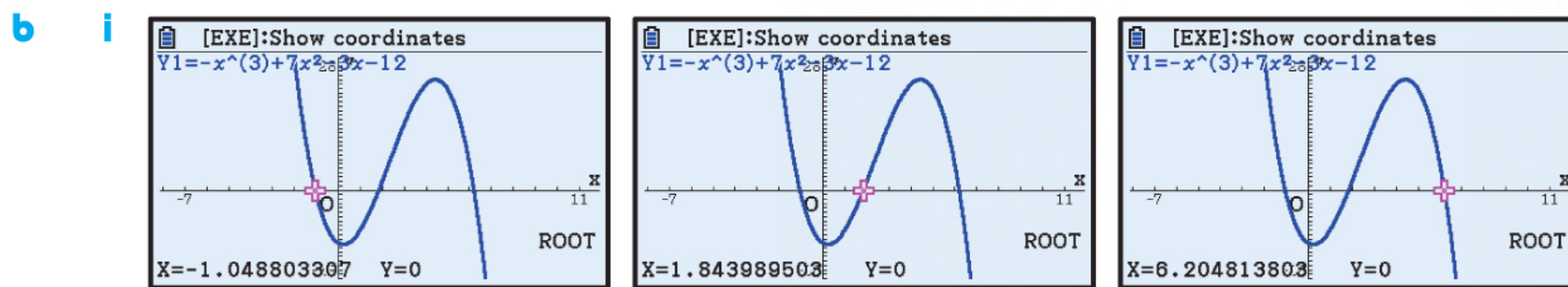


EXERCISE 3D

1 a When $x = 0$, $y = -(0)^3 + 7(0)^2 - 3(0) - 12$
 $= -12$

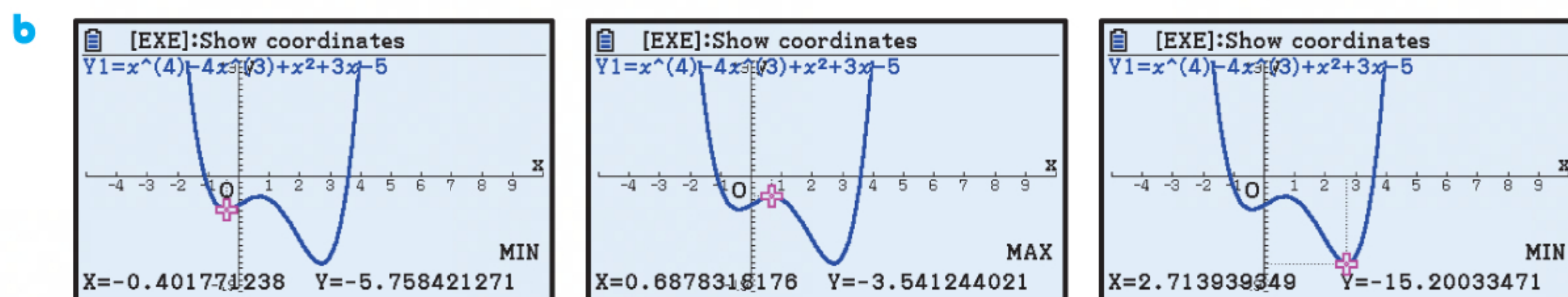
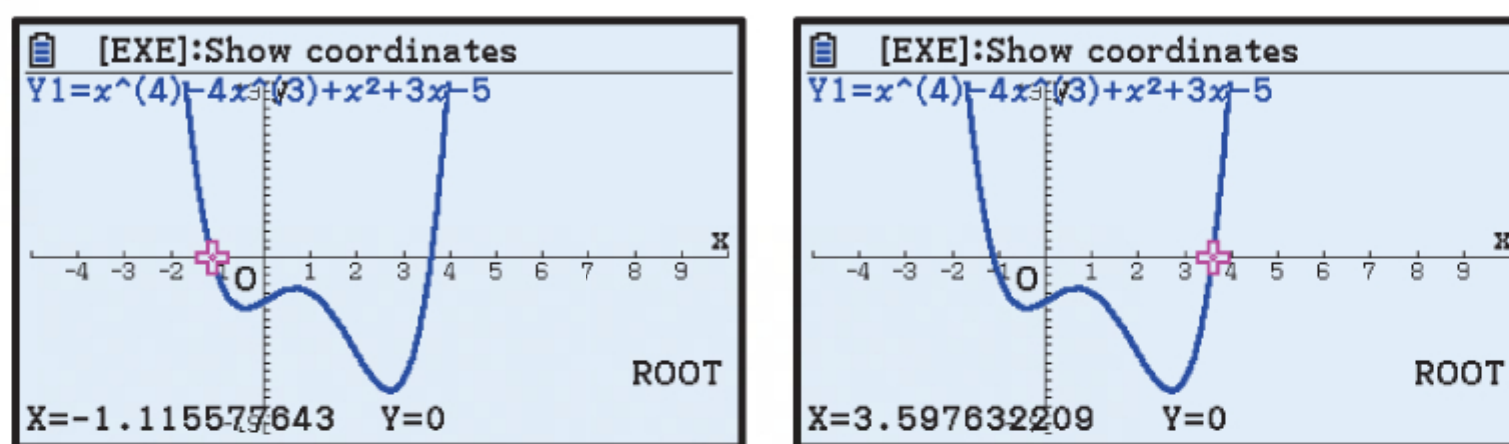
\therefore the y -intercept is -12 .



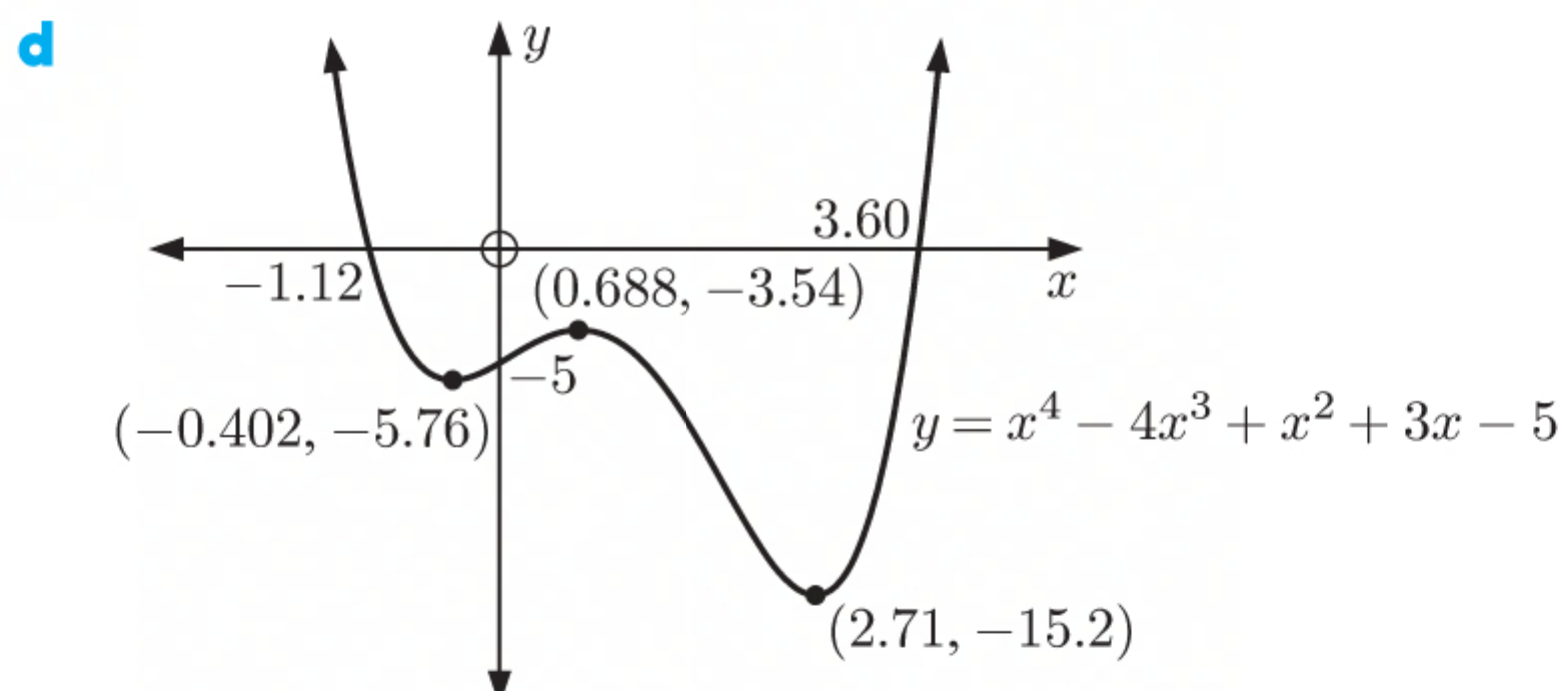


2 a When $x = 0$, $y = (0)^4 - 4(0)^3 + (0)^2 + 3(0) - 5$
 $= -5$

\therefore the y -intercept is -5 .



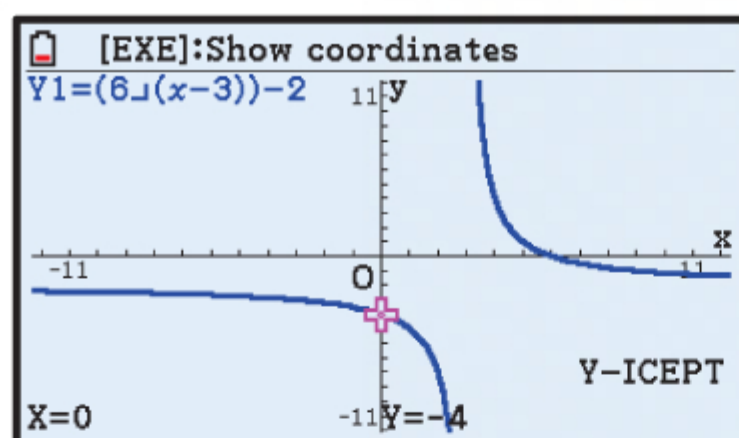
c As $x \rightarrow \infty$, $y \rightarrow \infty$ and as $x \rightarrow -\infty$, $y \rightarrow \infty$.



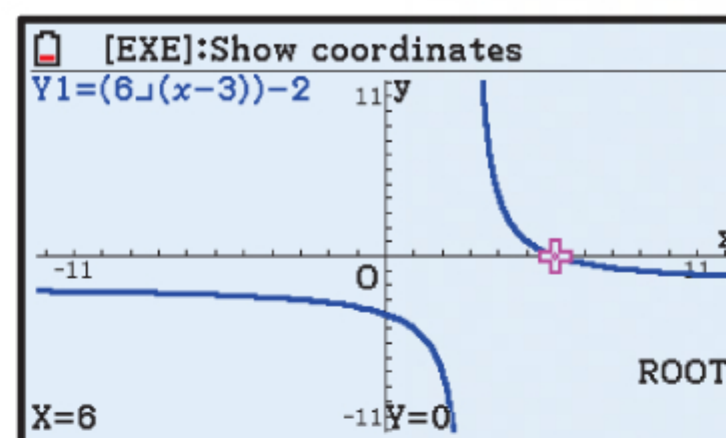
e The range is $\{y \mid y \geq -15.2\}$.

3

a



The y -intercept is -4 .



The x -intercept is 6 .

b

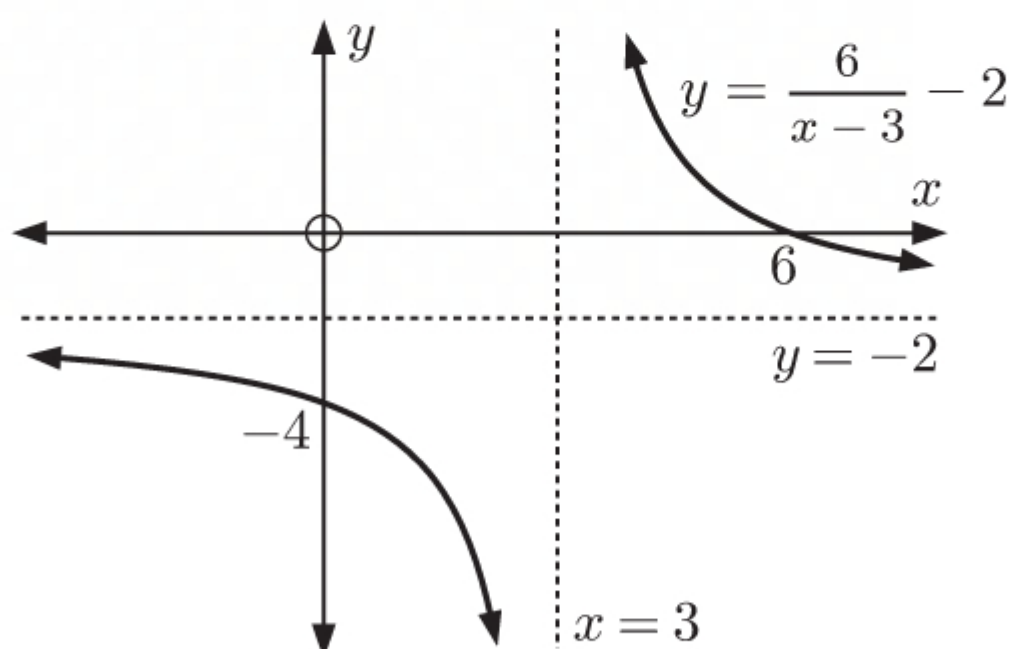
The denominator is zero when $x = 3$

\therefore there is a vertical asymptote at $x = 3$.

As $x \rightarrow \pm\infty$, the graph gets closer to the line $y = -2$.

So, $y = -2$ is a horizontal asymptote.

c



d

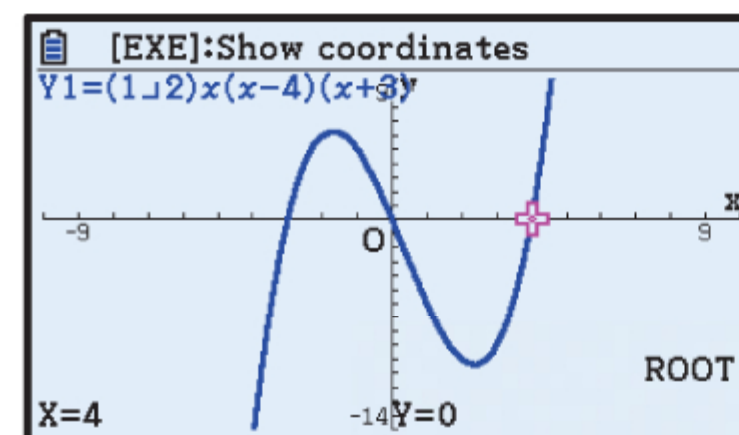
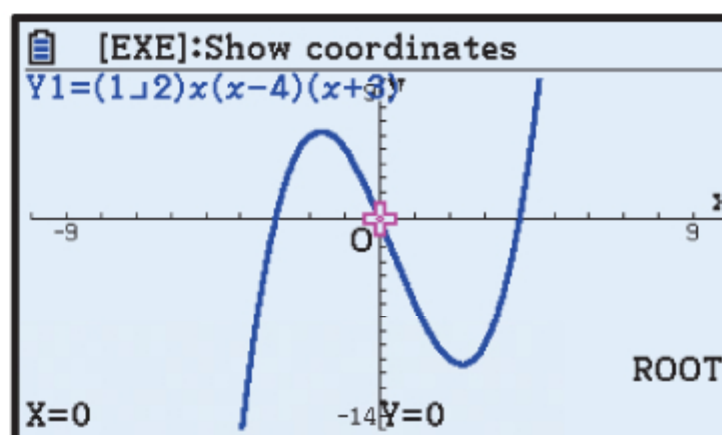
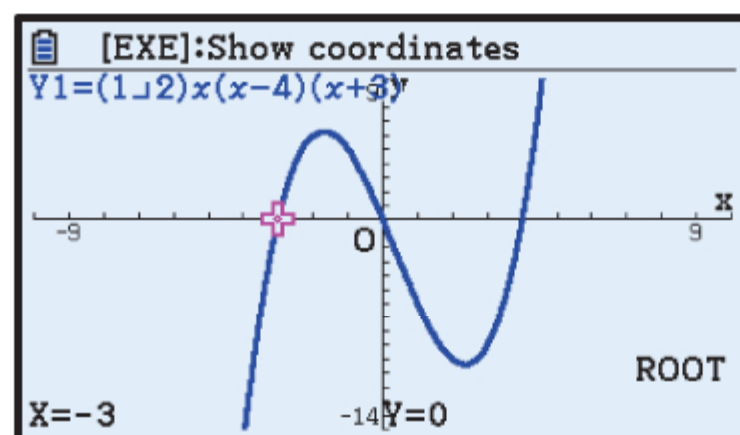
The domain is $\{x \mid x \neq 3\}$.

The range is $\{y \mid y \neq -2\}$.

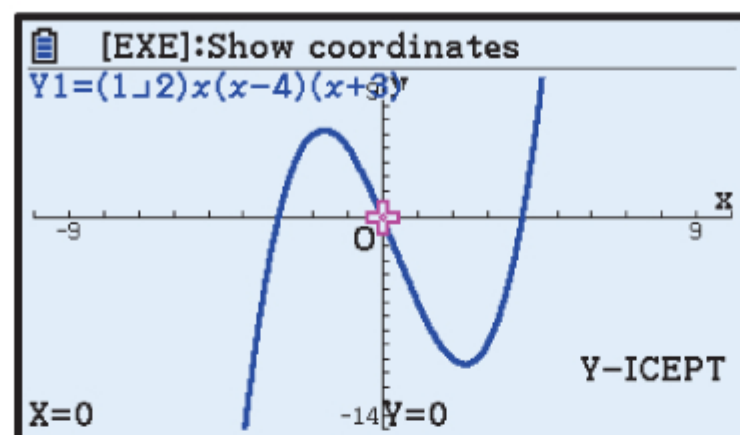
4

a

i

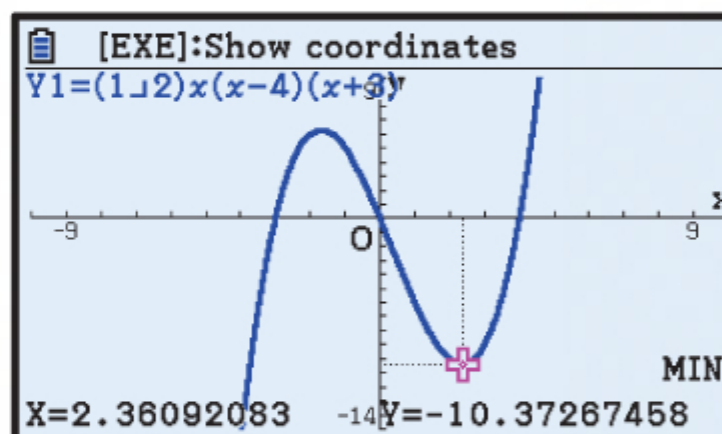
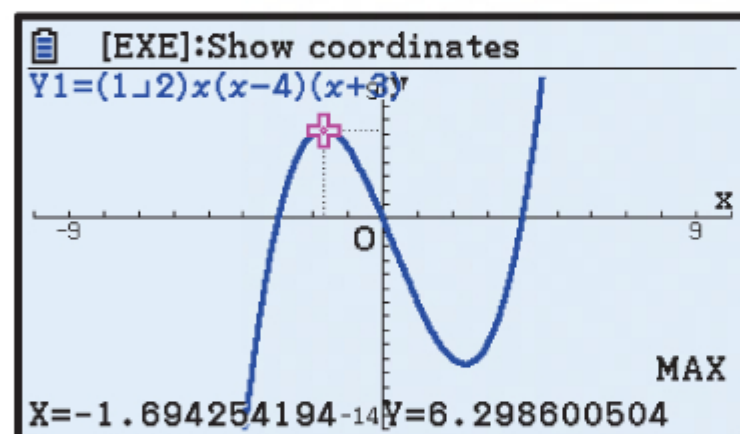


The x -intercepts are -3 , 0 , and 4 .



The y -intercept is 0 .

ii

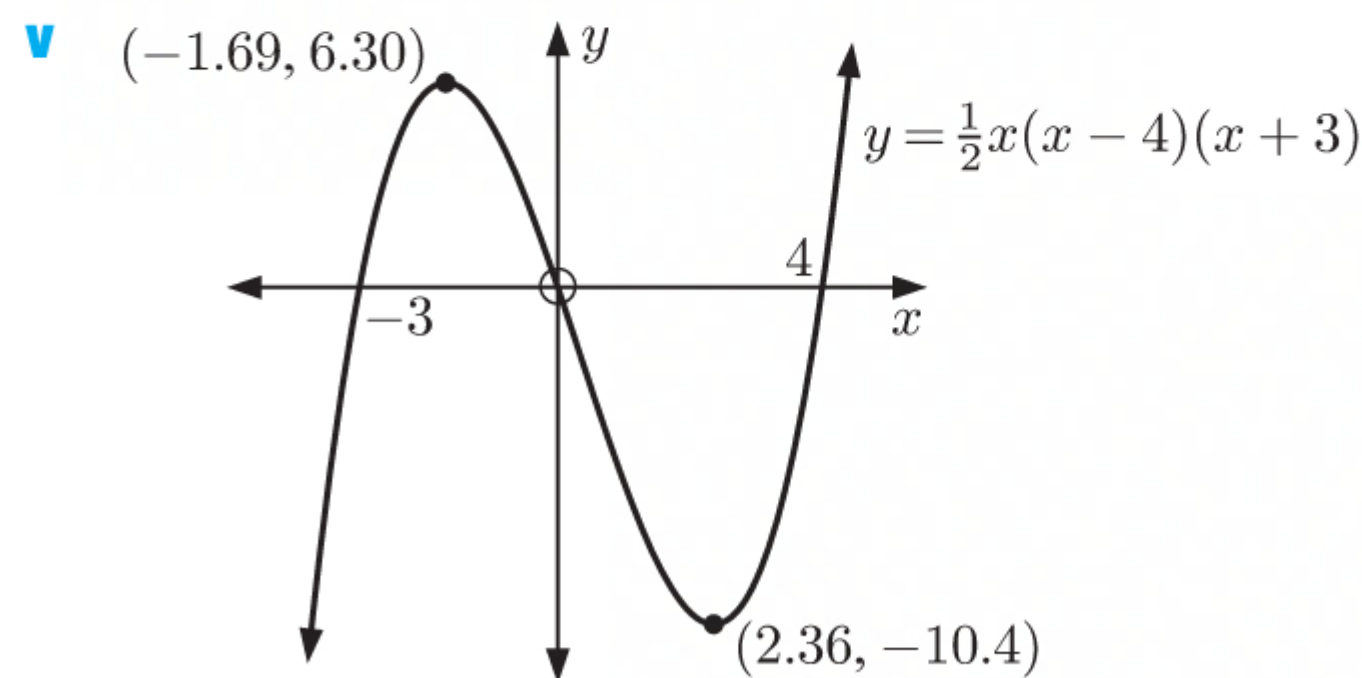


There is a maximum turning point at $(-1.69, 6.30)$ and a minimum turning point at $(2.36, -10.4)$.

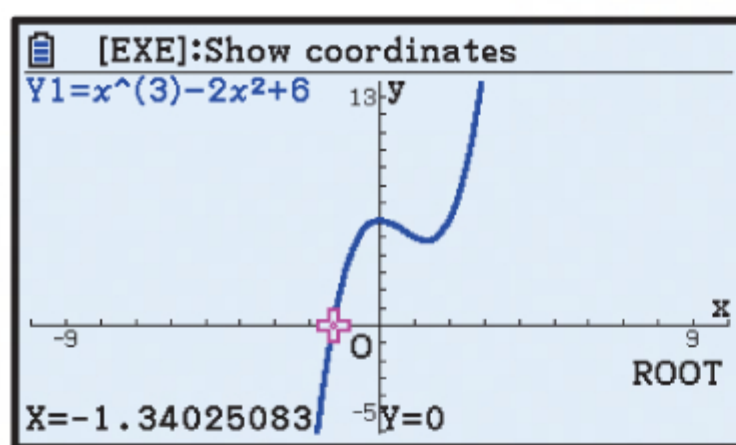
iii

The graph has no asymptotes.

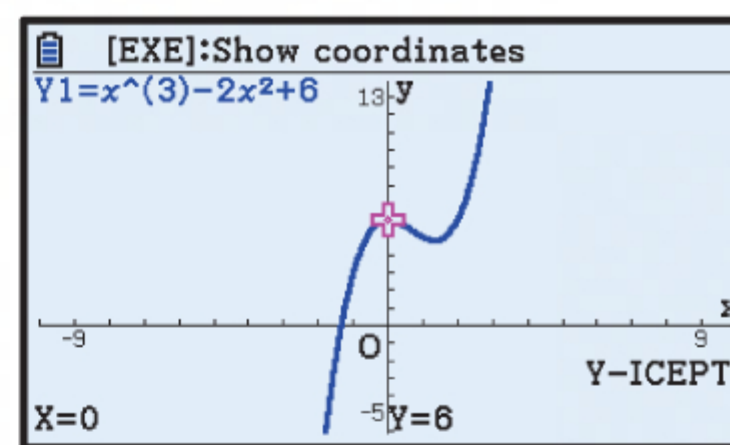
iv The domain is $\{x \mid x \in \mathbb{R}\}$. The range is $\{y \mid y \in \mathbb{R}\}$.



b i

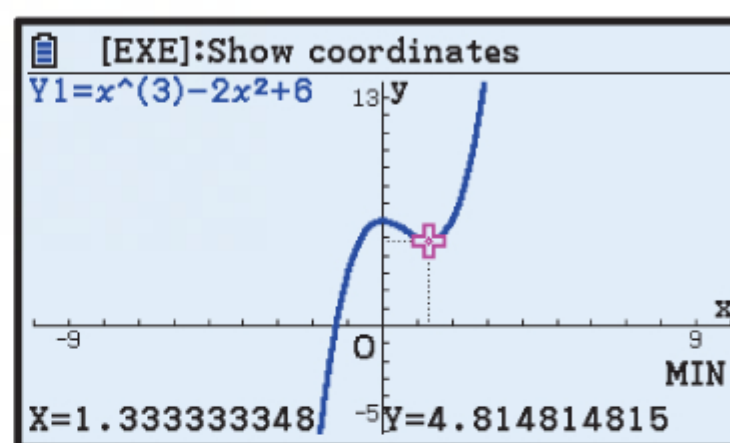
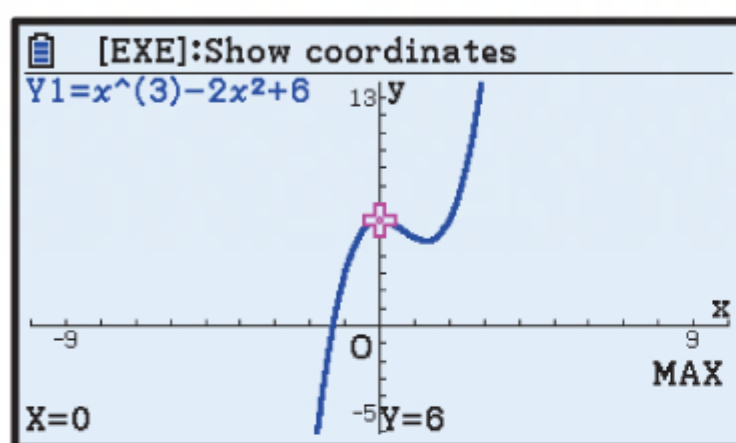


The x -intercept is ≈ -1.34 .



The y -intercept is 6.

ii

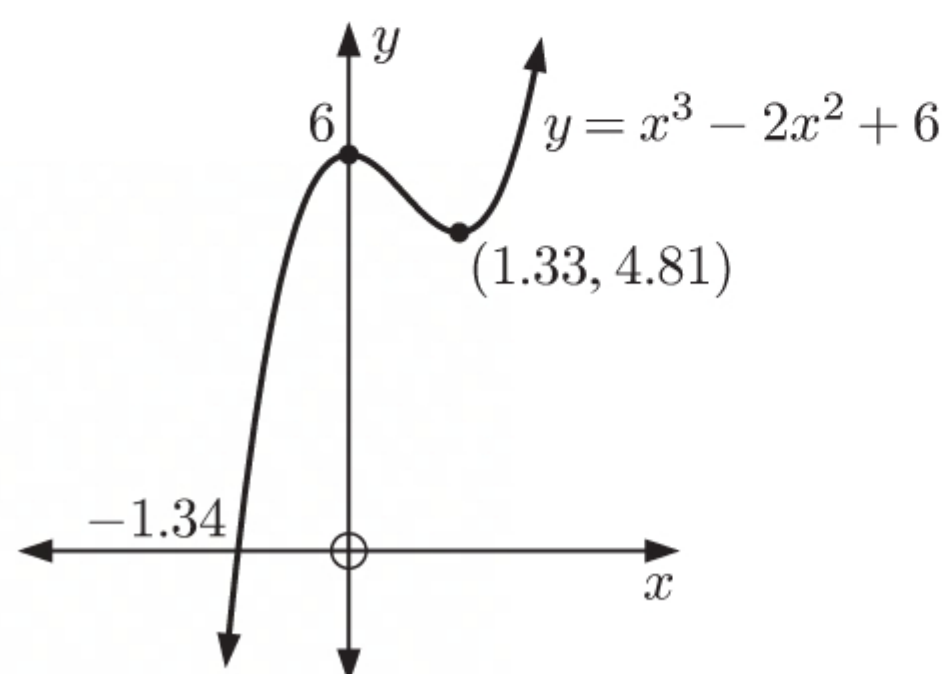


There is a maximum turning point at $(0, 6)$ and a minimum turning point at $(1.33, 4.81)$.

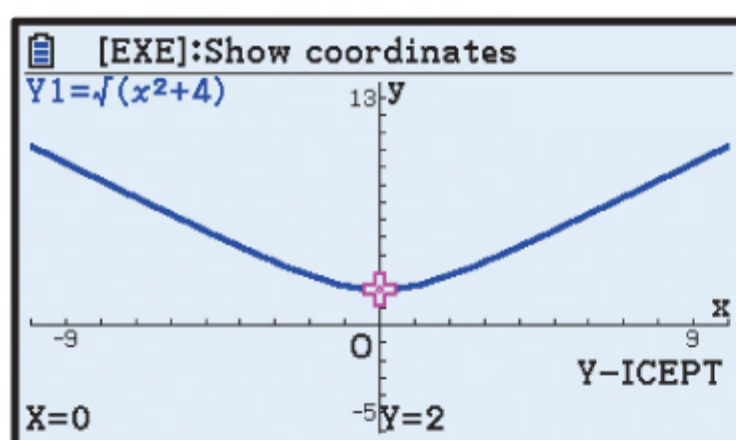
iii The graph has no asymptotes.

iv The domain is $\{x \mid x \in \mathbb{R}\}$. The range is $\{y \mid y \in \mathbb{R}\}$.

v



c i

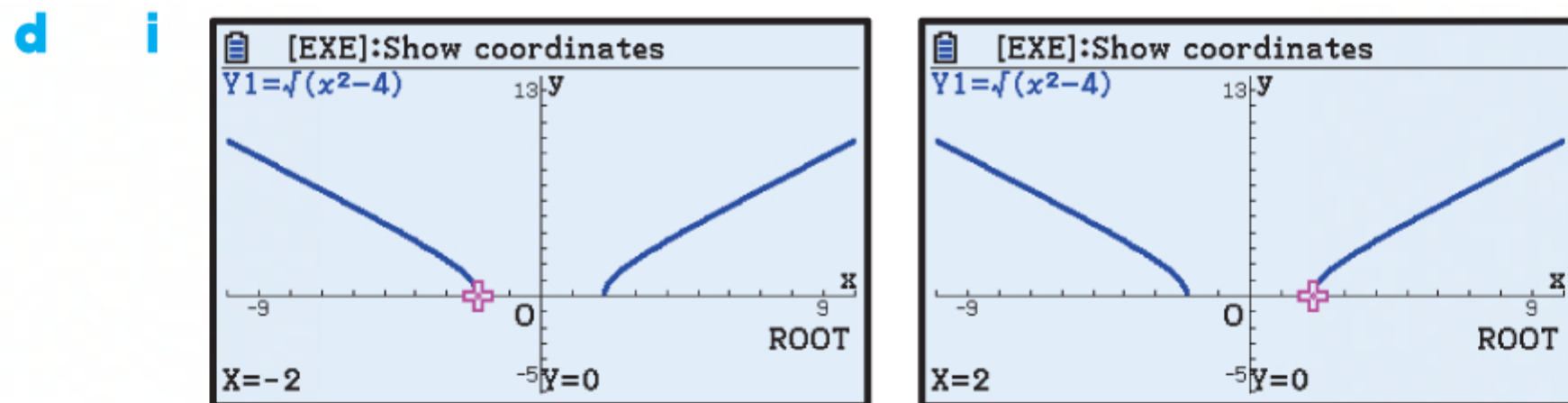
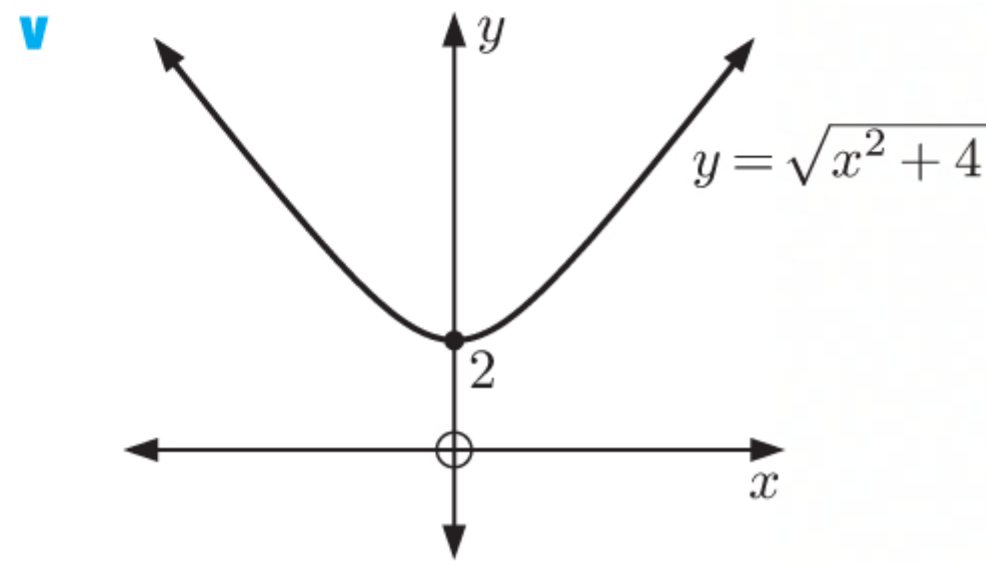


The y -intercept is 2. There are no x -intercepts.

ii There is a minimum turning point at $(0, 2)$.

iii The graph has no asymptotes.

iv The domain is $\{x \mid x \in \mathbb{R}\}$. The range is $\{y \mid y \geq 2\}$.

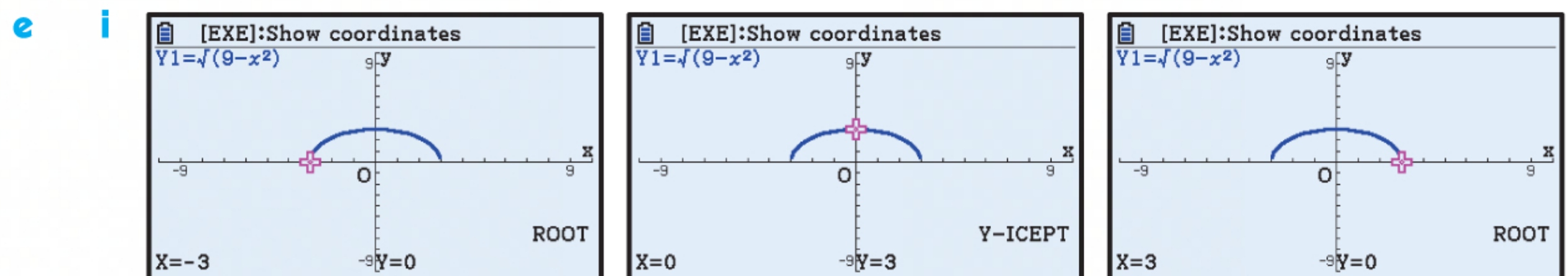
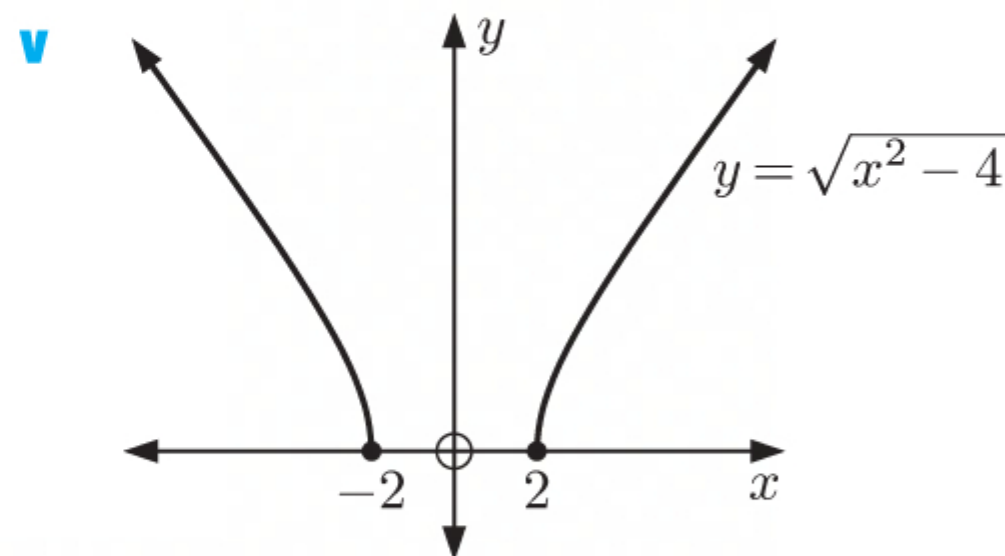


The x -intercepts are -2 and 2 . There is no y -intercept.

ii The graph has no turning points.

iii The graph has no asymptotes.

iv The domain is $\{x \mid x \leq -2 \text{ or } x \geq 2\}$. The range is $\{y \mid y \geq 0\}$.

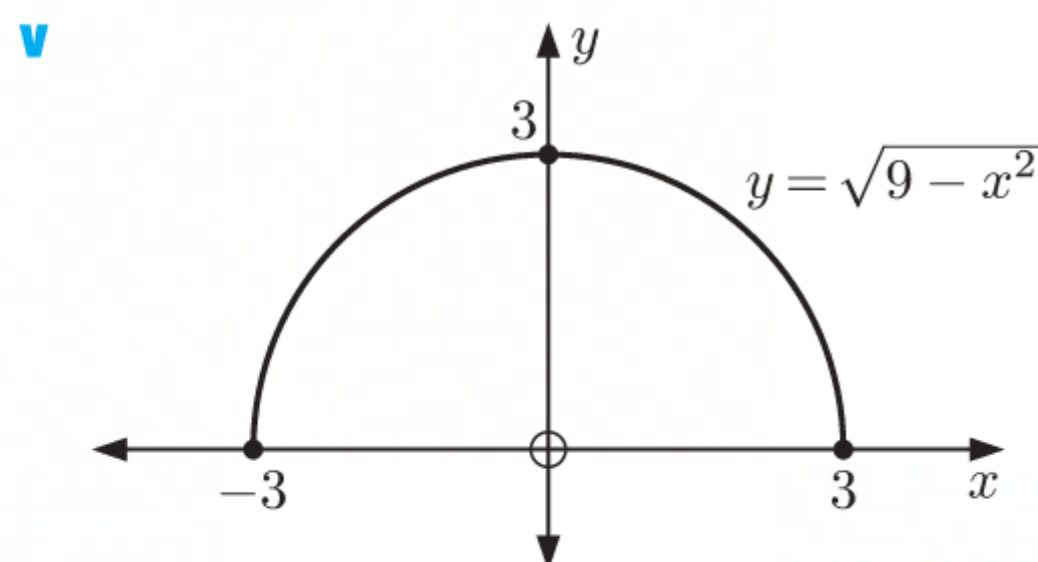


The x -intercepts are -3 and 3 . The y -intercept is 3 .

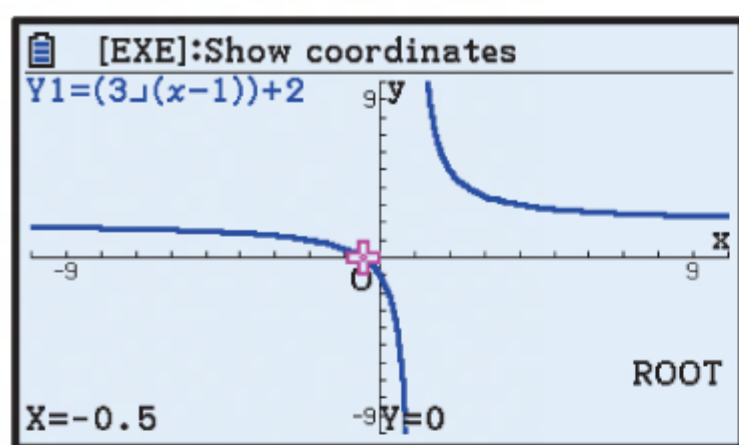
ii There is a maximum turning point at $(0, 3)$.

iii The graph has no asymptotes.

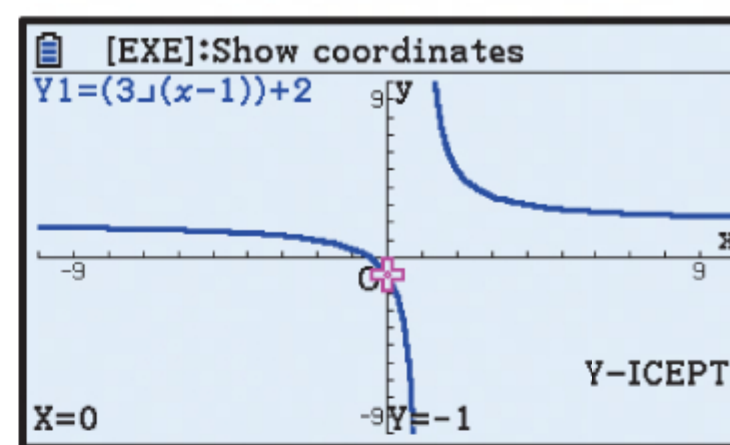
iv The domain is $\{x \mid -3 \leq x \leq 3\}$. The range is $\{y \mid 0 \leq y \leq 3\}$.



f i



The x -intercept is $-\frac{1}{2}$.



The y -intercept is -1 .

ii The graph has no turning points.

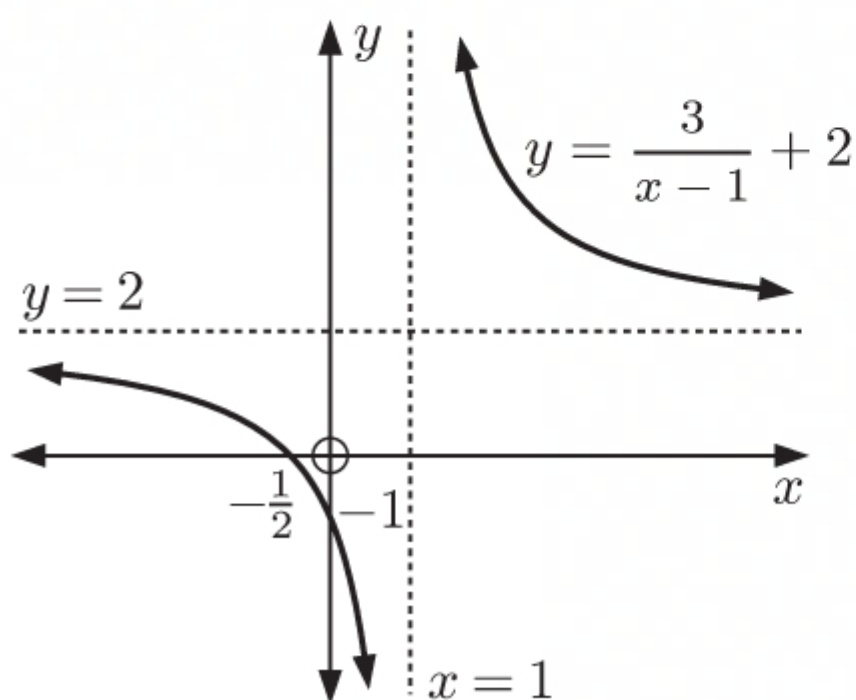
iii The graph appears to have a vertical asymptote at $x = 1$.

This is confirmed by the fact that y is undefined when $x = 1$.

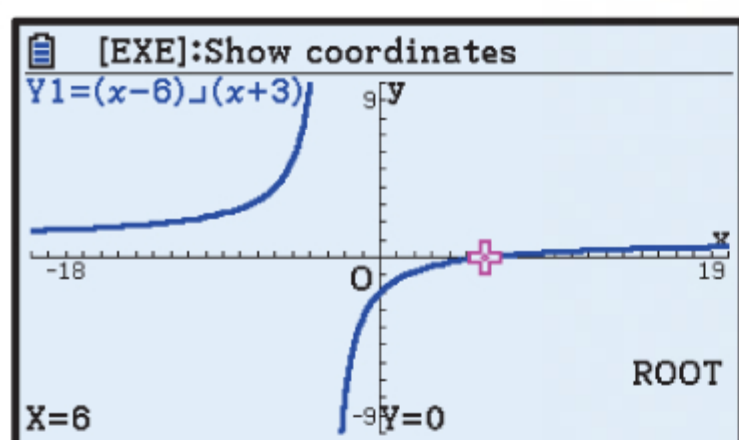
As $x \rightarrow \pm\infty$, the graph gets closer to the line $y = 2$. So, $y = 2$ is a horizontal asymptote.

iv The domain is $\{x \mid x \neq 1\}$. The range is $\{y \mid y \neq 2\}$.

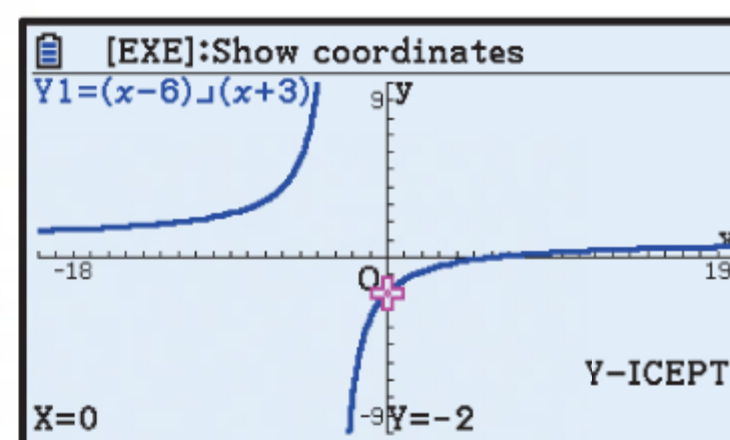
v



g i



The x -intercept is 6.



The y -intercept is -2 .

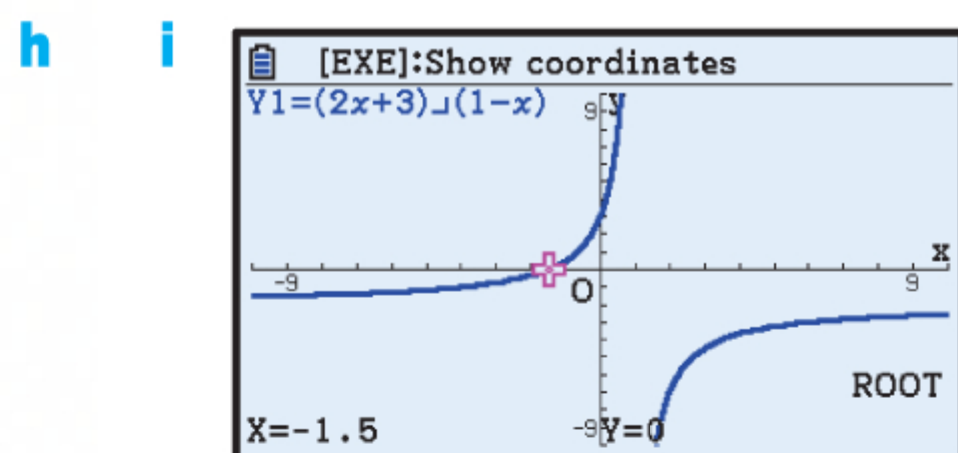
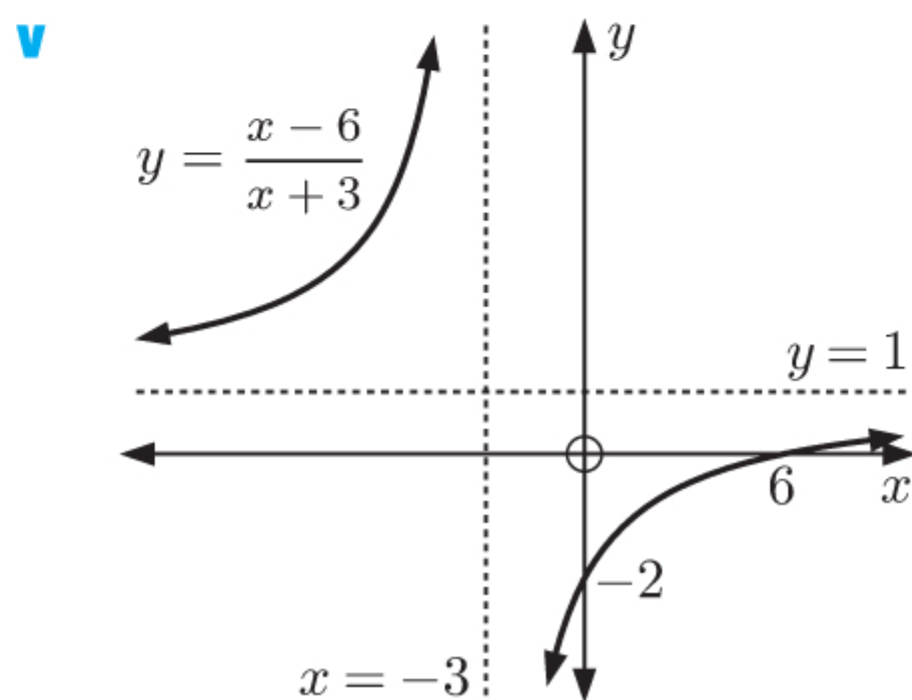
ii The graph has no turning points.

iii The graph appears to have a vertical asymptote at $x = -3$.

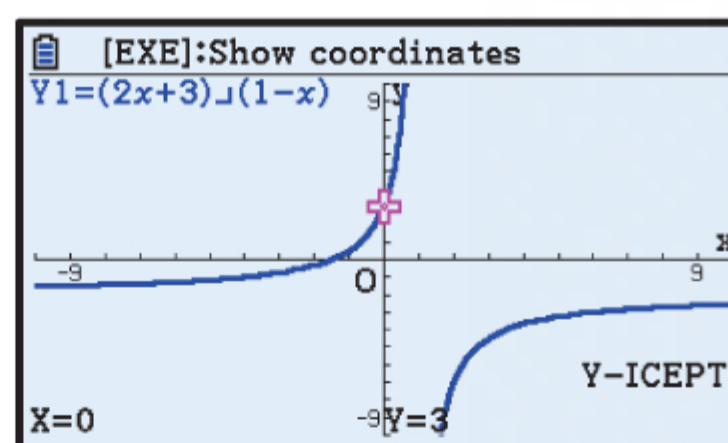
This is confirmed by the fact that y is undefined when $x = -3$.

As $x \rightarrow \pm\infty$, the graph gets closer to the line $y = 1$. So, $y = 1$ is a horizontal asymptote.

iv The domain is $\{x \mid x \neq -3\}$. The range is $\{y \mid y \neq 1\}$.



The x -intercept is $-\frac{3}{2}$.



The y -intercept is 3.

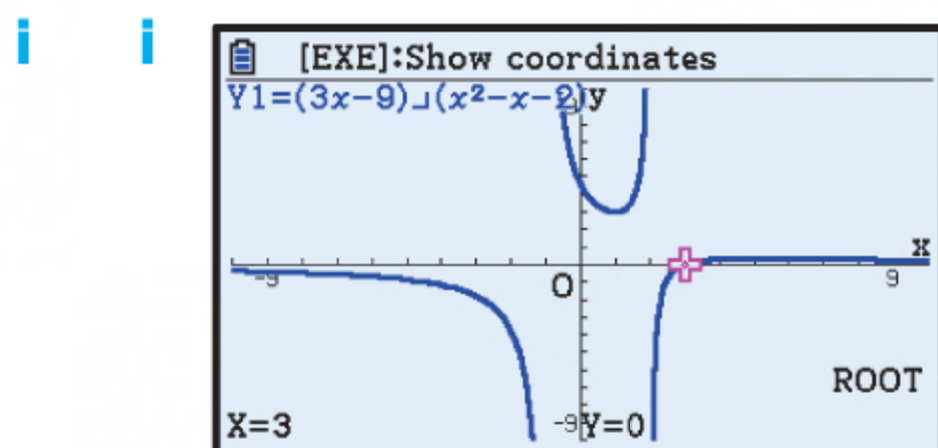
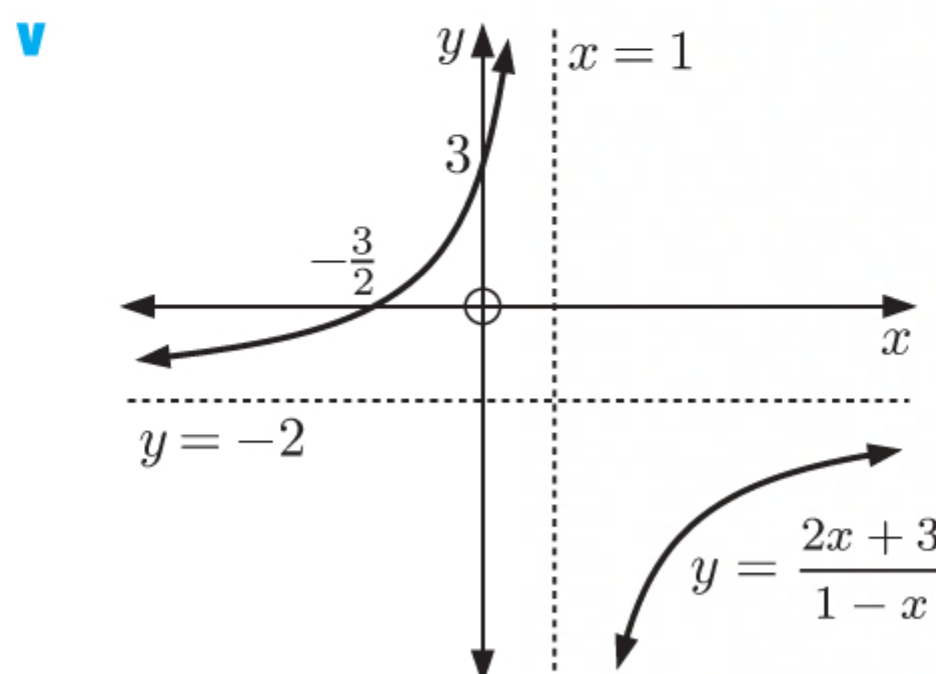
ii The graph has no turning points.

iii The graph appears to have a vertical asymptote at $x = 1$.

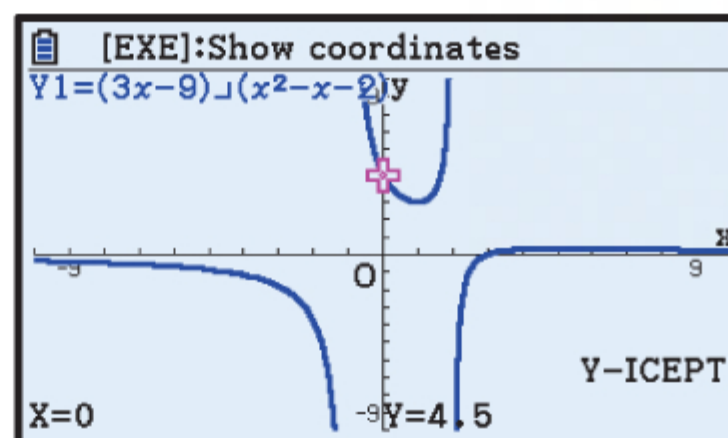
This is confirmed by the fact that y is undefined when $x = 1$.

As $x \rightarrow \pm\infty$, the graph gets closer to the line $y = -2$. So, $y = -2$ is a horizontal asymptote.

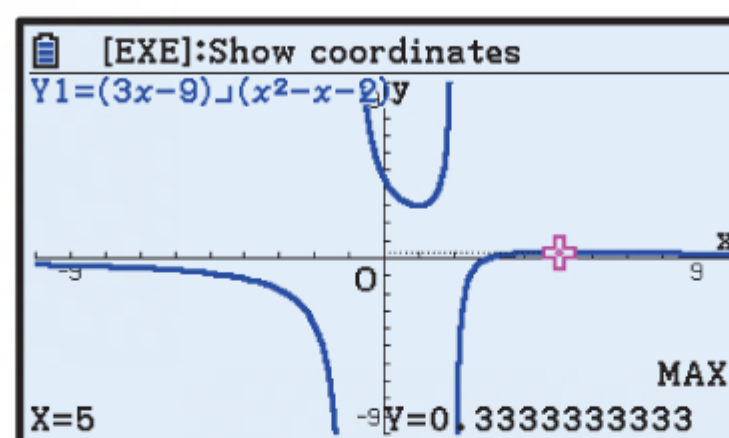
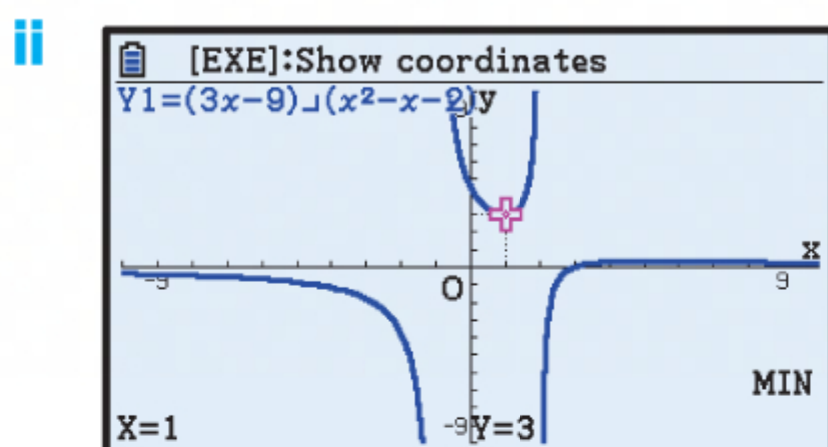
iv The domain is $\{x \mid x \neq 1\}$. The range is $\{y \mid y \neq -2\}$.



The x -intercept is 3.



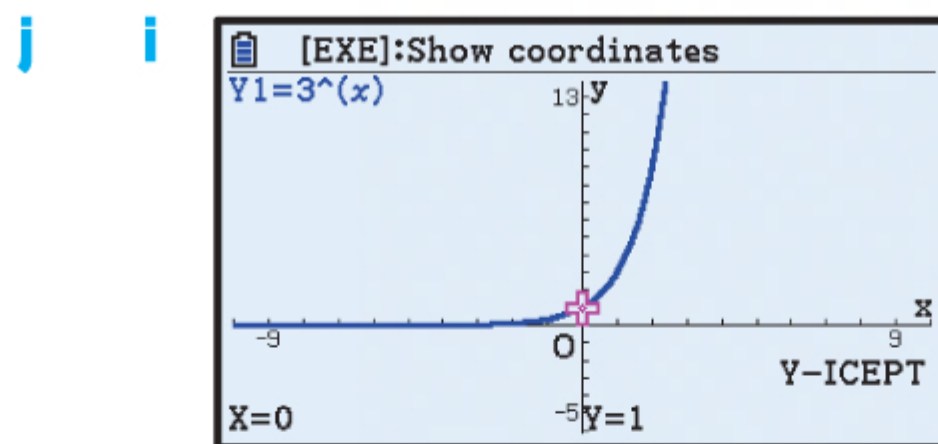
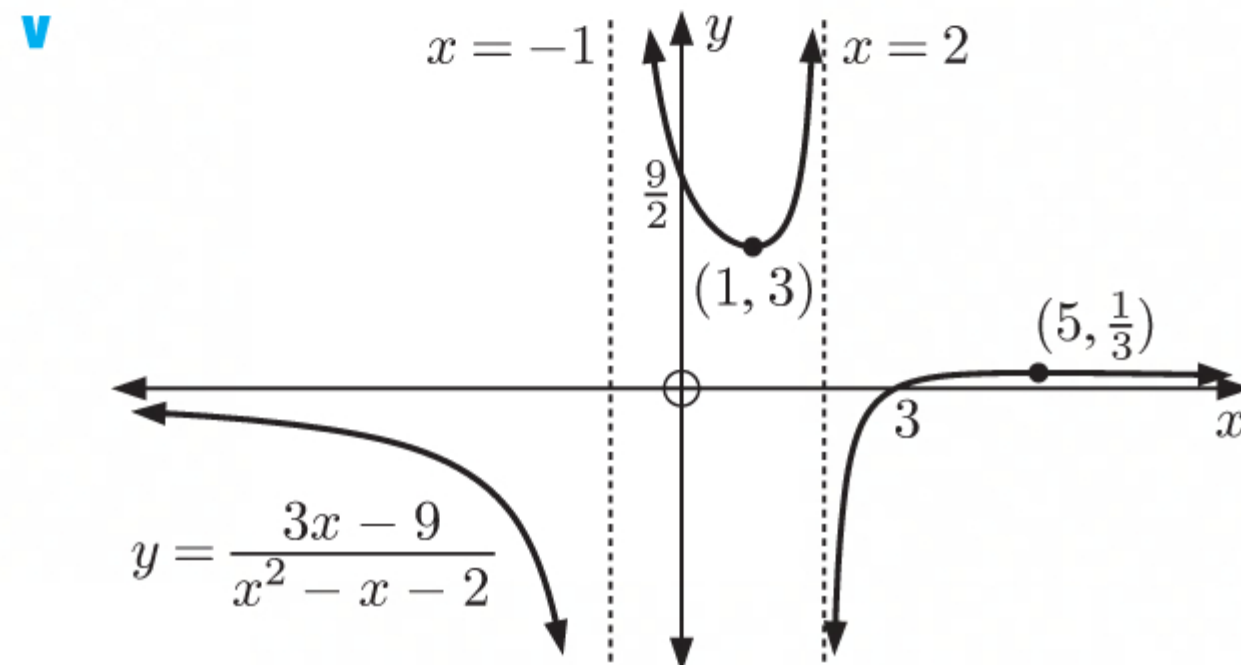
The y -intercept is $\frac{9}{2}$.



There is a minimum turning point at $(1, 3)$, and a maximum turning point at $(5, \frac{1}{3})$.

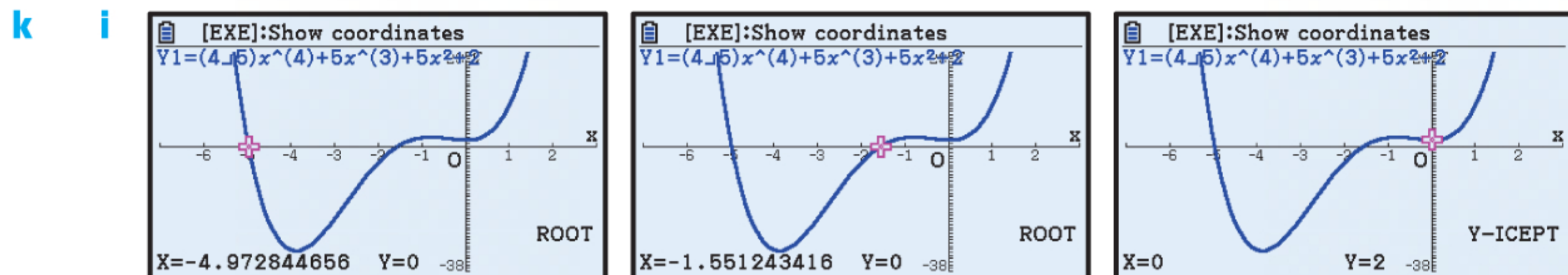
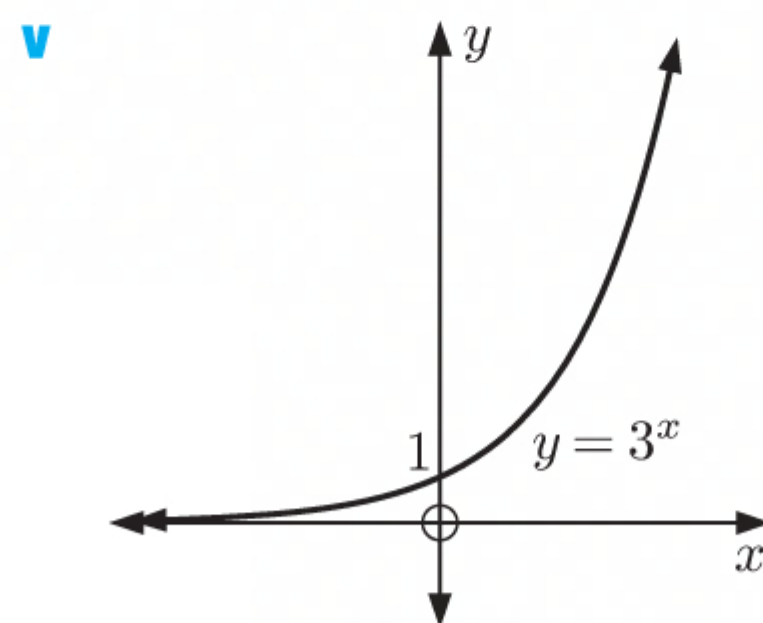
- iii The graph appears to have vertical asymptotes at $x = -1$ and $x = 2$.
This is confirmed by the fact that y is undefined for both of these values.
As $x \rightarrow \pm\infty$, the graph gets closer to the line $y = 0$. So, $y = 0$ is a horizontal asymptote.

- iv The domain is $\{x \mid x \neq -1 \text{ or } 2\}$. The range is $\{y \mid y \leq \frac{1}{3} \text{ or } y \geq 3\}$.

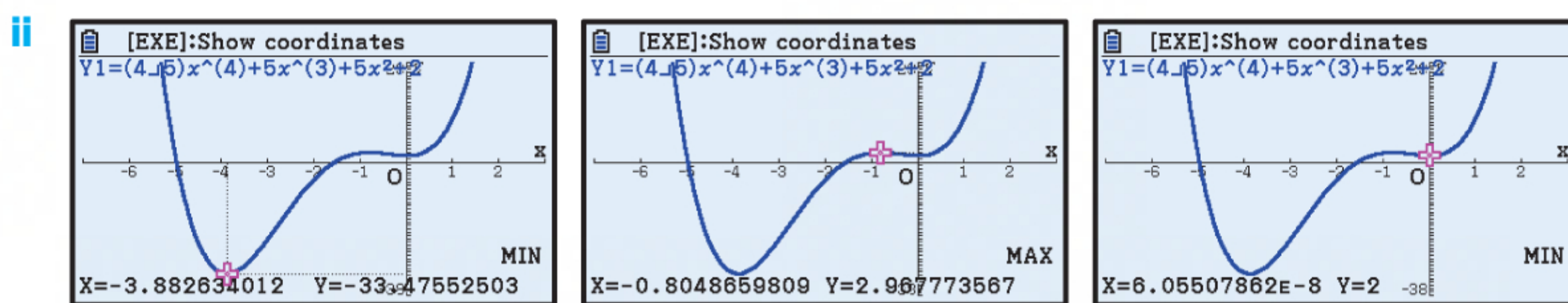


The y -intercept is 1. There are no x -intercepts.

- ii The graph has no turning points.
- iii As $x \rightarrow -\infty$, the graph gets closer to the line $y = 0$. So, $y = 0$ is a horizontal asymptote.
- iv The domain is $\{x \mid x \in \mathbb{R}\}$. The range is $\{y \mid y > 0\}$.



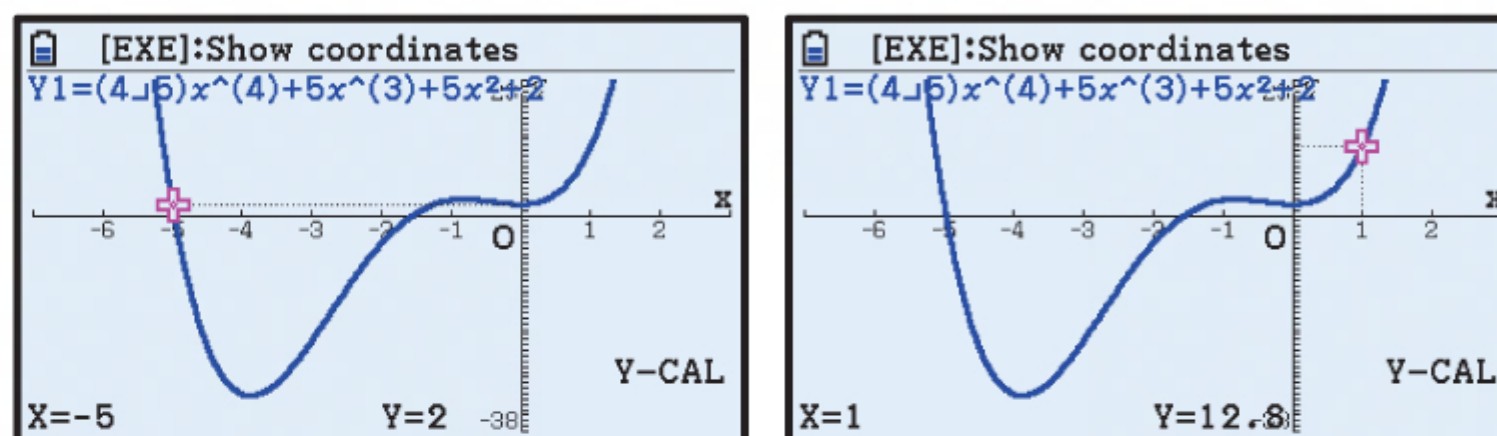
The x -intercepts are ≈ -4.97 and ≈ -1.55 . The y -intercept is 2.



There are minimum turning points at $(-3.88, -33.5)$ and $(0, 2)$, and a maximum turning point at $(-0.805, 2.97)$.

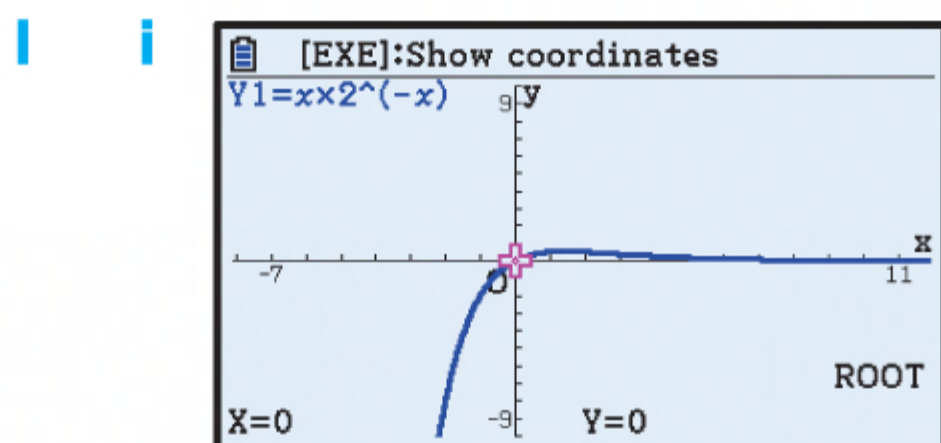
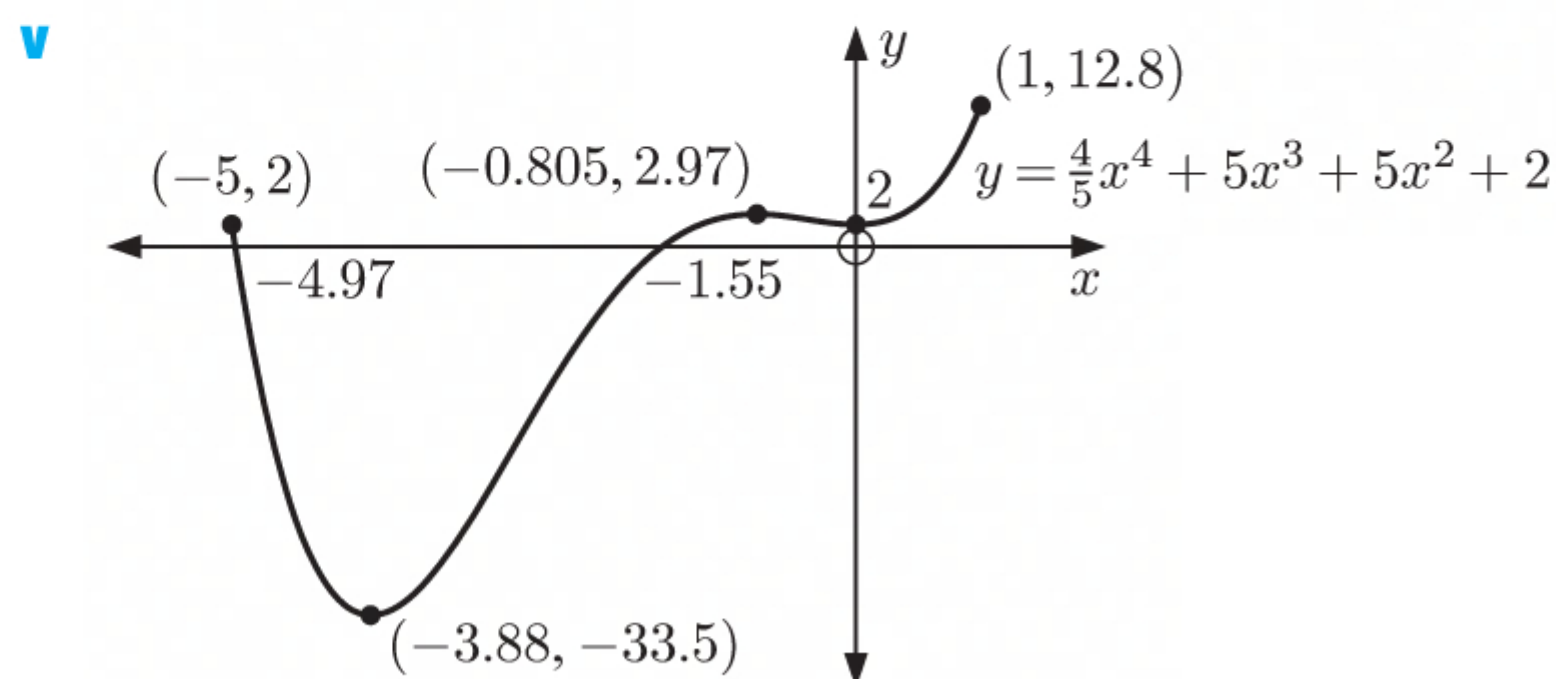
iii The graph has no asymptotes.

iv The domain is $\{x \mid -5 \leq x \leq 1\}$.



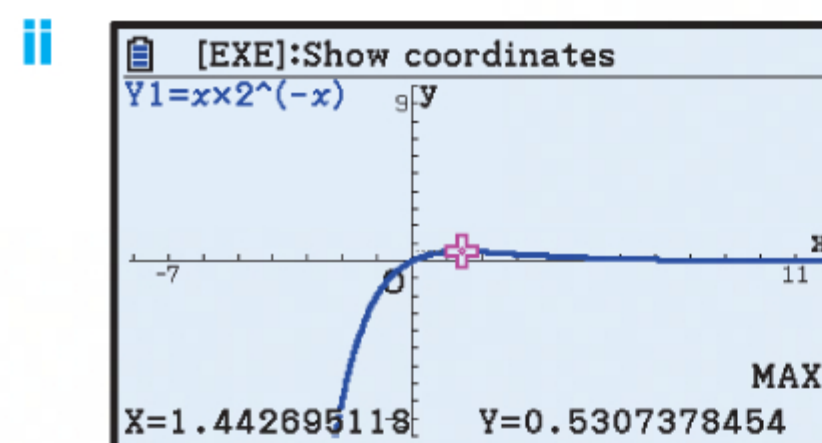
When $x = -5$, $y = 2$, and when $x = 1$, $y = 12.8$.

So, the range is $\{y \mid -33.5 \leq y \leq 12.8\}$. {using **ii**}



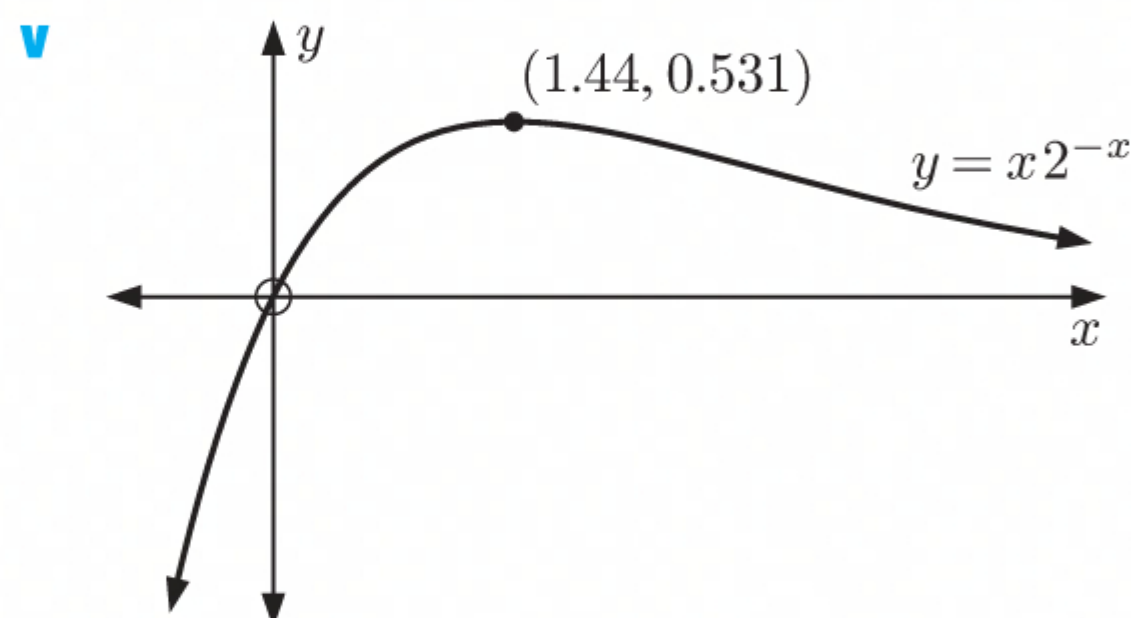
The x -intercept is 0. The y -intercept is 0.

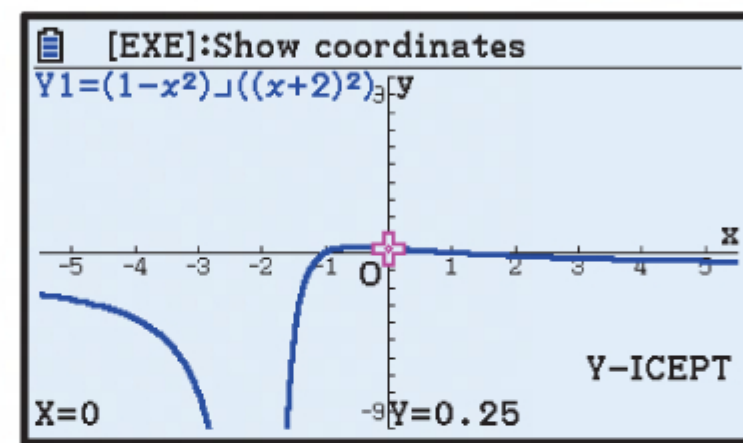
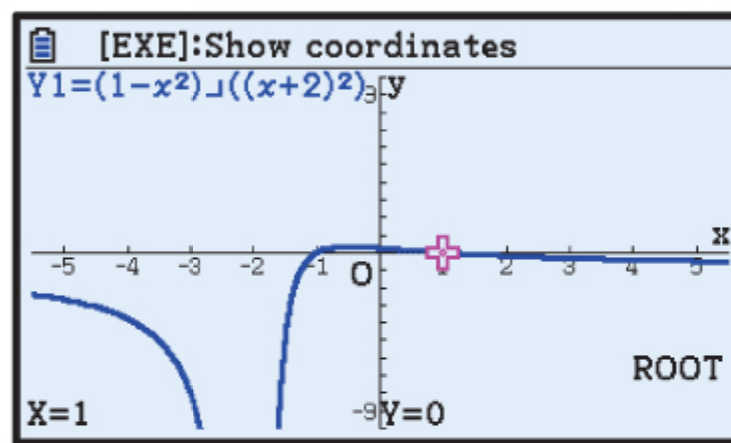
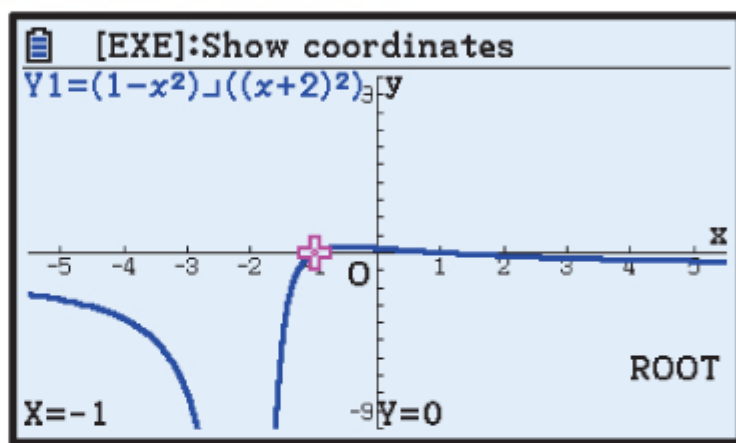
iii As $x \rightarrow \infty$, the graph gets closer to the line $y = 0$. So, $y = 0$ is a horizontal asymptote.



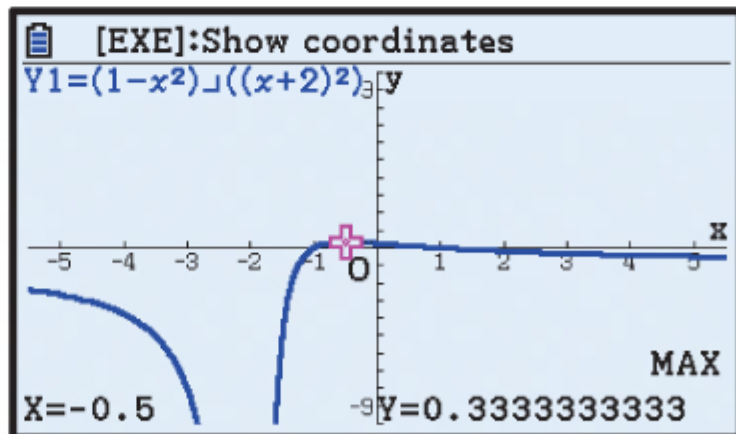
There is a maximum turning point at $(1.44, 0.531)$.

iv The domain is $\{x \mid x \in \mathbb{R}\}$.
The range is $\{y \mid y \leq 0.531\}$.



m i

The x -intercepts are -1 and 1 . The y -intercept is $\frac{1}{4}$.

ii

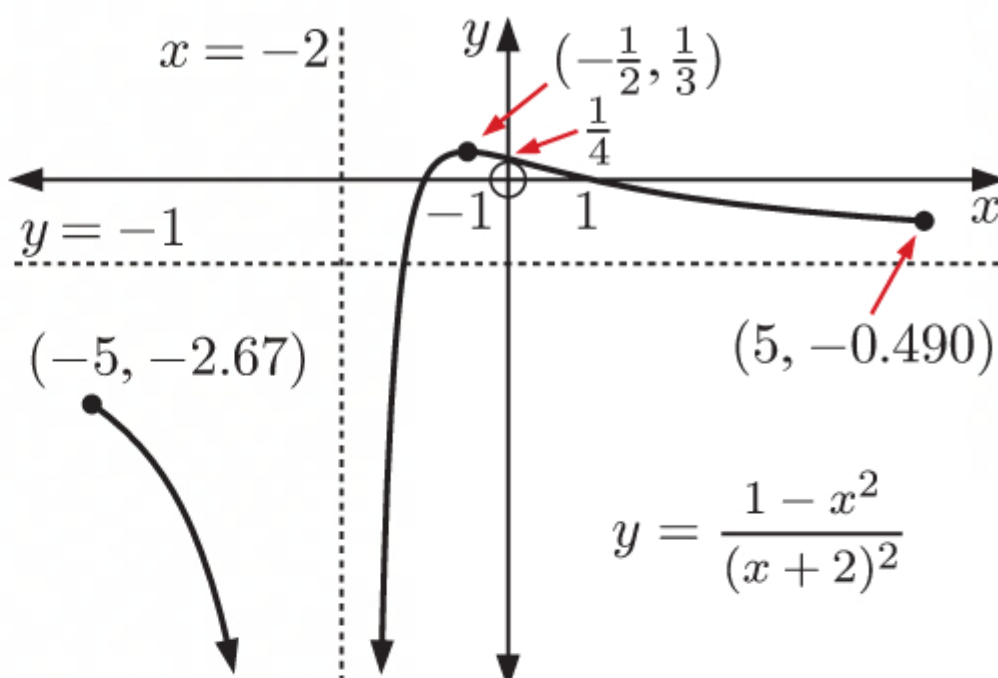
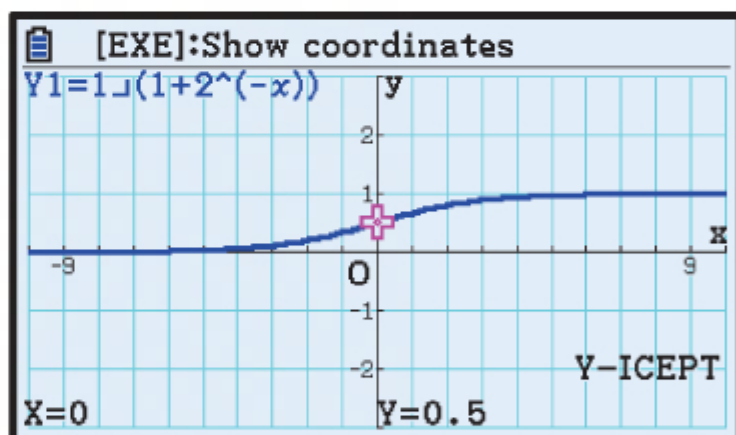
There is a maximum turning point at $(-\frac{1}{2}, \frac{1}{3})$.

iii The graph appears to have a vertical asymptote at $x = -2$.

This is confirmed by the fact that y is undefined when $x = -2$.

As $x \rightarrow \pm\infty$, the graph gets closer to the line $y = -1$. So, $y = -1$ is a horizontal asymptote.

iv The domain is $\{x \mid -5 \leq x \leq 5, x \neq -2\}$. The range is $\{y \mid y \leq \frac{1}{3}\}$.

v**n i**

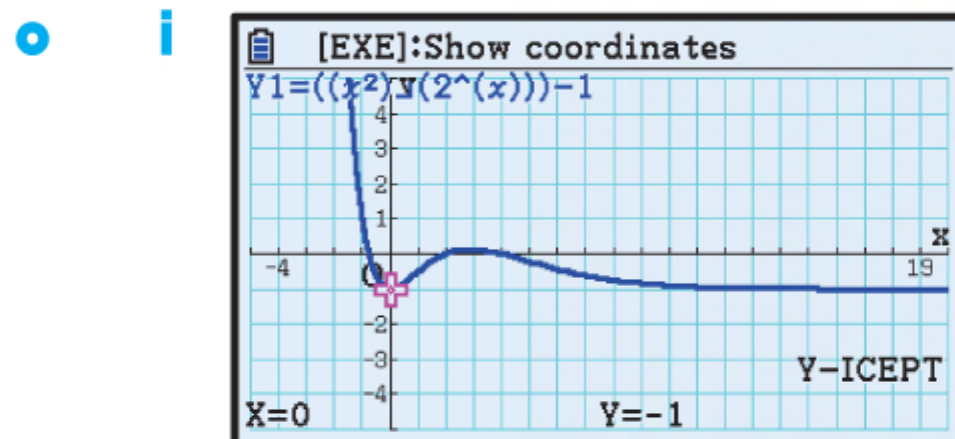
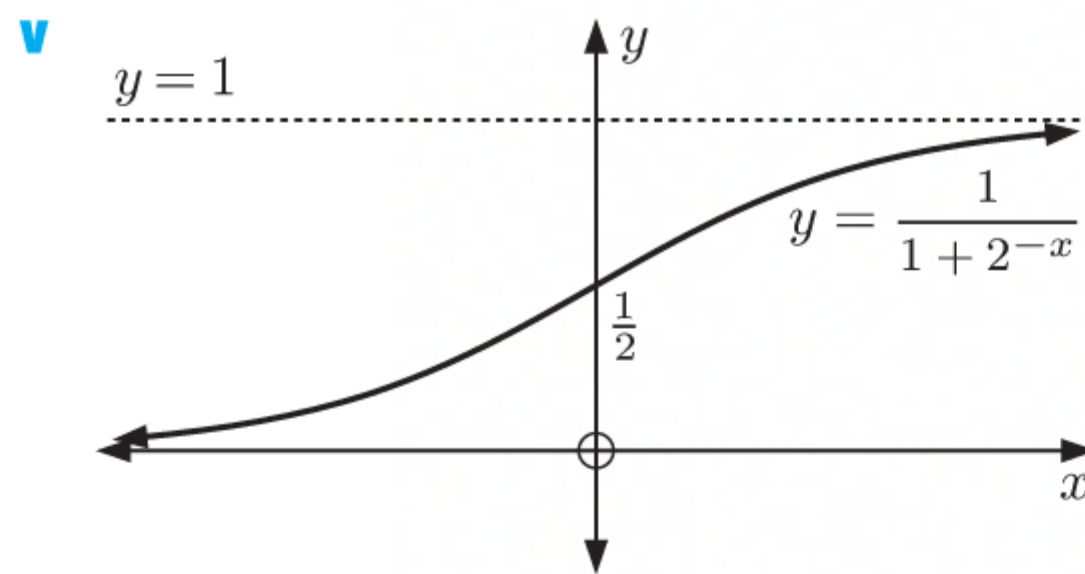
There are no x -intercepts. The y -intercept is $\frac{1}{2}$.

ii The graph has no turning points.

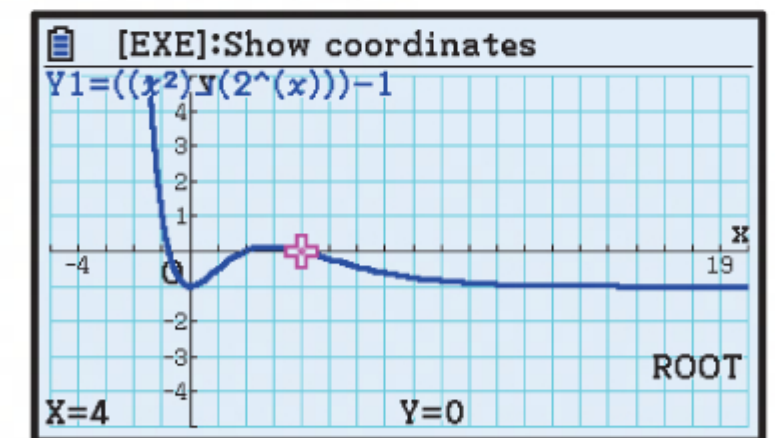
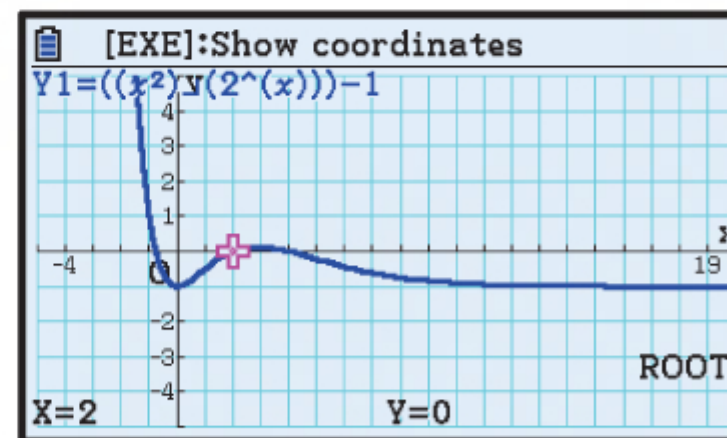
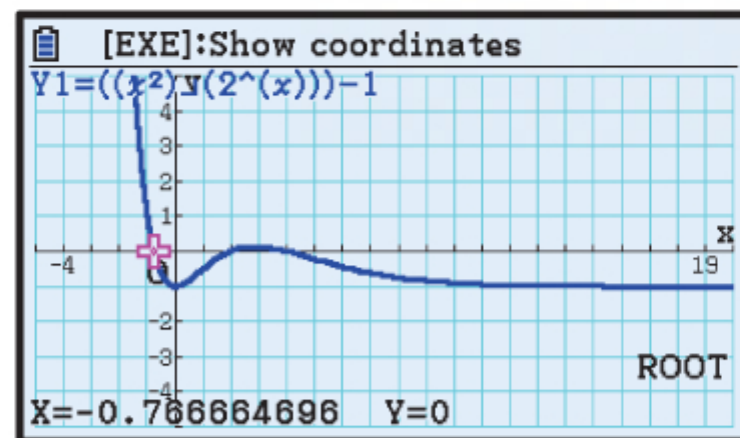
iii As $x \rightarrow \infty$, the graph gets closer to the line $y = 1$. So, $y = 1$ is a horizontal asymptote.

As $x \rightarrow -\infty$, the graph gets closer to the line $y = 0$. So, $y = 0$ is also a horizontal asymptote.

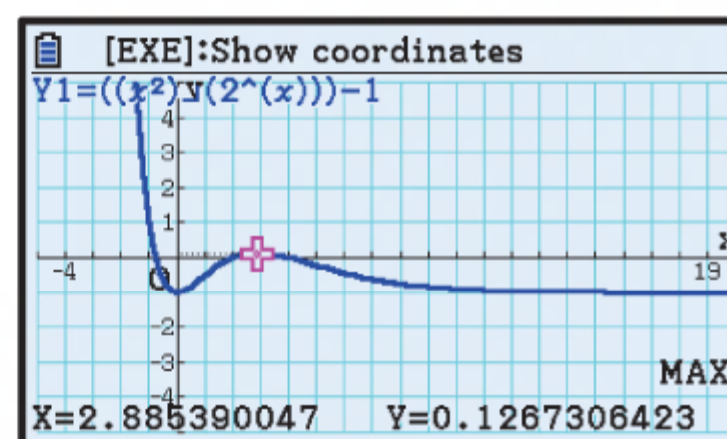
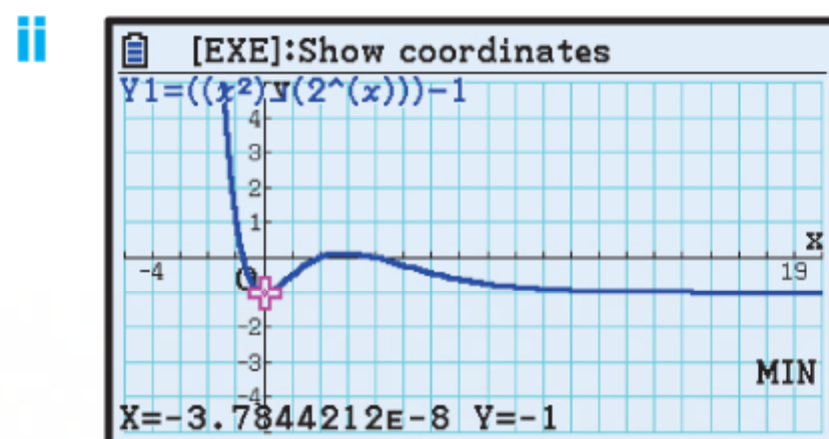
iv The domain is $\{x \mid x \in \mathbb{R}\}$. The range is $\{y \mid 0 < y < 1\}$.



The y -intercept is -1 .



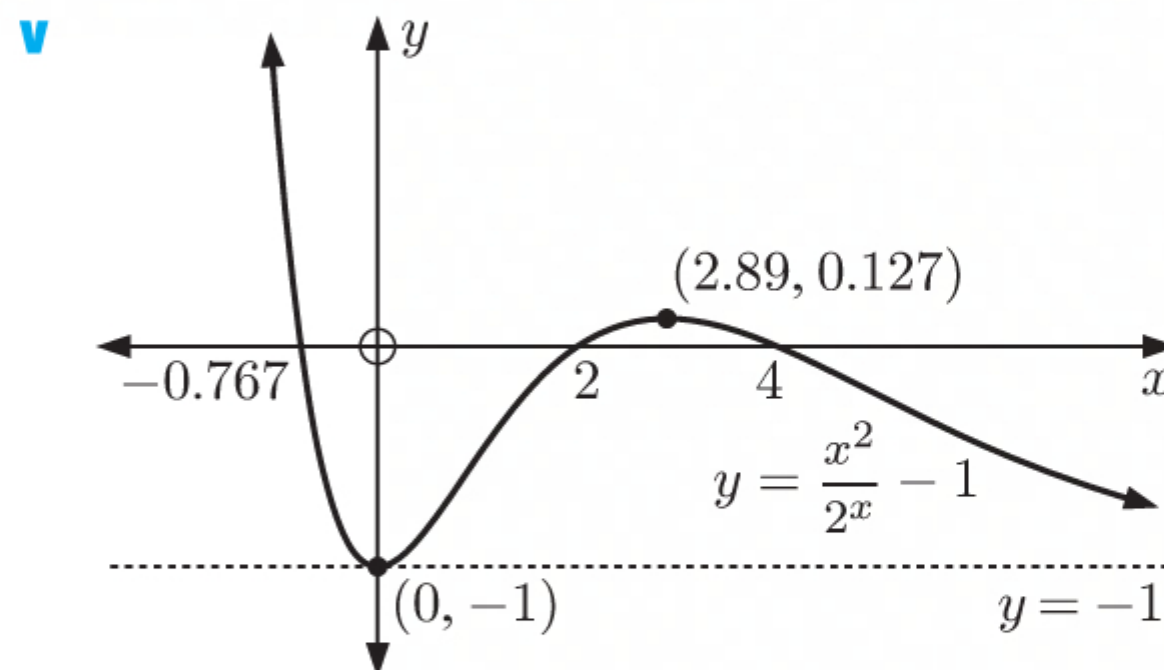
The x -intercepts are ≈ -0.767 , 2 , and 4 .

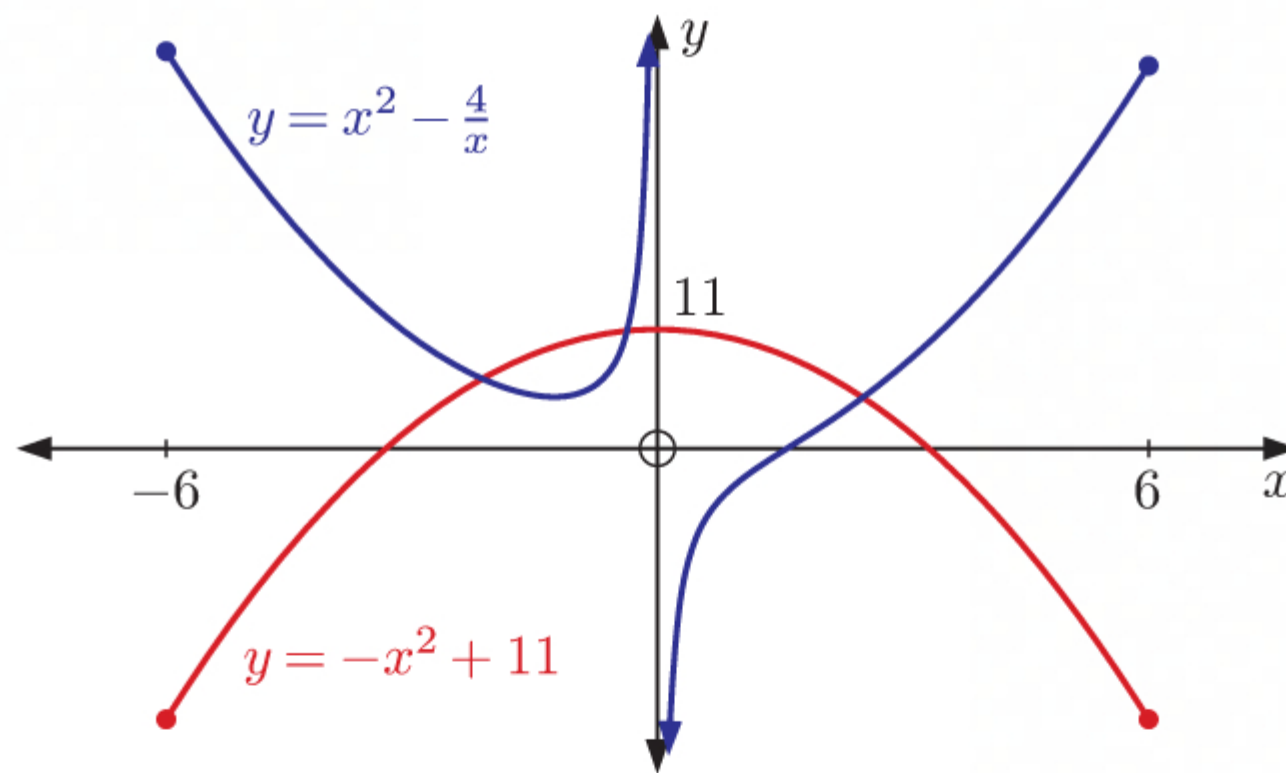
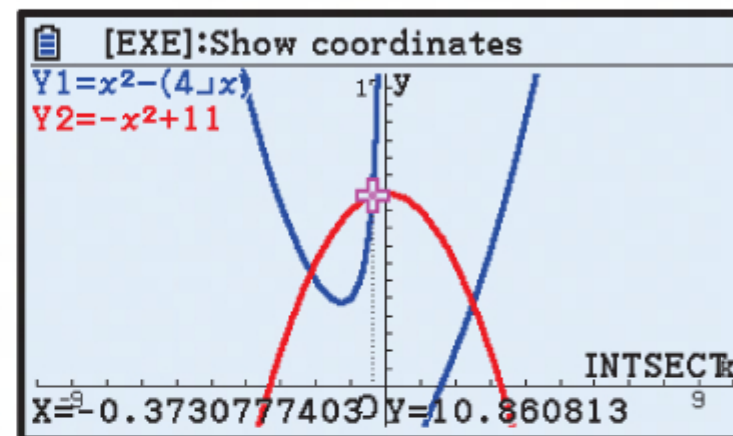
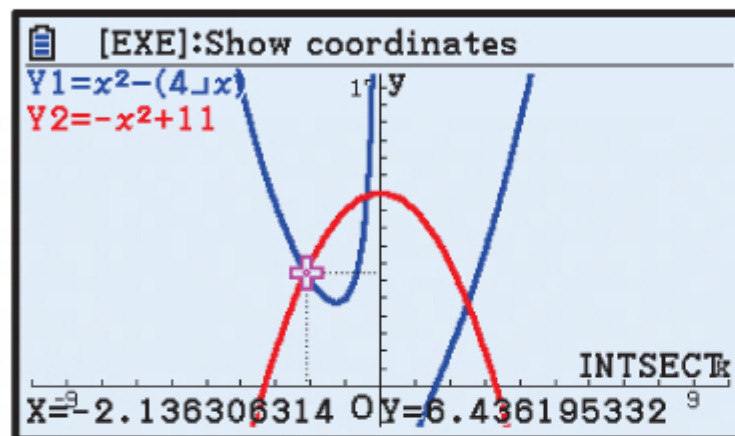


There is a minimum turning point at $(0, -1)$, and a maximum turning point at $(2.89, 0.127)$.

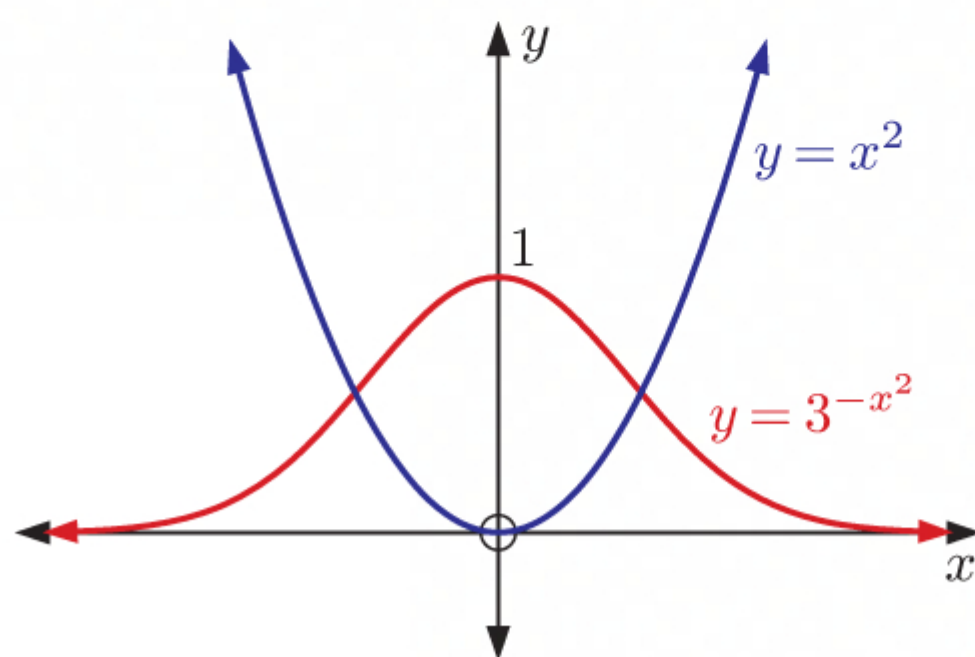
iii As $x \rightarrow \infty$, the graph gets closer to the line $y = -1$. So, $y = -1$ is a horizontal asymptote.

iv The domain is $\{x \mid x \in \mathbb{R}\}$. The range is $\{y \mid y \geq -1\}$.

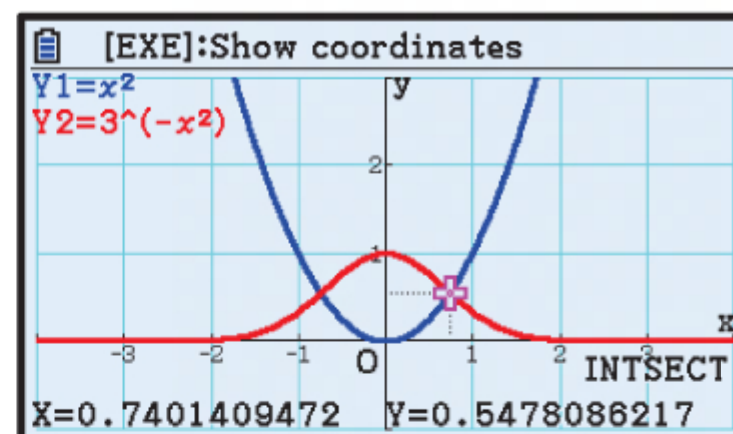
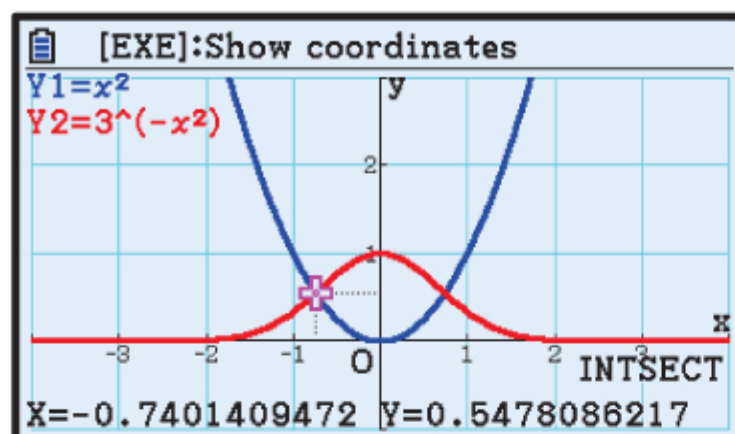


5 a**b**

The two negative solutions are $x \approx -2.14$ and $x \approx -0.373$.

6 a**b** $y = 3^{-x^2}$ has:

- no x -intercepts
- maximum turning point $(0, 1)$
- domain $\{x \mid x \in \mathbb{R}\}$
- y -intercept 1
- horizontal asymptote $y = 0$
- range $\{y \mid 0 < y \leq 1\}$.

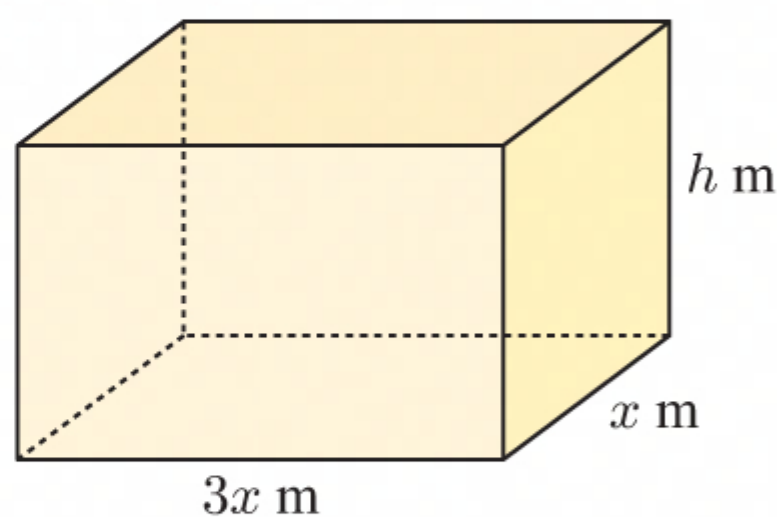
c

If $x^2 = 3^{-x^2}$, then $x \approx -0.740$ or $x \approx 0.740$.

7 a Volume = $3x \times x \times h$

$$\therefore 6 = 3x^2h$$

$$\therefore h = \frac{2}{x^2}$$



b Surface area $= 2 \times (3x \times x) + 2 \times (3x \times h) + 2(x \times h) \text{ m}^2$

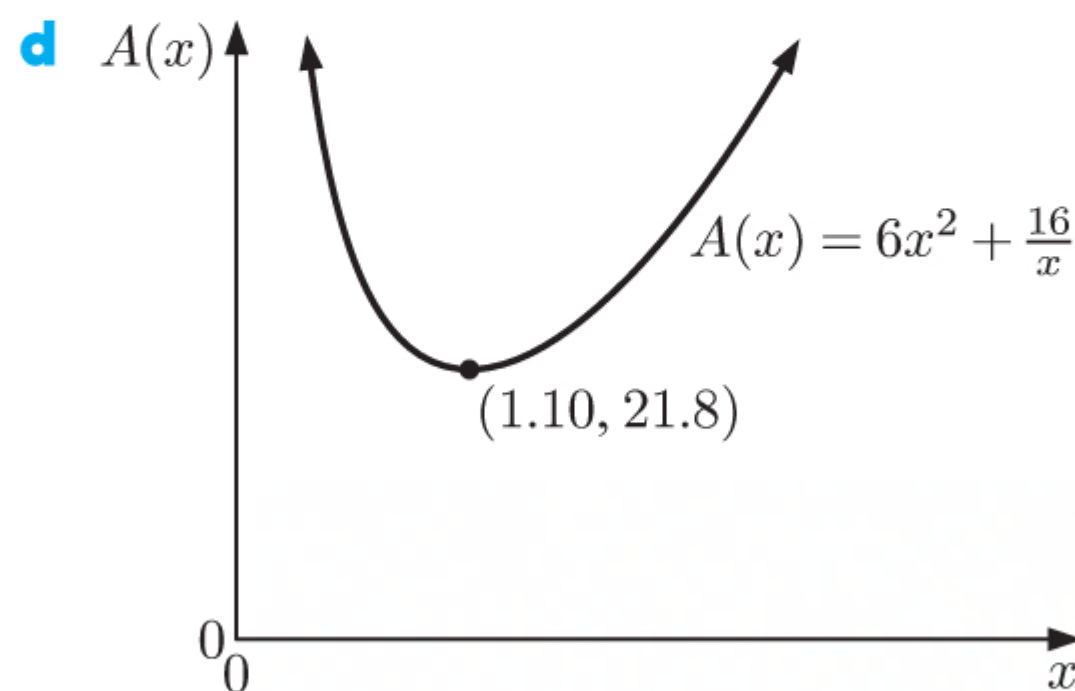
$$= 2 \times (3x \times x) + 2 \times \left(3x \times \frac{2}{x^2}\right) + 2 \left(x \times \frac{2}{x^2}\right) \text{ m}^2 \quad \{\text{using a}\}$$

$$= 2(3x^2) + 2\left(\frac{6}{x}\right) + 2\left(\frac{2}{x}\right) \text{ m}^2$$

$$= 6x^2 + \frac{12}{x} + \frac{4}{x} \text{ m}^2$$

$$\therefore A(x) = 6x^2 + \frac{16}{x} \text{ m}^2$$

- c** $A(x)$ is undefined when $x = 0$.
 $\therefore y = A(x)$ has vertical asymptote $x = 0$.



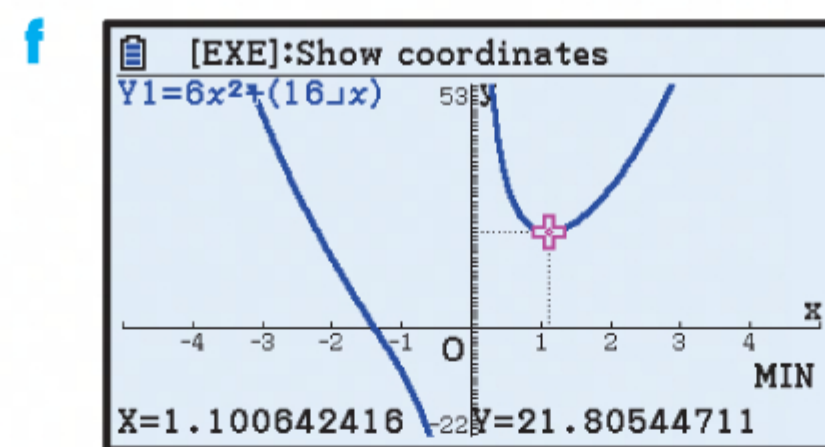
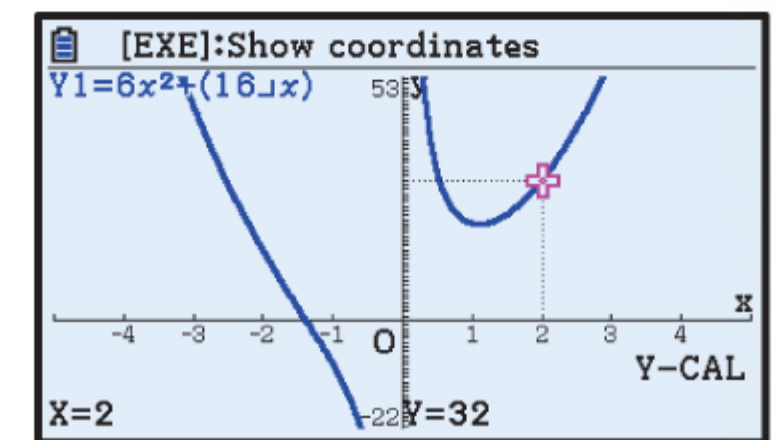
e $A(2) = 6(2)^2 + \frac{16}{2}$

$$= 6(4) + 8$$

$$= 24 + 8$$

$$= 32$$

\therefore the surface area of a box with width 2 metres is 32 m^2 .



The minimum turning point of $y = A(x)$ is $(1.10, 21.8)$. The box will have minimum surface area of about 21.8 m^2 when $x \approx 1.10$.

8 a $E(t) = 640t \times 4^{-t}$ units

$$E(1) = 640(1) \times 4^{-1}$$

$$= 640 \times \frac{1}{4}$$

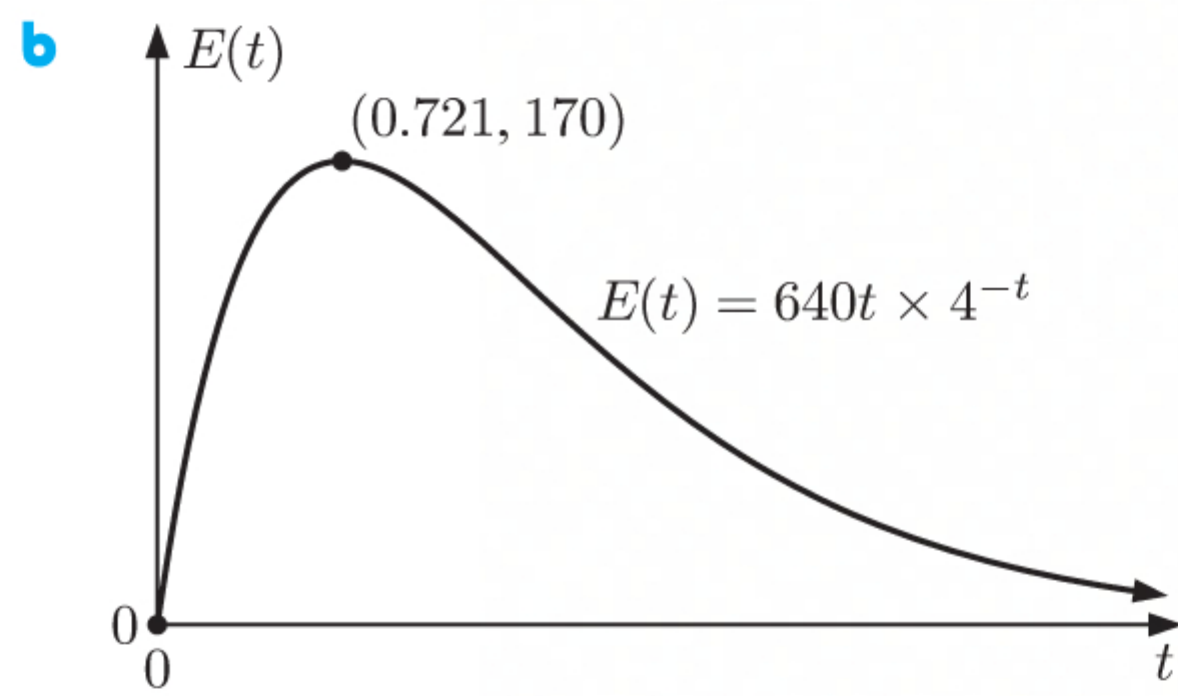
$$= 160$$

$$E(4) = 640(4) \times 4^{-4}$$

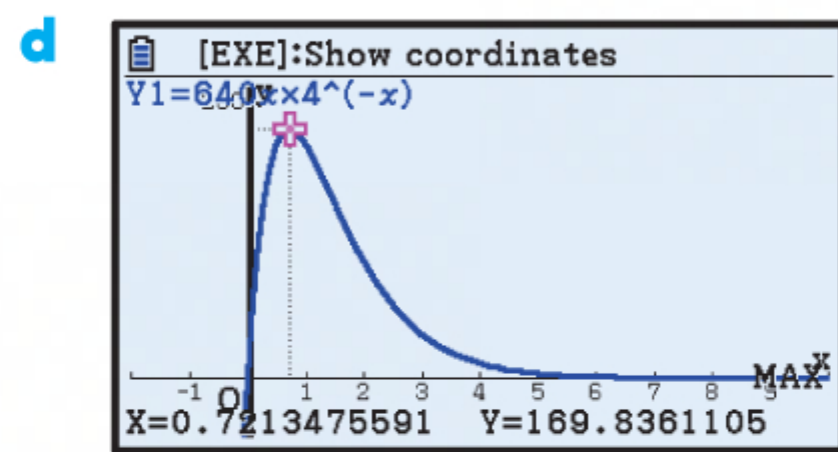
$$= 2560 \times \frac{1}{256}$$

$$= 10$$

These are the effects of the injection 1 and 4 hours after it was administered.



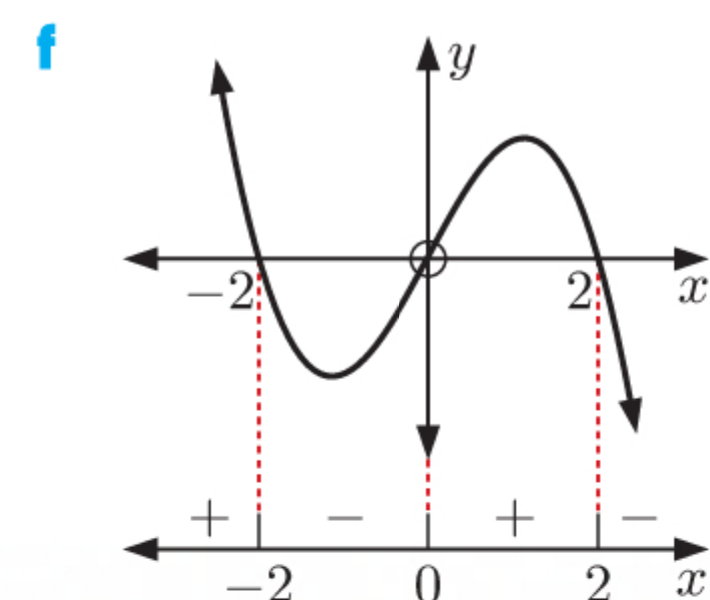
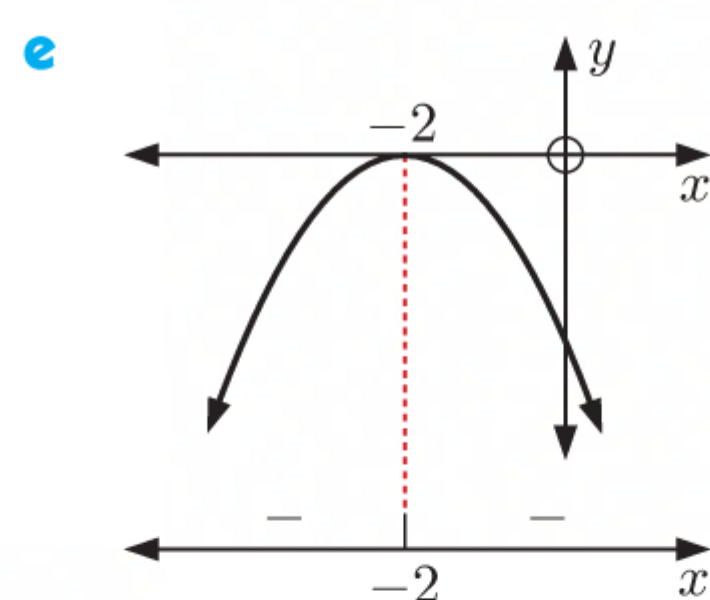
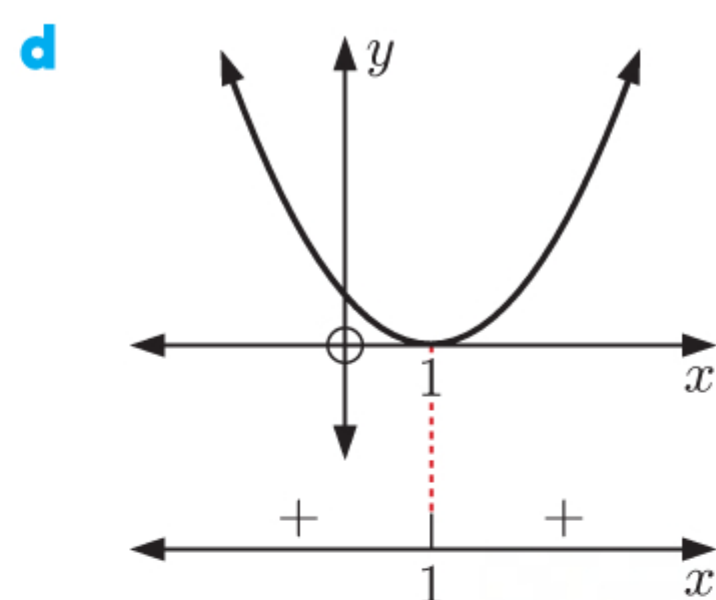
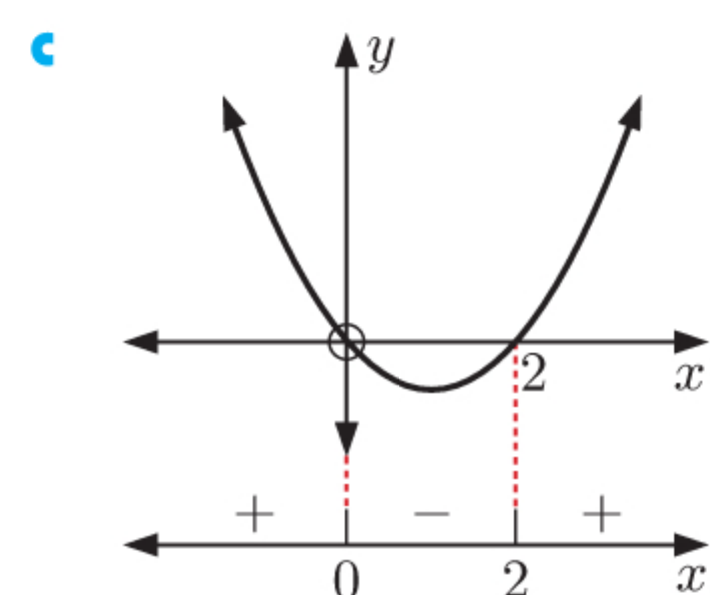
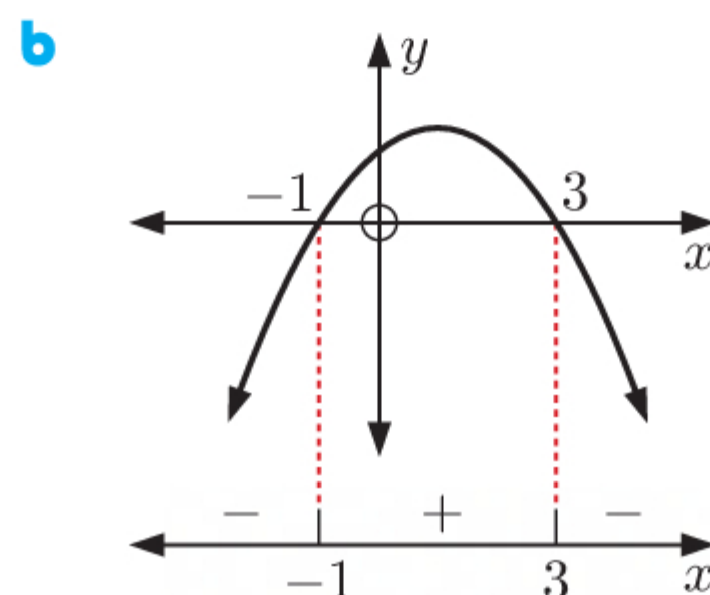
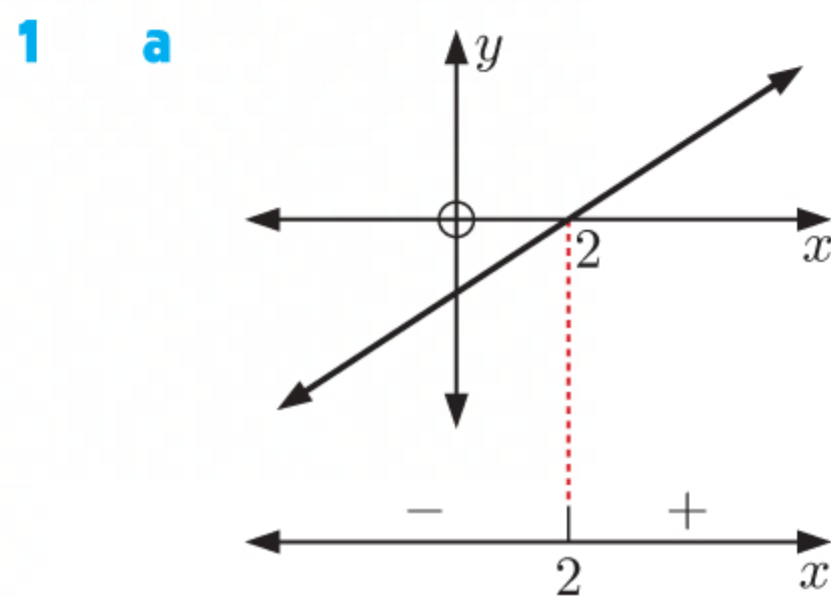
c As $t \rightarrow \infty$, $E(t)$ gets closer to 0. So the effectiveness of the drug approaches zero.

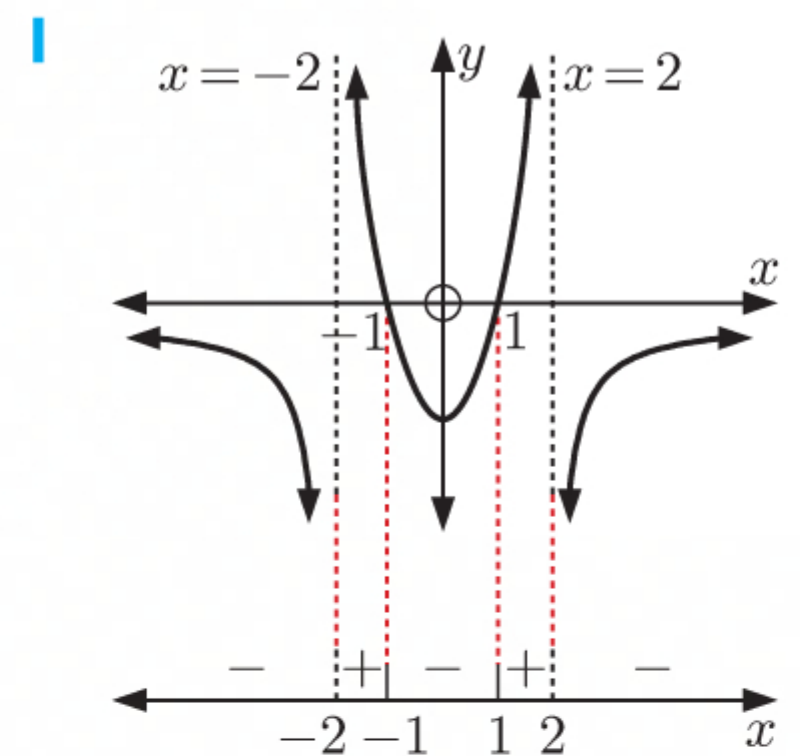
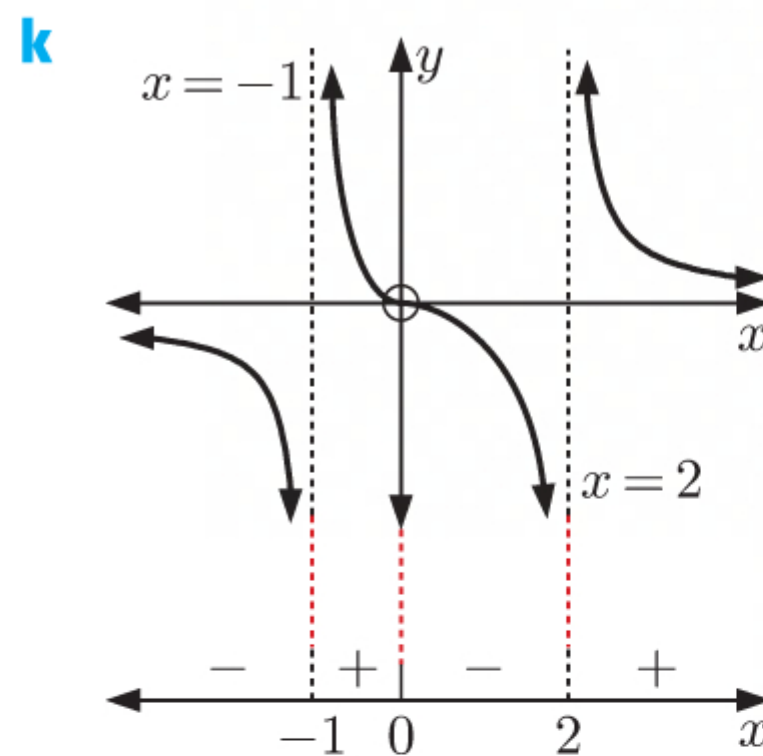
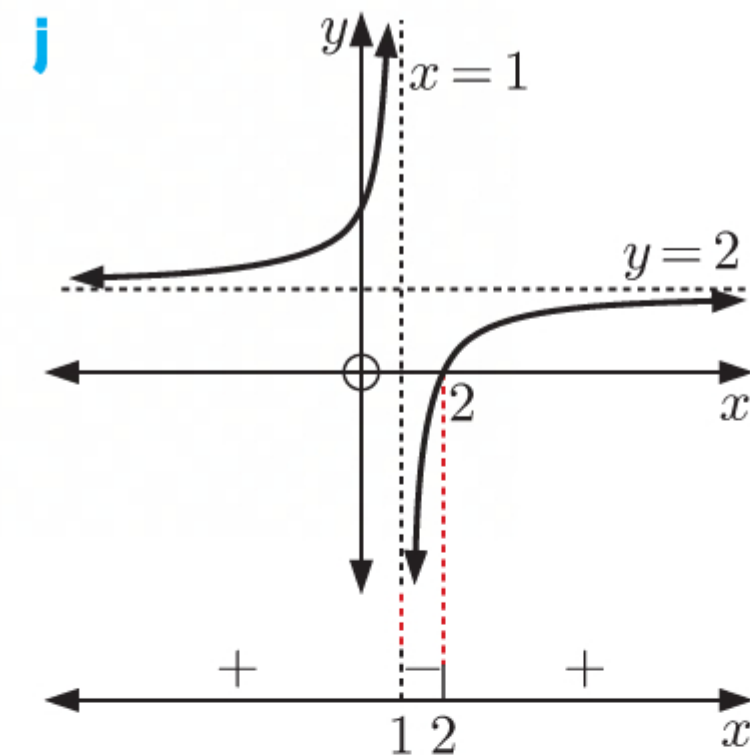
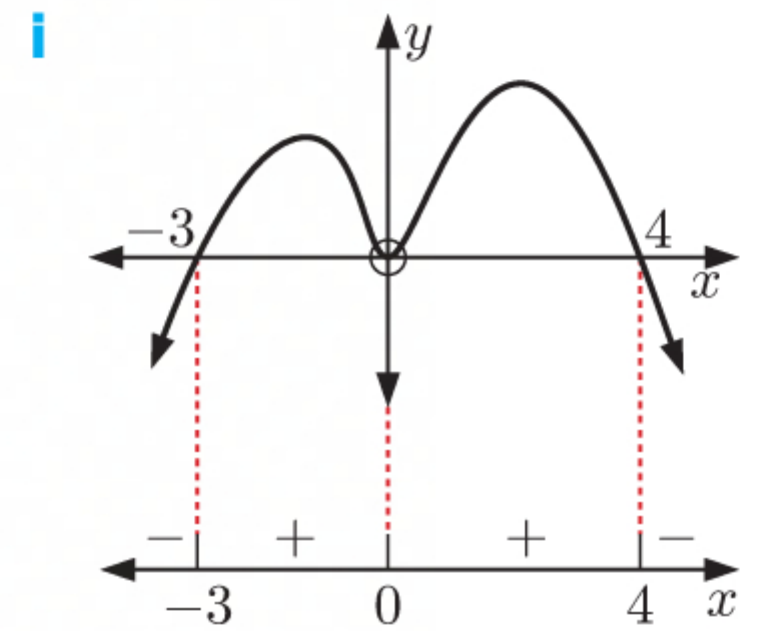
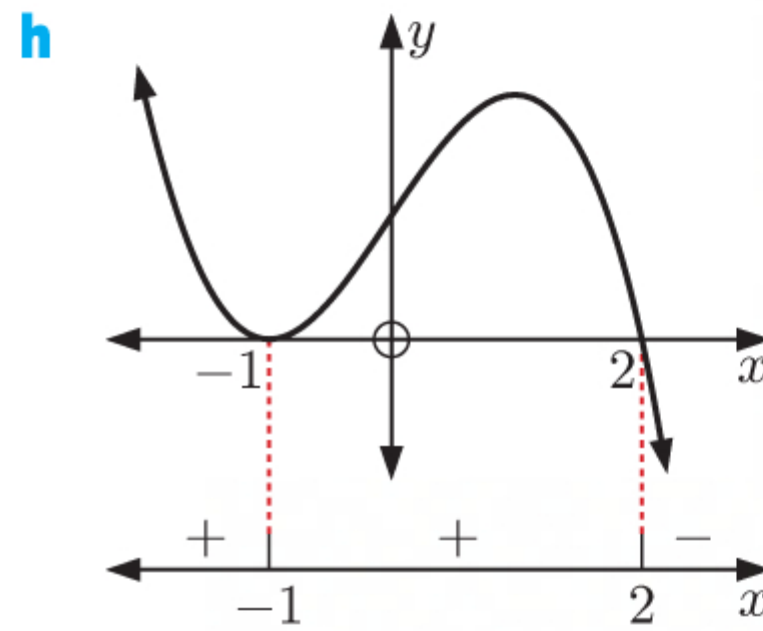
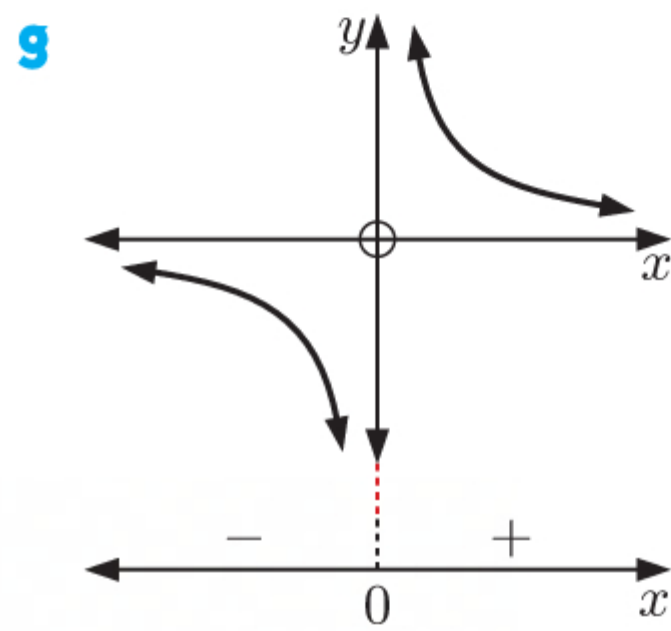


The graph has a maximum turning point at $(0.721, 170)$.

At about 0.721 hours after the injection, the drug has reached maximum effect of about 170 units and the effect will now begin to decrease.

EXERCISE 3E



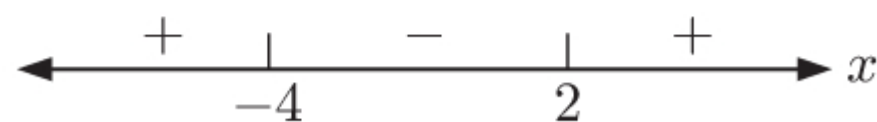


- 2 a** $(x+4)(x-2)$ has zeros -4 and 2 .

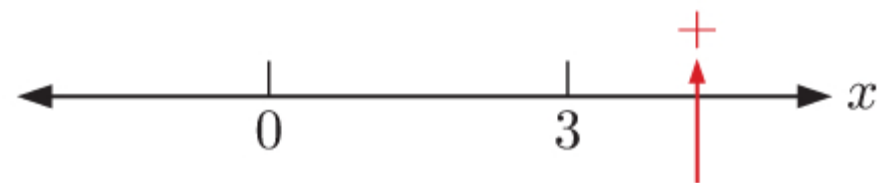


When $x = 3$ we have $(7)(1) > 0$,
so we put a $+$ sign here.

As the factors are single, the signs
alternate.

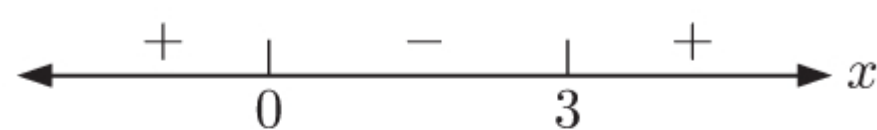


- c** $x(x-3)$ has zeros 0 and 3 .



When $x = 4$ we have $(4)(1) > 0$,
so we put a $+$ sign here.

As the factors are single, the signs
alternate.

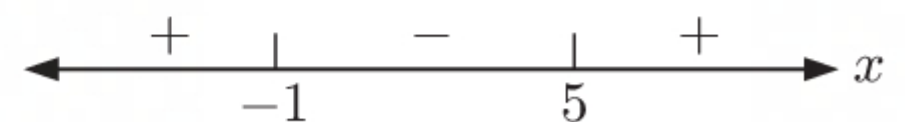


- b** $(x+1)(x-5)$ has zeros -1 and 5 .

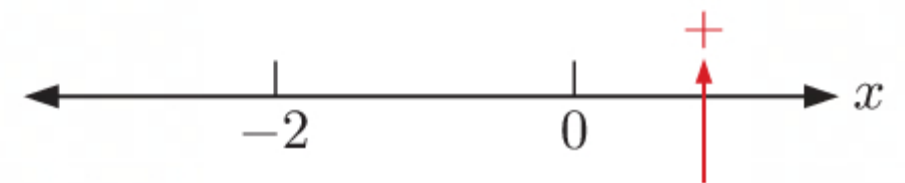


When $x = 6$ we have $(7)(1) > 0$,
so we put a $+$ sign here.

As the factors are single, the signs
alternate.

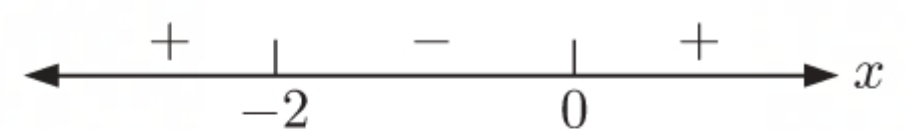


- d** $x(x+2)$ has zeros 0 and -2 .



When $x = 1$ we have $(1)(3) > 0$,
so we put a $+$ sign here.

As the factors are single, the signs
alternate.



e $(2x + 1)(x - 4)$ has zeros $-\frac{1}{2}$ and 4.

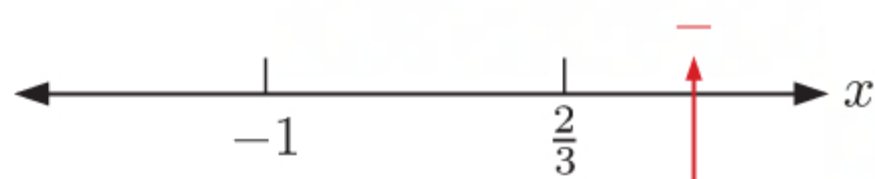


When $x = 5$ we have $(11)(1) > 0$,
so we put a + sign here.

As the factors are single, the signs alternate.

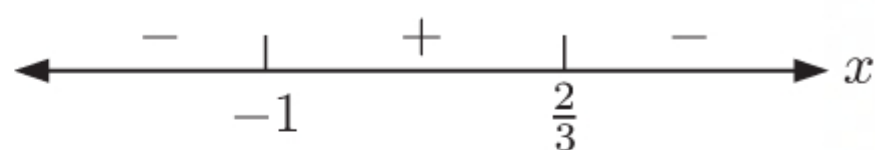


g $-(3x - 2)(x + 1)$ has zeros $\frac{2}{3}$ and -1 .



When $x = 1$ we have $-(1)(2) < 0$,
so we put a - sign here.

As the factors are single, the signs alternate.



i $(5 - x)(1 - 2x)$ has zeros 5 and $\frac{1}{2}$.

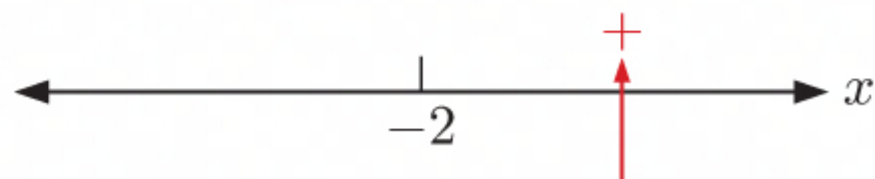


When $x = 6$ we have $(-1)(-11) > 0$,
so we put a + sign here.

As the factors are single, the signs alternate.

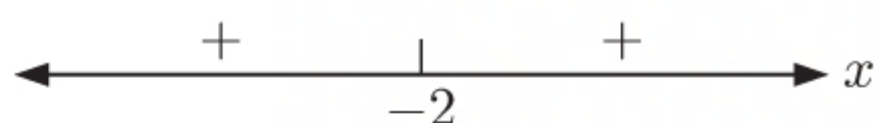


3 a $(x + 2)^2$ has zero -2 .



When $x = 0$ we have $(2)^2 > 0$,
so we put a + sign here.

As the factor is squared, the signs do not change.

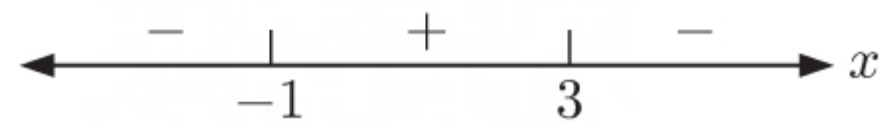


f $-(x + 1)(x - 3)$ has zeros -1 and 3.



When $x = 4$ we have $-(5)(1) < 0$,
so we put a - sign here.

As the factors are single, the signs alternate.

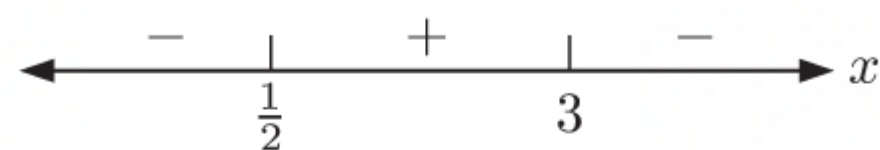


h $(2x - 1)(3 - x)$ has zeros $\frac{1}{2}$ and 3.

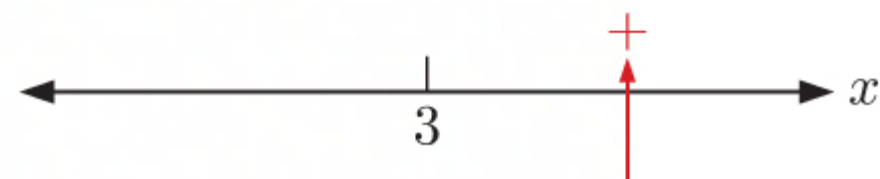


When $x = 4$ we have $(7)(-1) < 0$,
so we put a - sign here.

As the factors are single, the signs alternate.

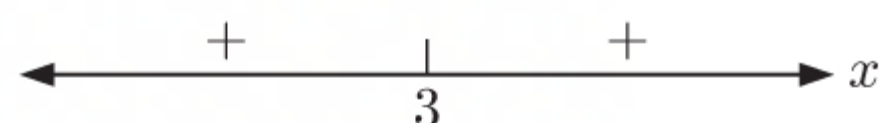


b $(x - 3)^2$ has zero 3.

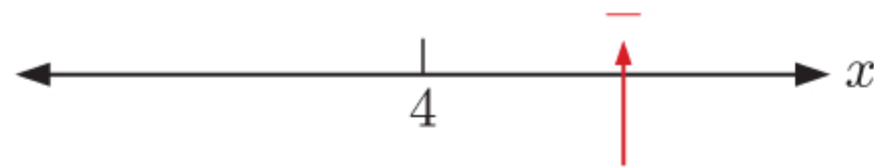


When $x = 4$ we have $(1)^2 > 0$,
so we put a + sign here.

As the factor is squared, the signs do not change.

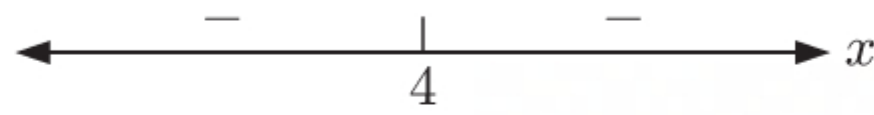


- c $-(x-4)^2$ has zero 4.

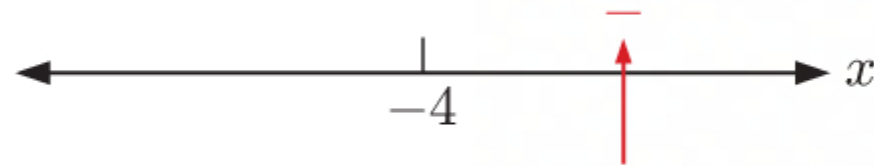


When $x = 5$ we have $-(1)^2 < 0$,
so we put a $-$ sign here.

As the factor is squared, the signs do not change.

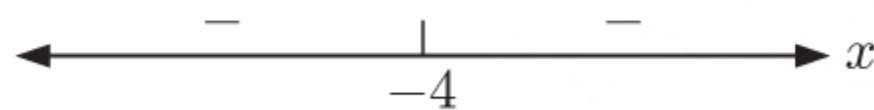


- e $-3(x+4)^2$ has zero -4 .

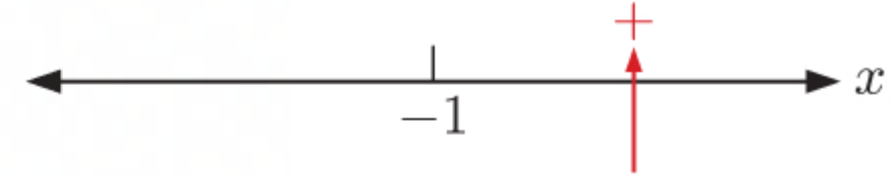


When $x = 0$ we have $-3(4)^2 < 0$,
so we put a $-$ sign here.

As the factor is squared, the signs do not change.

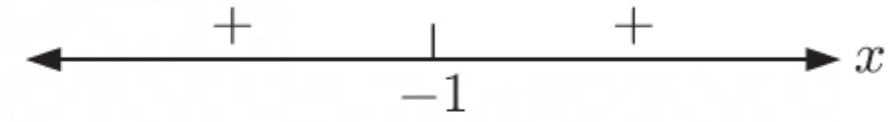


- d $2(x+1)^2$ has zero -1 .

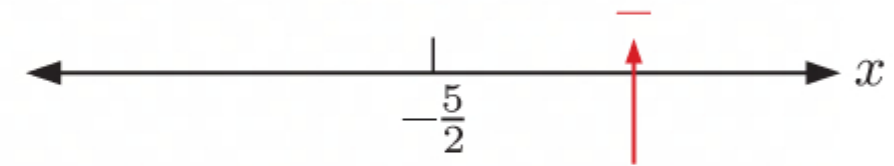


When $x = 0$ we have $2(1)^2 > 0$,
so we put a $+$ sign here.

As the factor is squared, the signs do not change.

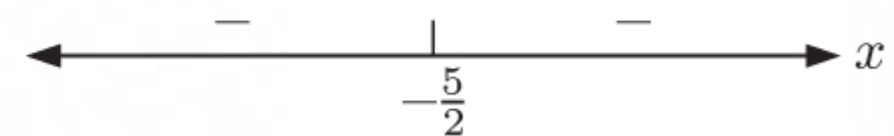


- f $-\frac{1}{2}(2x+5)^2$ has zero $-\frac{5}{2}$.



When $x = 0$ we have $-\frac{1}{2}(5)^2 < 0$,
so we put a $-$ sign here.

As the factor is squared, the signs do not change.

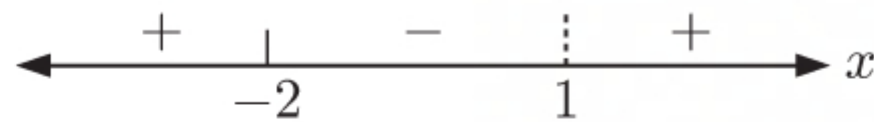


- 4 a $\frac{x+2}{x-1}$ is zero when $x = -2$ and
undefined when $x = 1$.

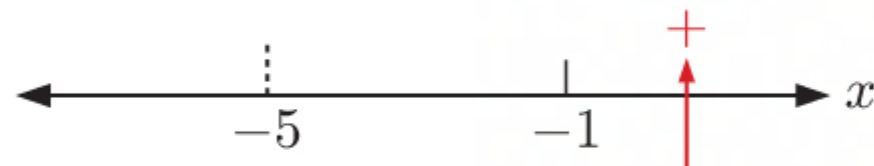


When $x = 2$ we have $\frac{4}{1} > 0$.

Since $(x+2)$ and $(x-1)$ are single factors, the signs alternate.



- c $\frac{x+1}{x+5}$ is zero when $x = -1$ and
undefined when $x = -5$.



When $x = 0$ we have $\frac{1}{5} > 0$.

Since $(x+1)$ and $(x+5)$ are single factors, the signs alternate.

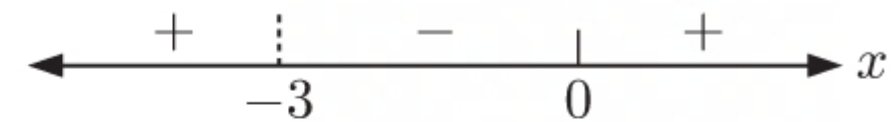


- b $\frac{x}{x+3}$ is zero when $x = 0$ and
undefined when $x = -3$.



When $x = 1$ we have $\frac{1}{4} > 0$.

Since x and $(x+3)$ are single factors, the signs alternate.

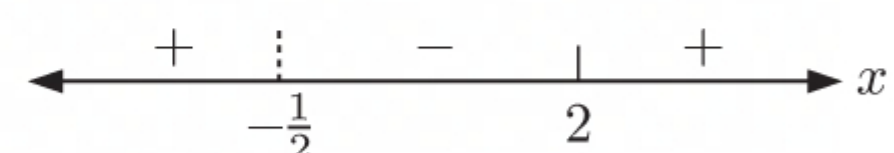


- d $\frac{x-2}{2x+1}$ is zero when $x = 2$ and
undefined when $x = -\frac{1}{2}$.

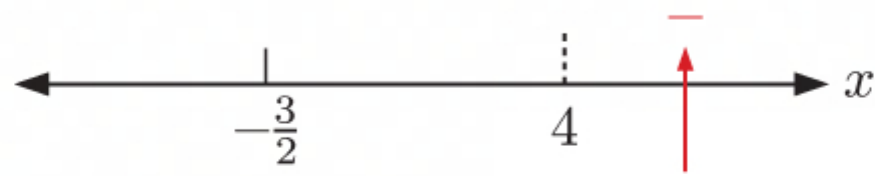


When $x = 3$ we have $\frac{1}{7} > 0$.

Since $(x-2)$ and $(2x+1)$ are single factors, the signs alternate.

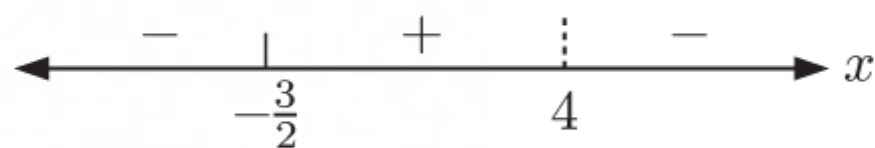


- e** $\frac{2x+3}{4-x}$ is zero when $x = -\frac{3}{2}$ and undefined when $x = 4$.

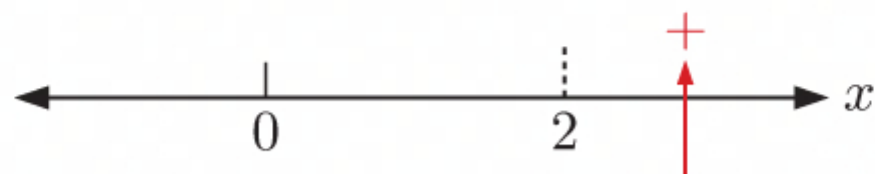


When $x = 5$ we have $\frac{13}{-1} < 0$.

Since $(2x+3)$ and $(4-x)$ are single factors, the signs alternate.



- g** $\frac{3x}{x-2}$ is zero when $x = 0$ and undefined when $x = 2$.



When $x = 3$ we have $\frac{9}{1} > 0$.

Since $3x$ and $(x-2)$ are single factors, the signs alternate.

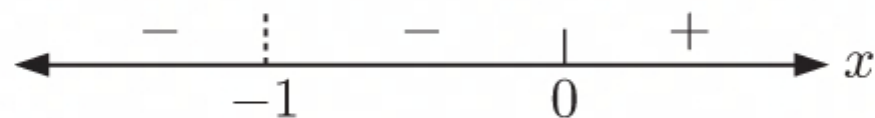


- i** $\frac{4x}{(x+1)^2}$ is zero when $x = 0$ and undefined when $x = -1$.

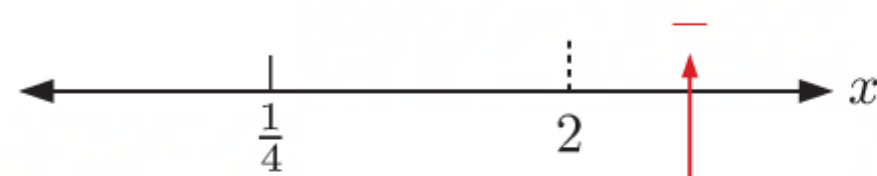


When $x = 1$ we have $\frac{4}{4} > 0$.

Since $4x$ is a single factor, the sign changes about $x = 0$. Since $(x+1)^2$ is a squared factor, the sign stays the same about $x = -1$.

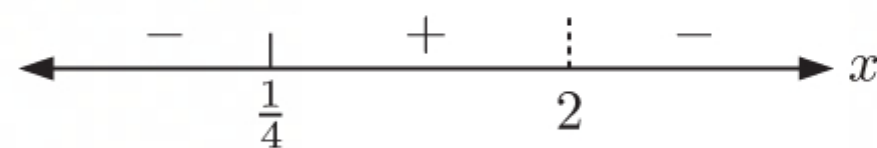


- f** $\frac{4x-1}{2-x}$ is zero when $x = \frac{1}{4}$ and undefined when $x = 2$.



When $x = 3$ we have $\frac{11}{-1} < 0$.

Since $(4x-1)$ and $(2-x)$ are single factors, the signs alternate.

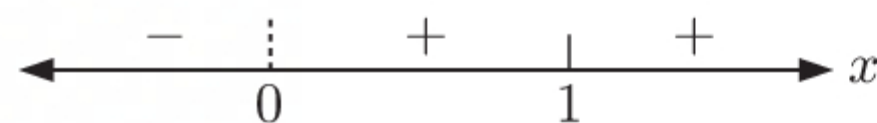


- h** $\frac{(x-1)^2}{x}$ is zero when $x = 1$ and undefined when $x = 0$.

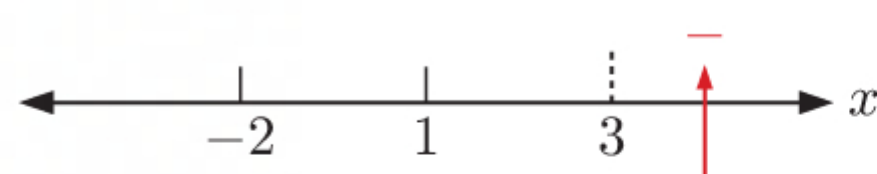


When $x = 2$ we have $\frac{1}{2} > 0$.

Since $(x-1)^2$ is a squared factor, the sign stays the same about $x = 1$. Since x is a single factor, the sign changes about $x = 0$.

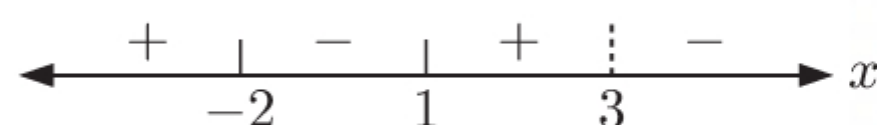


- j** $\frac{(x+2)(x-1)}{3-x}$ is zero when $x = -2$ or $x = 1$ and undefined when $x = 3$.

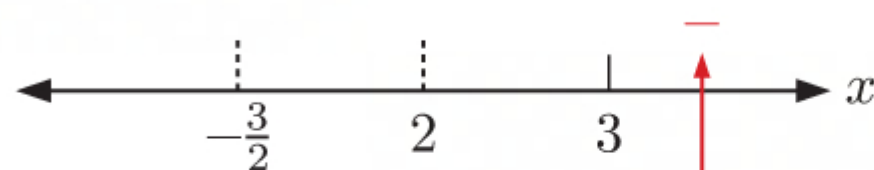


When $x = 4$ we have $\frac{(6)(3)}{-1} < 0$.

Since all of the factors are single, the signs alternate.

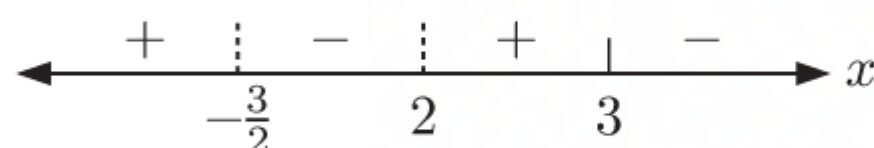


- k** $\frac{3-x}{(2x+3)(x-2)}$ is zero when $x = 3$
and undefined when $x = -\frac{3}{2}$ or 2 .

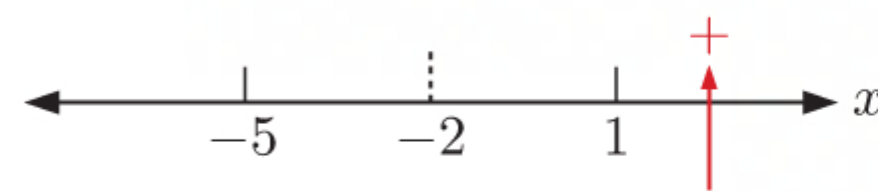


When $x = 4$ we have $\frac{-1}{(11)(2)} < 0$.

Since all of the factors are single, the signs alternate.



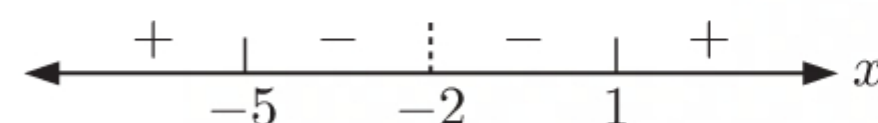
- l** $\frac{(x+5)(x-1)}{(x+2)^2}$ is zero when $x = -5$
or 1 and undefined when $x = -2$.



When $x = 2$ we have $\frac{(7)(1)}{16} > 0$.

Since $(x+5)$ and $(x-1)$ are single factors, the signs alternate about $x = -5$ and $x = 1$.

Since $(x+2)^2$ is a squared factor, the sign stays the same about $x = -2$.



INVESTIGATION

TRANSFORMATIONS OF GRAPHS

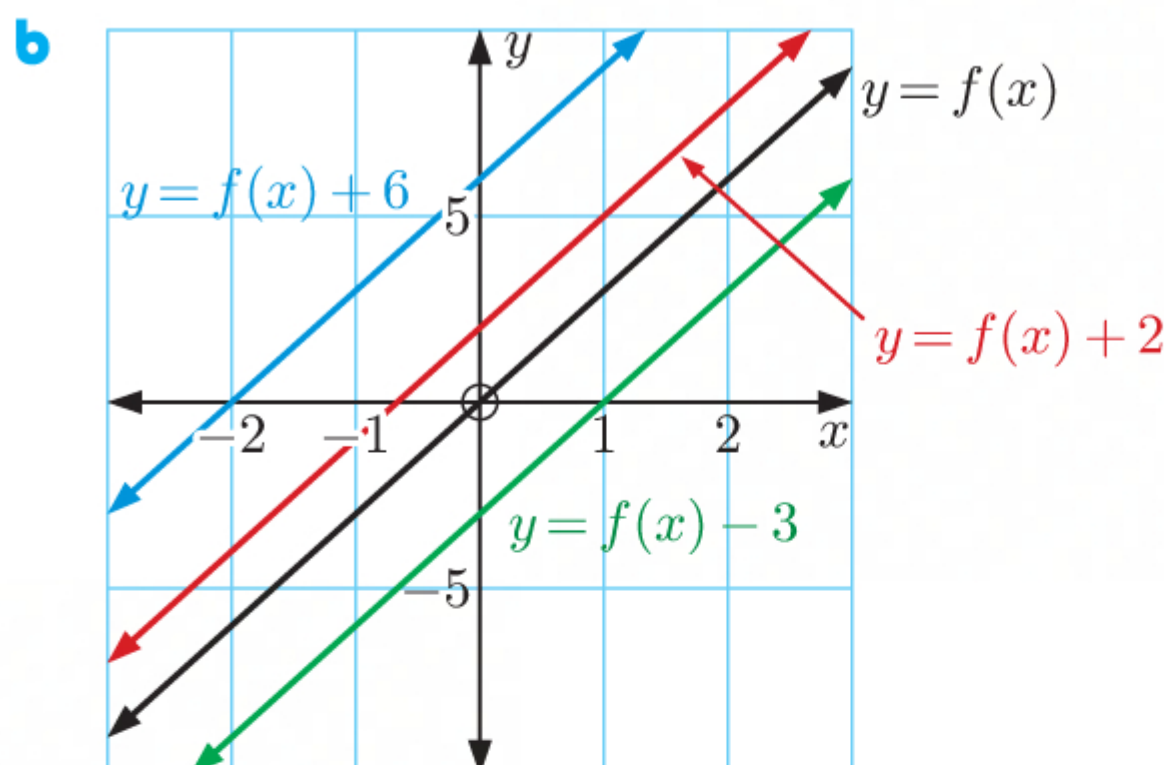
PART 1: GRAPHS OF THE FORM $y = f(x) + d$

1 a $f(x) = 3x$

i $f(x) + 2 = 3x + 2$

ii $f(x) - 3 = 3x - 3$

iii $f(x) + 6 = 3x + 6$

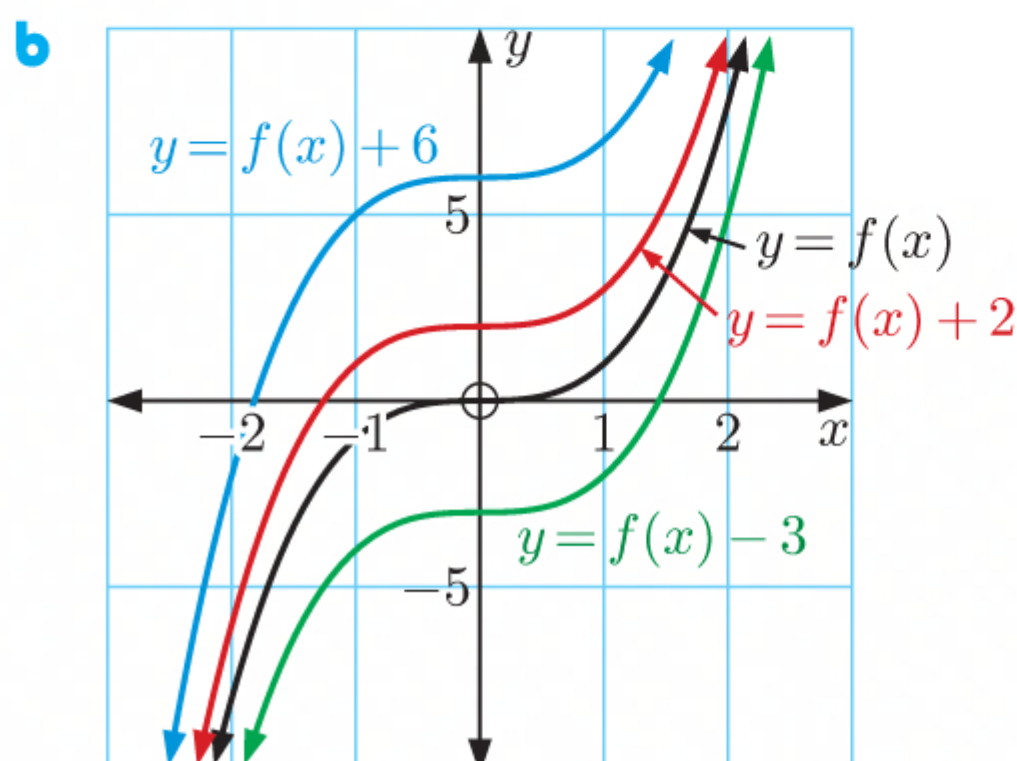


2 a $f(x) = x^3$

i $f(x) + 2 = x^3 + 2$

ii $f(x) - 3 = x^3 - 3$

iii $f(x) + 6 = x^3 + 6$



- 3** For $y = f(x) = d$, the effect of d is to **translate** the graph **vertically** through d units.
- If $d > 0$ it moves **upwards**.
 - If $d < 0$ it moves **downwards**.

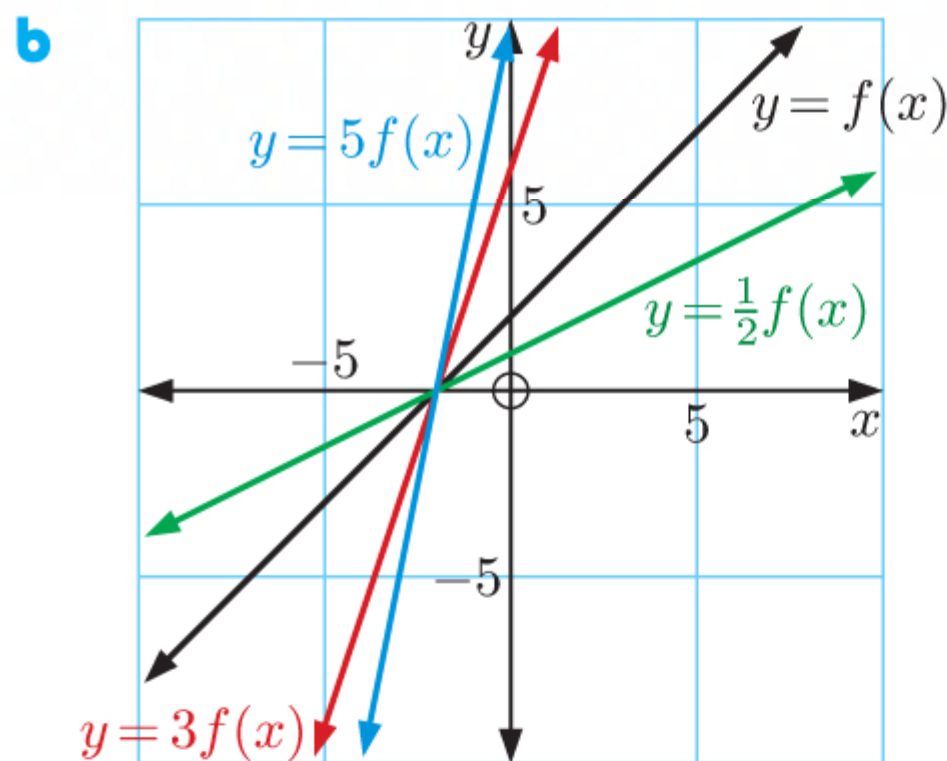
PART 2: GRAPHS OF THE FORM $y = pf(x)$, $p > 0$ AND $y = f(qx)$, $q > 0$

1 $f(x) = x + 2$

a i $3f(x) = 3(x + 2)$
 $= 3x + 6$

ii $\frac{1}{2}f(x) = \frac{1}{2}(x + 2)$
 $= \frac{1}{2}x + 1$

iii $5f(x) = 5(x + 2)$
 $= 5x + 10$



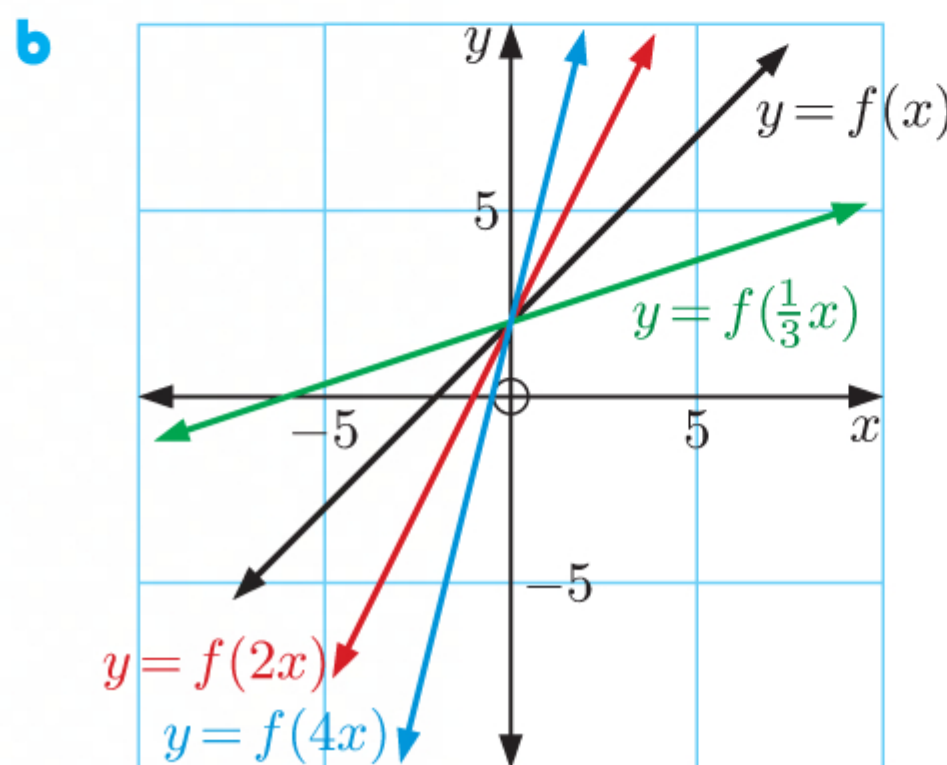
- c** For the transformation $y = pf(x)$, $p > 0$, each point becomes p times its previous distance from the x -axis.
- d** $(-2, 0)$ is 0 units from the x -axis, so it becomes $0 \times p = 0$ units from the x -axis.
 \therefore all transformations of the form $pf(x)$, $p > 0$ do not move the point $(-2, 0)$.

2 $f(x) = x + 2$

a i $f(2x) = 2x + 2$

ii $f(\frac{1}{3}x) = \frac{1}{3}x + 2$

iii $f(4x) = 4x + 2$



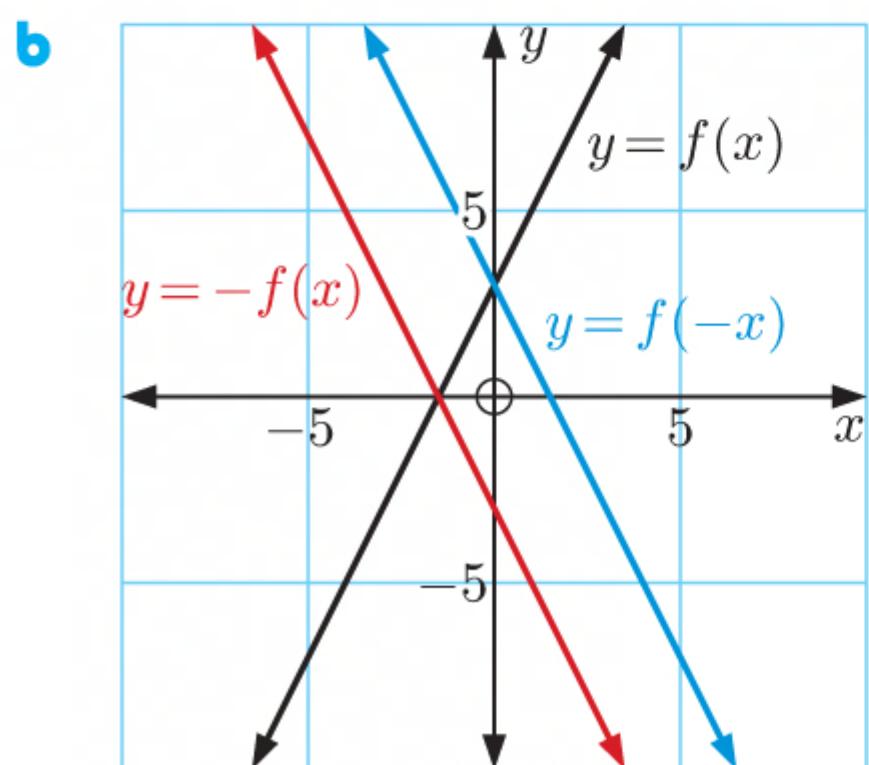
- c** For the transformation $y = f(qx)$, $q > 0$, each point becomes $\frac{1}{q}$ times its previous distance from the y -axis.
- d** $(0, 2)$ is 0 units from the y -axis, so it becomes $0 \times \frac{1}{q} = 0$ units from the y -axis.
 \therefore all transformations of the form $f(qx)$, $q > 0$ do not move the point $(0, 2)$.

PART 3: GRAPHS OF THE FORM $y = -f(x)$ AND $y = f(-x)$

1 $f(x) = 2x + 3$

a **i** $-f(x) = -(2x + 3)$
 $= -2x - 3$

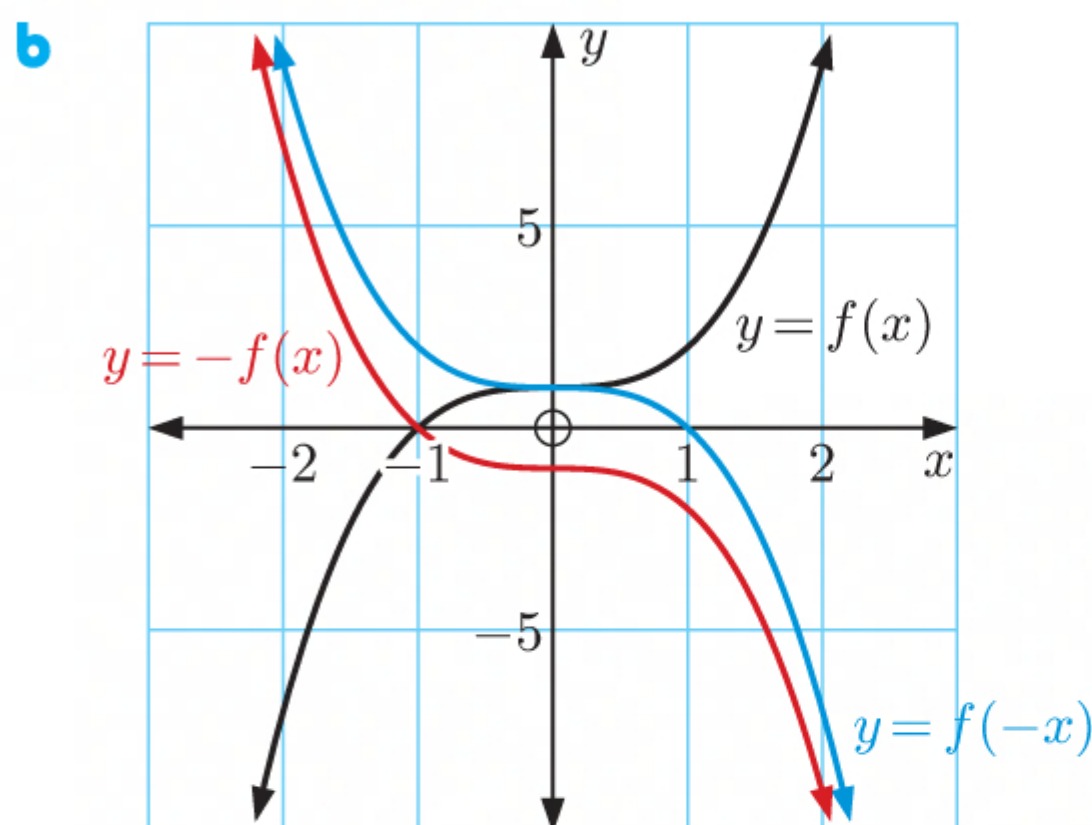
ii $f(-x) = 2(-x) + 3$
 $= -2x + 3$



2 $f(x) = x^3 + 1$

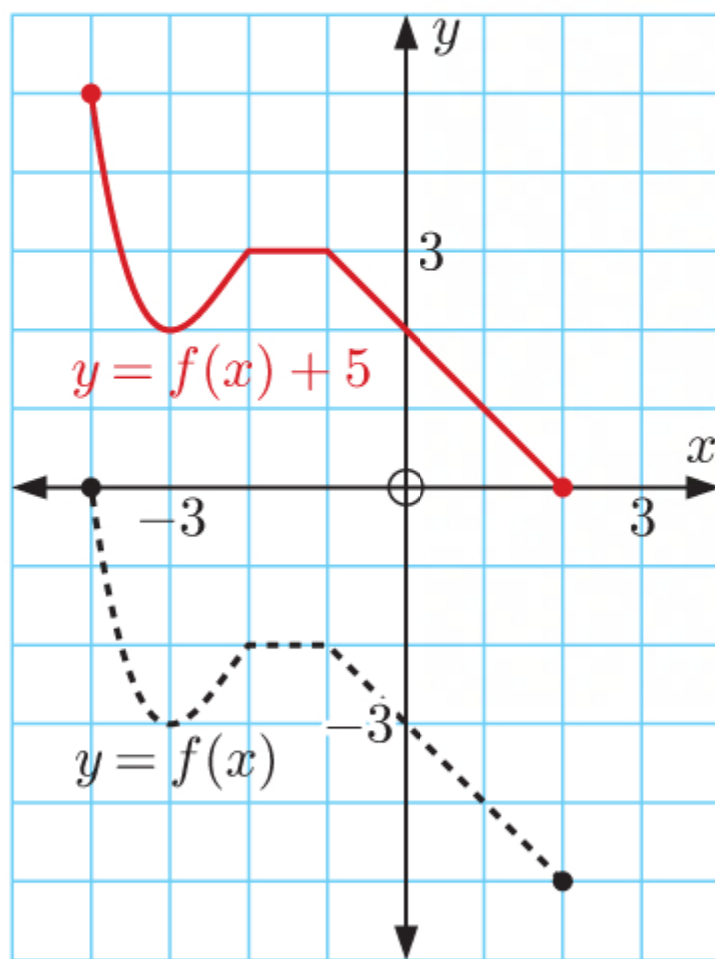
a **i** $-f(x) = -(x^3 + 1)$
 $= -x^3 - 1$

ii $f(-x) = (-x)^3 + 1$
 $= -x^3 + 1$

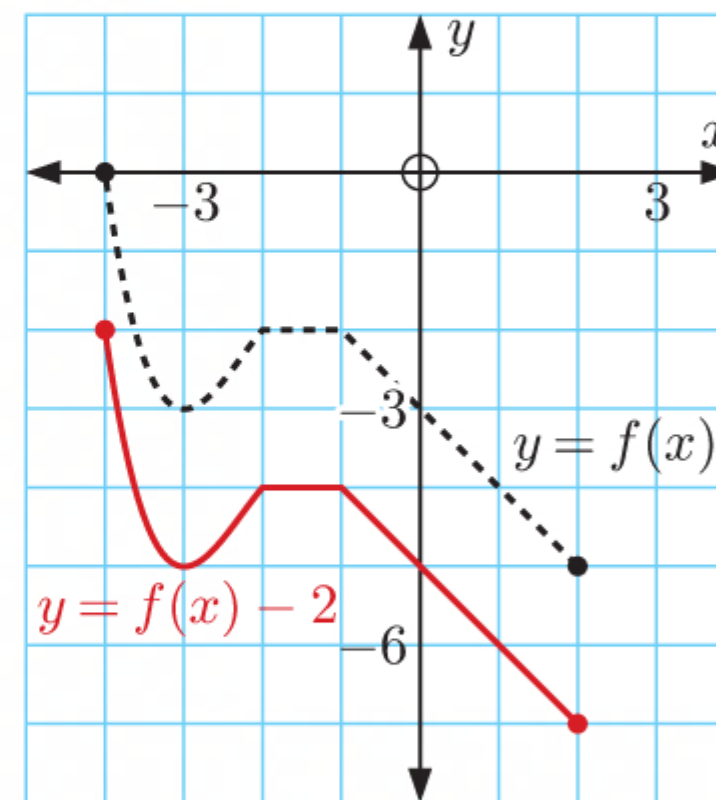
**3** **a** A reflection of the graph in the **x-axis** moves $y = f(x)$ to $y = -f(x)$.**b** A reflection of the graph in the **y-axis** moves $y = f(x)$ to $y = f(-x)$.

EXERCISE 3F

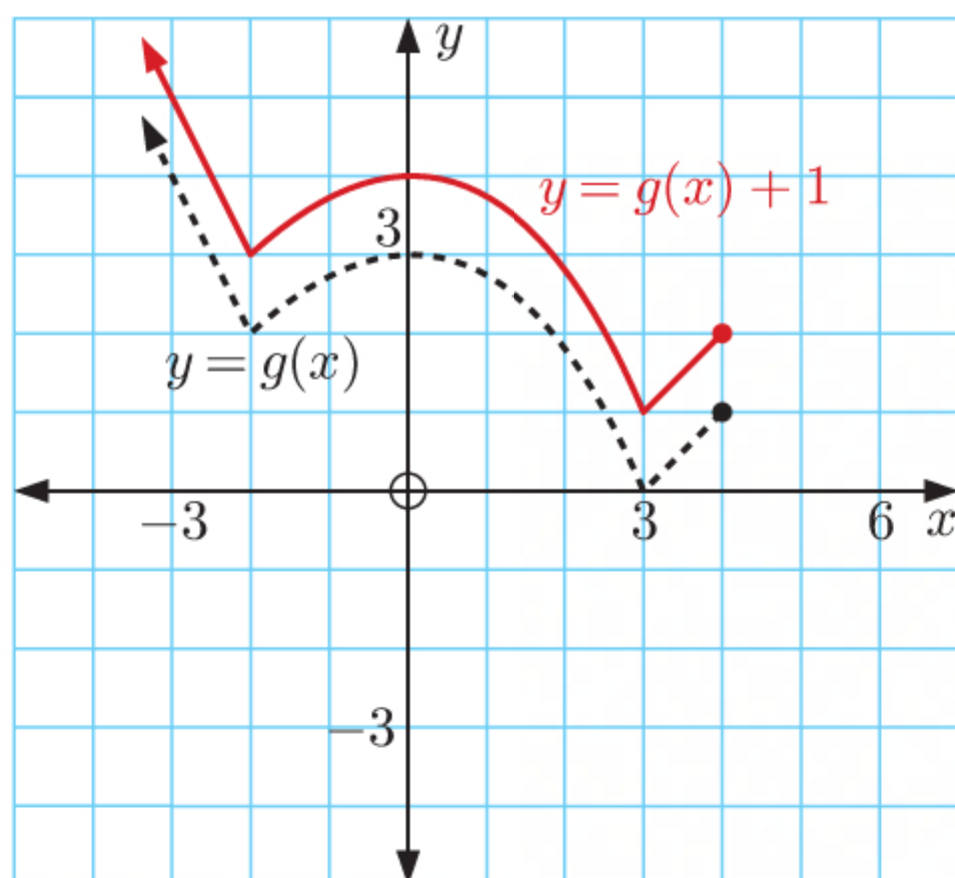
- 1 a The graph of $y = f(x) + 5$ is found by translating $y = f(x)$ 5 units upwards.



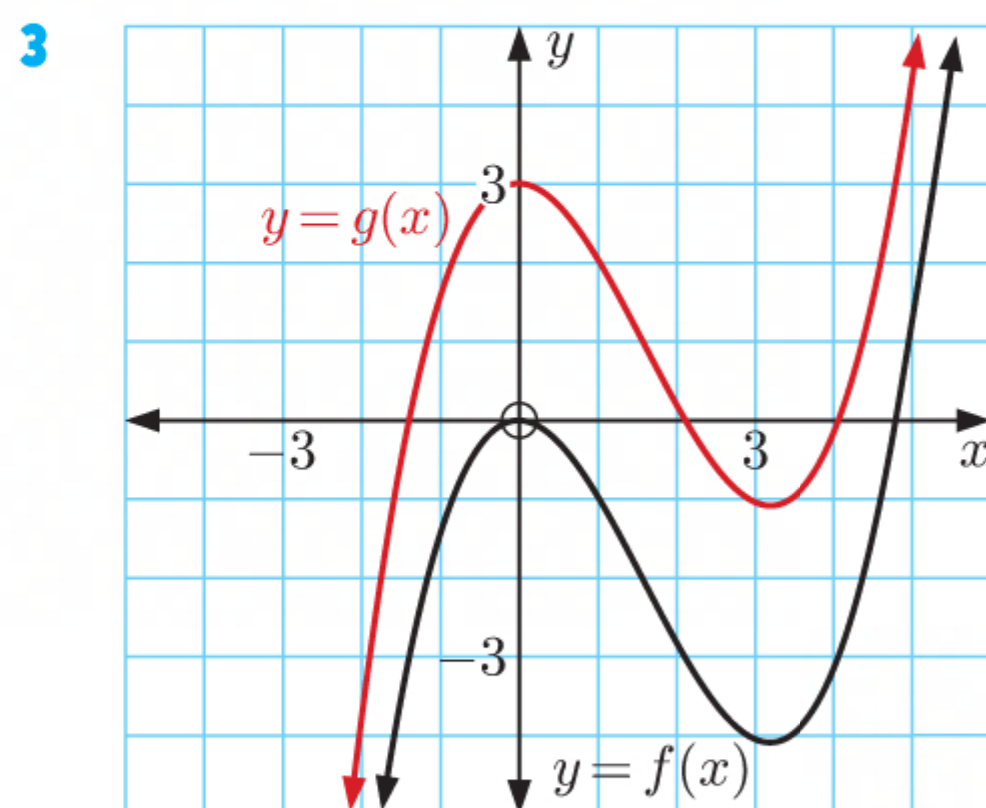
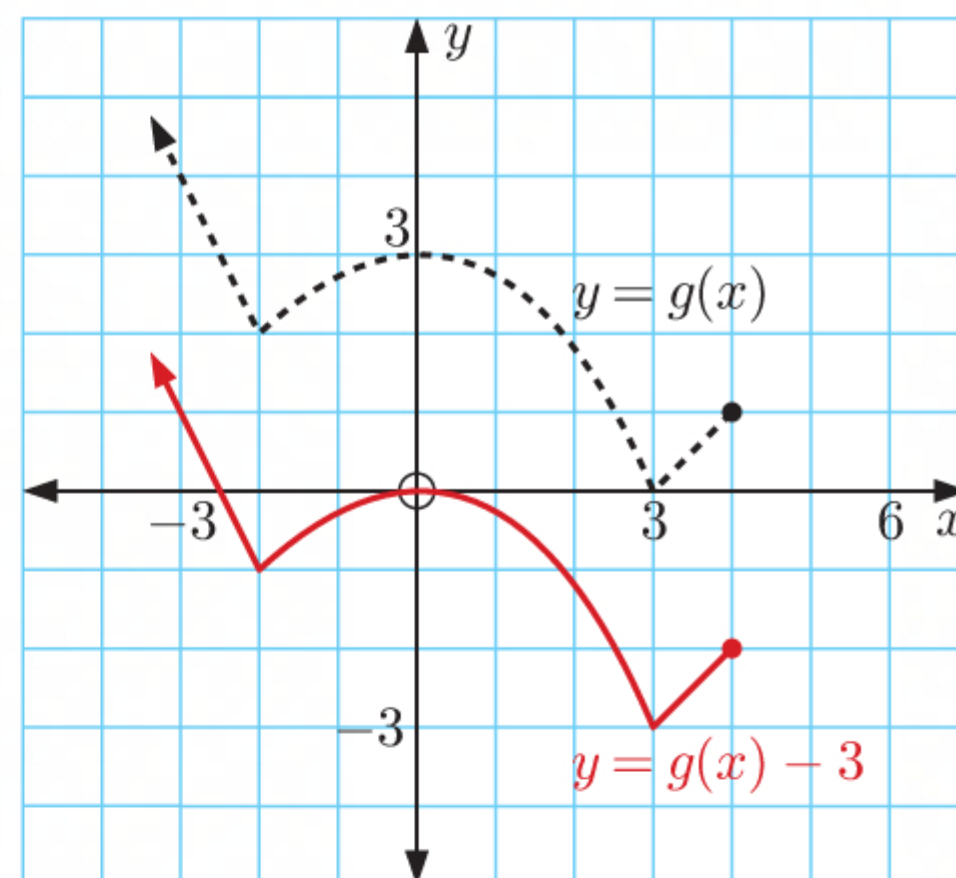
- b The graph of $y = f(x) - 2$ is found by translating $y = f(x)$ 2 units downwards.



- 2 a The graph of $y = g(x) + 1$ is found by translating $y = g(x)$ 1 unit upwards.



- b The graph of $y = g(x) - 3$ is found by translating $y = g(x)$ 3 units downwards.



The graph of $y = f(x)$ has been translated 3 units upwards to result in $y = g(x)$.
So, $g(x) = f(x) + 3$.

- 4 a** The graph of $y = g(x)$ is found by translating $y = f(x)$ 4 units downwards.

$$\therefore g(x) = f(x) - 4$$

$$\therefore g(x) = (2x + 3) - 4 \quad \{\text{since } f(x) = 2x + 3\}$$

$$\therefore g(x) = 2x - 1$$

- b** The graph of $y = g(x)$ is found by translating $y = f(x)$ 3 units upwards.

$$\therefore g(x) = f(x) + 3$$

$$\therefore g(x) = (-x^2 + 5x - 7) + 3 \quad \{\text{since } f(x) = -x^2 + 5x - 7\}$$

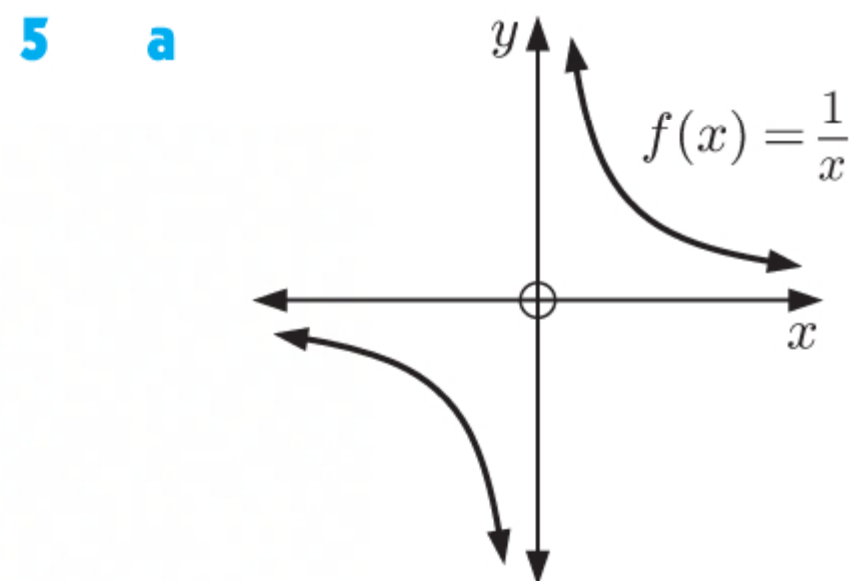
$$\therefore g(x) = -x^2 + 5x - 4$$

- c** The graph of $y = g(x)$ is found by translating $y = f(x)$ 6 units downwards.

$$\therefore g(x) = f(x) - 6$$

$$\therefore g(x) = \left(x^3 + 2 - \frac{1}{x}\right) - 6 \quad \left\{\text{since } f(x) = x^3 + 2 - \frac{1}{x}\right\}$$

$$\therefore g(x) = x^3 - 4 - \frac{1}{x}$$



b i $g(x) = f(x) + k$

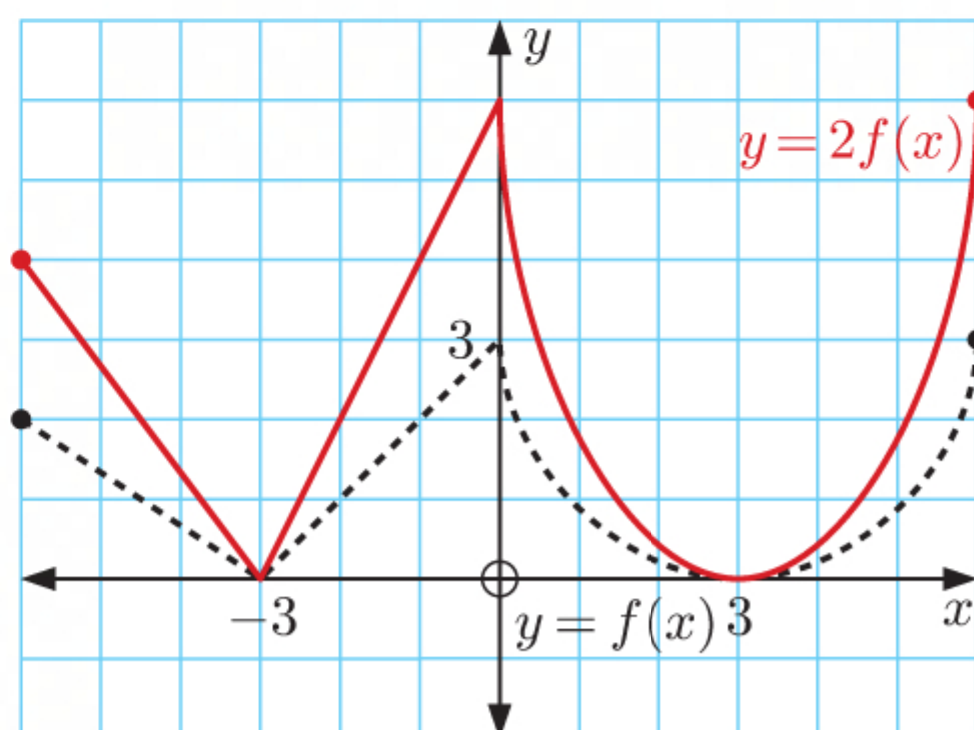
$$\therefore g(x) = \frac{1}{x} + k$$

ii As $x \rightarrow \pm\infty$, $\frac{1}{x} \rightarrow 0$

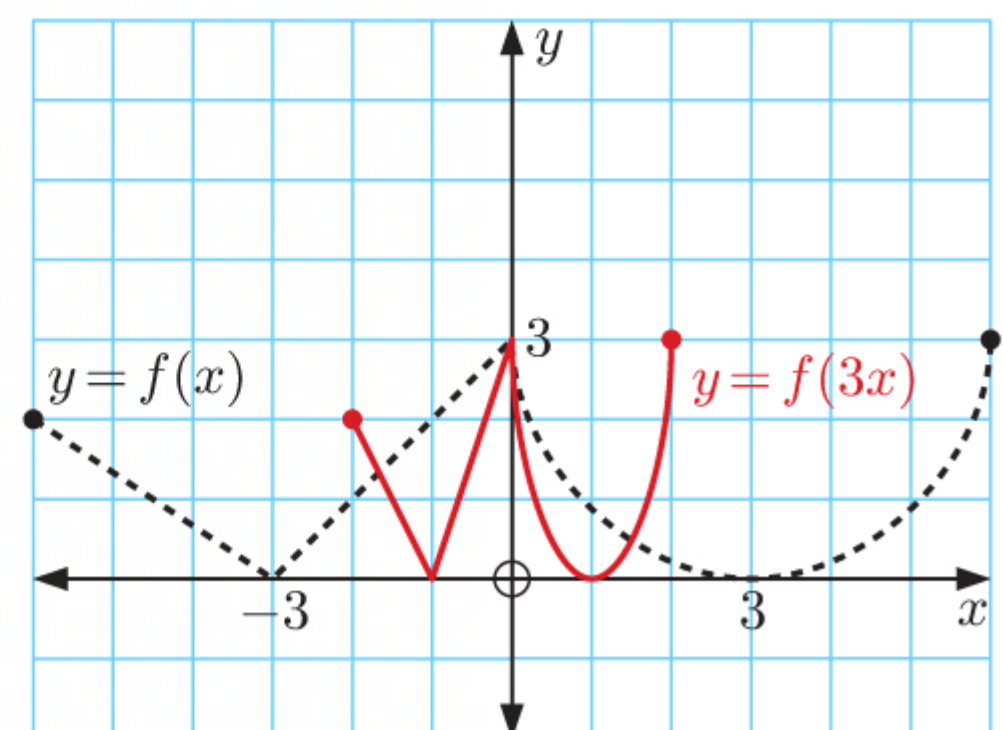
$$\therefore \text{as } x \rightarrow \pm\infty, \frac{1}{x} + k \rightarrow k$$

\therefore the horizontal asymptote of $g(x)$ is $y = k$.

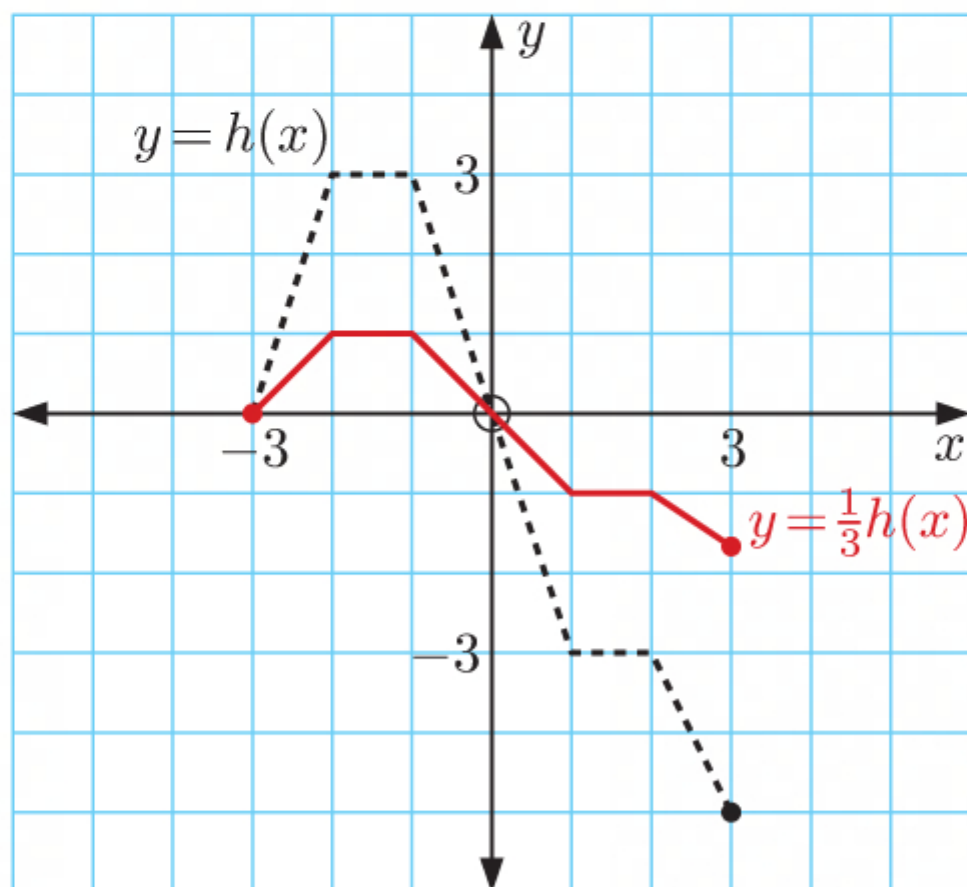
- 6 a** The graph of $y = 2f(x)$ is a vertical stretch of $y = f(x)$ with scale factor 2.



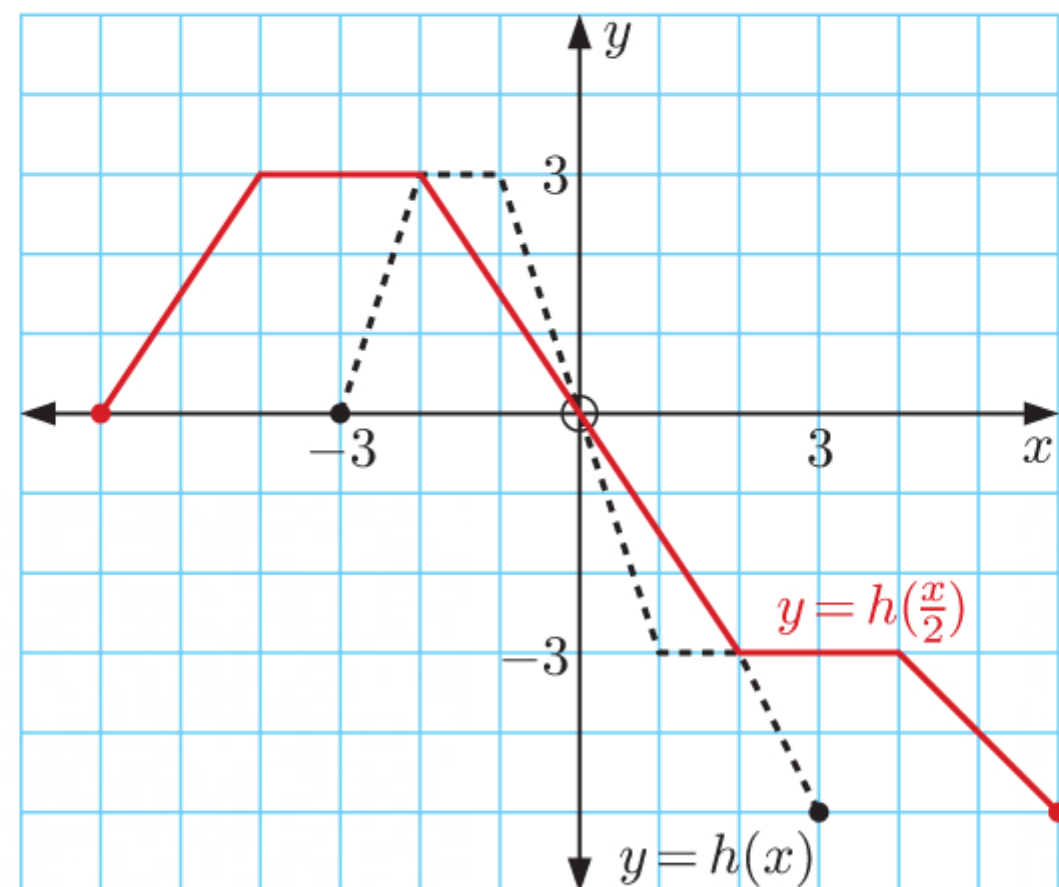
- b** The graph of $y = f(3x)$ is a horizontal stretch of $y = f(x)$ with scale factor $\frac{1}{3}$.



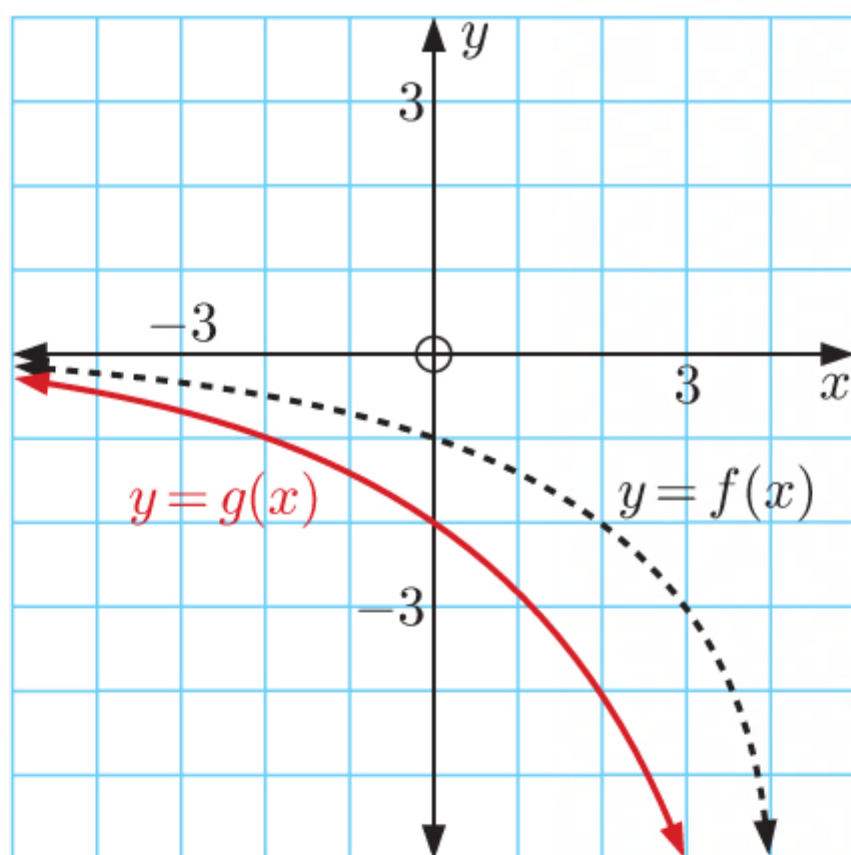
- 7 a** The graph of $y = \frac{1}{3}h(x)$ is a vertical stretch of $y = h(x)$ with scale factor $\frac{1}{3}$.



- b** The graph of $y = h\left(\frac{x}{2}\right)$ is a horizontal stretch of $y = h(x)$ with scale factor 2.



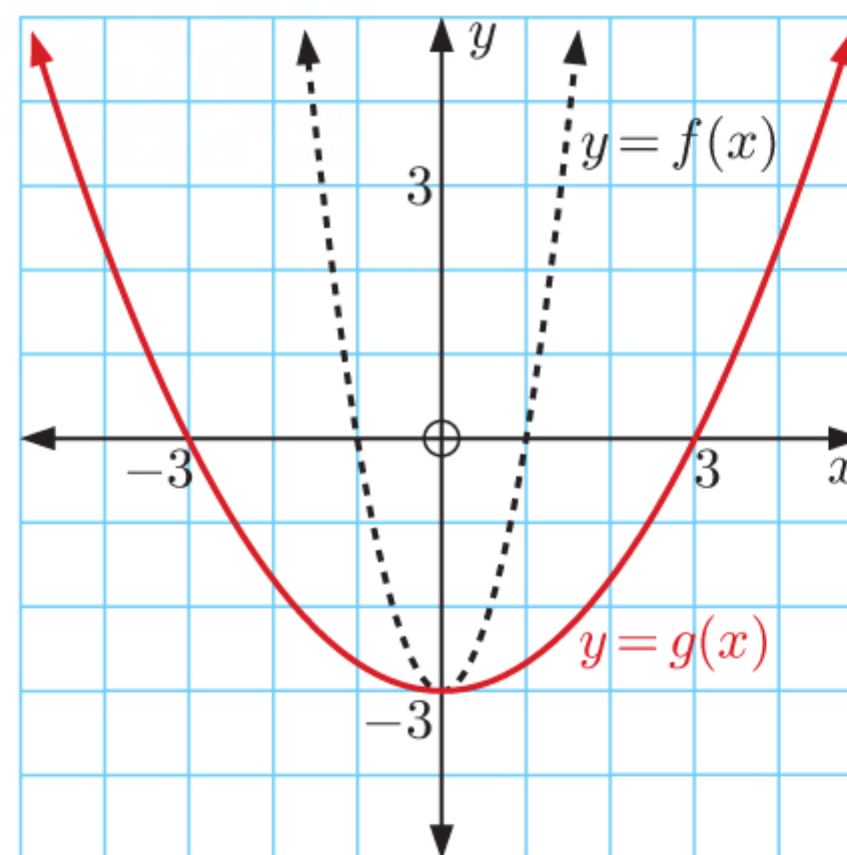
8 a



The graph of $y = f(x)$ has been vertically stretched with scale factor 2 to give $y = g(x)$.

So, $g(x) = 2f(x)$.

b



The graph of $y = f(x)$ has been horizontally stretched with scale factor 3 to give $y = g(x)$.

So, $g(x) = f\left(\frac{x}{3}\right)$.

- 9** Let the original linear function be $y = f(x) = mx + a$.

When $y = f(x)$ is vertically stretched with scale factor c , the resulting function is

$$y = c(f(x))$$

$$\therefore y = c(mx + a)$$

$$\therefore y = cmx + ac$$

So, the resulting line has gradient cm .

- 10 a** The graph of $y = g(x)$ is a vertical stretch of $y = f(x)$ with scale factor 2.

$$\therefore g(x) = 2f(x)$$

$$\therefore g(x) = 2(x^2 + 2) \quad \{\text{since } f(x) = x^2 + 2\}$$

$$\therefore g(x) = 2x^2 + 4$$

- b** The graph of $y = g(x)$ is a horizontal stretch of $y = f(x)$ with scale factor 3.

$$\therefore g(x) = f\left(\frac{x}{3}\right)$$

$$\therefore g(x) = 5 - 3\left(\frac{x}{3}\right) \quad \{\text{since } f(x) = 5 - 3x\}$$

$$\therefore g(x) = 5 - x$$

- c** The graph of $y = g(x)$ is a vertical dilation of $y = f(x)$ with scale factor $\frac{1}{4}$.

$$\therefore g(x) = \frac{1}{4}f(x)$$

$$\therefore g(x) = \frac{1}{4}(x^3 + 8x^2 - 2) \quad \{\text{since } f(x) = x^3 + 8x^2 - 2\}$$

$$\therefore g(x) = \frac{1}{4}x^3 + 2x^2 - \frac{1}{2}$$

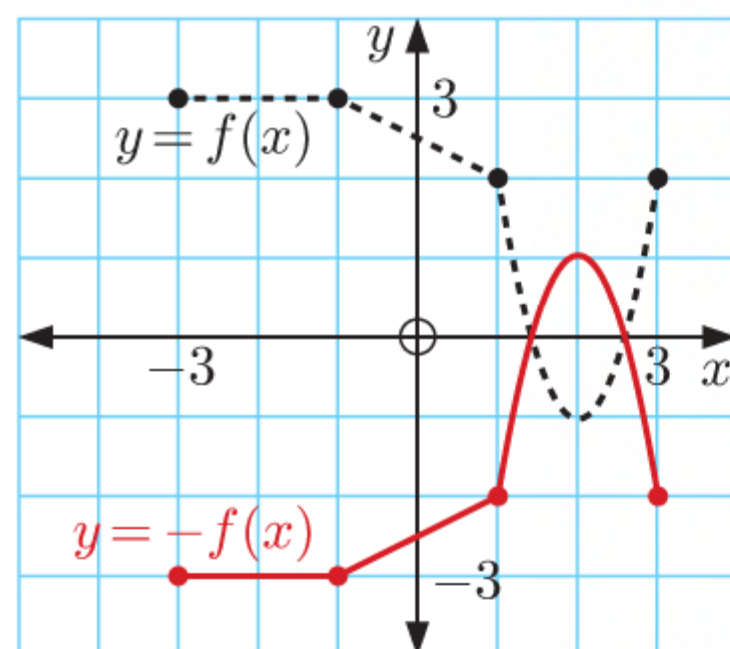
- d** The graph of $y = g(x)$ is a horizontal dilation of $y = f(x)$ with scale factor $\frac{1}{2}$.

$$\therefore g(x) = f(2x)$$

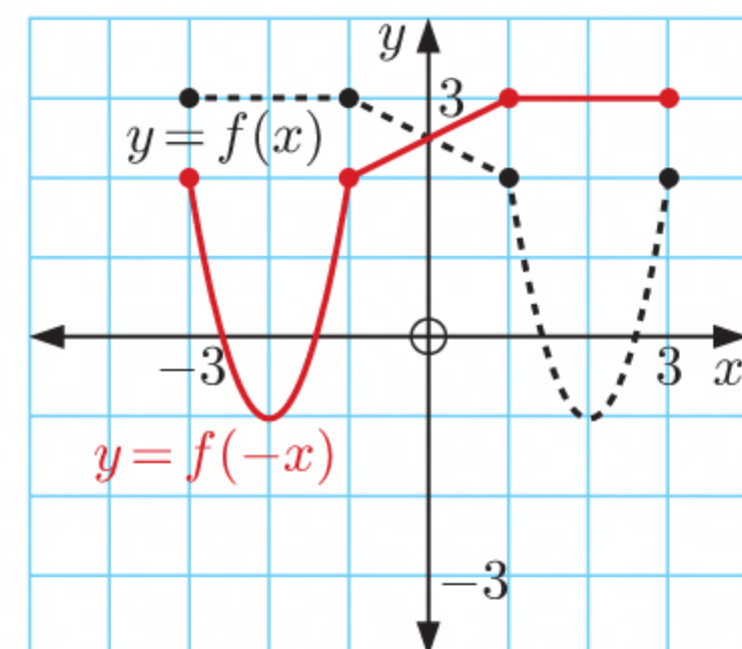
$$\therefore g(x) = 2(2x)^2 + (2x) - 3 \quad \{\text{since } f(x) = 2x^2 + x - 3\}$$

$$\therefore g(x) = 8x^2 + 2x - 3$$

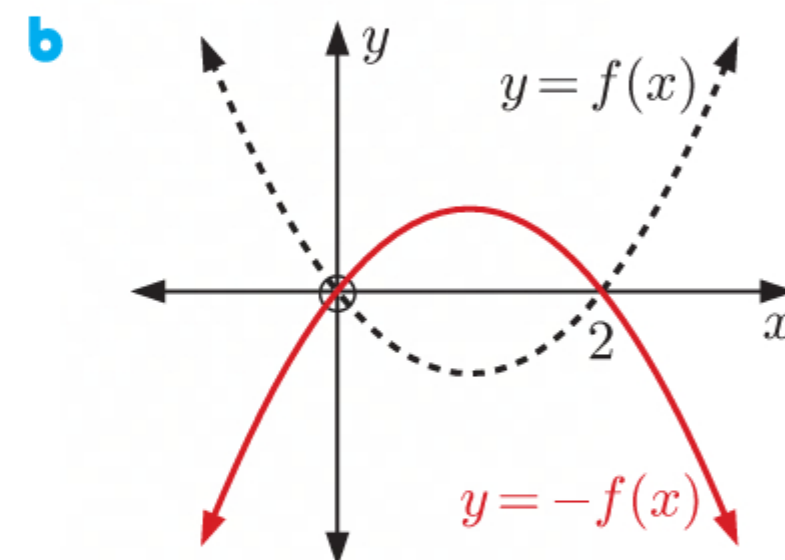
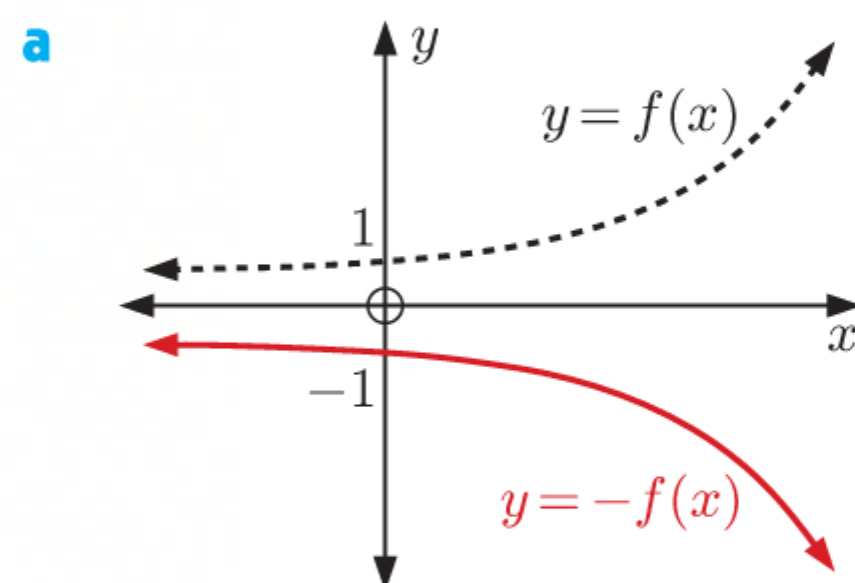
- 11 a** The graph of $y = -f(x)$ is found by reflecting $y = f(x)$ in the x -axis.

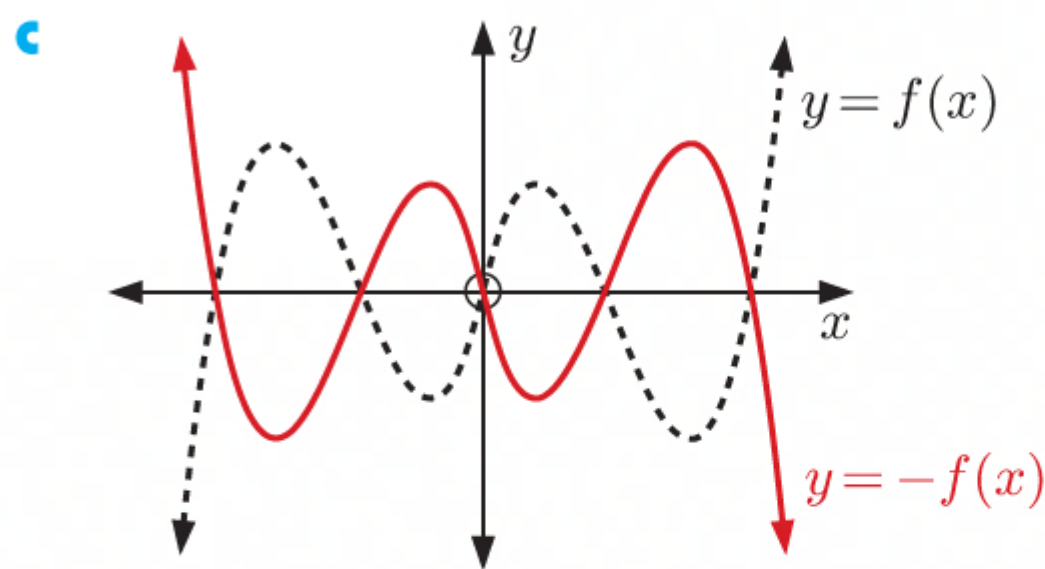


- b** The graph of $y = f(-x)$ is found by reflecting $y = f(x)$ in the y -axis.

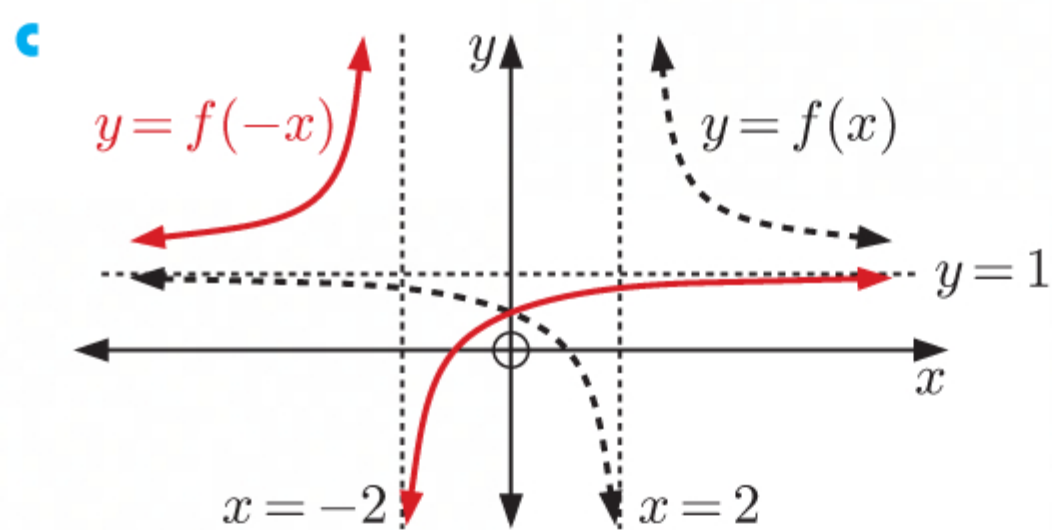
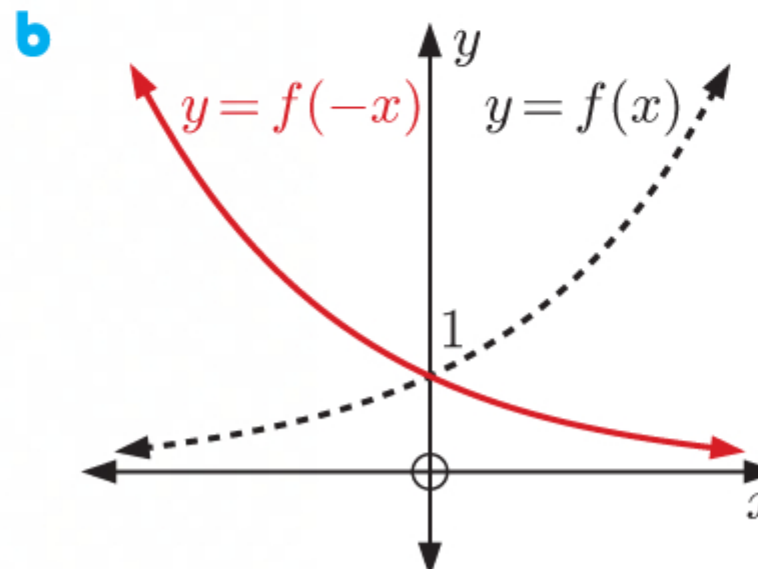
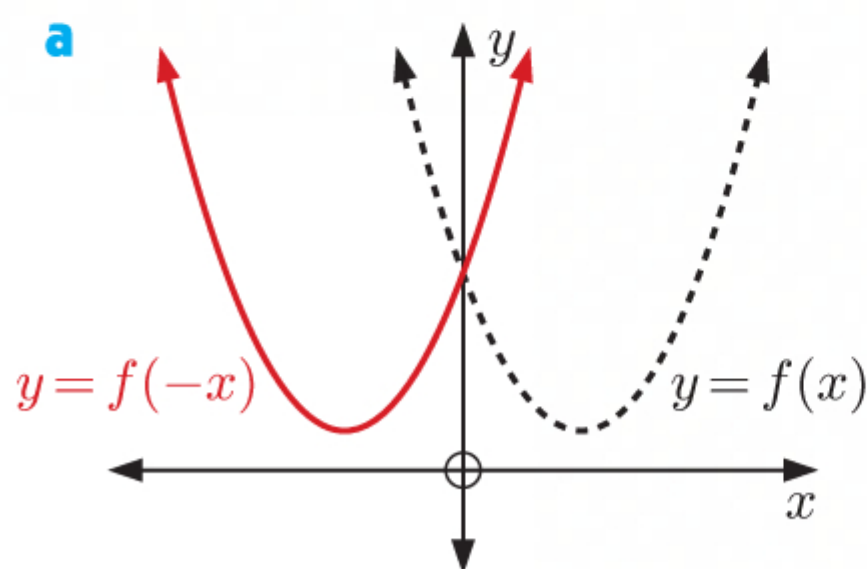


- 12** The graph of $y = -f(x)$ is found by reflecting $y = f(x)$ in the x -axis.





- 13** The graph of $y = f(-x)$ is found by reflecting $y = f(x)$ in the y -axis.



- 14 a** The graph of $y = g(x)$ is a reflection of $y = f(x)$ in the x -axis.

$$\begin{aligned}\therefore g(x) &= -f(x) \\ \therefore g(x) &= -(5x + 7) \quad \{\text{since } f(x) = 5x + 7\} \\ \therefore g(x) &= -5x - 7\end{aligned}$$

- b** The graph of $y = g(x)$ is a reflection of $y = f(x)$ in the y -axis.

$$\begin{aligned}\therefore g(x) &= f(-x) \\ \therefore g(x) &= 2^{-x} \quad \{\text{since } f(x) = 2^x\}\end{aligned}$$

- c** The graph of $y = g(x)$ is a reflection of $y = f(x)$ in the x -axis.

$$\begin{aligned}\therefore g(x) &= -f(x) \\ \therefore g(x) &= -(2x^2 + 1) \quad \{\text{since } f(x) = 2x^2 + 1\} \\ \therefore g(x) &= -2x^2 - 1\end{aligned}$$

- d** The graph of $y = g(x)$ is a reflection of $y = f(x)$ in the y -axis.

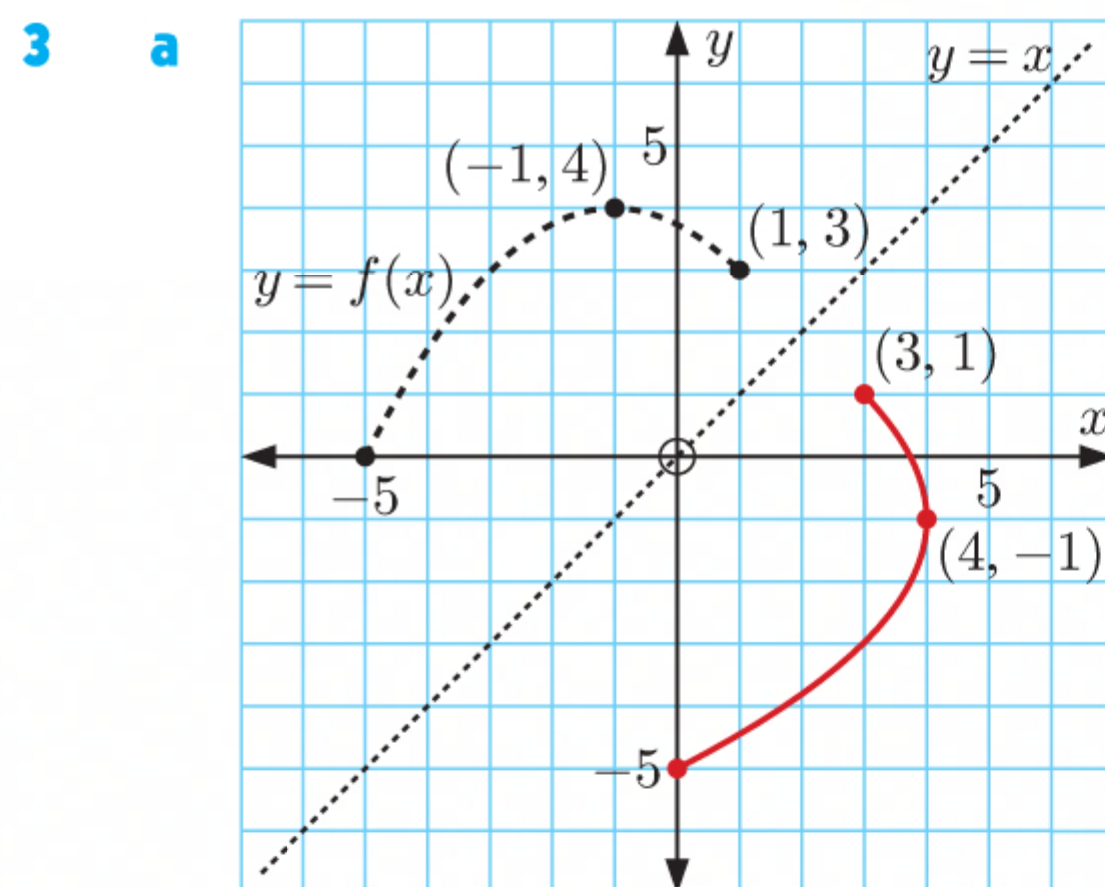
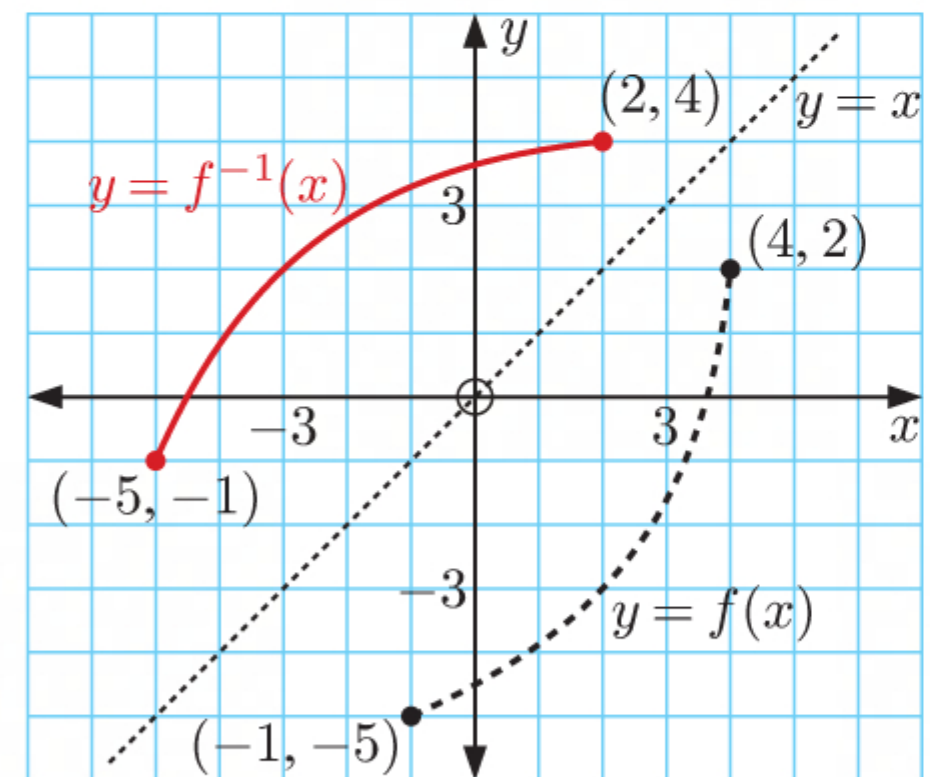
$$\begin{aligned}\therefore g(x) &= f(-x) \\ \therefore g(x) &= (-x)^4 - 2(-x)^3 - 3(-x)^2 + 5(-x) - 7 \\ &\quad \{\text{since } f(x) = x^4 - 2x^3 - 3x^2 + 5x - 7\} \\ \therefore g(x) &= x^4 + 2x^3 - 3x^2 - 5x - 7\end{aligned}$$

EXERCISE 3G

- 1 $(-3, 7)$, $(0, 4)$, and $(2, -6)$ lie on f .
 $\therefore (7, -3)$, $(4, 0)$, and $(-6, 2)$ lie on f^{-1} .

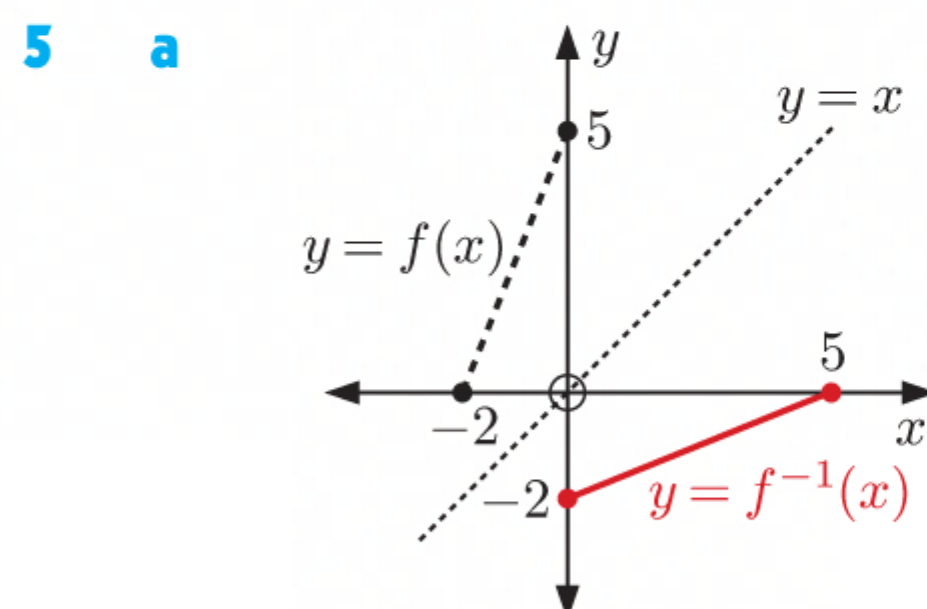
- 2 a An inverse function exists since $f(x)$ passes both the vertical and horizontal line tests.

- b $y = f(x)$ passes through $(-1, -5)$ and $(4, 2)$
 $\therefore y = f^{-1}(x)$ passes through $(-5, -1)$ and $(2, 4)$.

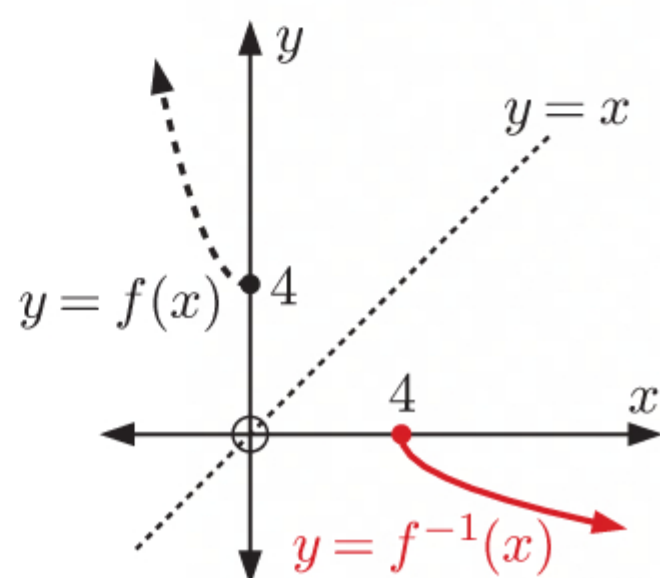
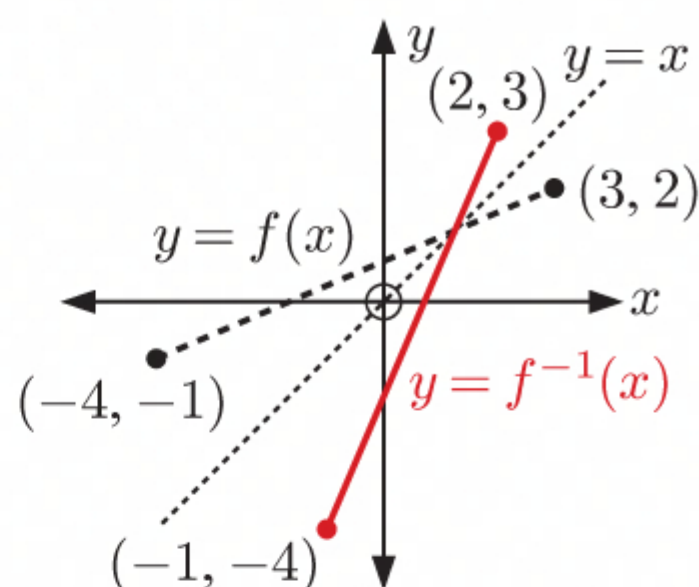
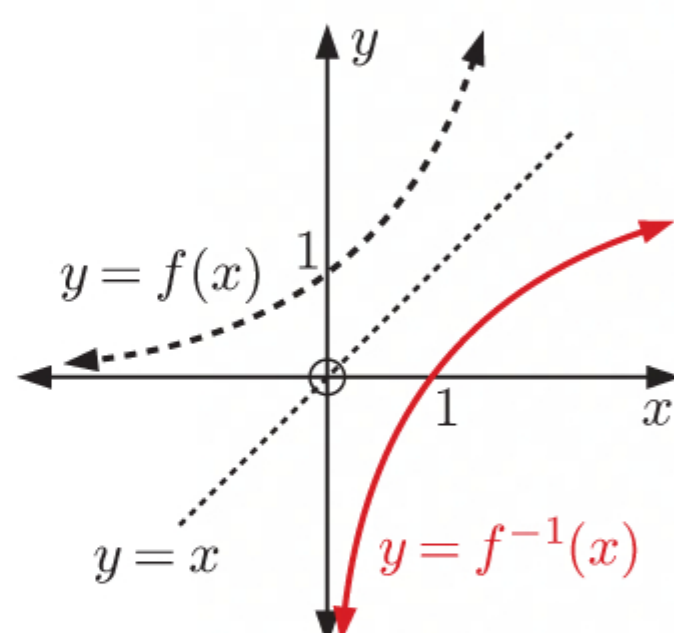
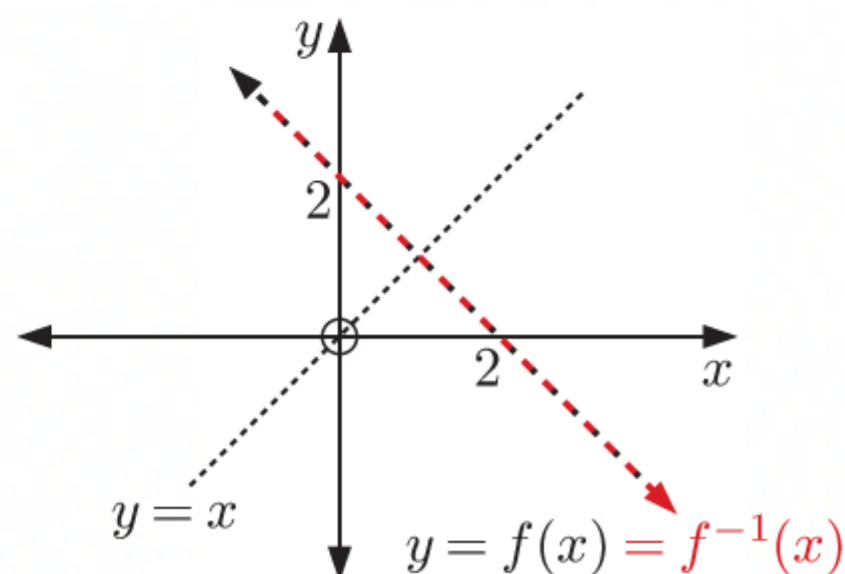
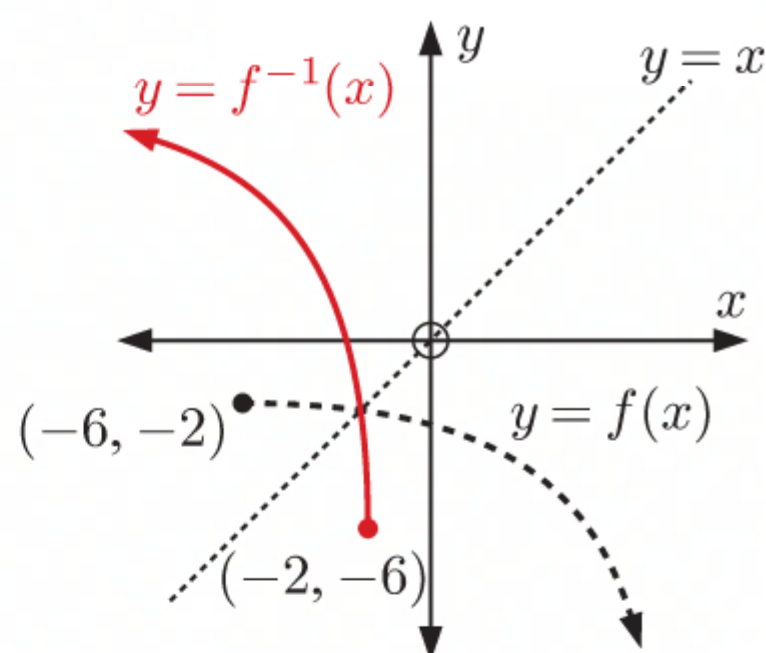


- b No, $f(x)$ does not have an inverse since it does not pass the horizontal line test.

- 4 $f(x)$ has domain $\{x \mid -2 \leq x < 3\}$.
 $\therefore f^{-1}(x)$ has range $\{y \mid -2 \leq y < 3\}$.



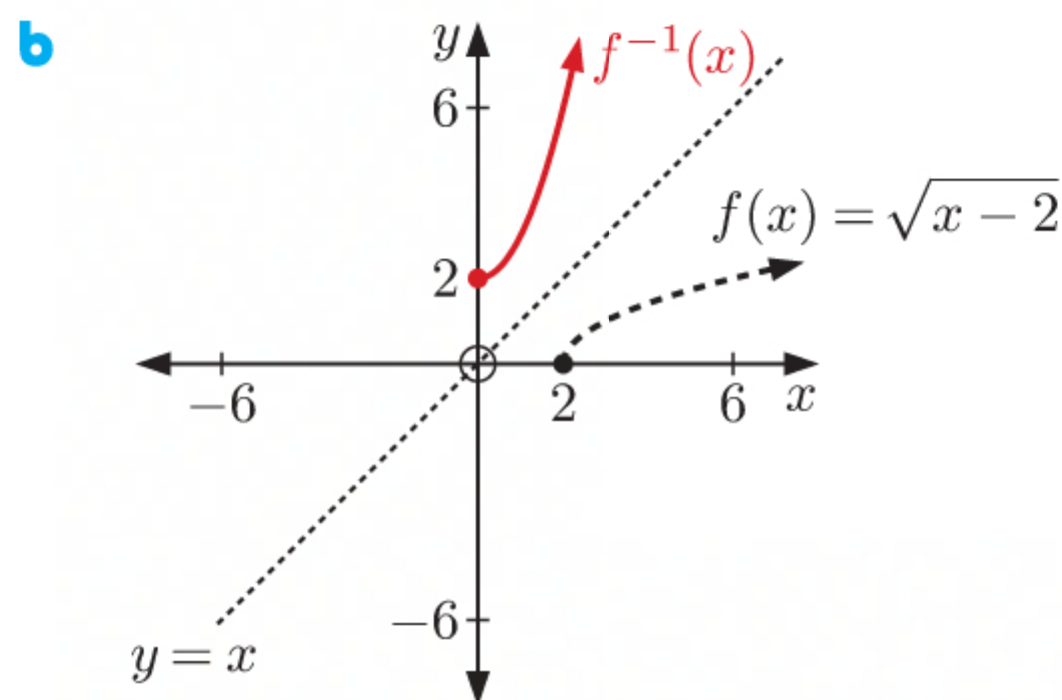
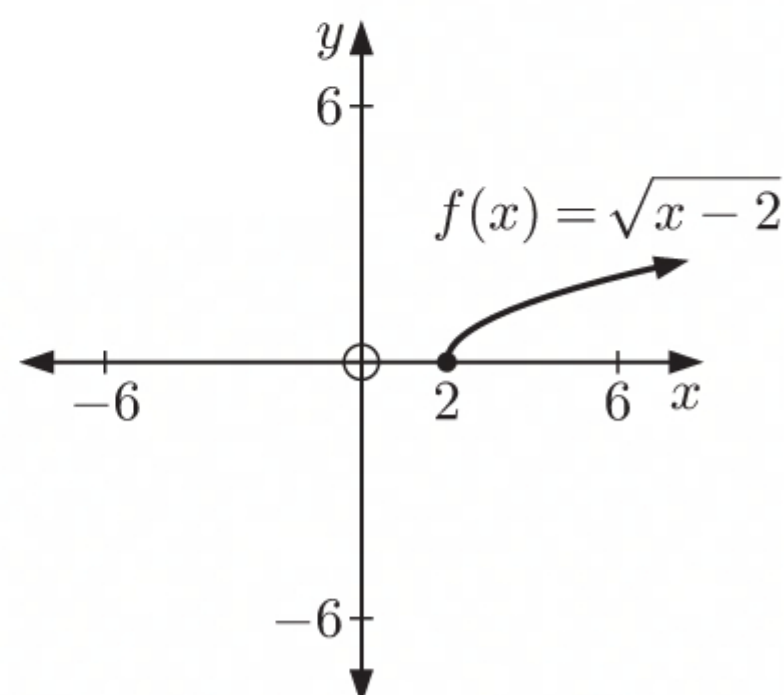
- f : Domain is $\{x \mid -2 \leq x \leq 0\}$
 Range is $\{y \mid 0 \leq y \leq 5\}$
 f^{-1} : Domain is $\{x \mid 0 \leq x \leq 5\}$
 Range is $\{y \mid -2 \leq y \leq 0\}$

b f : Domain is $\{x \mid x \leq 0\}$ Range is $\{y \mid y \geq 4\}$ f^{-1} : Domain is $\{x \mid x \geq 4\}$ Range is $\{y \mid y \leq 0\}$ **c** f : Domain is $\{x \mid -4 \leq x \leq 3\}$ Range is $\{y \mid -1 \leq y \leq 2\}$ f^{-1} : Domain is $\{x \mid -1 \leq x \leq 2\}$ Range is $\{y \mid -4 \leq y \leq 3\}$ **d** f : Domain is $\{x \mid x \in \mathbb{R}\}$ Range is $\{y \mid y > 0\}$ f^{-1} : Domain is $\{x \mid x > 0\}$ Range is $\{y \mid y \in \mathbb{R}\}$ **e** f : Domain is $\{x \mid x \in \mathbb{R}\}$ Range is $\{y \mid y \in \mathbb{R}\}$ f^{-1} : Domain is $\{x \mid x \in \mathbb{R}\}$ Range is $\{y \mid y \in \mathbb{R}\}$ **f** f : Domain is $\{x \mid x \geq -6\}$ Range is $\{y \mid y \leq -2\}$ f^{-1} : Domain is $\{x \mid x \leq -2\}$ Range is $\{y \mid y \geq -6\}$ **6** $g(x)$ has x -intercept 5 and y -intercept -3 . $\therefore y = g(x)$ passes through $(5, 0)$ and $(0, -3)$. $\therefore y = g^{-1}(x)$ passes through $(0, 5)$ and $(-3, 0)$. $\therefore g^{-1}(x)$ has x -intercept -3 and y -intercept 5.

- 7** $f(x)$ does not pass the horizontal line test ($y = 0$ passes through the two x -intercepts).
 $\therefore f(x)$ is not invertible.

- 8 a** $\sqrt{x-2}$ is defined when $x-2 \geq 0$
 $\therefore x \geq 2$

$\therefore f(x) = \sqrt{x-2}$ has domain $\{x \mid x \geq 2\}$
 and range $\{y \mid y \geq 0\}$.



- c** $f(x)$ has domain $\{x \mid x \geq 2\}$
 and range $\{y \mid y \geq 0\}$.
 $\therefore f^{-1}(x)$ has domain $\{x \mid x \geq 0\}$
 and range $\{y \mid y \geq 2\}$.

- d** If $f^{-1}(x) = 3$, then $y = f^{-1}(x)$ passes through $(x, 3)$.

So, $y = f(x)$ passes through $(3, x)$.

$$\therefore f(3) = x$$

$$\therefore x = \sqrt{3-2}$$

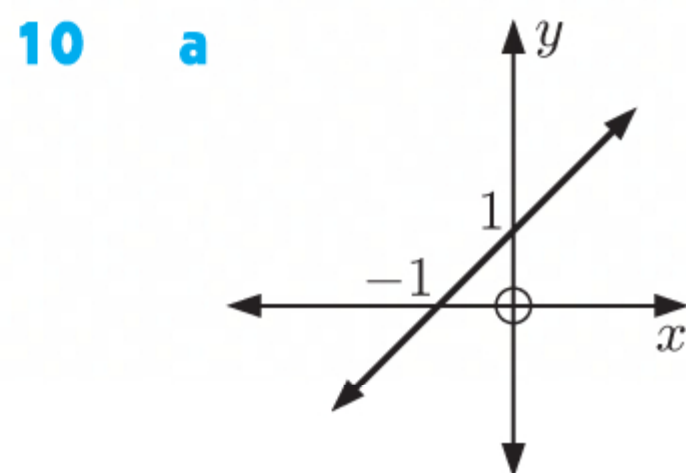
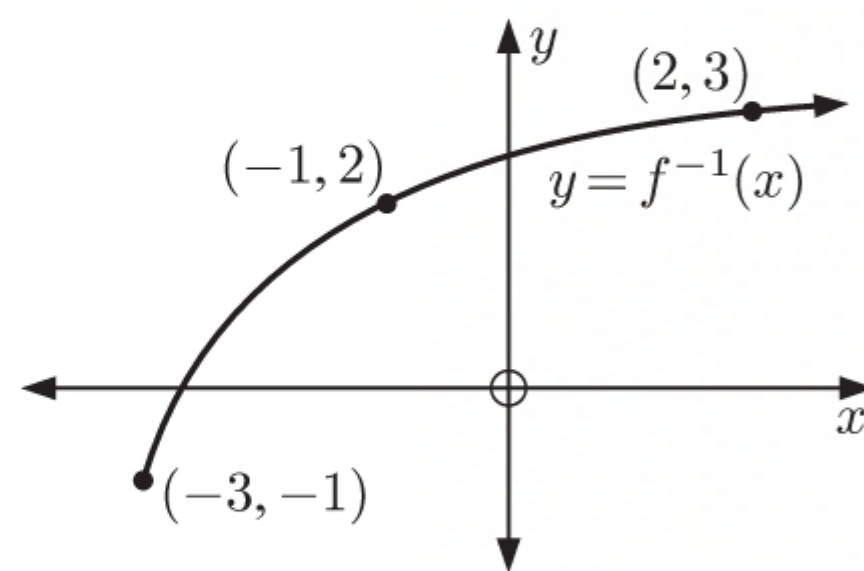
$$\therefore x = \sqrt{1} = 1$$

- 9 a** $f^{-1}(x)$ has domain $\{x \mid x \geq -3\}$
 and range $\{y \mid y \geq -1\}$.
 $\therefore f(x)$ has domain $\{x \mid x \geq -1\}$
 and range $\{y \mid y \geq -3\}$.

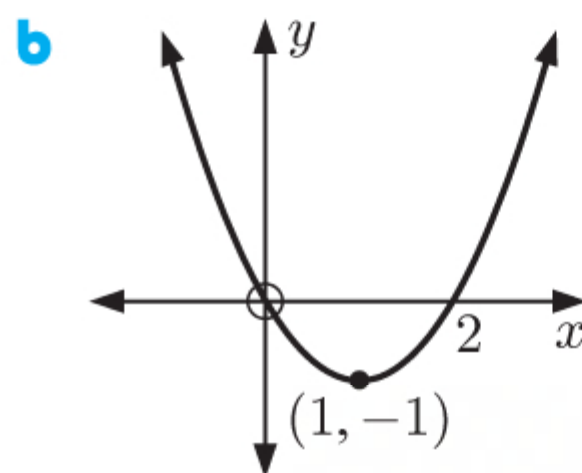
- b** If $f(x) = 2$, then $y = f(x)$ passes through $(x, 2)$.

So, $y = f^{-1}(x)$ passes through $(2, x)$.

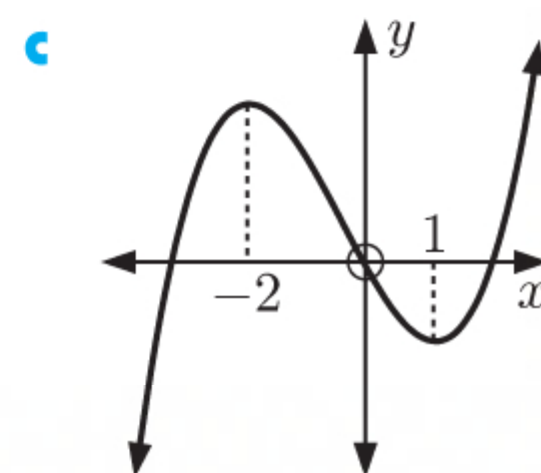
Now $f^{-1}(2) = 3$ {from the graph}
 $\therefore x = 3$



The graph passes the horizontal and vertical line tests.
 \therefore the function has an inverse.



The graph fails the horizontal line test.
 \therefore the function does not have an inverse.



The graph fails the horizontal line test.
 \therefore the function does not have an inverse.

$$\begin{aligned}
 11 \quad a \quad f(-2) &= (-2)^2 - 2(-2) + 5 \\
 &= 4 + 4 + 5 \\
 &= 13
 \end{aligned}$$

$$\begin{aligned}
 f(0) &= (0)^2 - 2(0) + 5 \\
 &= 5
 \end{aligned}$$

$$\begin{aligned}
 f(2) &= (2)^2 - 2(2) + 5 \\
 &= 4 - 4 + 5 \\
 &= 5
 \end{aligned}$$

x	-2	-1	0	1	2
$f(x)$	13	8	5	4	5

$$\begin{aligned}
 f(-1) &= (-1)^2 - 2(-1) + 5 \\
 &= 1 + 2 + 5 \\
 &= 8
 \end{aligned}$$

$$\begin{aligned}
 f(1) &= (1)^2 - 2(1) + 5 \\
 &= 1 - 2 + 5 \\
 &= 4
 \end{aligned}$$

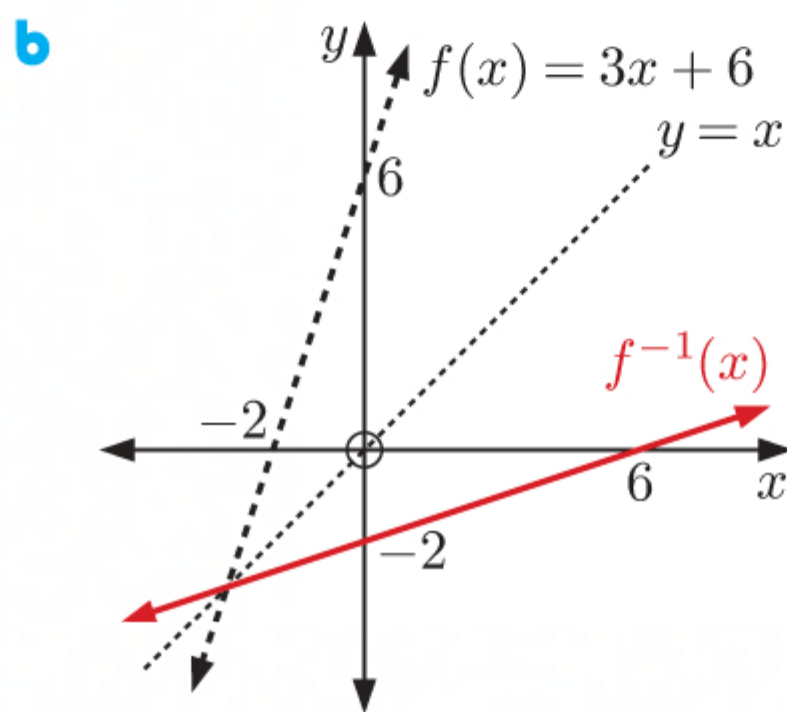
- b** There are two points with the same y -coordinate, $(0, 5)$ and $(2, 5)$, so $f(x)$ does not pass the horizontal line test and hence is not invertible.

$$\begin{aligned}
 12 \quad a \quad f(x) &= 3x + 6 \\
 \therefore f(0) &= 3(0) + 6 \\
 &= 6
 \end{aligned}$$

\therefore the y -intercept is 6.

$$\begin{aligned}
 \text{When } f(x) = 0, \quad 3x + 6 &= 0 \\
 \therefore 3x &= -6 \\
 \therefore x &= -2
 \end{aligned}$$

\therefore the x -intercept is -2 .



- c** $f(x)$ is a line through $(-2, 0)$ and $(0, 6)$.
 $\therefore f^{-1}(x)$ is a line through $(0, -2)$ and $(6, 0)$.

$$\begin{aligned}
 \text{The line through } (0, -2) \text{ and } (6, 0) \text{ has gradient } &\frac{0 - (-2)}{6 - 0} = \frac{2}{6} \\
 &= \frac{1}{3}
 \end{aligned}$$

and the y -intercept is -2 .

$$\therefore f^{-1}(x) = \frac{1}{3}x - 2.$$

$$\begin{aligned}
 13 \quad f(x) &= mx + c \\
 \therefore f(0) &= m(0) + c \\
 &= c
 \end{aligned}$$

\therefore the y -intercept is c .

$$\text{When } f(x) = 0, \quad mx + c = 0$$

$$\therefore mx = -c$$

$$\therefore x = -\frac{c}{m} \quad \{m \neq 0\}$$

\therefore the x -intercept is $-\frac{c}{m}$.

$f(x)$ is a line through $\left(-\frac{c}{m}, 0\right)$ and $(0, c)$.

$\therefore f^{-1}(x)$ is a line through $\left(0, -\frac{c}{m}\right)$ and $(c, 0)$.

The line through $\left(0, -\frac{c}{m}\right)$ and $(c, 0)$ has gradient $\frac{0 - \left(-\frac{c}{m}\right)}{c - 0} = \frac{\frac{c}{m}}{c}$
 $= \frac{1}{m}$

and the y -intercept is $-\frac{c}{m}$.

$$\begin{aligned}\therefore f^{-1}(x) &= \frac{1}{m}x - \frac{c}{m} \\ &= \frac{x - c}{m}, \quad m \neq 0\end{aligned}$$

REVIEW SET 3A

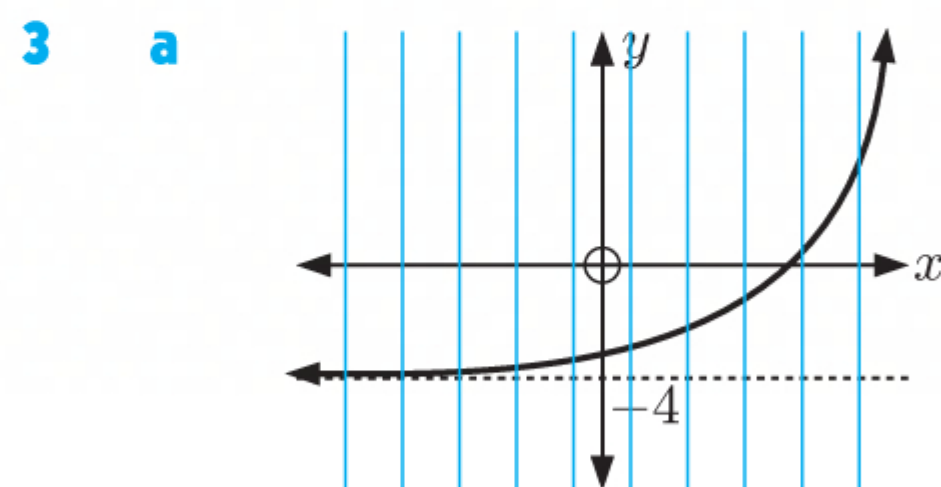
- 1 a** $\{(1, 5), (4, 2), (2, 5), (6, -1)\}$ is a function since no two ordered pairs have the same x -coordinate.
- b** $\{(-4, 0), (3, 2), (0, -2), (3, 5)\}$ is not a function since two of the ordered pairs, $(3, 2)$ and $(3, 5)$, have the same x -coordinate 3.

2 $f(x) = 2x - x^2$

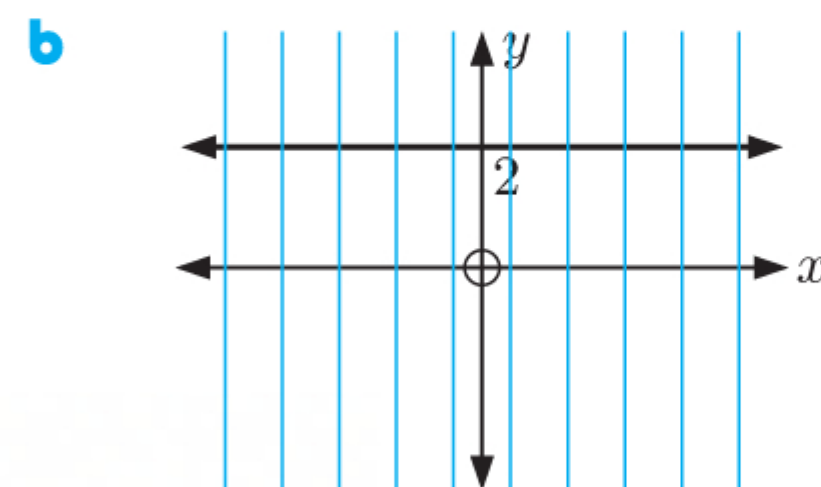
a $f(2) = 2(2) - 2^2$
 $= 4 - 4$
 $= 0$

b $f(-3) = 2(-3) - (-3)^2$
 $= -6 - 9$
 $= -15$

c $f\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{2}\right) - \left(-\frac{1}{2}\right)^2$
 $= -1 - \frac{1}{4}$
 $= -\frac{5}{4}$

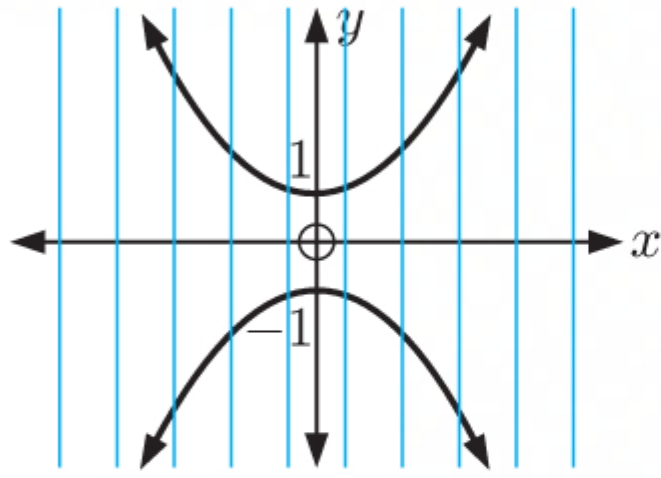


- i** Domain is $\{x \mid x \in \mathbb{R}\}$.
- ii** Range is $\{y \mid y > -4\}$.
- iii** Each vertical line cuts the graph at most once, so the graph shows a function.



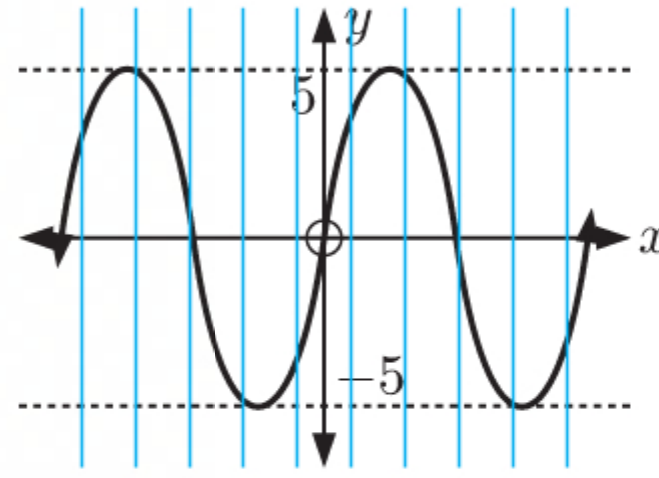
- i** Domain is $\{x \mid x \in \mathbb{R}\}$.
- ii** Range is $\{y \mid y = 2\}$.
- iii** Each vertical line cuts the graph at most once, so the graph shows a function.

c



- i Domain is $\{x \mid x \in \mathbb{R}\}$.
- ii Range is $\{y \mid y \leq -1 \text{ or } y \geq 1\}$.
- iii All vertical lines cut the graph more than once, so the graph does not show a function.

d

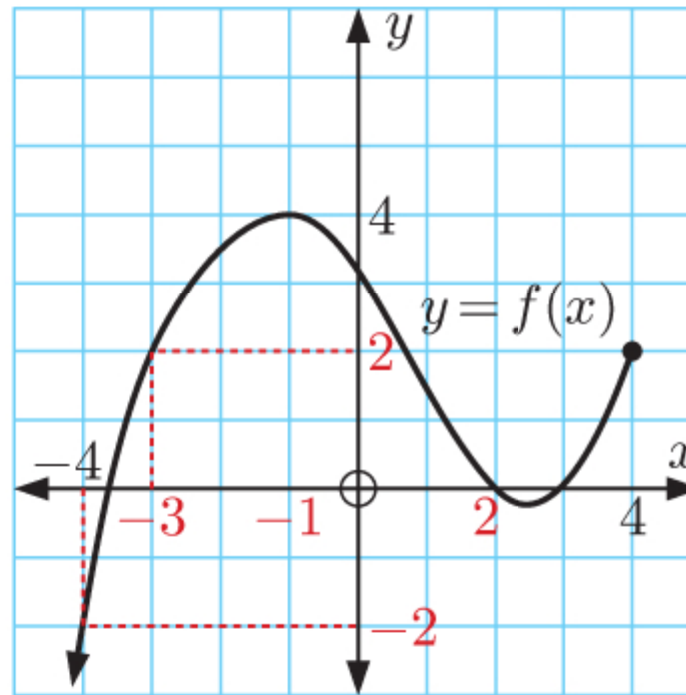


- i Domain is $\{x \mid x \in \mathbb{R}\}$.
- ii Range is $\{y \mid -5 \leq y \leq 5\}$.
- iii Each vertical line cuts the graph at most once, so the graph shows a function.

4 a i $f(-3) = 2$

ii $f(2) = 0$

b When $y = f(x) = -2$, $x = -4$.



5 $f(x) = ax + b$ where $f(1) = 7$ and $f(3) = -5$

So, $a(1) + b = 7$ and $a(3) + b = -5$

$\therefore a + b = 7$ $\therefore 3a + b = -5$

$\therefore b = 7 - a$ (*) $\therefore 3a + (7 - a) = -5$ {using (*)}

$\therefore 2a = -12$

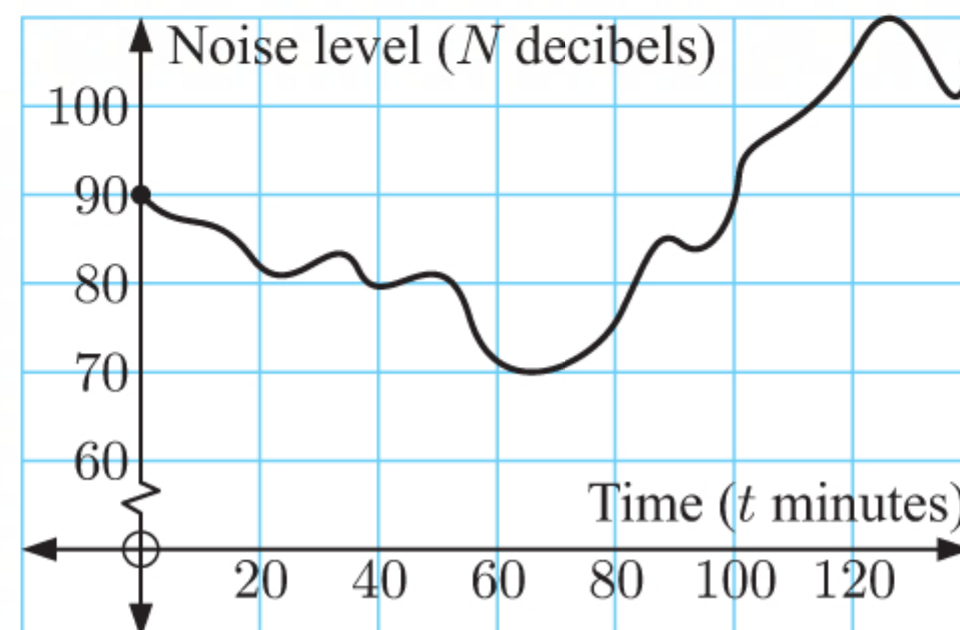
$\therefore a = -6$

Substituting $a = -6$ into (*) gives $b = 7 - (-6) = 13$

So, $a = -6$, $b = 13$.

6 The domain is $\{t \mid 0 \leq t \leq 140\}$.

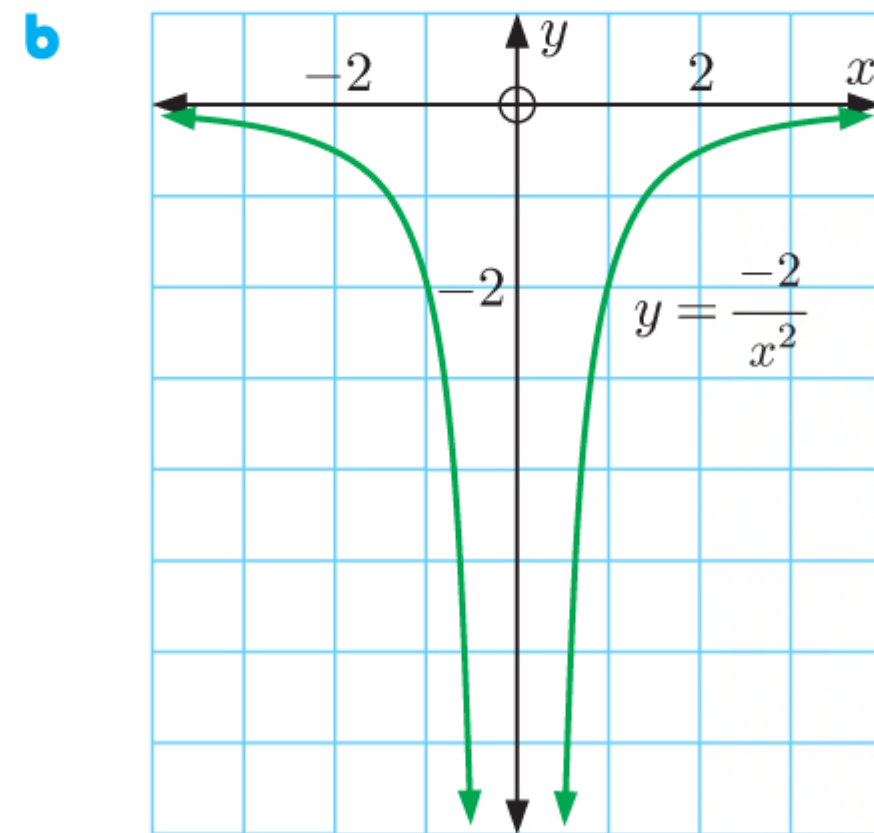
The range is $\{N \mid 70 \leq N \leq 110\}$.



7 $f(x) = \frac{-2}{x^2}$

a $f(x) = \frac{-2}{x^2}$ is undefined when $x^2 = 0$
 $\therefore x = 0$

c The domain is $\{x \mid x \neq 0\}$.
 The range is $\{y \mid y < 0\}$.



8 $f(x) = x^2$ and $g(x) = 1 - 6x$

a $f(-3) = (-3)^2$ and $g(-\frac{4}{3}) = 1 - 6(-\frac{4}{3})$
 $= 9$ $= 1 + 8$
 $= 9$

$\therefore f(-3) = g(-\frac{4}{3})$ as required

b $g(x) = f(5)$
 $\therefore 1 - 6x = 5^2$
 $\therefore 1 - 6x = 25$
 $\therefore -6x = 24$
 $\therefore x = -4$

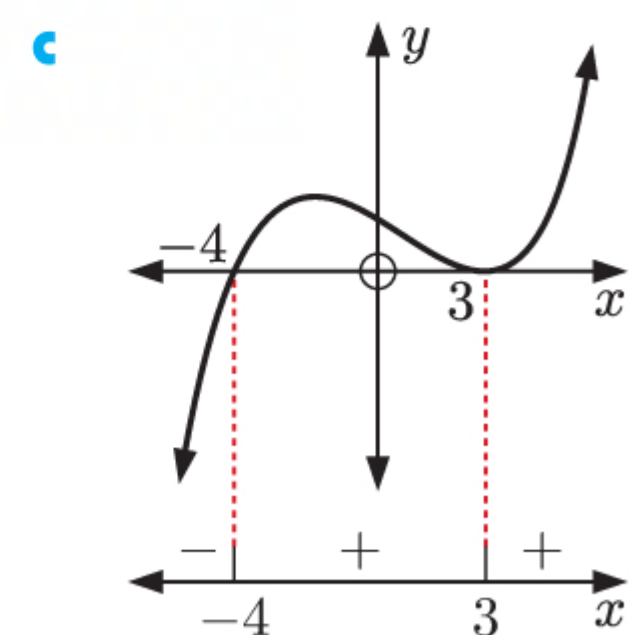
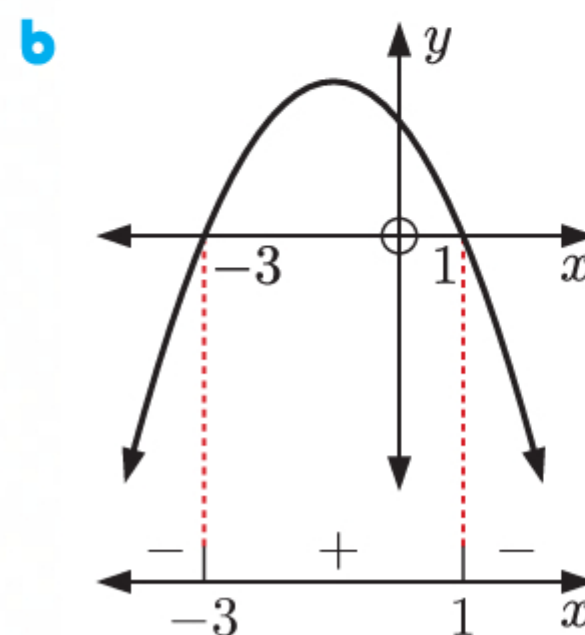
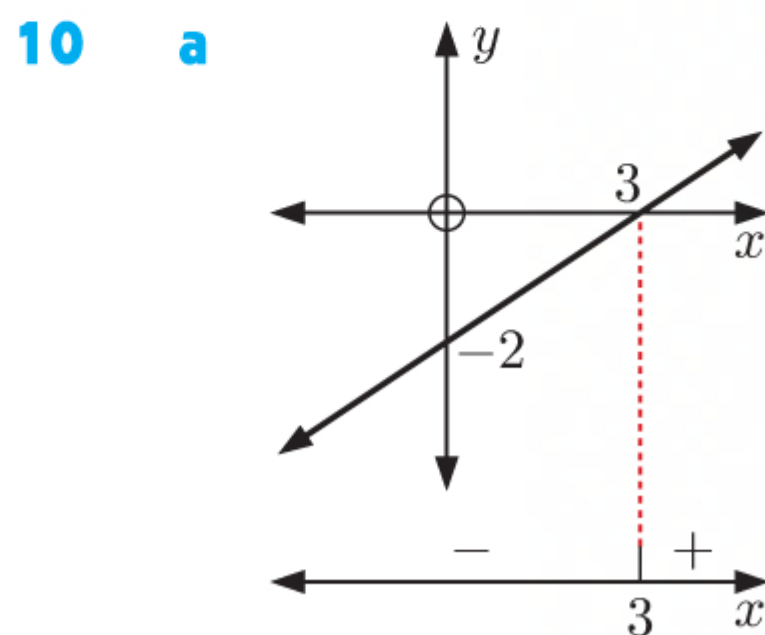
9 $f(x) = x^2 - 2x$

a $f(x) + 5 = x^2 - 2x + 5$

b $3f(x) = 3(x^2 - 2x)$
 $= 3x^2 - 6x$

c $f(2x) = (2x)^2 - 2(2x)$
 $= 4x^2 - 4x$

d $f(-x) = (-x)^2 - 2(-x)$
 $= x^2 + 2x$

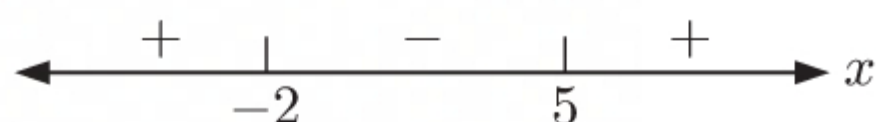


- 11 a** $(x - 5)(x + 2)$ has zeros 5 and -2 .

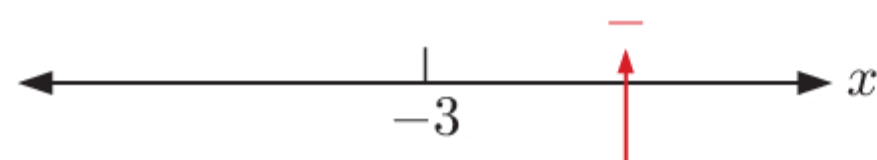


When $x = 6$ we have $(1)(8) > 0$,
so we put a $+$ sign here.

As the factors are single, the signs alternate.

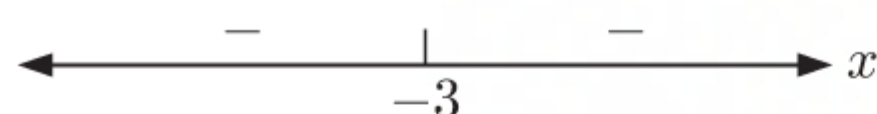


- b** $-(x + 3)^2$ has zero -3 .



When $x = 0$ we have $-(3)^2 < 0$,
so we put a $-$ sign here.

As the factor is squared, the signs do not change.

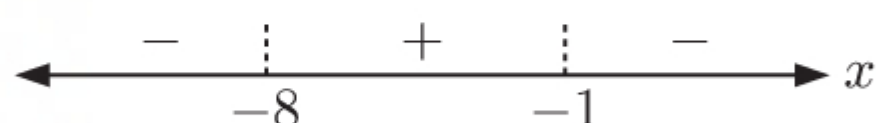


- c** $\frac{-11}{(x + 1)(x + 8)}$ is undefined when
 $x = -1$ or -8 .

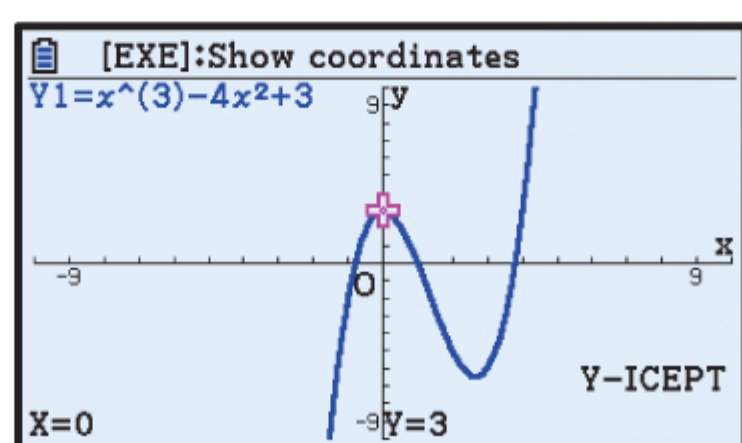


When $x = 0$ we have $\frac{-11}{(1)(8)} < 0$,
so we put a $-$ sign here.

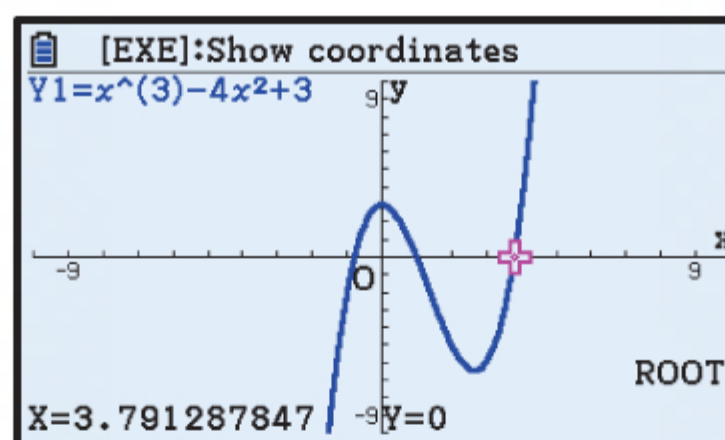
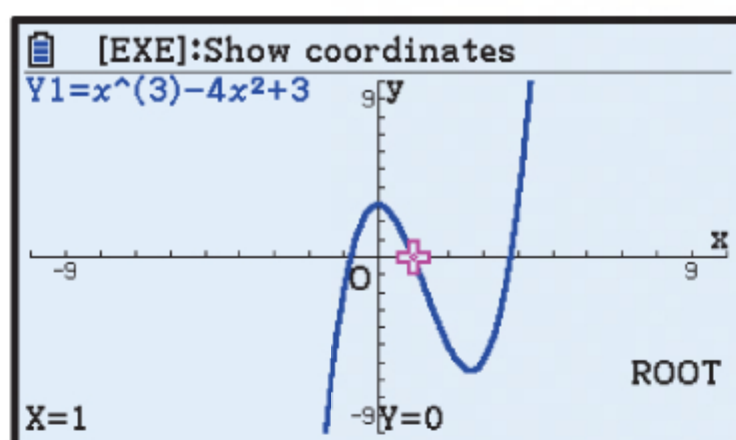
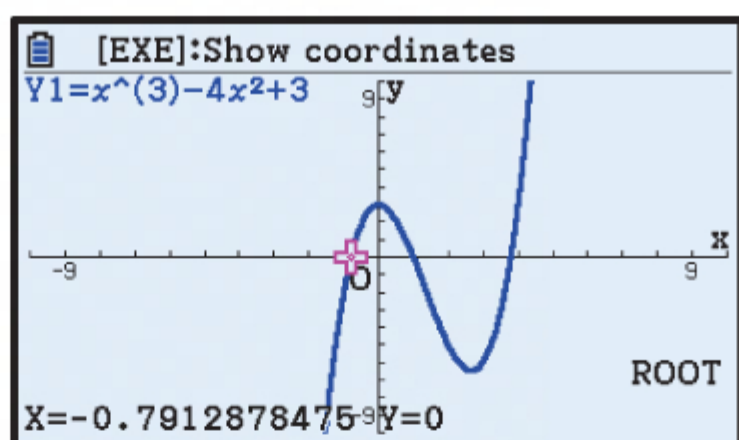
As the factors are single, the signs alternate.



- 12 a i**

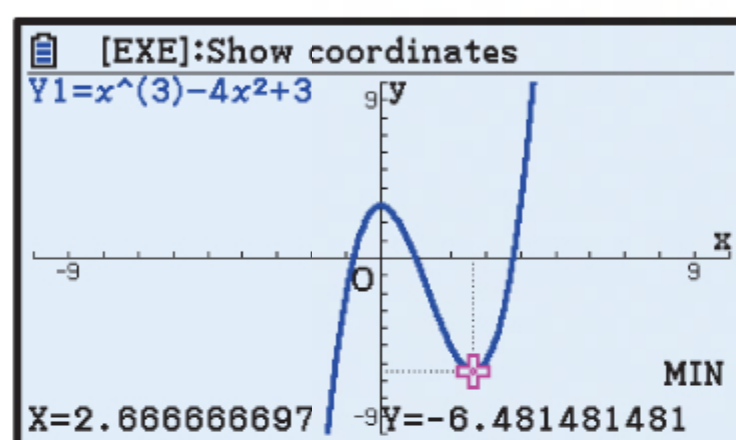
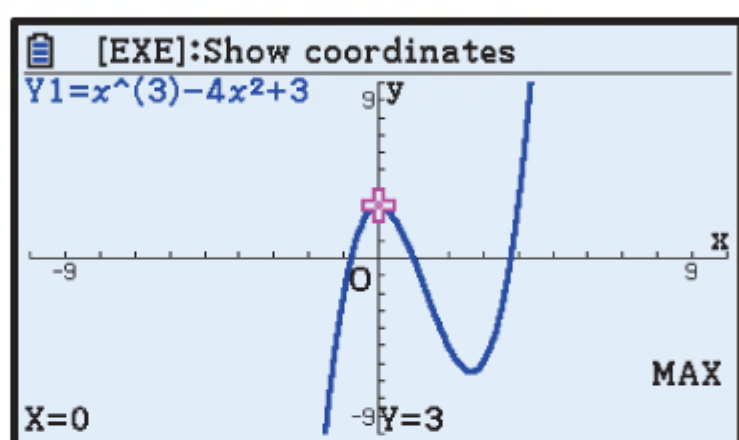


The y -intercept is 3.



The x -intercepts are ≈ -0.791 , 1 , and ≈ 3.79 .

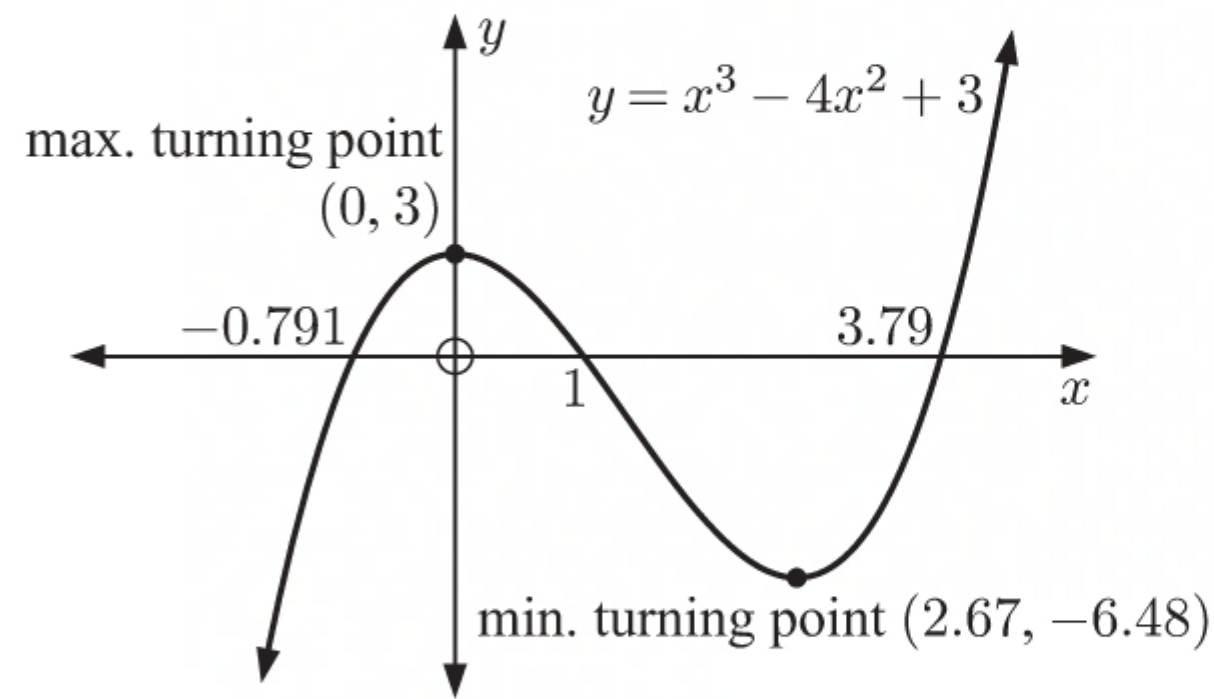
- ii**



There is a maximum turning point at $(0, 3)$ and a minimum turning point at $(2.67, -6.48)$.

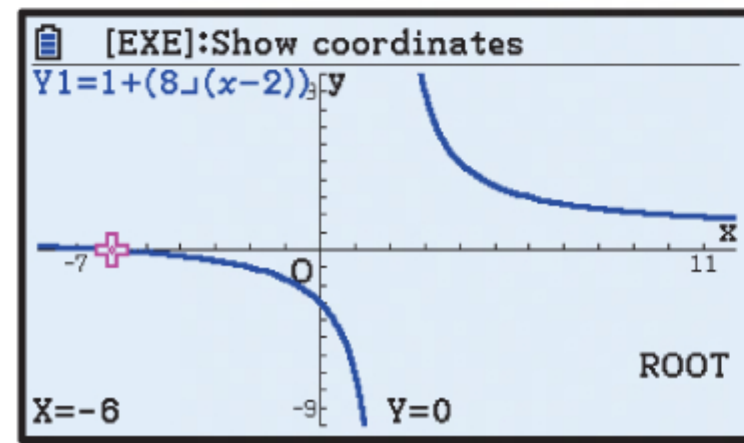
iii The graph has no asymptotes.

iv

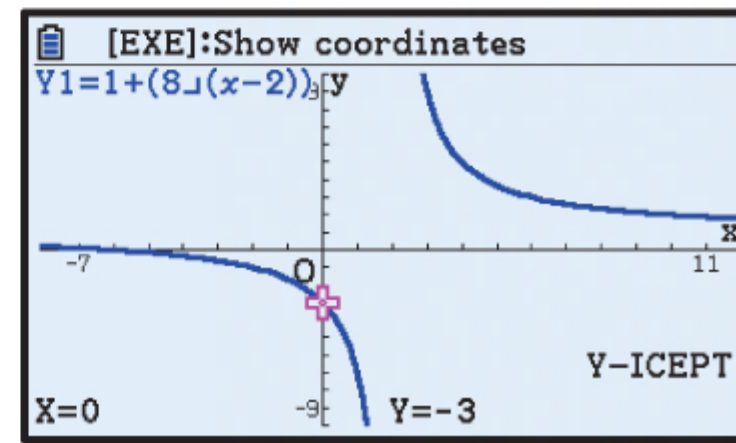


b

i



The x -intercept is -6 .



The y -intercept is -3 .

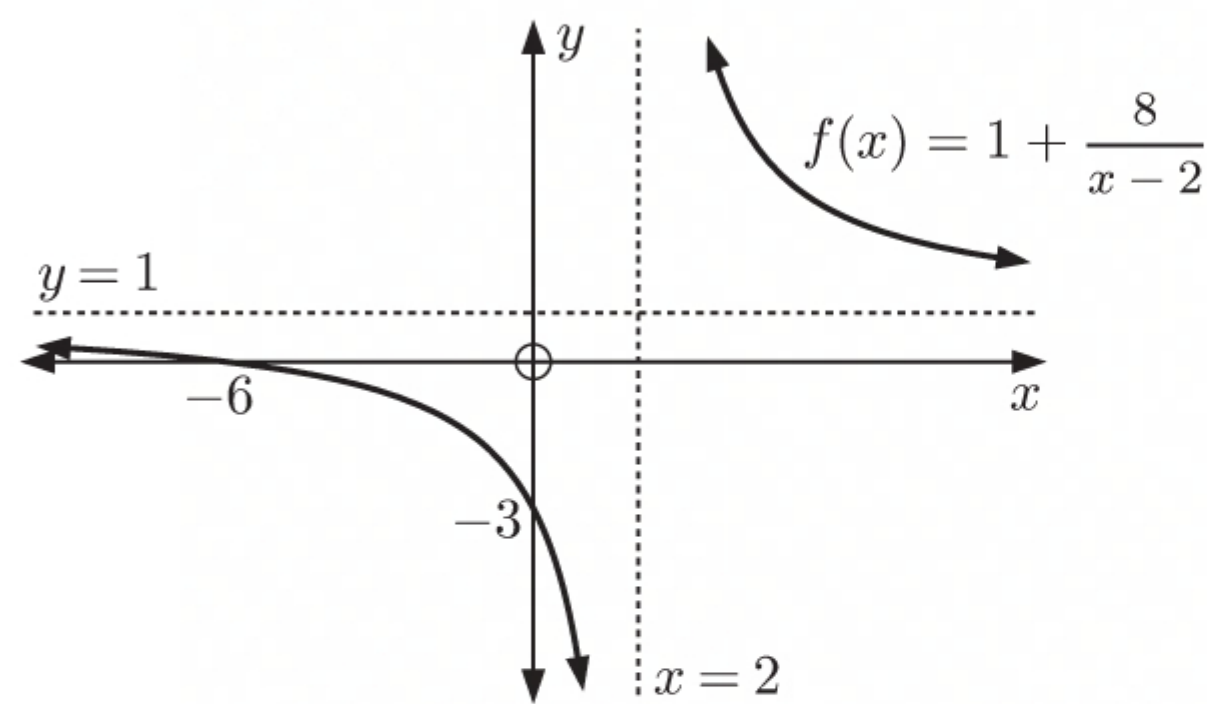
ii The graph has no turning points.

iii The graph appears to have a vertical asymptote at $x = 2$.

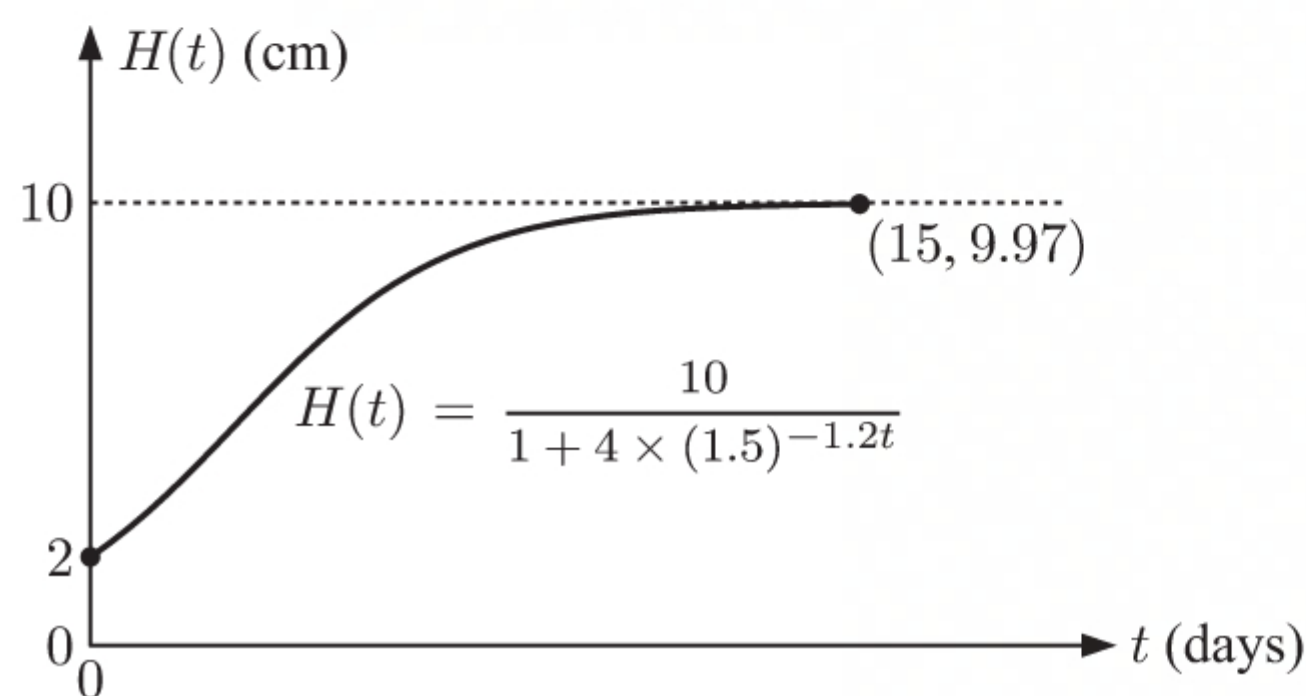
This is confirmed by the fact that y is undefined when $x = 2$.

As $x \rightarrow \pm\infty$, the graph gets closer to the line $y = 1$. So, $y = 1$ is a horizontal asymptote.

iv



13 a

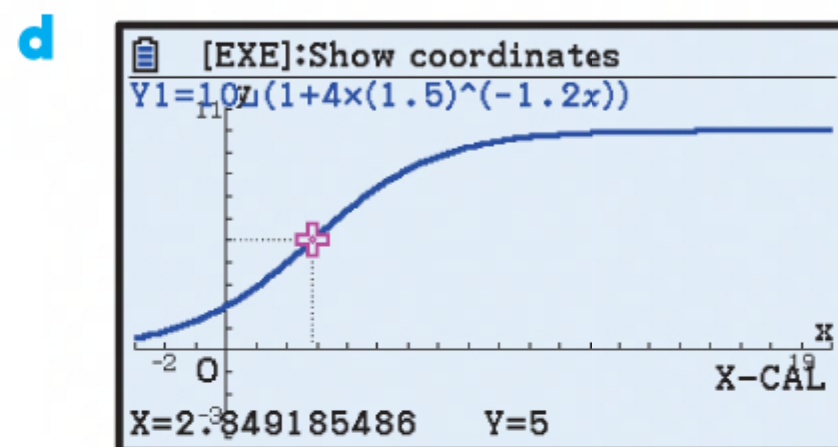


$$\begin{aligned}
 \text{b} \quad H(t) &= \frac{10}{1 + 4 \times (1.5)^{-1.2t}} \\
 \therefore H(0) &= \frac{10}{1 + 4 \times (1.5)^0} \\
 &= \frac{10}{1 + 4 \times 1} \\
 &= \frac{10}{1 + 4} \\
 &= \frac{10}{5} \\
 &= 2
 \end{aligned}$$

\therefore the seedling was 2 cm high when it was planted.

$$\begin{aligned}
 \text{c} \quad H(7) &= \frac{10}{1 + 4 \times (1.5)^{-1.2(7)}} \\
 &\approx 8.83
 \end{aligned}$$

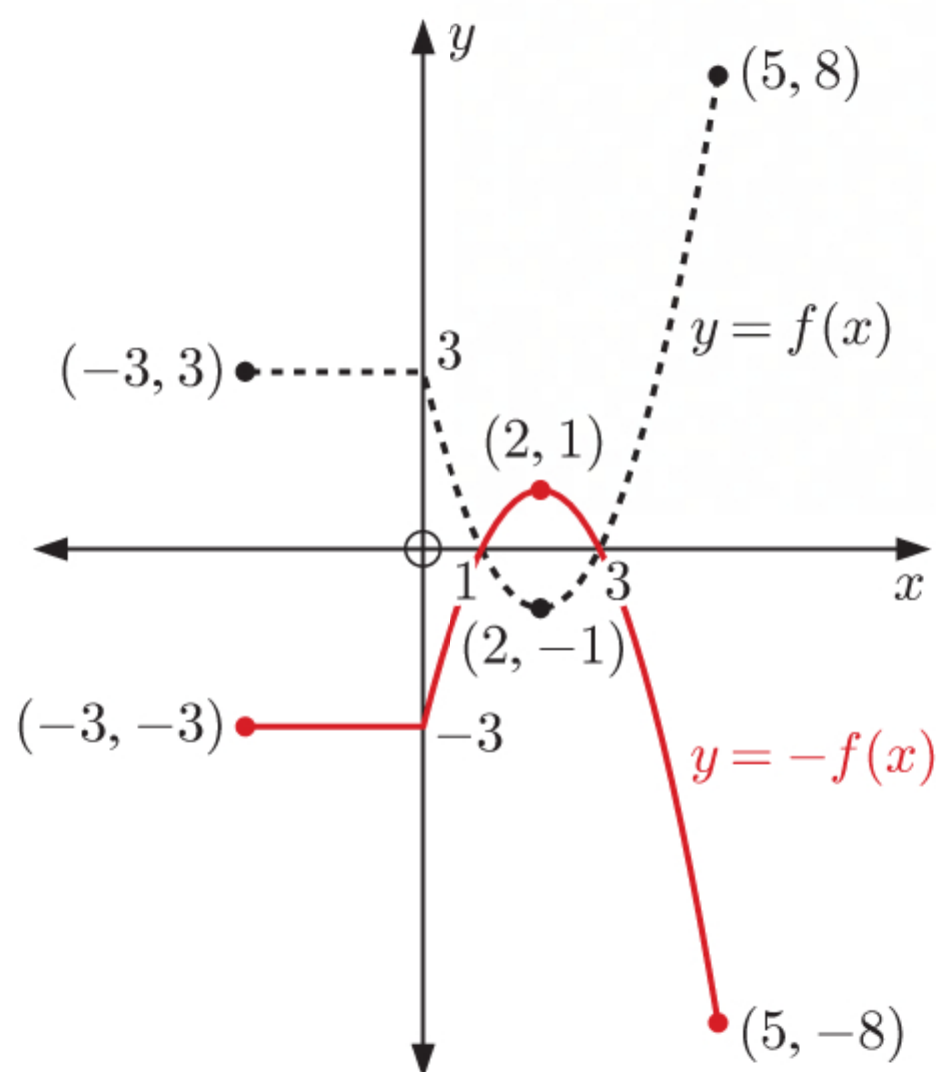
\therefore the seedling was about 8.83 cm high after 1 week.



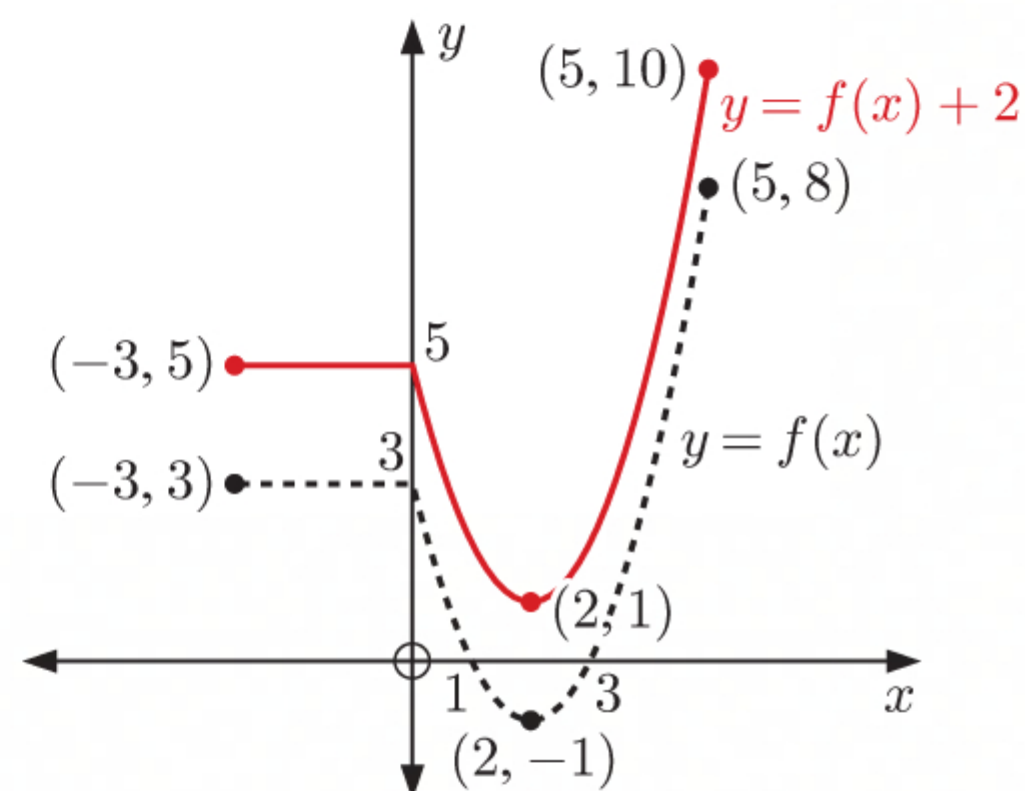
Using technology, the seedling was 5 cm high after about 2.85 days.

- e** As $t \rightarrow \infty$, the graph gets closer to the line $H = 10$.
 \therefore yes, the seedling grows to a limit of 10 cm.

- 14 a** The graph of $y = -f(x)$ is found by reflecting $y = f(x)$ in the x -axis.



- b** The graph of $y = f(x) + 2$ is found by translating $y = f(x)$ 2 units upwards.



15 a $g(x) = f(x) - 3$
 $= 4x - 7 - 3$
 $= 4x - 10$

c $g(x) = f\left(\frac{1}{3}x\right)$
 $= 2\left(\frac{1}{3}x\right)^2 - \left(\frac{1}{3}x\right) + 4$
 $= \frac{2}{9}x^2 - \frac{1}{3}x + 4$

b $g(x) = 5f(x)$
 $= 5(x^2 + 6)$
 $= 5x^2 + 30$

d $g(x) = f(-x)$
 $= (-x)^3$
 $= -x^3$

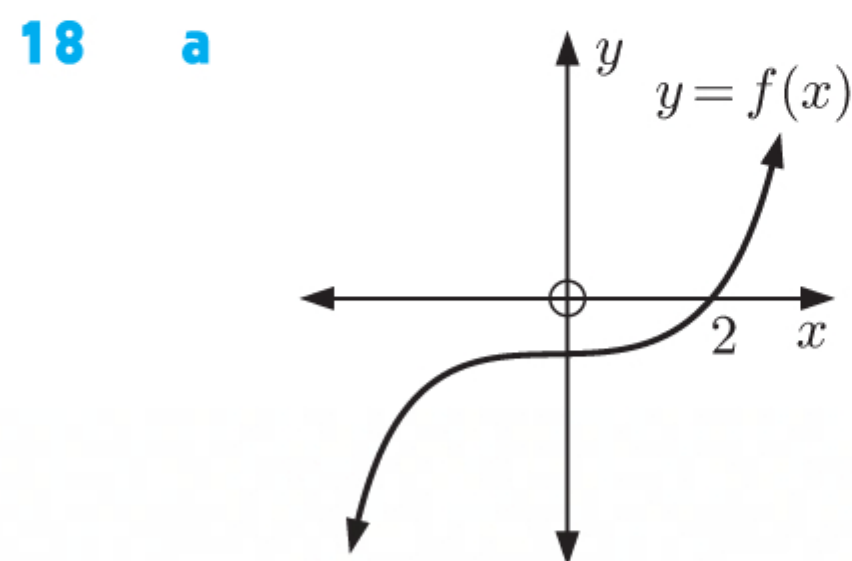
16 $f(x)$ has domain $\{x \mid -2 \leq x \leq 3\}$ and range $\{y \mid -1 \leq y \leq 7\}$.

$y = g(x) = f(x) - 4$ is the graph of $y = f(x)$ translated 4 units downwards.

$\therefore g(x)$ has domain $\{x \mid -2 \leq x \leq 3\}$ and range $\{y \mid -5 \leq y \leq 3\}$.

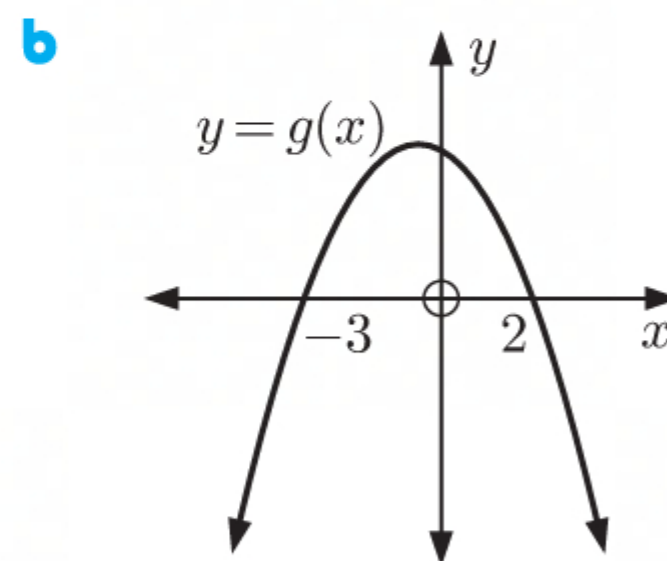
17 $(-1, 4)$ and $(6, -2)$ lie on f .

$\therefore (4, -1)$ and $(-2, 6)$ lie on f^{-1} .



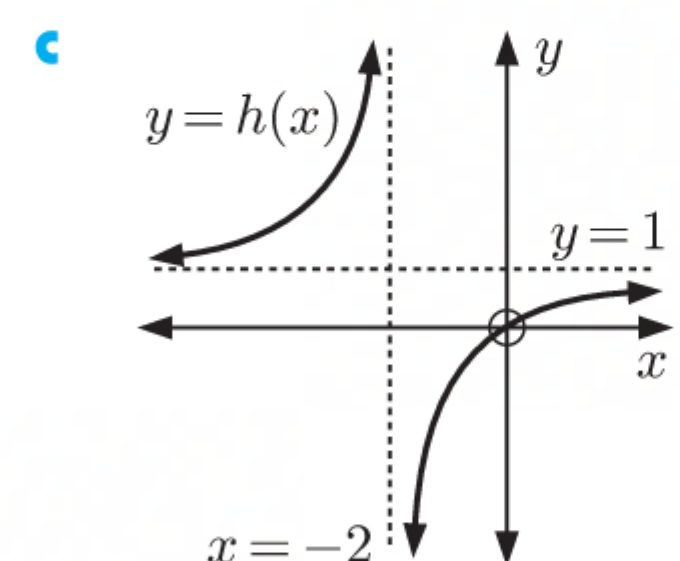
The graph passes the horizontal and vertical line tests.

\therefore the function has an inverse.



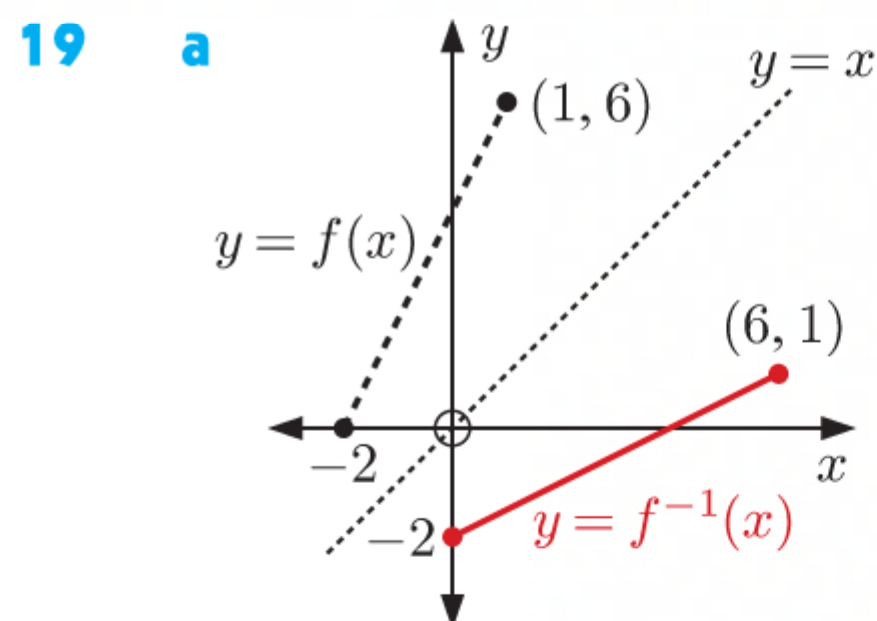
The graph fails the horizontal line test.

\therefore the function does not have an inverse.



The graph passes the horizontal and vertical line tests.

\therefore the function has an inverse.

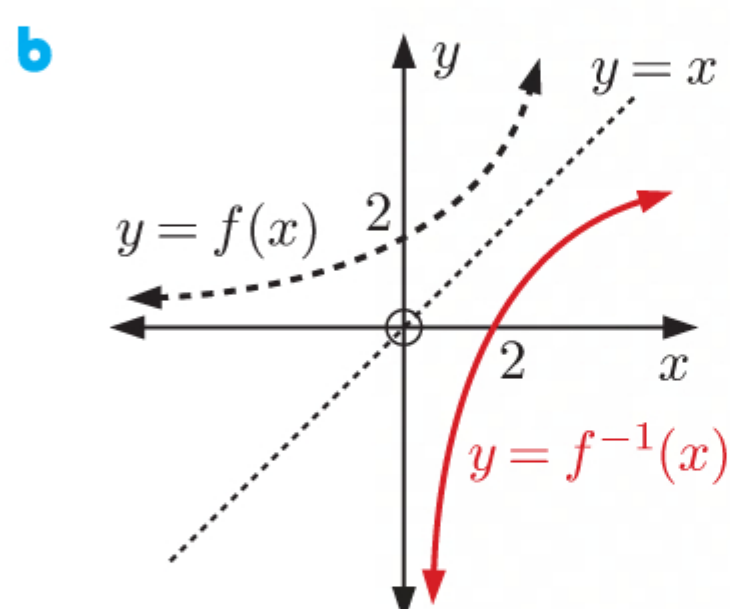


f : Domain is $\{x \mid -2 \leq x \leq 1\}$

Range is $\{y \mid 0 \leq y \leq 6\}$

f^{-1} : Domain is $\{x \mid 0 \leq x \leq 6\}$

Range is $\{y \mid -2 \leq y \leq 1\}$

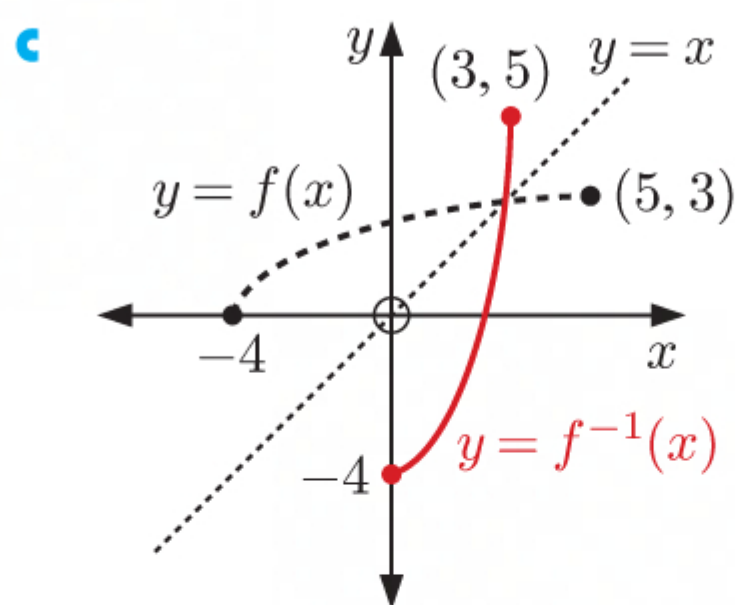


f : Domain is $\{x \mid x \in \mathbb{R}\}$

Range is $\{y \mid y > 0\}$

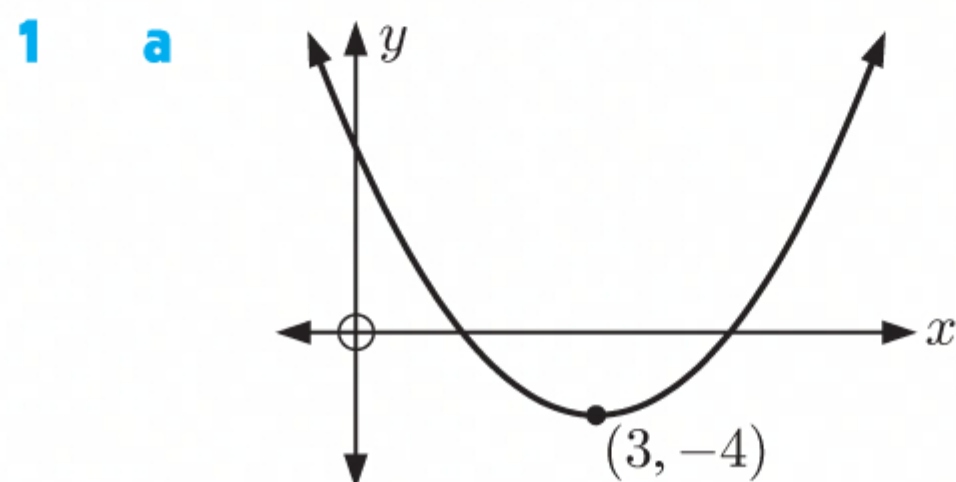
f^{-1} : Domain is $\{x \mid x > 0\}$

Range is $\{y \mid y \in \mathbb{R}\}$



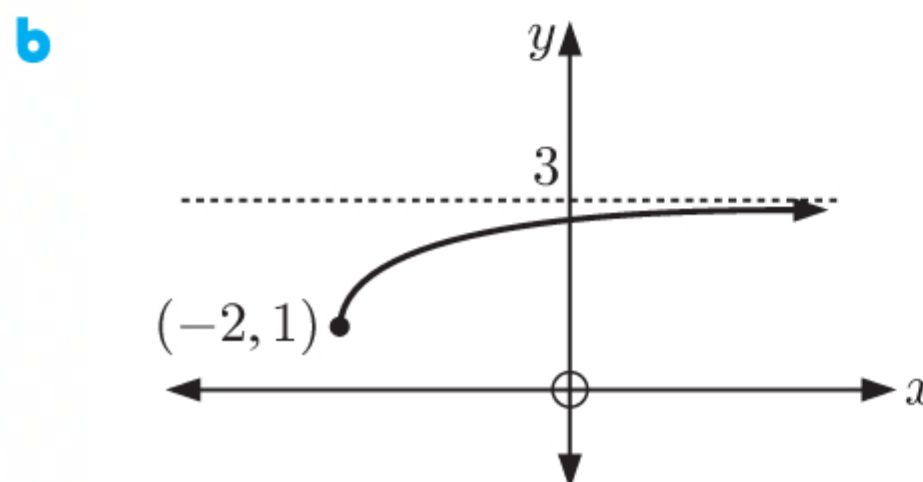
f : Domain is $\{x \mid -4 \leq x \leq 5\}$
 Range is $\{y \mid 0 \leq y \leq 3\}$
 f^{-1} : Domain is $\{x \mid 0 \leq x \leq 3\}$
 Range is $\{y \mid -4 \leq y \leq 5\}$

REVIEW SET 3B

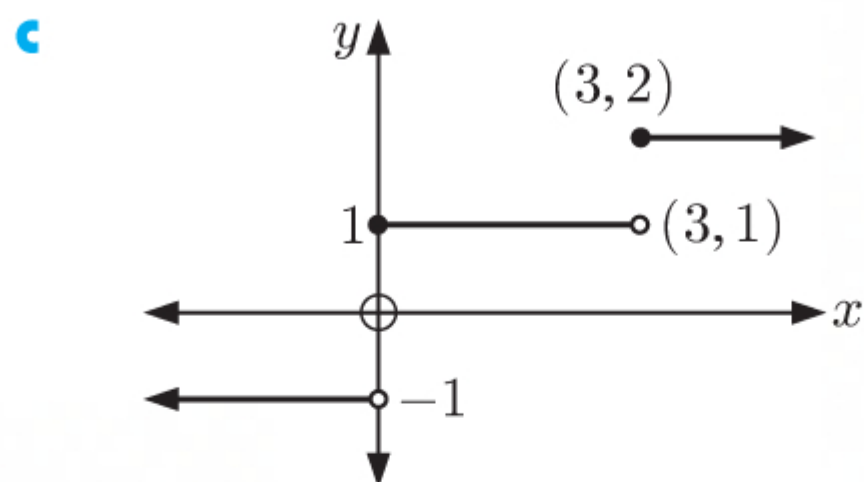


The minimum y -value is -4 and there is no maximum y -value.

\therefore the domain is $\{x \mid x \in \mathbb{R}\}$,
 and the range is $\{y \mid y \geq -4\}$.



The domain is $\{x \mid x \geq -2\}$,
 and the range is $\{y \mid 1 \leq y < 3\}$.



x can take any value.

\therefore the domain is $\{x \mid x \in \mathbb{R}\}$.

The possible values of y are -1 , 1 , or 2 .

\therefore the range is $\{y \mid y = -1, 1, \text{ or } 2\}$.

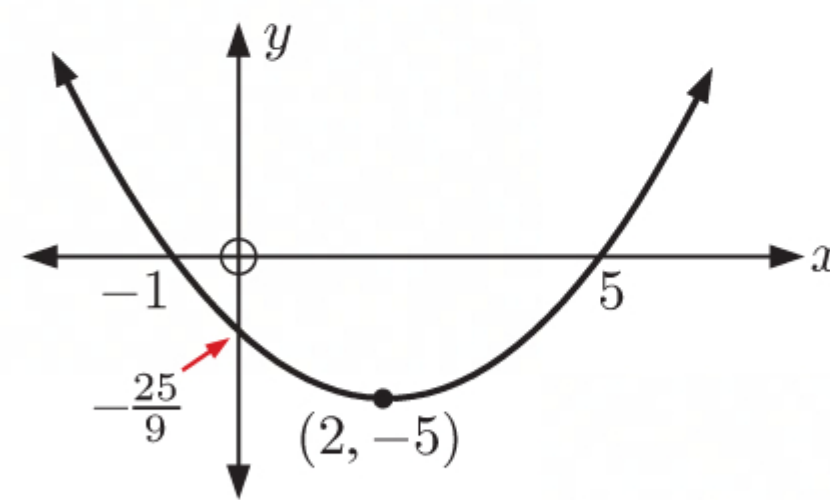
2 $g(x) = x^2 - 3x$

$a \quad g(2) = (2)^2 - 3(2)$
 $\quad \quad = 4 - 6$
 $\quad \quad = -2$

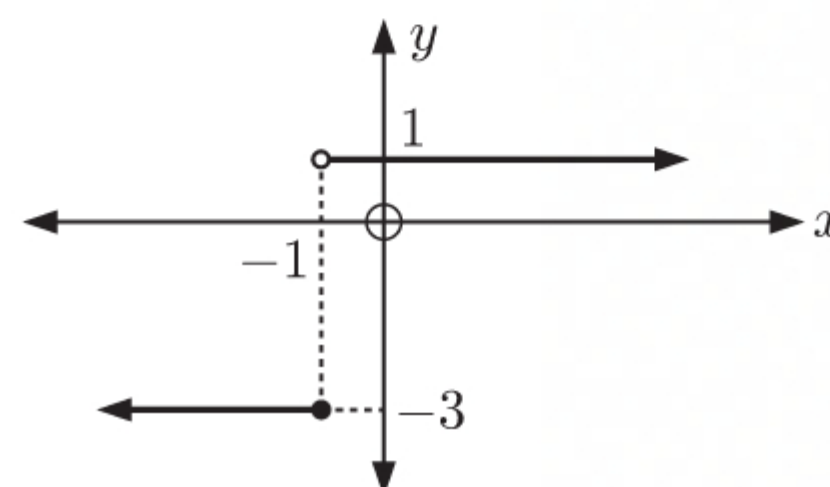
$c \quad g(4x) = (4x)^2 - 3(4x)$
 $\quad \quad = 16x^2 - 12x$

$b \quad g(x+1) = (x+1)^2 - 3(x+1)$
 $\quad \quad = x^2 + 2x + 1 - 3x - 3$
 $\quad \quad = x^2 - x - 2$

- 3 a**
- i** Domain is $\{x \mid x \in \mathbb{R}\}$.
Range is $\{y \mid y \geq -5\}$.
 - ii** x -intercepts are -1 and 5 , y -intercept is $-\frac{25}{9}$
 - iii** The graph passes the vertical line test, so it is therefore a function.



- b**
- i** Domain is $\{x \mid x \in \mathbb{R}\}$.
Range is $\{y \mid y = 1 \text{ or } -3\}$.
 - ii** There are no x -intercepts, y -intercept is 1 .
 - iii** The graph passes the vertical line test, so it is therefore a function.



- 4 a** $x + 2y = 10$ is a function, since for any value of x there is at most one value of y .

- b** $x + y^2 = 10$ is not a function.

If $x + y^2 = 10$, then $y = \pm\sqrt{10 - x}$. So, for example, for $x = 1$, $y = \pm 3$.

5 $f(x) = \frac{3x - 1}{x + 2}$

a i $f(-1) = \frac{3(-1) - 1}{-1 + 2}$
 $= -4$

ii $f(0) = \frac{3(0) - 1}{0 + 2}$
 $= -\frac{1}{2}$

iii $f(5) = \frac{3(5) - 1}{5 + 2}$
 $= \frac{14}{7}$
 $= 2$

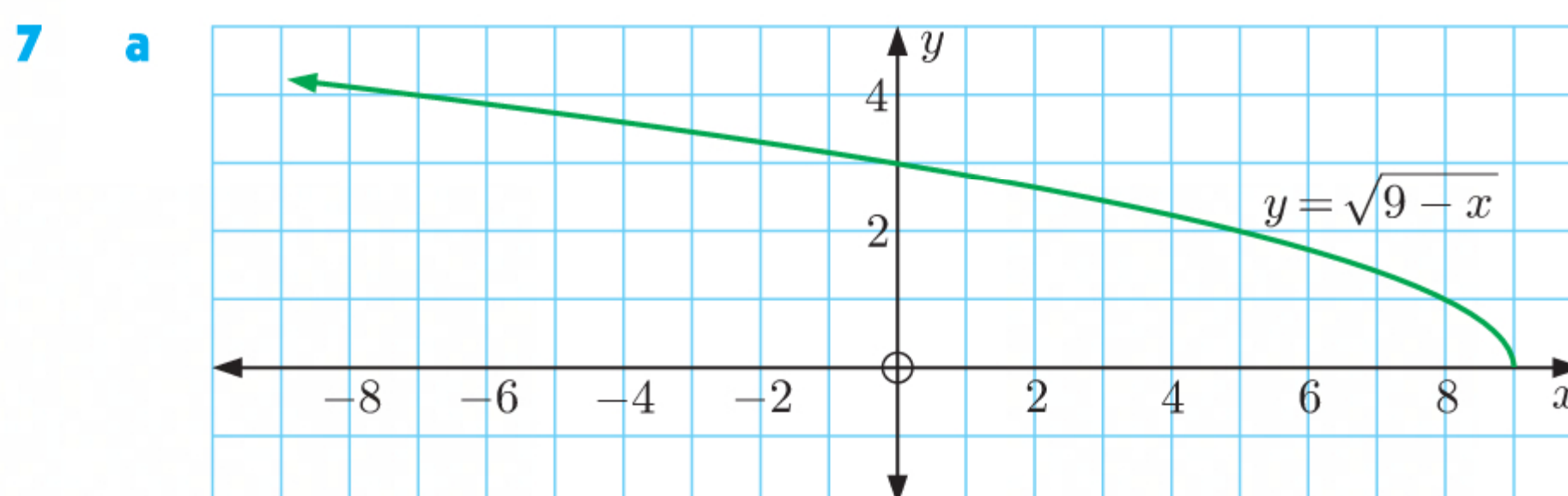
b $f(x) = \frac{3x - 1}{x + 2}$ is undefined when $x = -2$.

c $f(x - 1) = \frac{3(x - 1) - 1}{(x - 1) + 2}$
 $= \frac{3x - 3 - 1}{x + 1}$
 $= \frac{3x - 4}{x + 1}$

d If $f(x) = 4$, then $\frac{3x - 1}{x + 2} = 4$
 $\therefore 3x - 1 = 4(x + 2)$
 $\therefore 3x - 1 = 4x + 8$
 $\therefore x = -9$

6 a $f(x) = x^2 + 3$
 $\therefore f(-3) = (-3)^2 + 3$
 $= 9 + 3$
 $= 12$

b $f(x) = 4$
 $\therefore x^2 + 3 = 4$
 $\therefore x^2 = 1$
 $\therefore x = \pm 1$



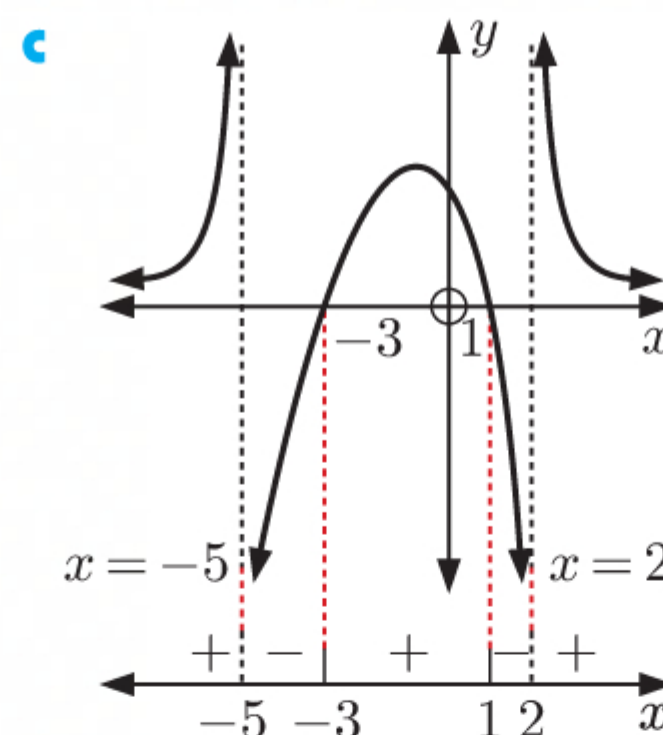
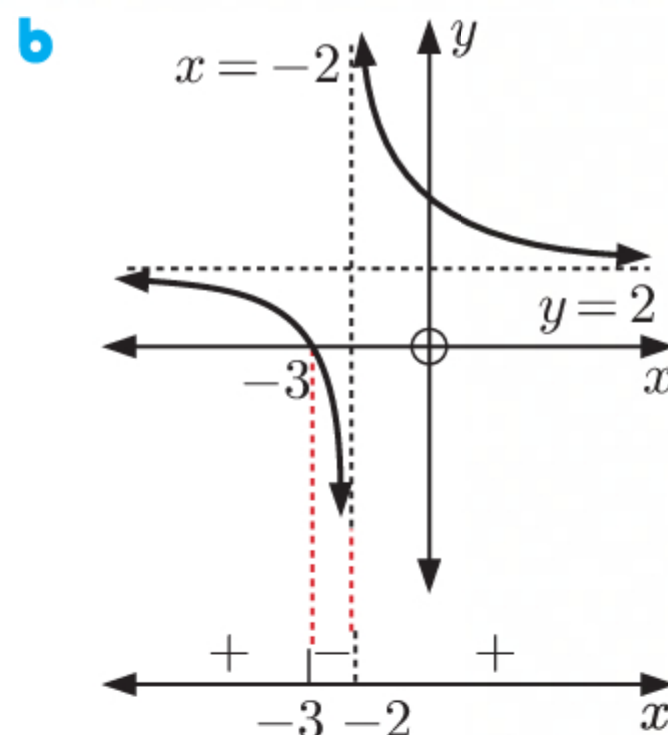
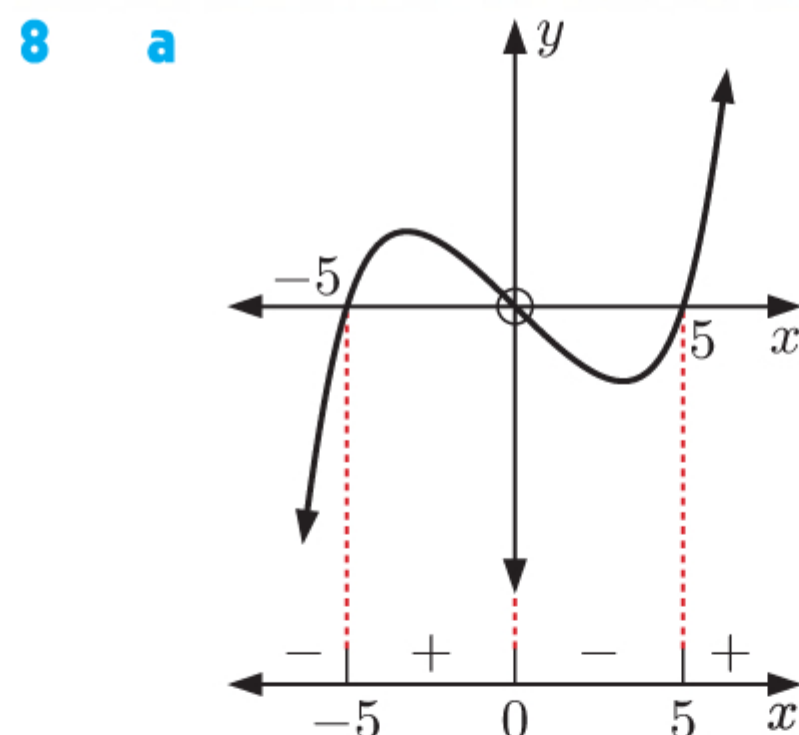
- b** The graph passes the vertical line test. There is at most one value of y for any one value of x .
 \therefore the relation is a function.

$\sqrt{9-x}$ is defined when $9-x \geq 0$
 $\therefore x \leq 9$

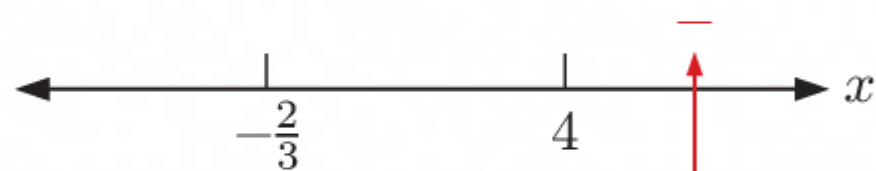
\therefore the domain is $\{x \mid x \leq 9\}$.

A square root cannot be negative.

\therefore the range is $\{y \mid y \geq 0\}$.

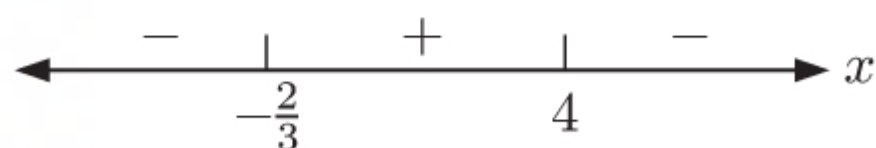


9 a $(3x+2)(4-x)$ has zeros $-\frac{2}{3}$ and 4.



When $x = 5$ we have $(17)(-1) < 0$,
so we put a $-$ sign here.

As the factors are single, the signs alternate.

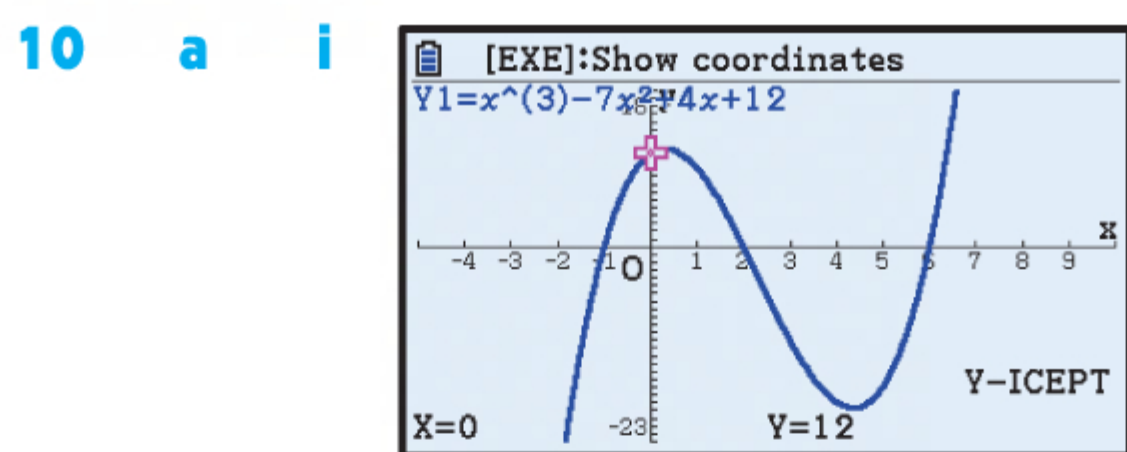


b $\frac{x-3}{(x+2)^2}$ is zero when $x = 3$ and undefined when $x = -2$.

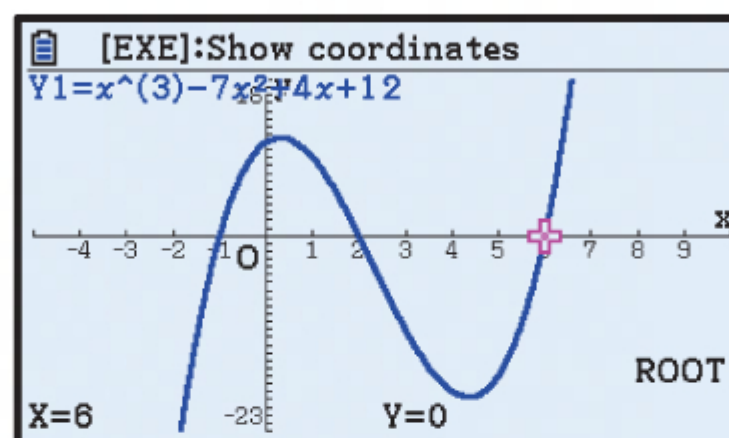
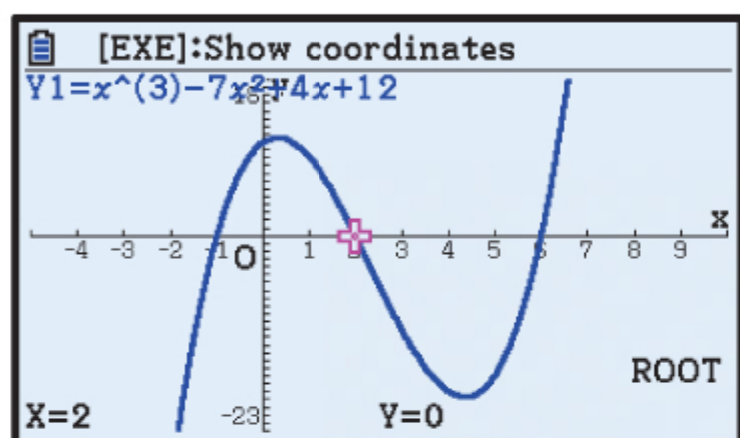
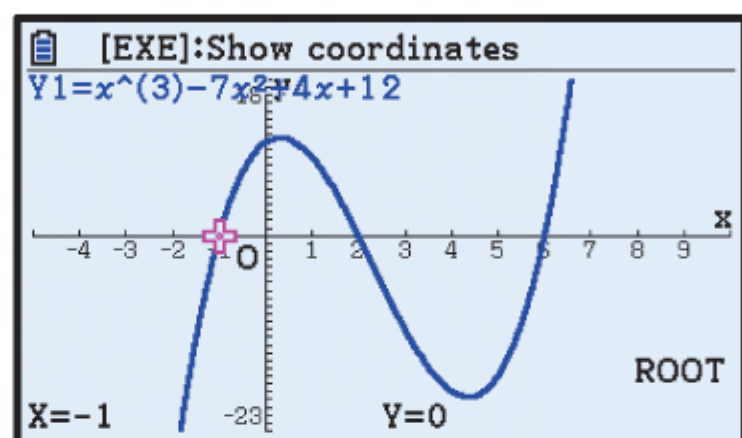


When $x = 4$ we have $\frac{1}{(6)^2} > 0$,
so we put a $+$ sign here.

Since $(x-3)$ is a single factor, the sign changes about $x = 3$. Since $(x+2)^2$ is a squared factor, the sign stays the same about $x = -2$.

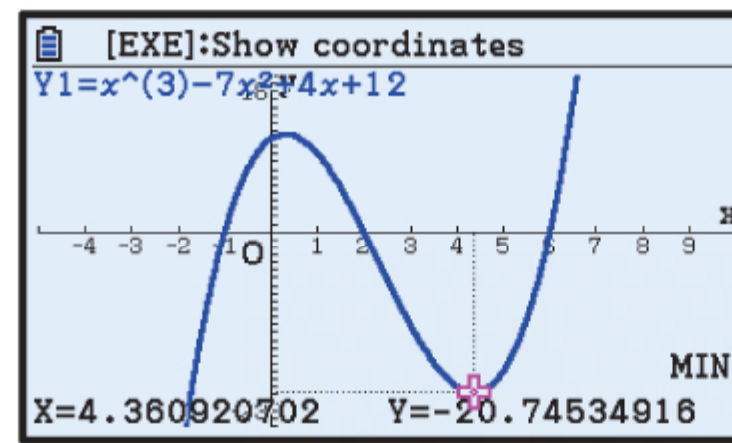
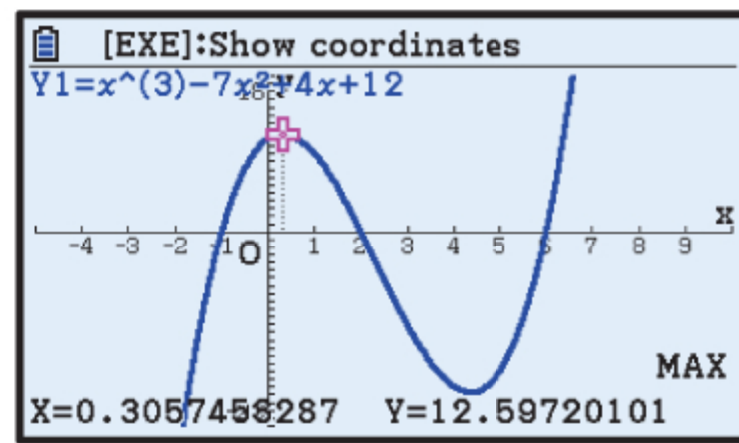


The y -intercept is 12.



The x -intercepts are -1 , 2 , and 6 .

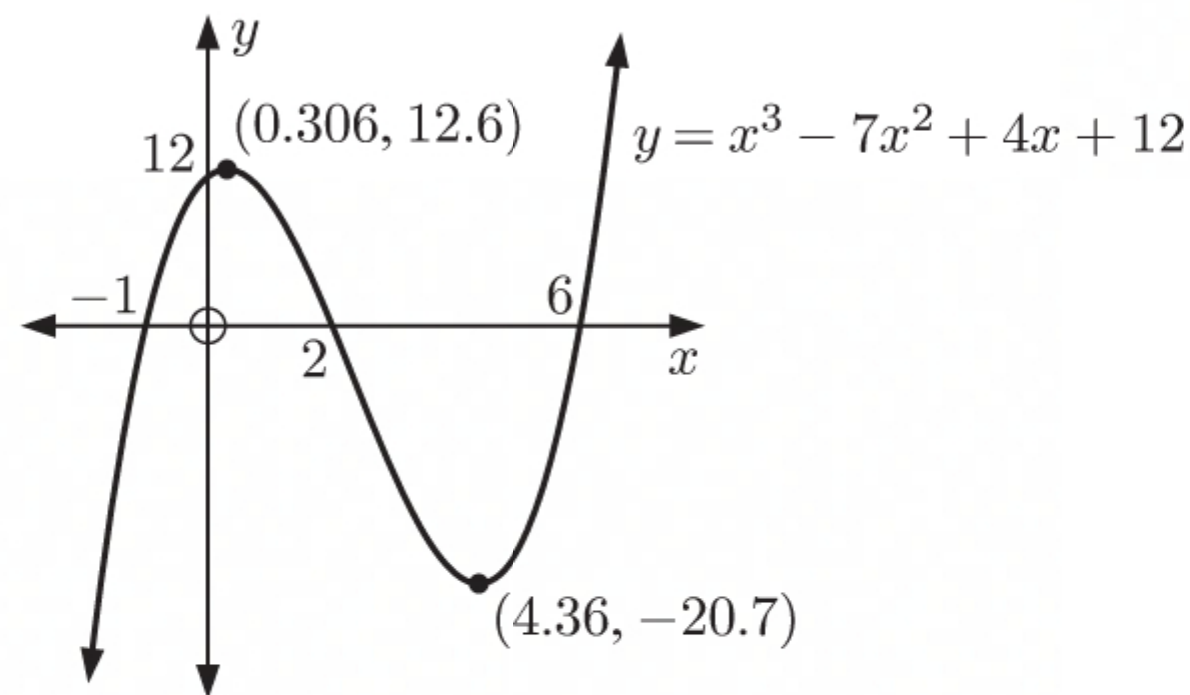
ii



There is a maximum turning point at $(0.306, 12.6)$ and a minimum turning point at $(4.36, -20.7)$.

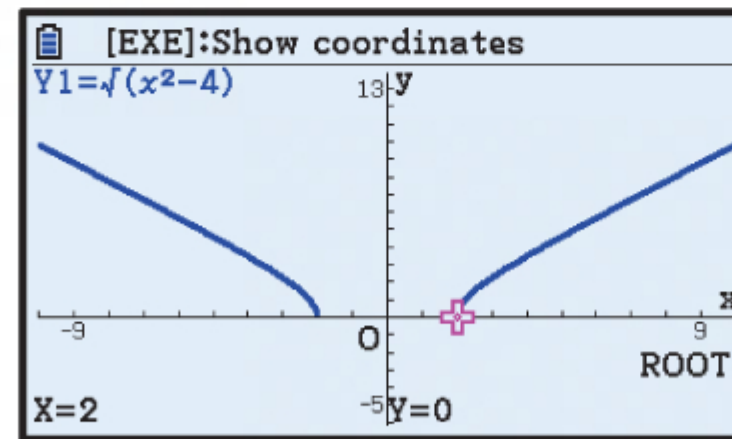
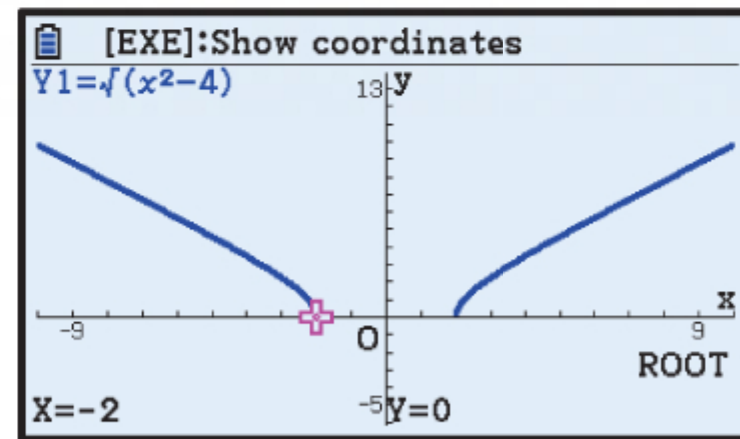
iii The domain is $\{x \mid x \in \mathbb{R}\}$. The range is $\{y \mid y \in \mathbb{R}\}$.

iv



b

i

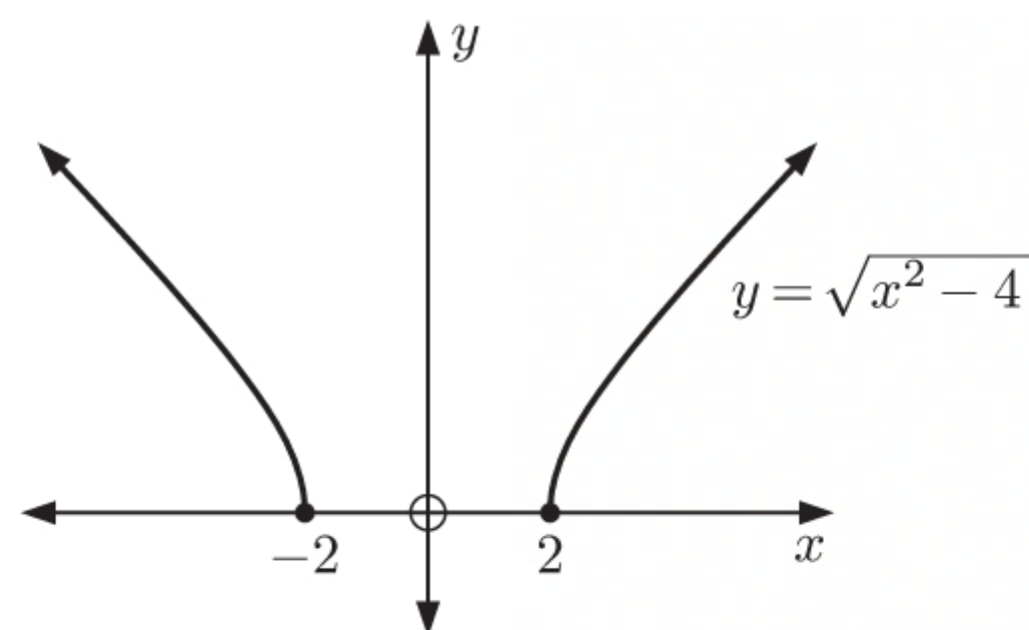


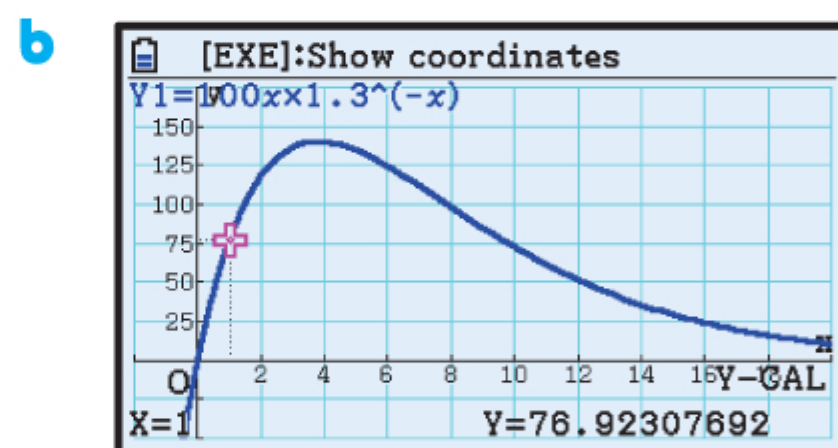
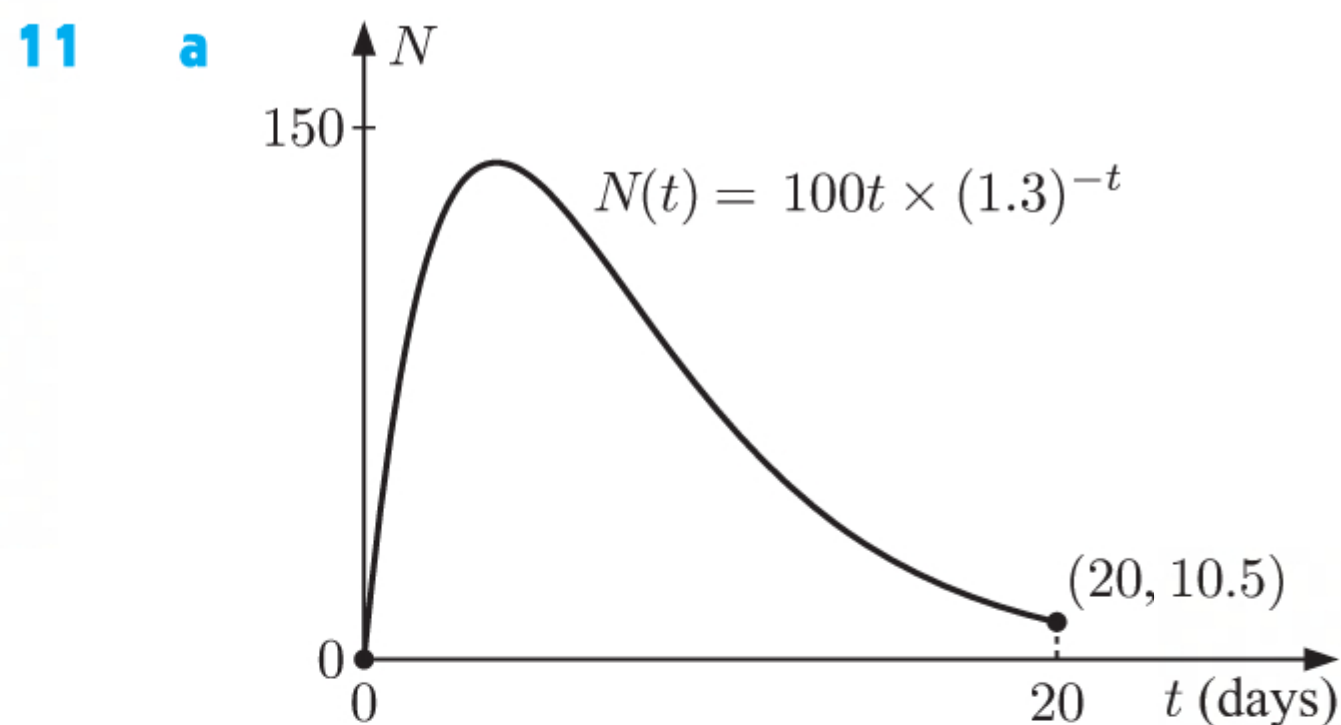
The x -intercepts are -2 and 2 . There is no y -intercept.

ii The graph has no turning points.

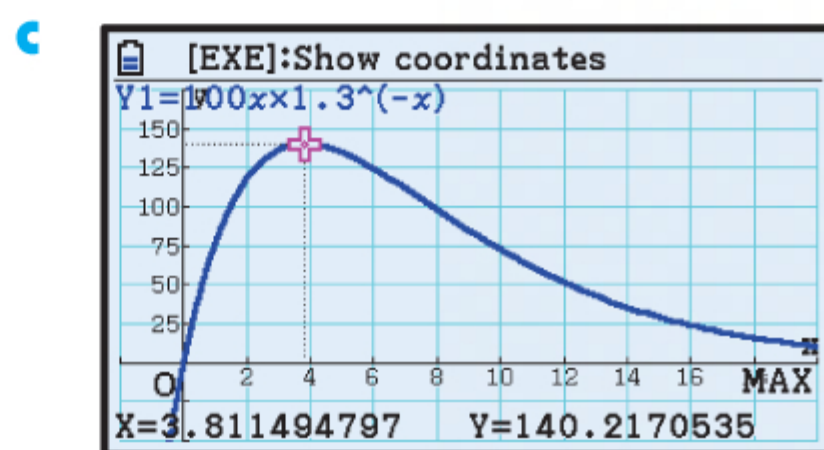
iii The domain is $\{x \mid x \leq -2 \text{ or } x \geq 2\}$. The range is $\{y \mid y \geq 0\}$.

iv

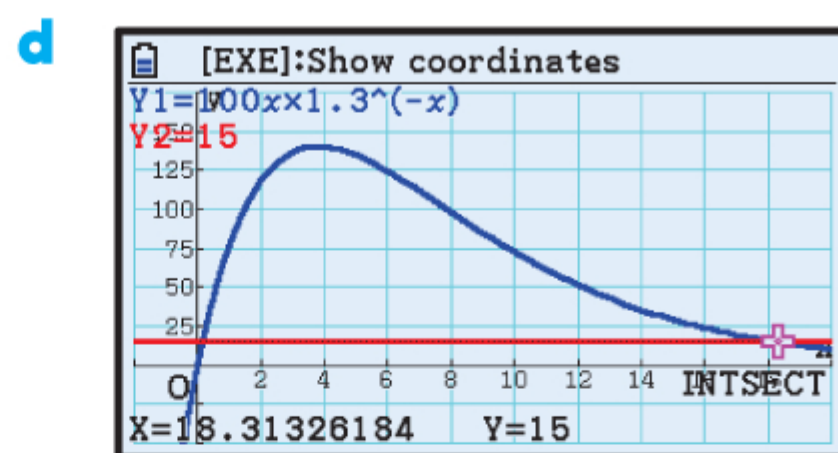




Using technology, when $t = 1$, $N \approx 76.9$.
 \therefore about 77 people are infected after 1 day.



Using technology, the maximum value of N is ≈ 140 , when $t \approx 3.81$.
 \therefore the outbreak was greatest after about 3.81 days, when about 140 people were infected.



Using technology, N reduces to 15 when $t \approx 18.3$.
 \therefore the outbreak was contained to 15 people after about 18.3 days.

12 a

$$\begin{aligned} g(x) &= -f(x) \\ &= -(x^2 - 3x) \\ &= 3x - x^2 \end{aligned}$$

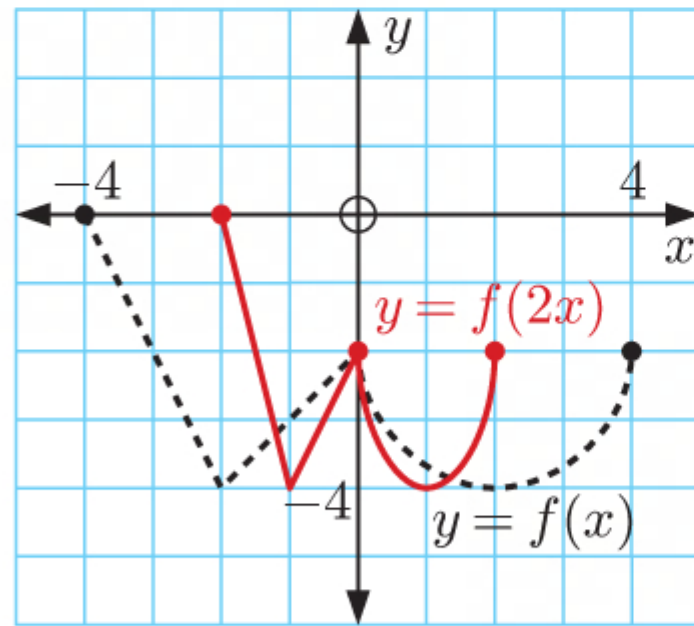
b

$$\begin{aligned} g(x) &= f(x) + 2 \\ &= 14 - x + 2 \\ &= 16 - x \end{aligned}$$

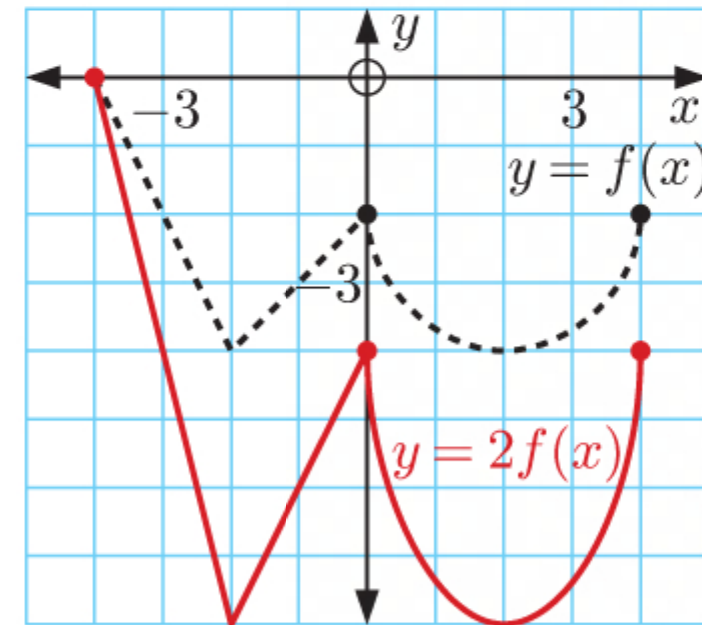
c

$$\begin{aligned} g(x) &= f\left(\frac{1}{4}x\right) \\ &= \frac{1}{3}\left(\frac{1}{4}x\right) + 2 \\ &= \frac{1}{12}x + 2 \end{aligned}$$

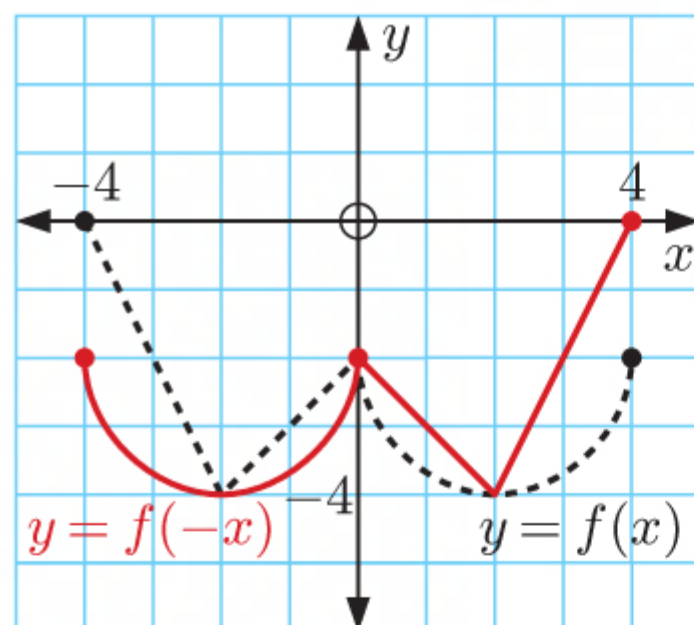
- 13 a** The graph of $y = f(2x)$ is found by horizontally stretching $y = f(x)$ with scale factor $\frac{1}{2}$.



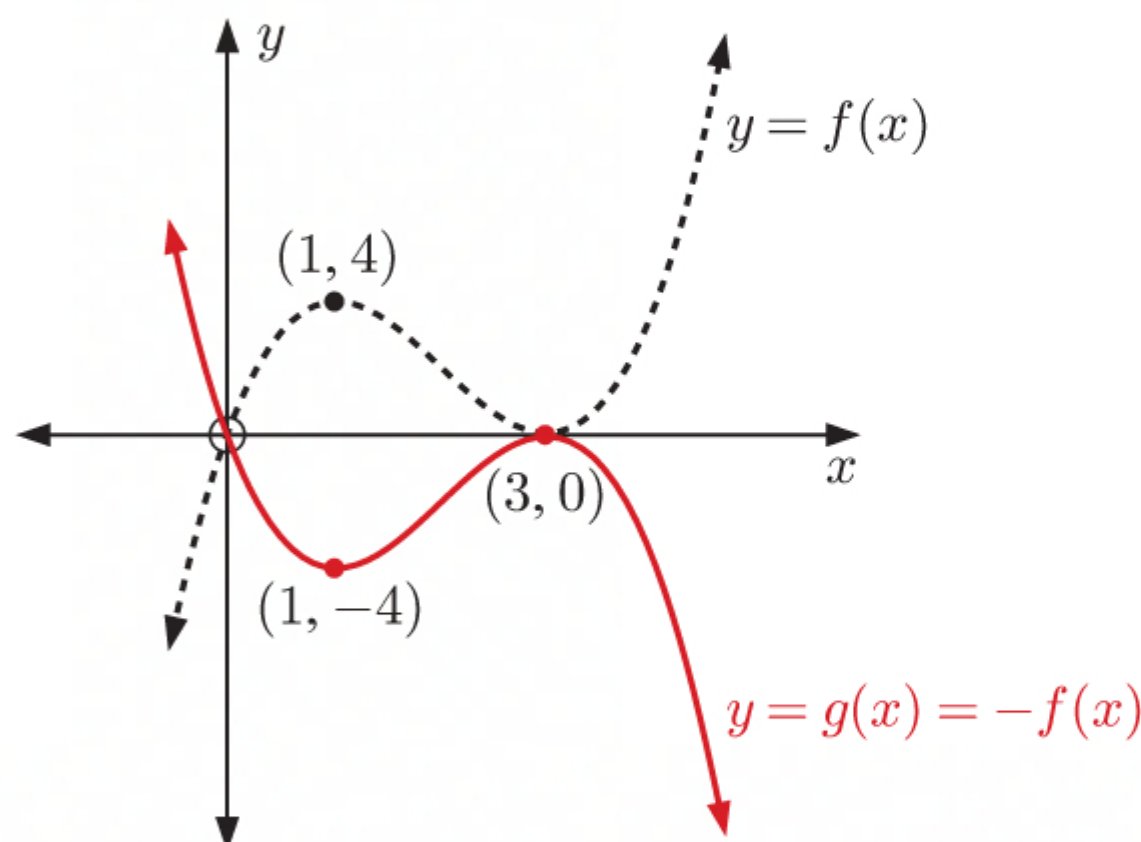
- b** The graph of $y = 2f(x)$ is found by vertically stretching $y = f(x)$ with scale factor 2.



- c** The graph of $y = f(-x)$ is found by reflecting $y = f(x)$ in the y -axis.



- 14 a** The graph of $y = g(x) = -f(x)$ is found by reflecting $y = f(x)$ in the x -axis.



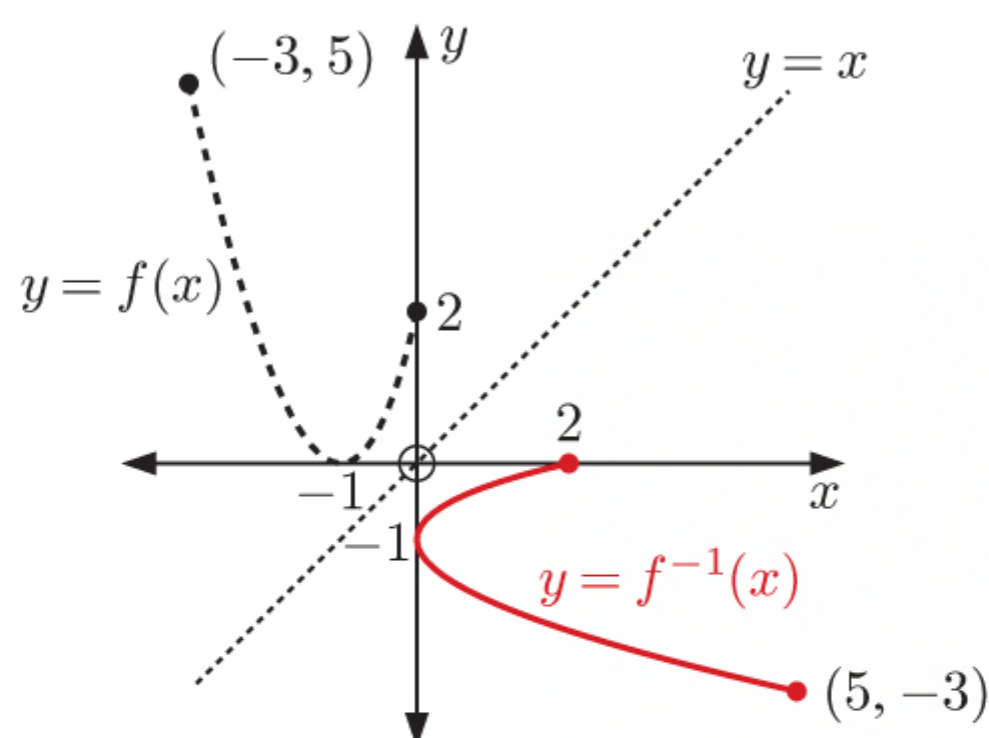
- b** The maximum turning point of $y = g(x)$ is $(3, 0)$. The minimum turning point of $y = g(x)$ is $(1, -4)$.

- 15 a** The graph of $y = f(-x)$ is found by reflecting $y = f(x)$ in the y -axis.

If $y = f(x)$ has x -intercept 4 and y -intercept -1 ,
then $y = f(-x)$ has x -intercept -4 and y -intercept -1 .

- b** $(4, 0)$ and $(0, -1)$ lie on the graph of $y = f(x)$.
 $\therefore (0, 4)$ and $(-1, 0)$ lie on the graph of $y = f^{-1}(x)$.
 $\therefore y = f^{-1}(x)$ has x -intercept -1 and y -intercept 4.

16 a



b No, $f(x)$ does not have an inverse since it does not pass the horizontal line test.

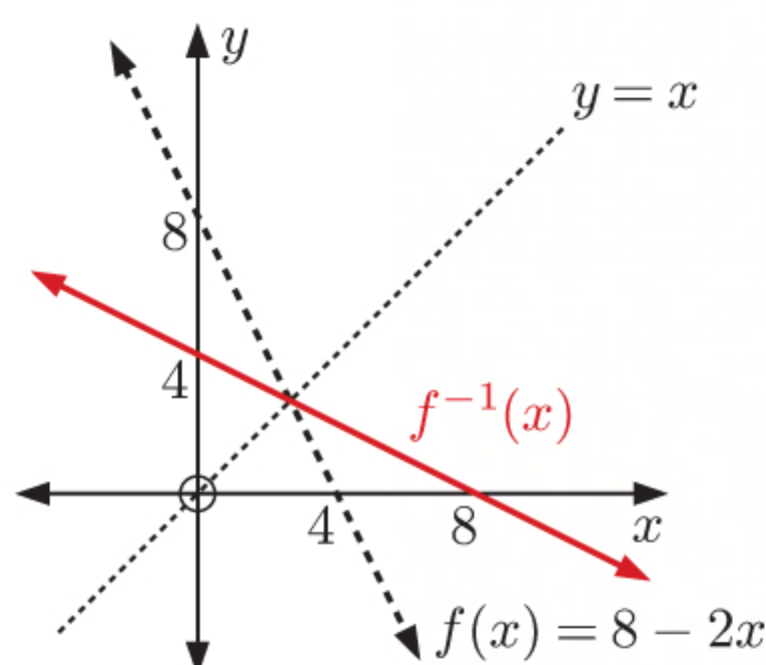
17 a, b

$$\begin{aligned} f(x) &= 8 - 2x \\ \therefore f(0) &= 8 - 2(0) \\ &= 8 \end{aligned}$$

\therefore the y -intercept is 8.

$$\begin{aligned} \text{When } f(x) &= 0, \quad 8 - 2x = 0 \\ \therefore 2x &= 8 \\ \therefore x &= 4 \end{aligned}$$

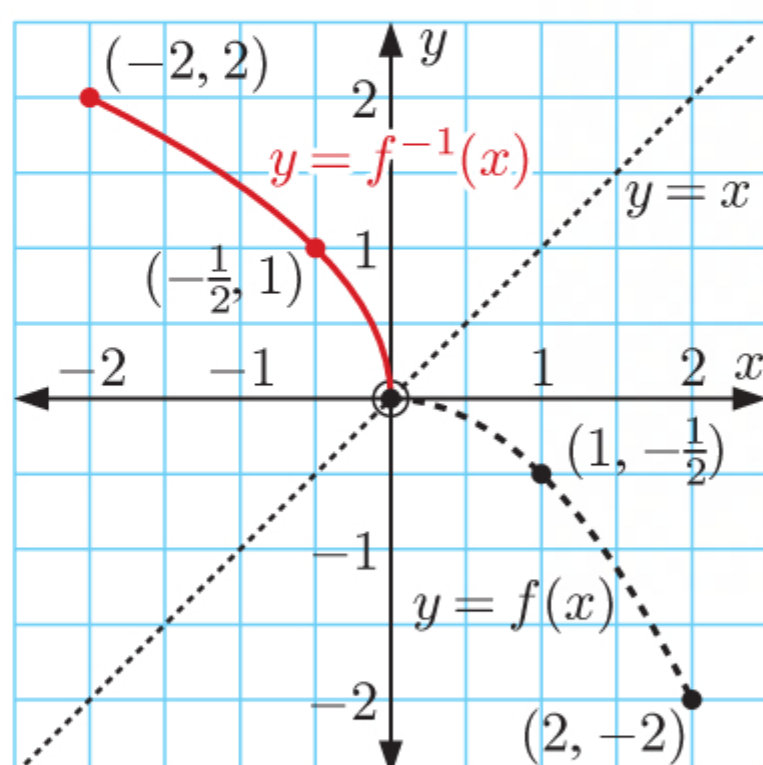
\therefore the x -intercept is 4.



c $y = f^{-1}(x)$ is a line through $(8, 0)$ and $(0, 4)$, which has gradient $\frac{4 - 0}{0 - 8} = \frac{4}{-8} = -\frac{1}{2}$ and y -intercept 4.

$$\therefore f^{-1}(x) = -\frac{1}{2}x + 4$$

18 a



b f has domain $\{x \mid 0 \leq x \leq 2\}$
 $\therefore f^{-1}$ has range $\{y \mid 0 \leq y \leq 2\}$

c i $f(x) = -\frac{3}{2}$
 $\therefore -\frac{1}{2}x^2 = -\frac{3}{2}$
 $\therefore x^2 = 3$
 $\therefore x = \sqrt{3} \quad \{0 \leq x \leq 2\}$

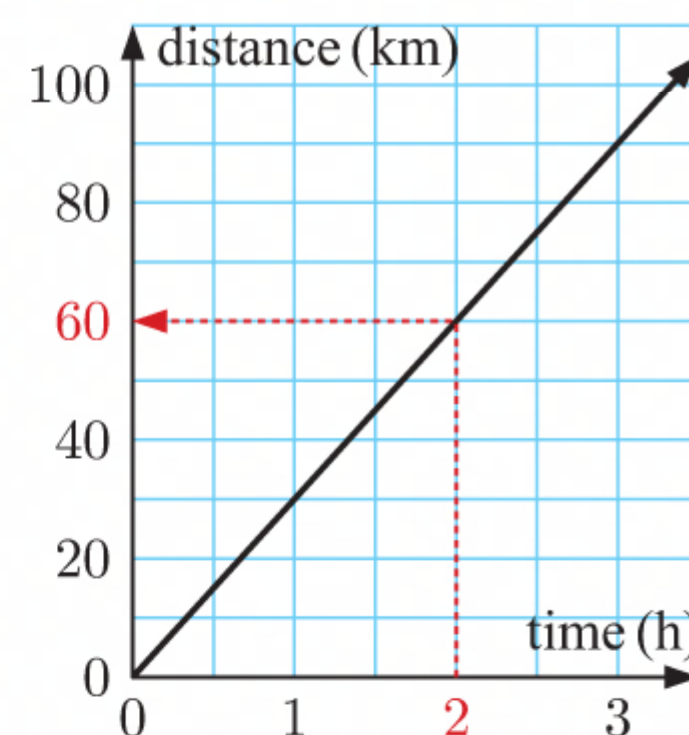
ii From the graph in a, if $f^{-1}(x) = 1$,
then $x = -\frac{1}{2}$.

Chapter 4

MODELLING

EXERCISE 4A

- 1
 - a We have assumed that the cyclist travelled at a constant speed of 30 km h^{-1} the entire time. This is not realistic, as the cyclist will travel at different speeds uphill, downhill, and on flat ground.
 - b From the diagram, the model predicts that the cyclist will travel 60 km in 2 hours.



- 2
 - a Briony constructed her model by finding the equation of the line through $(0, 8)$ and $(12, 23)$. Briony has assumed that the laptop will charge at a constant rate, and indefinitely. These assumptions are not realistic, but they may be satisfactory for this problem.

b $0 \leq C \leq 100, \quad t \geq 0$

Briony's model suggests that it is possible to have charge greater than 100%, which is not possible.

c The gradient of the line $= \frac{23 - 8}{12 - 0}$
 $= \frac{15}{12}$
 $= \frac{5}{4}$

The C -intercept is 8.

$$\therefore C = \frac{5}{4}t + 8$$

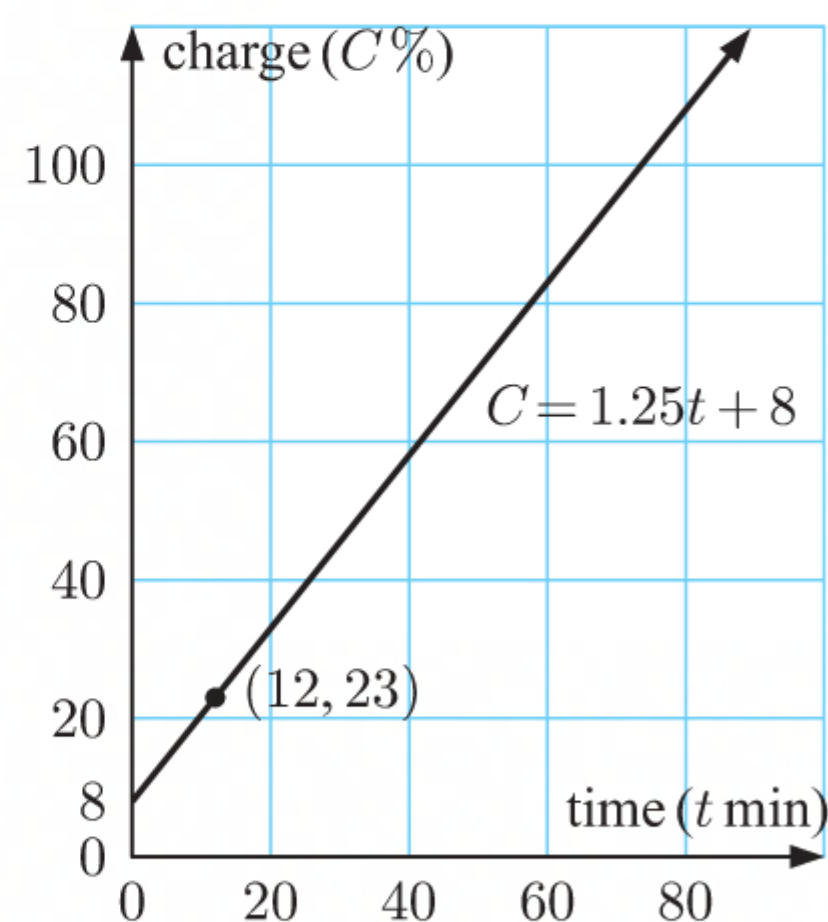
When $C = 100$, $\frac{5}{4}t + 8 = 100$

$$\therefore \frac{5}{4}t = 92$$

$$\therefore t = 73.6$$

\therefore Briony's model predicts that it will take 73.6 minutes for the laptop to be fully charged.

- d
 - i The laptop likely charges at a faster rate earlier on, then at a slower rate as it approaches a full charge.
 - ii Yes, 73.6 minutes was a useful estimate.



- 3 a** Rick takes 15 seconds to run 100 metres.

We assume that Rick runs at a constant speed of $\frac{100}{15} = \frac{20}{3} \text{ m s}^{-1}$.

Let t be the time in seconds it takes Rick to run d metres.

$$\text{Now speed} = \frac{\text{distance}}{\text{time}}$$

$$\therefore \text{time} = \frac{\text{distance}}{\text{speed}}$$

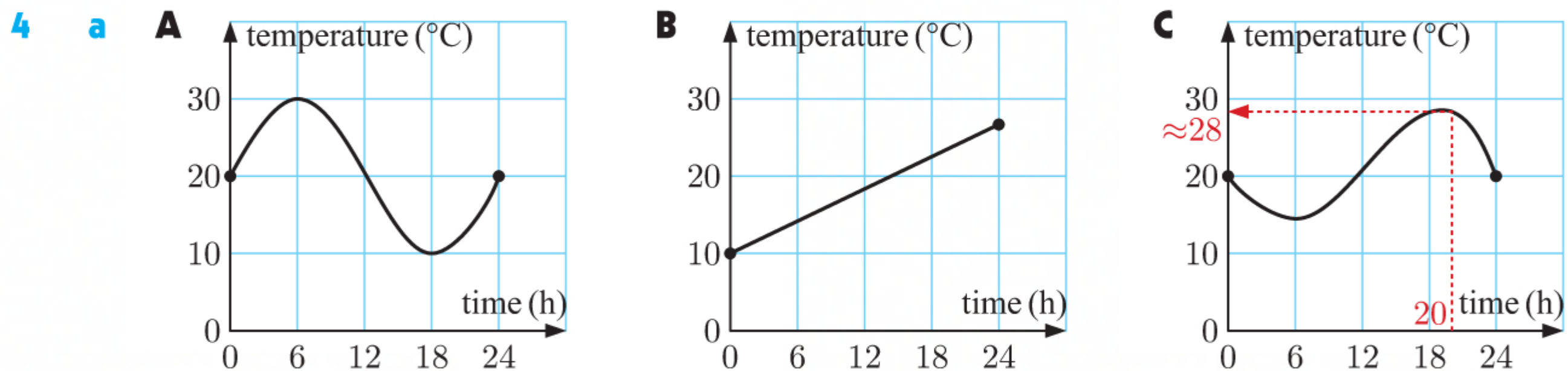
$$\therefore t = \frac{d}{\frac{20}{3}}$$

$$\therefore t = \frac{3}{20}d \text{ seconds}$$

- b** When $d = 500$, $t = \frac{3}{20}(500)$
 $= 75$

\therefore we predict that Rick will take 75 seconds to run 500 metres.

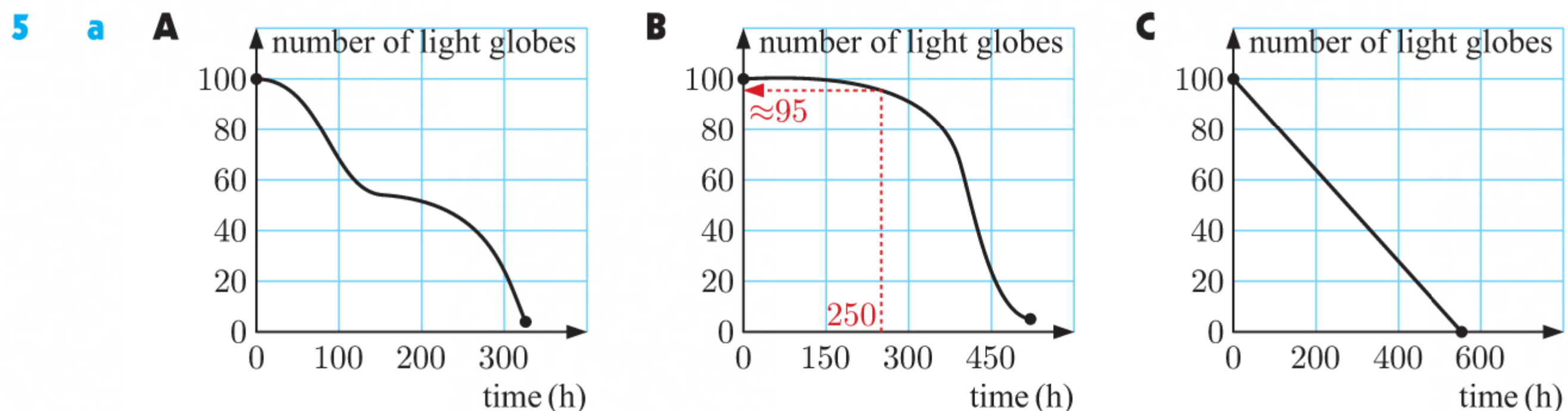
- c** Rick will take a longer time to run 500 metres than our prediction. He will not be able to run 500 metres at the same pace that he runs 100 metres.



The most appropriate model for the temperature of a city on a particular day is **C**. It is usually coldest at dawn, and warmest in the afternoon. Daily temperature should be roughly periodic.

- b** 8 pm is 20 hours after midnight.

Using model **C**, we predict that it will be about 28°C at 8 pm.



The most appropriate model for the number of light globes still working at any given time is **B**. Most light globes last for 200 hours, then the number of working globes quickly decreases.

- b** Using model **B**, we predict that about 95 globes will still be working after 250 hours.

- 6 a** Darren has assumed that the Earth is perfectly spherical, and that the lighthouse is perpendicular to the Earth's surface. These are reasonable assumptions.

b $r \approx 6370 \text{ km}, \quad h = 40 \text{ m}$
 $\quad \quad \quad = 0.04 \text{ km}$

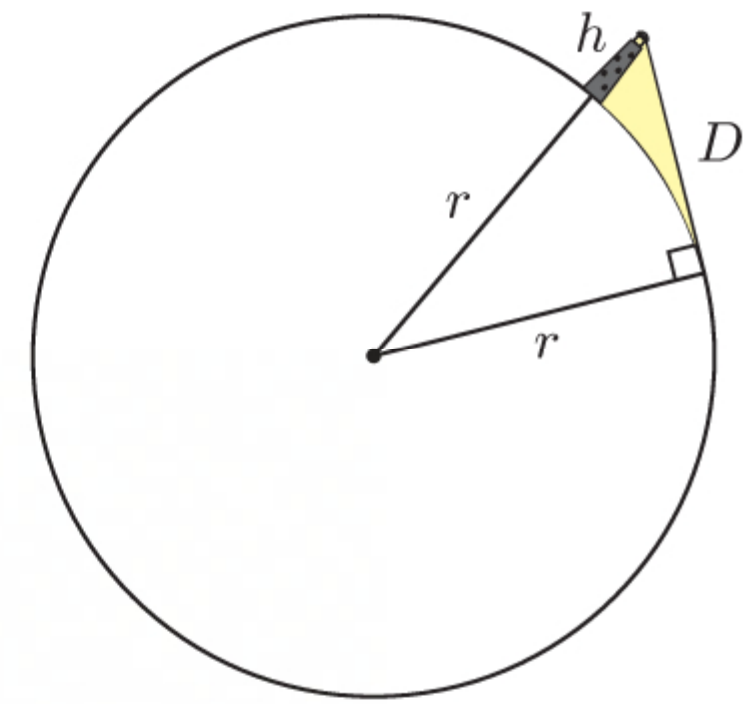
$$\therefore r^2 + D^2 = (r + h)^2 \quad \{\text{Pythagoras}\}$$

$$\therefore 6370^2 + D^2 \approx (6370 + 0.04)^2$$

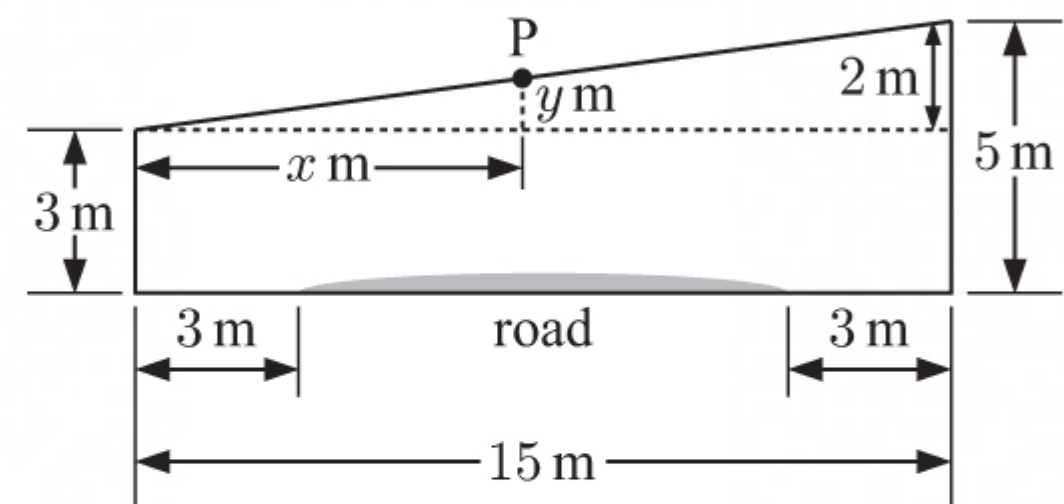
$$\therefore 6370^2 + D^2 \approx 6370.04^2$$

$$\therefore D \approx \sqrt{6370.04^2 - 6370^2} \quad \{D > 0\}$$

$$\therefore D \approx 22.6 \text{ km}$$



- 7 a** Using similar triangles, $\frac{y}{2} = \frac{x}{15}$
 $\therefore y = \frac{2}{15}x$
 \therefore the height of P above ground level is
 $h = y + 3 \text{ m}$
 $\quad = (\frac{2}{15}x + 3) \text{ m}$



We have assumed that the power line is hanging perfectly straight and that the poles are directly opposite.

- b** It is reasonable to apply this model for x such that $0 \leq x \leq 15$.

- c** The middle of the road corresponds to $x = \frac{15}{2}$.

$$\begin{aligned} \text{When } x = \frac{15}{2}, \quad h &= \frac{2}{15}(\frac{15}{2}) + 3 \text{ m} \\ &= 1 + 3 \text{ m} \\ &= 4 \text{ m} \end{aligned}$$

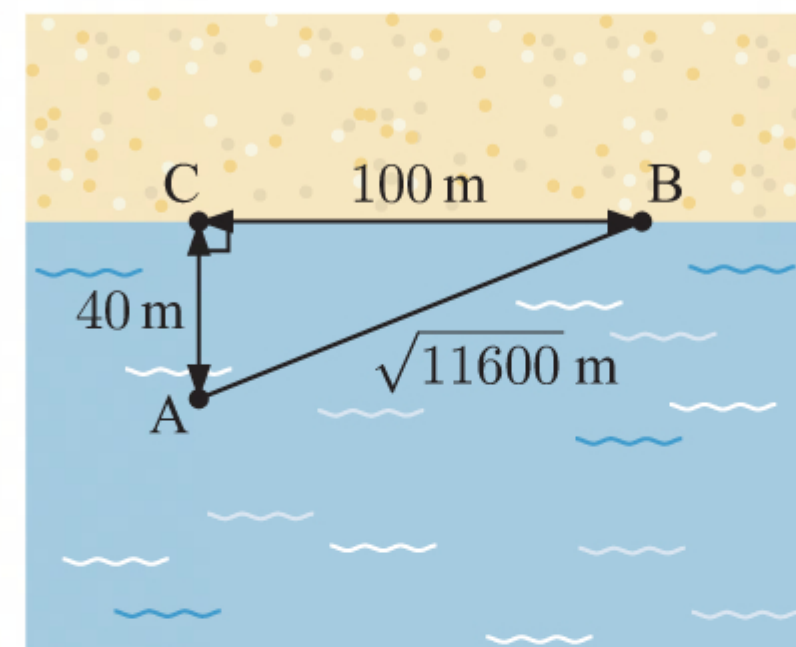
\therefore we predict that the height of the power line above ground level as it passes over the middle of the road is 4 m.

- d** Yes, the answer in **c** seems reasonable. If the poles are 3 m and 5 m high, we expect the height of the power line above the middle of the road to be 4 m.
- e** Yes, the model is useful to Hashni. Provided he knows the width and height of the truck, he should be able to reasonably decide whether or not the truck can fit.

8 a i $AB = \sqrt{40^2 + 100^2} \quad \{\text{Pythagoras}\}$
 $\quad \quad \quad = \sqrt{11\,600} \text{ m}$

Time taken to swim directly to B

$$\begin{aligned} &= \frac{\sqrt{11\,600} \text{ m}}{1.5 \text{ m s}^{-1}} \\ &\approx 71.8 \text{ s} \end{aligned}$$



$$\text{ii Time taken to swim to C} = \frac{40 \text{ m}}{1.5 \text{ m s}^{-1}} \\ \approx 26.7 \text{ s}$$

$$\text{Time taken to jog to B} = \frac{100 \text{ m}}{4 \text{ m s}^{-1}} \\ = 25 \text{ s}$$

$$\therefore \text{ total time taken} \approx 26.7 \text{ s} + 25 \text{ s} \\ \approx 51.7 \text{ s}$$

We have assumed that there is no current, and that Antonio can swim/jog at a constant speed, in all situations.

b Swimming to C then jogging to B appears to be quicker.

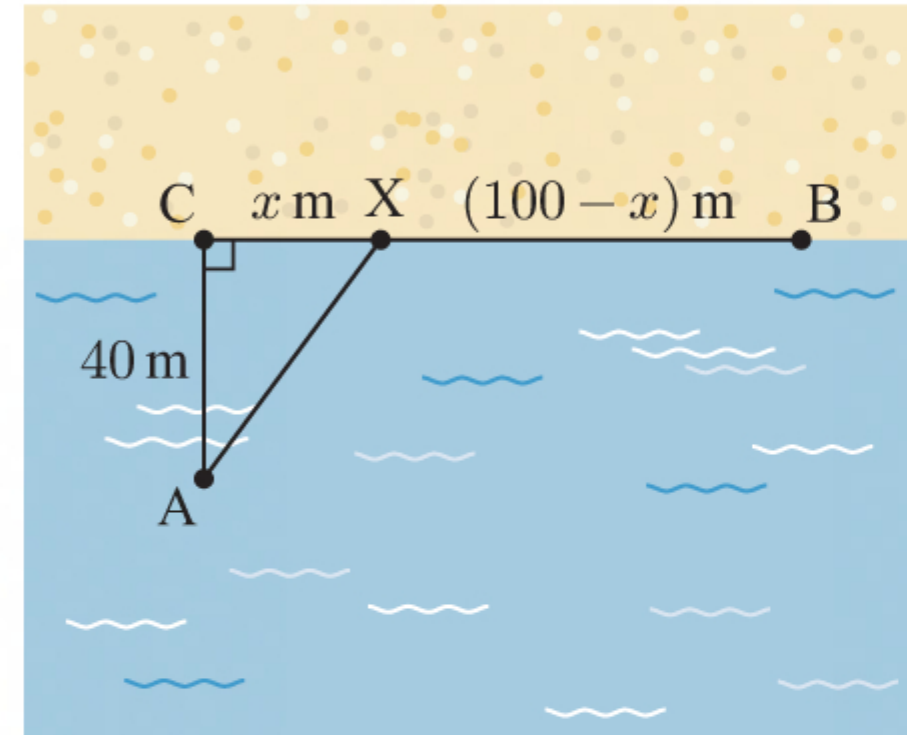
c Suppose Antonio swims to a point X, which is x m from C, then jogs to B.

$$AX = \sqrt{40^2 + x^2} \quad \{\text{Pythagoras}\} \\ = \sqrt{1600 + x^2} \text{ m}$$

$$\text{Time taken to swim to X} = \frac{\sqrt{1600 + x^2}}{1.5} \text{ s.}$$

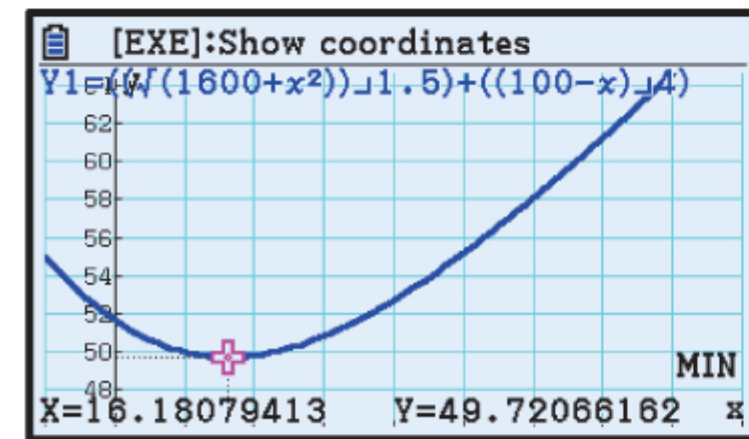
$$\text{Time taken to jog to B} = \frac{100 - x}{4} \text{ s.}$$

$$\therefore \text{ total time taken} = \left(\frac{\sqrt{1600 + x^2}}{1.5} + \frac{100 - x}{4} \right) \text{ s}$$



Using technology, $\frac{\sqrt{1600 + x^2}}{1.5} + \frac{100 - x}{4}$ has a minimum value of ≈ 49.7 , when $x \approx 16.2$.

So, Antonio could swim to the point about 16.2 m from C and jog about 83.8 m to B in about 49.7 seconds.



9 Kate can dig $\frac{1}{7}$ of a hole each hour, and Lenny can dig $\frac{1}{12}$ of a hole each hour.

We assume Kate and Lenny can work together without getting in each other's way. So, working together they will dig $\frac{1}{7} + \frac{1}{12} = \frac{19}{84}$ of a hole each hour.

\therefore it would take them $\frac{84}{19} = 4\frac{8}{19}$ hours ≈ 4 hours 25 minutes to dig a hole working together.

10 Pulling the plug would drain 1200 litres in 15 minutes, or $\frac{1200}{15} = 80$ litres in 1 minute.

A hole in the tank would drain 1200 litres in 25 minutes, or $\frac{1200}{25} = 48$ litres in 1 minute.

\therefore pulling the plug, and having a hole in the tank would drain $80 + 48 = 128$ litres each minute.

It would take $\frac{1200}{128} = 9\frac{3}{8}$ minutes ≈ 9 minutes 23 seconds for the tank to empty.

11 Independent of one another, Aaron can make 4 widgets per hour, Bonnie can make 3 widgets per hour, and Calum can make 2 widgets per hour.

In one hour they will make $4 + 3 + 2 = 9$ widgets.

\therefore it will take them $\frac{135}{9} = 15$ hours to make 135 widgets working together.

12 Suppose Beitidh can paint the room on her own in x hours.

\therefore Beitidh can paint $\frac{1}{x}$ of the room each hour.

Angus can paint $\frac{1}{3}$ of the room each hour.

Working together they can paint $\frac{1}{3} + \frac{1}{x}$ of the room each hour.

Now, $\frac{1}{\frac{1}{3} + \frac{1}{x}} = 2$ (it would take them 2 hours to paint the room working together)

$$\therefore \frac{1}{3} + \frac{1}{x} = \frac{1}{2}$$

$$\therefore \frac{1}{x} = \frac{1}{6}$$

$$\therefore x = 6$$

\therefore it would take Beitidh 6 hours to paint the room on her own.

INVESTIGATION

CALENDARS

- 1 a** Difference between a tropical year and 365 days $\approx 365.242\,199 - 365$
 $\approx 0.242\,199$ days

b $\frac{1}{0.242\,199} \approx 4.13$

\therefore it would take about 4.13 years for the difference between 365 days and a tropical year to accumulate to one whole day.

- c** Assume that six months = 183 days.

$$\frac{183}{0.242\,199} \approx 756$$

\therefore it would take about 756 years for the difference between 365 days and a tropical year to accumulate to six months.

- 2 a** $365.25 - 365.242\,199 = 0.007\,801$

\therefore the Julian calendar is inaccurate by about 0.007 801 days per year.

b i $\frac{1}{0.007\,801} \approx 128$

\therefore it would take about 128 years for the inaccuracy to accumulate to one whole day.

- ii** Assume that six months = 183 days.

$$\frac{183}{0.007\,801} \approx 23\,459$$

\therefore it would take about 23 459 years for the inaccuracy to accumulate to six months.

- 3 a** In a period of 400 years, the Gregorian calendar adds $100 - 3 = 97$ leap years (the 100th, 200th, and 300th years are not leap years).

- b** 303 years with 365 days and 97 years with 366 days gives a total of
 $303 \times 365 + 97 \times 366 = 146\,097$ days in a period of 400 years.

\therefore the average year in the Gregorian calendar has $\frac{146\,097}{400} = 365.2425$ days.

- c** $365.2425 - 365.242\,199 = 0.000\,301$

\therefore the Gregorian calendar is inaccurate by about 0.000 301 days per year.

$$\frac{1}{0.000\,301} \approx 3322$$

\therefore it would take about 3322 years for the inaccuracy to accumulate to one whole day.

EXERCISE 4B

- 1 a** The gradient of the graph is $\frac{5-0}{2-0} = 2.5$

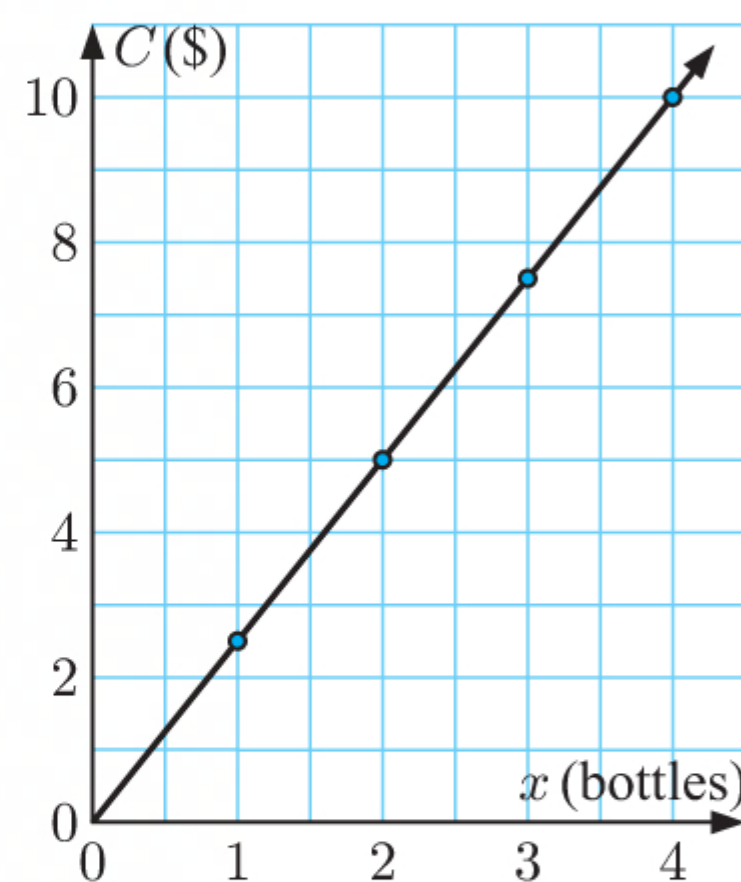
The C -intercept is 0.

$$\therefore C = 2.5x$$

- b** The model fits the data points exactly.
 \therefore the model is exact.

- c** Yes, if $x = 12$, $C = 2.5(12)$
 $= 30$

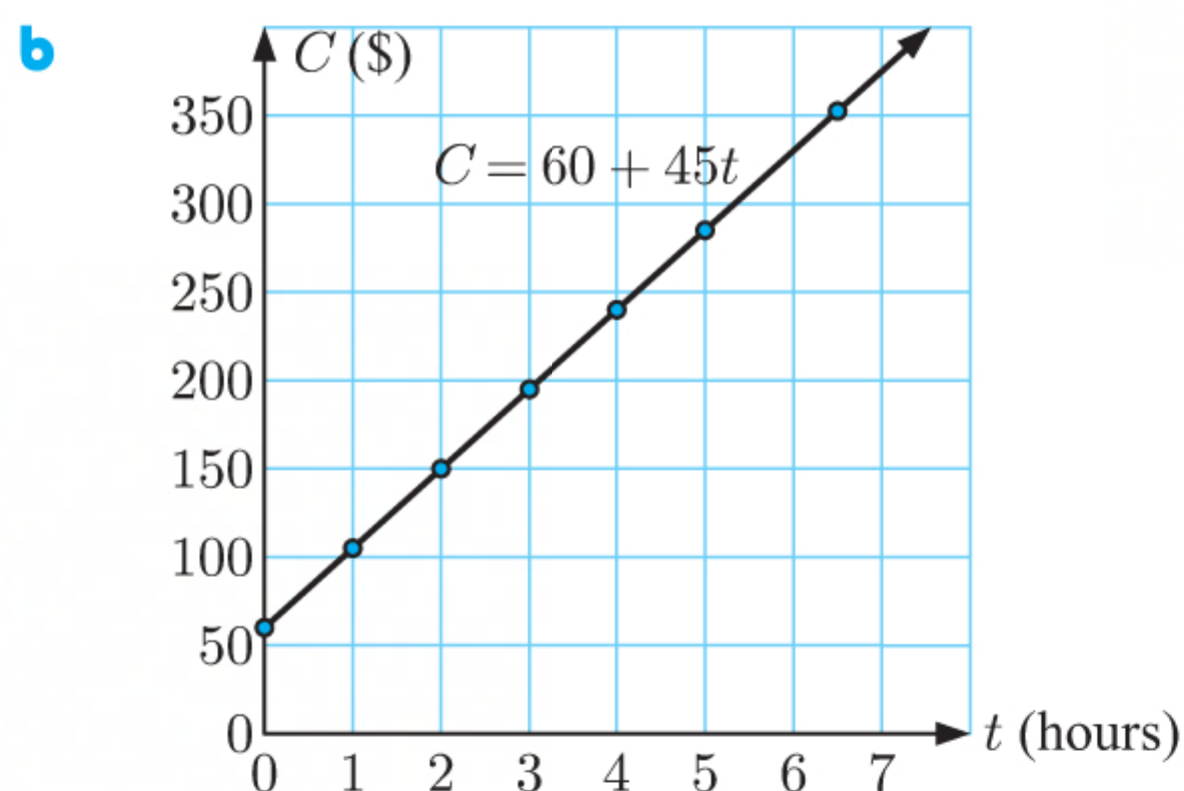
\therefore the cost of 12 bottles of juice is \$30.



2 a

Time (t hours)	0	1	2	3	4	5
Cost (\$ C)	60	105	150	195	240	285

$\overset{+45}{\curvearrowright}$
 $\overset{+45}{\curvearrowright}$
 $\overset{+45}{\curvearrowright}$
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- c** The gradient of the graph is 45.

The C -intercept is 60.

$$\therefore C = 45t + 60$$

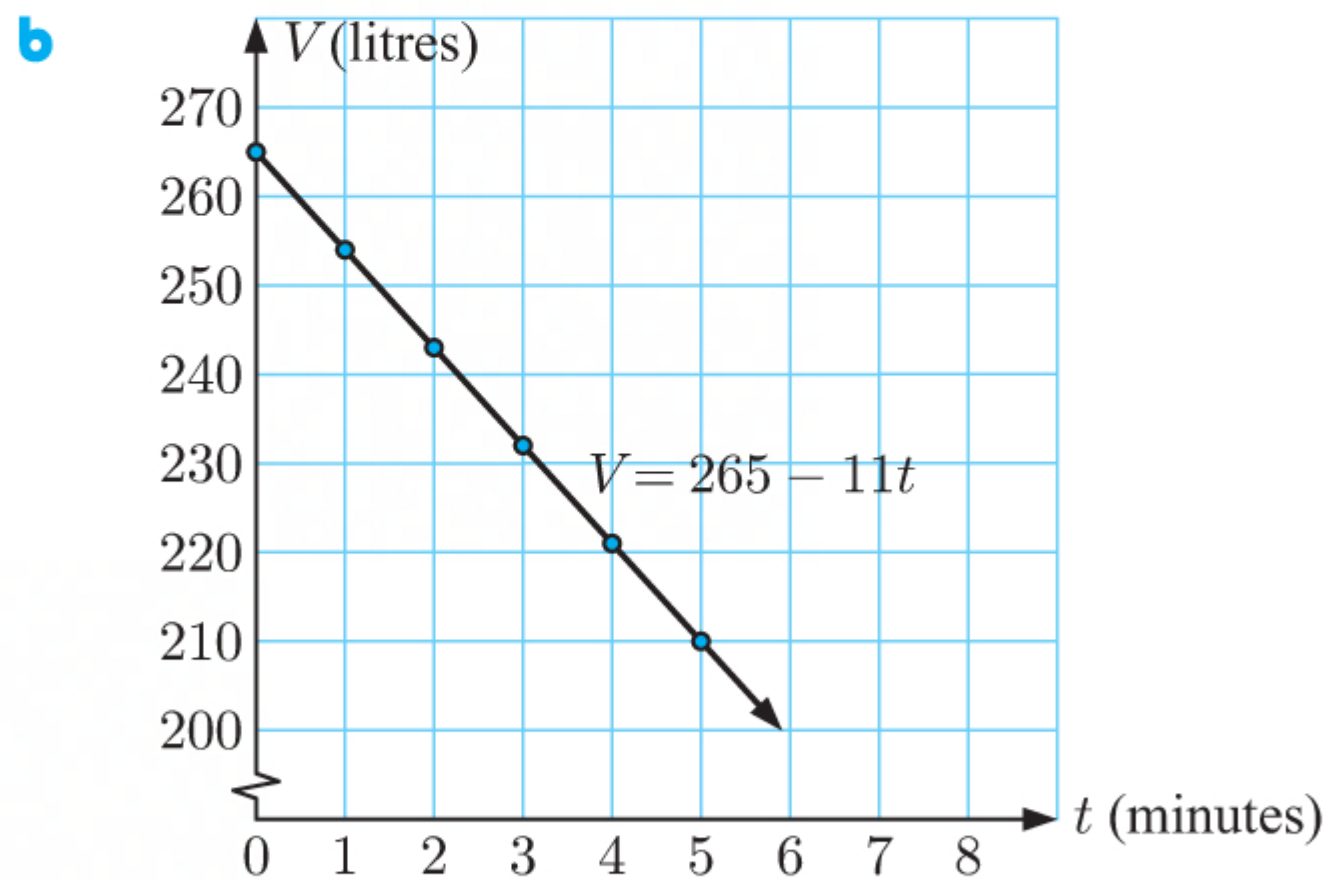
- d** When $t = 6\frac{1}{2} = \frac{13}{2}$, $C = 45(\frac{13}{2}) + 60$
 $= 292.5 + 60$
 $= 352.5$

\therefore the electrician's total cost for a job lasting $6\frac{1}{2}$ hours is \$352.50.

3 a

Time (t minutes)	0	1	2	3	4	5
Volume (V litres)	265	254	243	232	221	210

$\overset{-11}{\curvearrowright}$
 $\overset{-11}{\curvearrowright}$
 $\overset{-11}{\curvearrowright}$
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 $\overset{-11}{\curvearrowright}$



- c** The gradient of the graph is -11 .
The V -intercept is 265.
 $\therefore V = 265 - 11t$

- d i** When $t = 15$, $V = 265 - 11(15)$
 $= 265 - 165$
 $= 100$
 \therefore there are 100 litres of water left in the tank after 15 minutes.

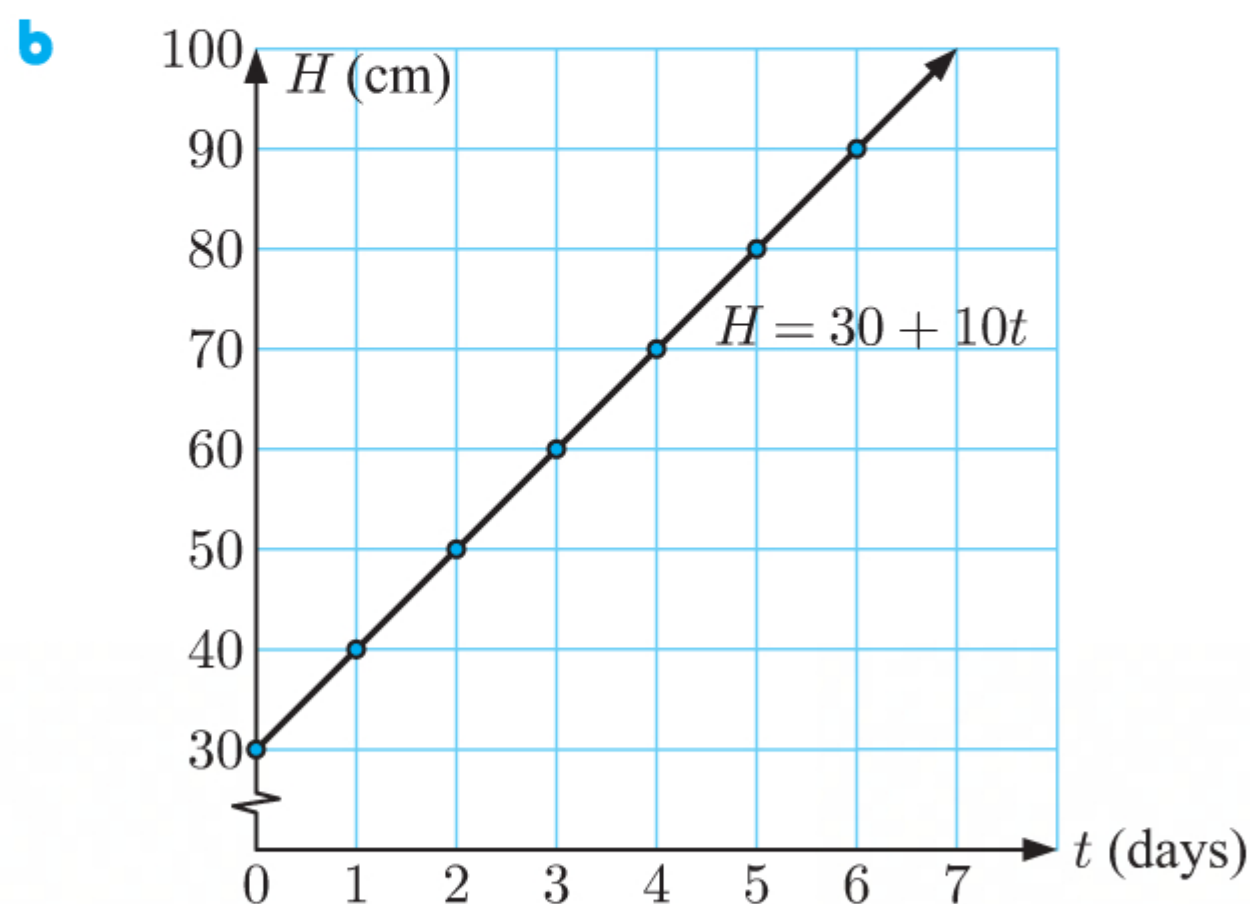
- ii** When $V = 0$, $265 - 11t = 0$
 $\therefore 11t = 265$
 $\therefore t = \frac{265}{11}$
 $\therefore t \approx 24.1$

\therefore it will take about 24.1 minutes for the tank to empty.

4 a

t (days)	0	1	2	3	4	5	6
H (cm)	30	40	50	60	70	80	90

$\overset{\curvearrowright}{+10}$ $\overset{\curvearrowright}{+10}$ $\overset{\curvearrowright}{+10}$ $\overset{\curvearrowright}{+10}$ $\overset{\curvearrowright}{+10}$ $\overset{\curvearrowright}{+10}$



- c** It does not make sense to extend the model for $t < 0$ since the number of days after planting cannot be negative. It does not make sense to extend the model indefinitely for $t > 6$ since the plant will not continue to grow at this rate forever. However, we will assume that it does for the purposes of this model. $H(t)$ has domain $\{t \mid t \geq 0\}$.

- d** The gradient of the graph is 10.
The H -intercept is 30.
 $\therefore H = 10t + 30$

- e** When $H = 100$, $10t + 30 = 100$
 $\therefore 10t = 70$
 $\therefore t = 7$

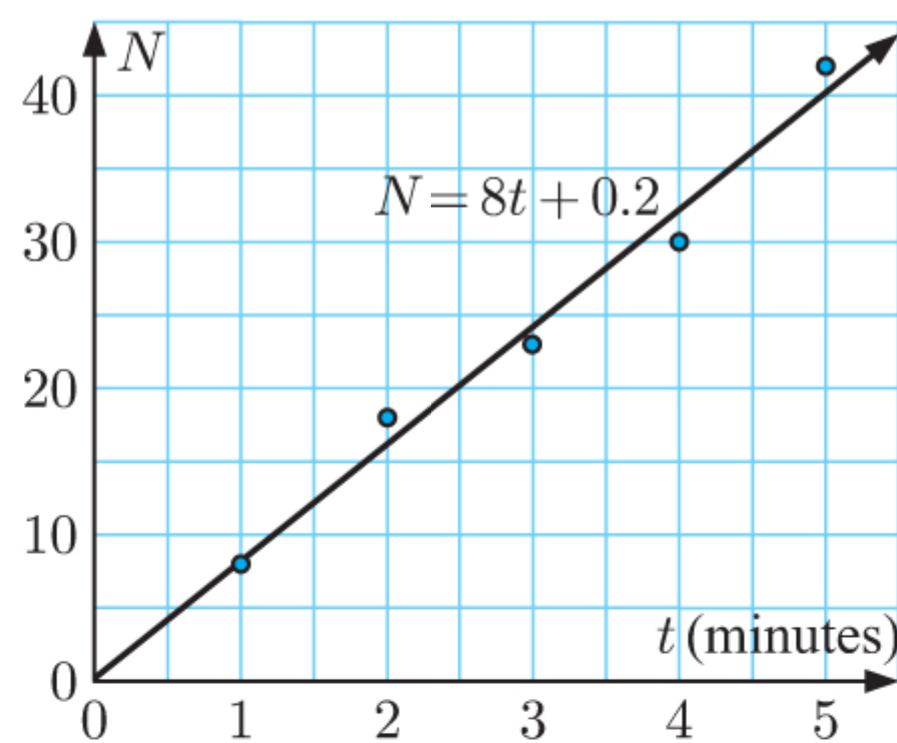
\therefore it will take 7 days for the bamboo to be 1 m high.

- 5 a** The points do not lie exactly on the line.
 \therefore the model is approximate.

b $N = 8t + 0.2$

When $t = 20$, $N = 8(20) + 0.2$
 $= 160.2$

\therefore the model predicts that Jack will pick up about 160 pieces of litter in 20 minutes. This estimate is likely to be inaccurate as it is an extrapolation. There would probably not be this much litter for Jack to pick up.

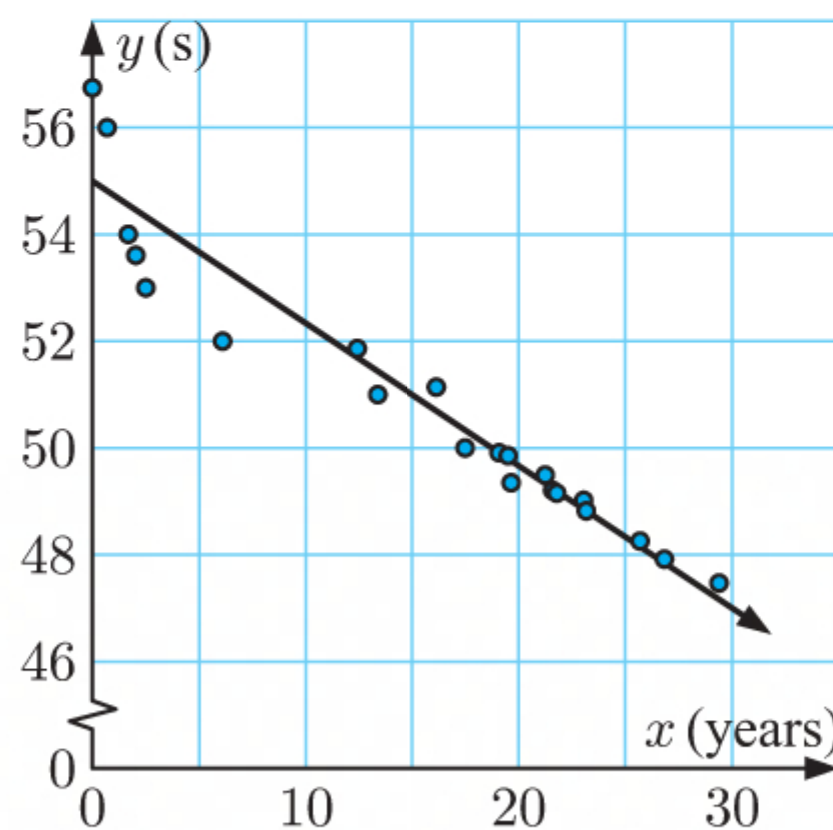


6 a $y = 54.85 - 0.2660x$

When $x = 0$, $y = 54.85 - 0.2660(0)$
 $= 54.85$

The model predicts that in 1957 ($x = 0$) the world record time will be 54.85 s, while the actual time was 57.0 s.

The values differ since the model is not exact. Some times will be higher or lower than what is predicted.



- b** The model would not be accurate for years much earlier than 1957, as the model would begin to predict unrealistically large times. The model should still be useful in estimating times a few years prior to 1957.

c When $y = 41.96$, $54.85 - 0.2660x = 41.96$
 $\therefore 0.2660x = 12.89$
 $\therefore x \approx 48.5$

The model predicts that the women's 400 m will be completed in 41.96 s in the 48th year after 1957, which is 2005.

- d** It is not suitable to extrapolate future times, since improvements in time will eventually stop following a linear trend.

EXERCISE 4C

- 1 a A is (0, 50) and B is (25, 100).

$$\therefore \text{gradient of [AB]} = \frac{100 - 50}{25 - 0} = 2$$

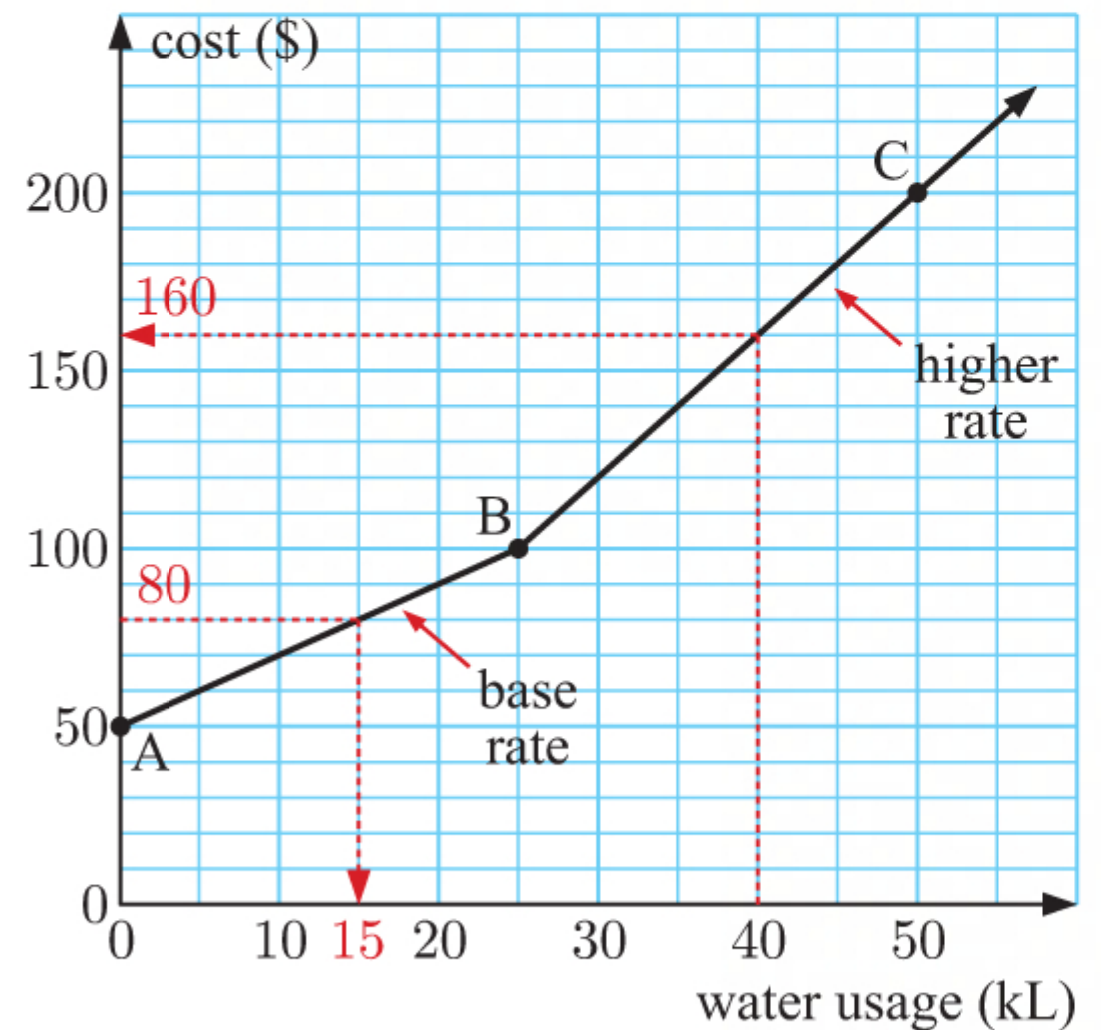
The first 25 kL of water used costs \$2 per kL.
This is the base rate.

B is (25, 100) and C is (50, 200).

$$\therefore \text{gradient of [BC]} = \frac{200 - 100}{50 - 25} = 4$$

Each kL of water used above 25 kL costs \$4 per kL. This is the higher rate.

- b The cost of using 40 kL of water is \$160.
c Kelly's last water bill of \$80 was for 15 kL of water used.



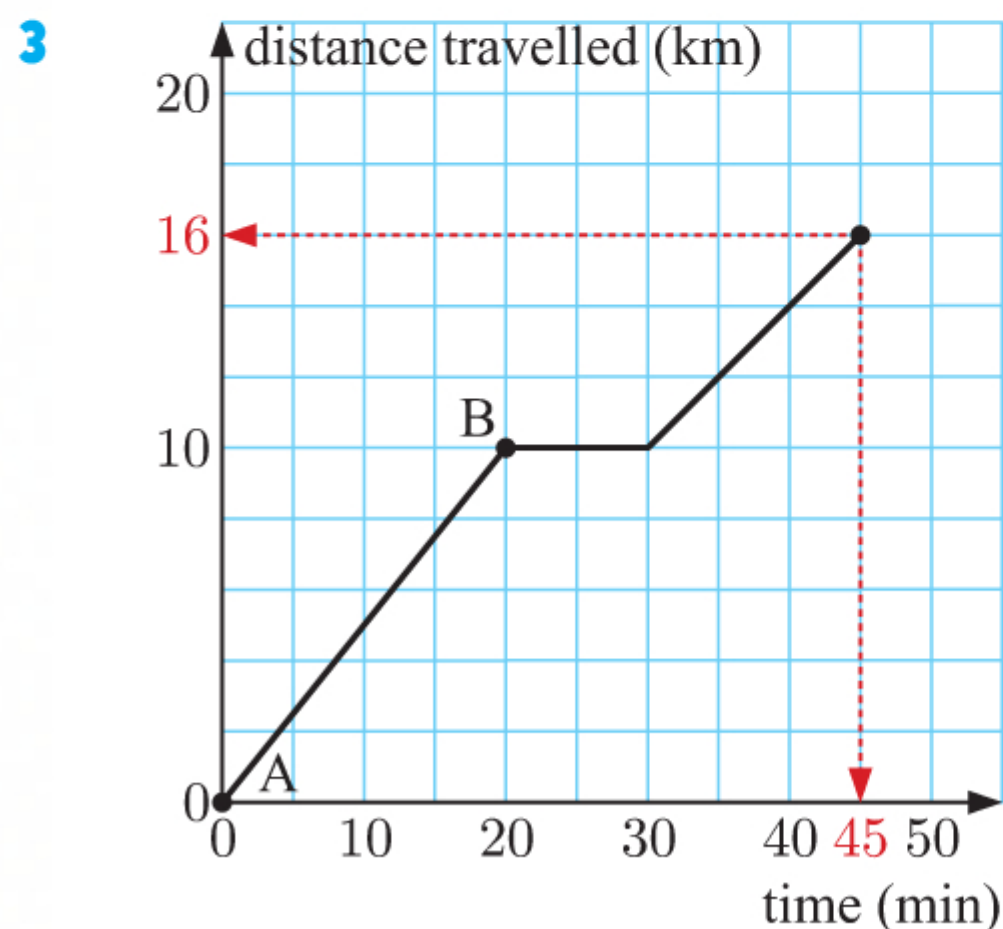
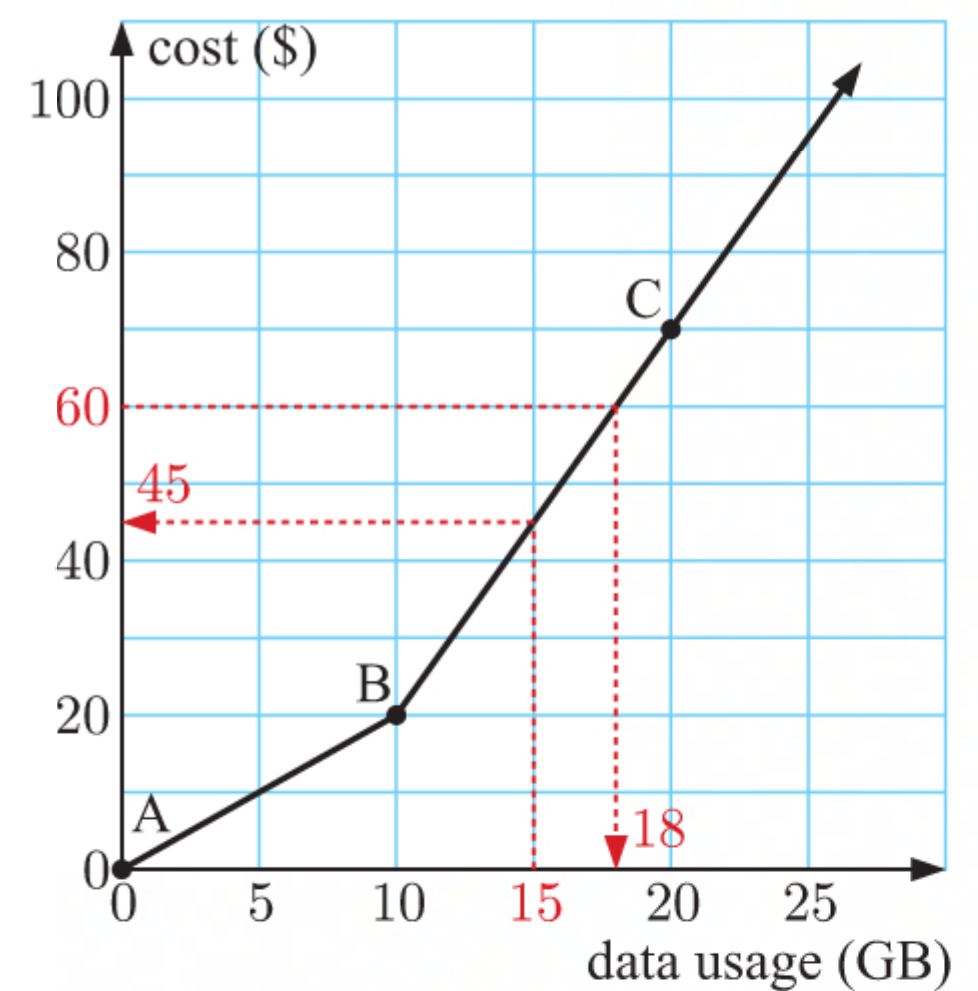
- 2 a Colin pays one rate up to his allowance and then a higher rate after that. The gradient of the graph changes at the point (10, 20), which is when Colin's allowance must end.
So, Colin has 10 GB of data allowance.

- b B is (10, 20) and C is (20, 70).

$$\therefore \text{gradient of [BC]} = \frac{70 - 20}{20 - 10} = 5$$

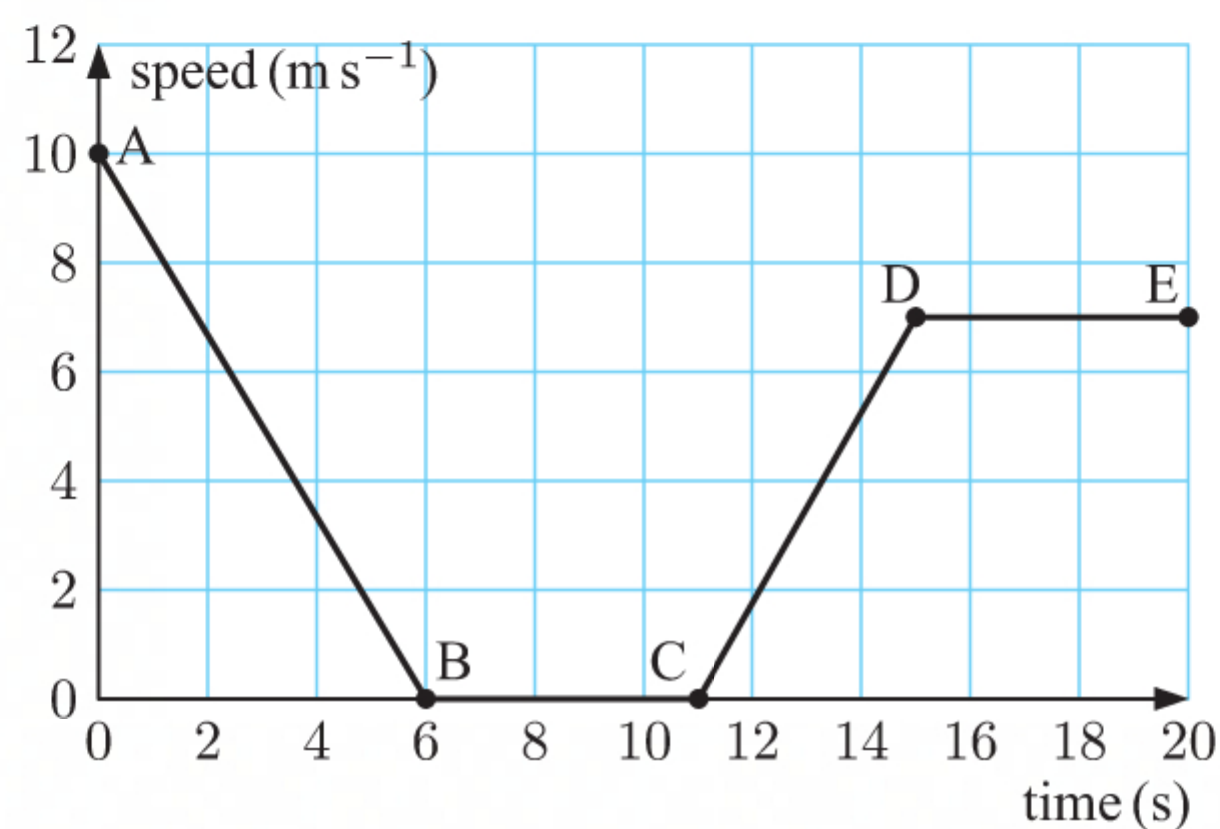
Colin is charged \$5 per GB for excess data.

- c Colin needs to pay \$45 for using 15 GB of data in a month.
d A bill of \$60 would result by using 18 GB of data in a month.



- a The end point of the graph is at (45, 16).
 \therefore it took the cyclist 45 minutes to ride to work.
b The cyclist travelled 16 km in total.
c A is (0, 0) and B is (20, 10).
 $\therefore \text{gradient of [AB]} = \frac{10 - 0}{20 - 0} = 0.5$
The cyclist was travelling at 0.5 km per min for the first 10 km.
d The distance travelled was constant between the 20th and 30th minute. So, the cyclist stopped for 10 minutes to fix her tyre.
e The cyclist had travelled 12 km after 35 minutes.

- 4 a** The car was initially travelling at 10 m s^{-1} , then braked for 6 seconds to a complete stop. The car stopped for 5 seconds before accelerating for 4 seconds to a speed of 7 m s^{-1} which it maintained for the remaining 5 seconds.
- b** We have assumed that the car accelerates and decelerates at a constant rate. These assumptions are reasonable.



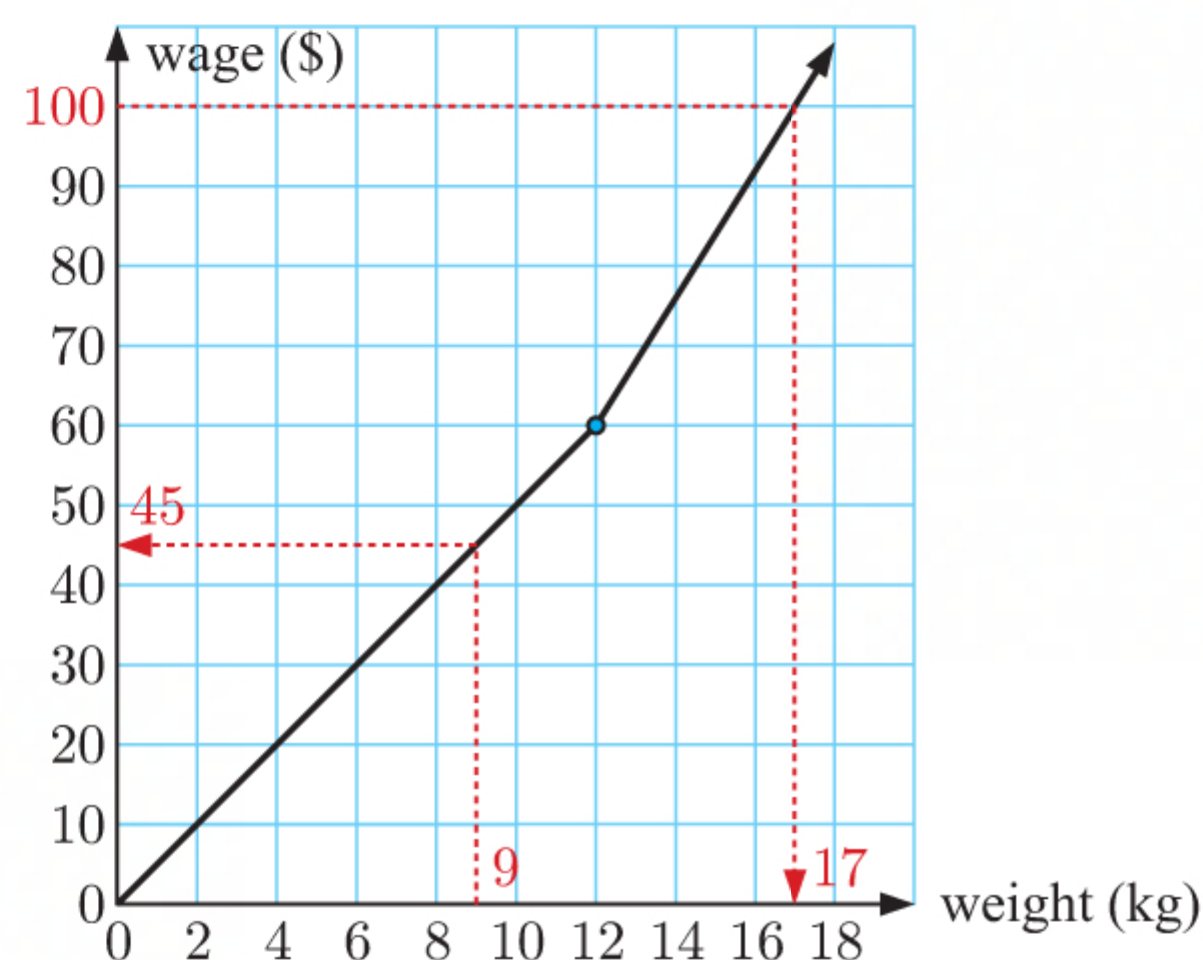
- c i** The car decelerates at a constant rate from A to B.
 \therefore average speed between A and B $= \frac{10 + 0}{2} = 5 \text{ m s}^{-1}$.
- ii** The car travels at an average speed of 5 m s^{-1} for 6 s.
 \therefore distance travelled between A and B $= 5 \times 6 = 30 \text{ m}$.

d

Segment	Average speed (m s^{-1})	Time (s)	Distance travelled (m)
AB	5	6	$5 \times 6 = 30$
BC	0	5	$0 \times 5 = 0$
CD	$\frac{7 + 0}{2} = 3.5$	4	$3.5 \times 4 = 14$
DE	7	5	$7 \times 5 = 35$

\therefore the car travelled a total distance of $30 + 0 + 14 + 35 = 79 \text{ m}$.

- 5 a** For the first 12 kg of berries picked, the gradient of the graph is 5.
 For any berries picked over 12 kg, the gradient of the graph is 8.



- b** Jasper earns \$45 for picking 9 kg of berries.
- c** Jasper must pick 17 kg of berries to earn \$100 in a day.

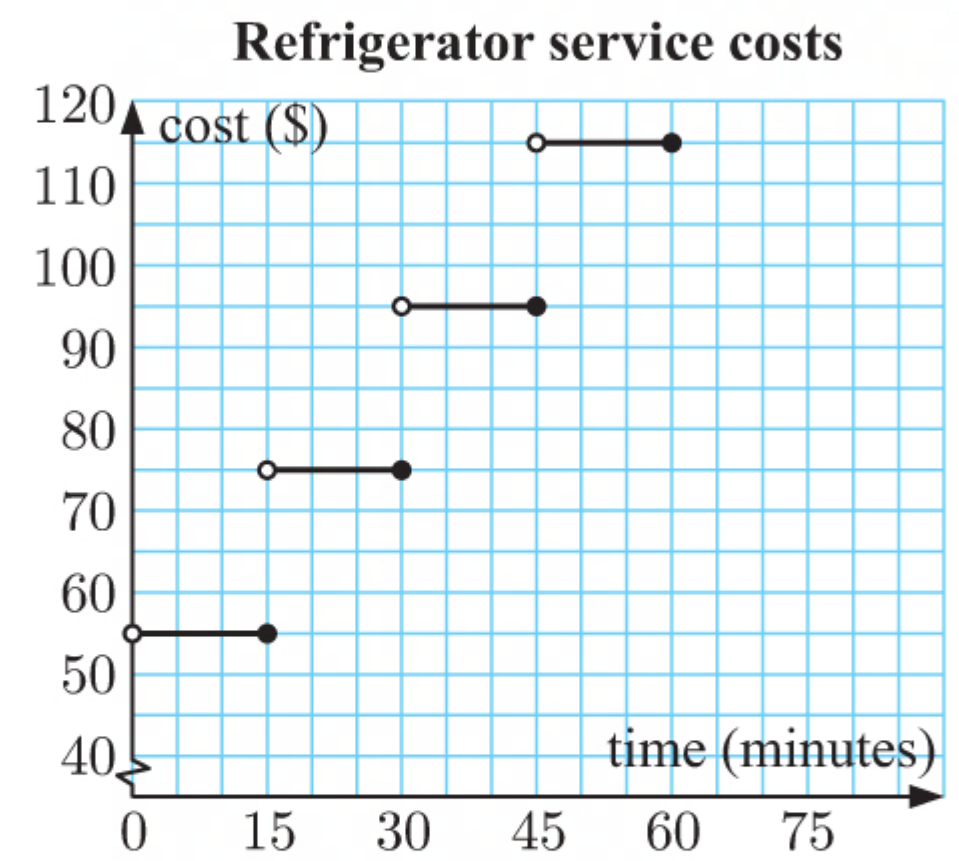
- 6 a**
- i** From the graph, a service which takes 20 minutes costs \$75.
 - ii** From the graph, a service which takes 45 minutes costs \$95.
- b**
- i** From the graph, the maximum time for a service costing \$55 is 15 minutes.
 - ii** From the graph, the maximum time for a service costing \$115 is 60 minutes.
- c** Looking at the graph, a service costs \$55 for the first 15 minutes, and an extra \$20 for each additional 15 minutes or part thereof.

We assume that this trend continues indefinitely.

$$80 \text{ minutes} = 1 \times 15 \text{ minutes} + 4\frac{1}{3} \times 15 \text{ minutes}$$

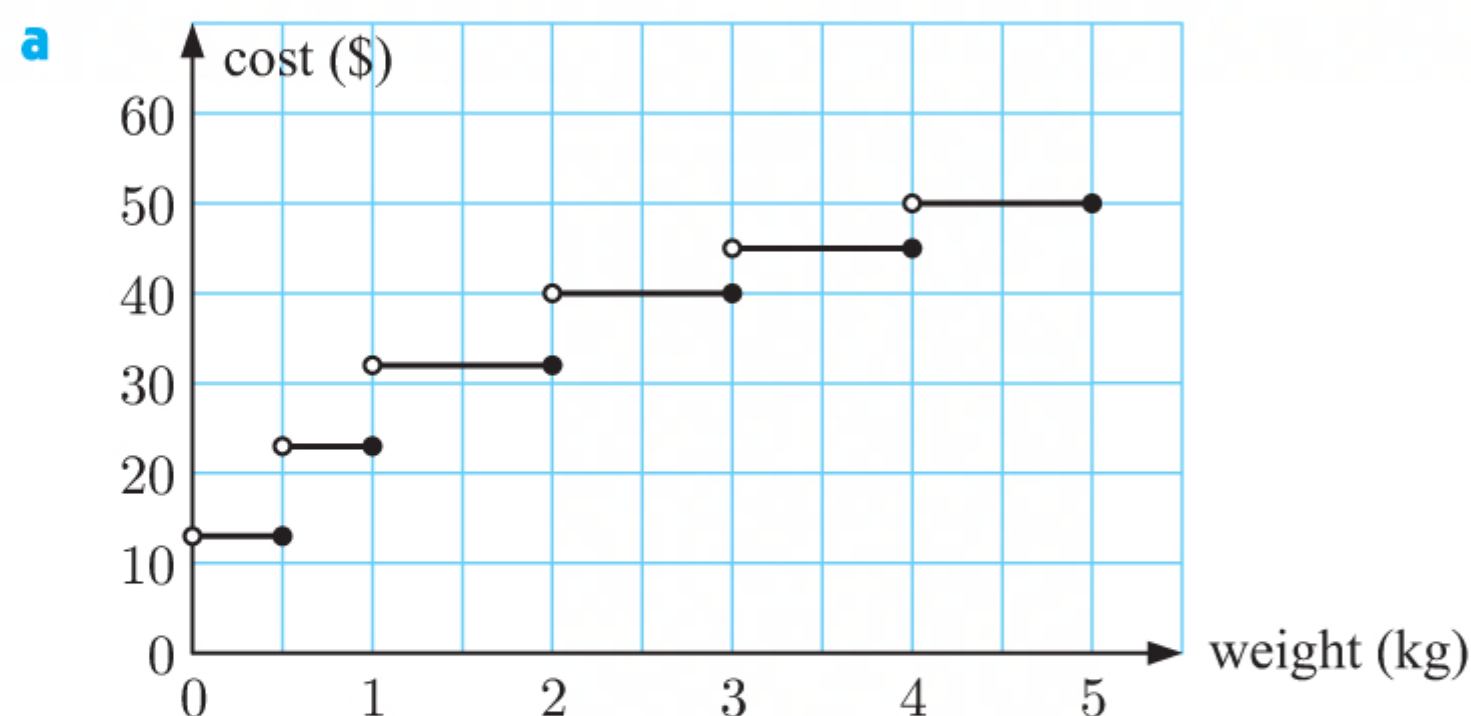
So a service which takes 80 minutes will be charged

$$\begin{aligned} & \$55 \text{ for the first 15 minutes} + 5 \times \$20 \text{ for the next 5 lots of 15 minutes} \\ &= \$55 + 5 \times \$20 \\ &= \$155 \end{aligned}$$



7

Weight	Cost
Up to 500 g	\$13
Over 500 g up to 1 kg	\$23
Over 1 kg up to 2 kg	\$32
Over 2 kg up to 3 kg	\$40
Extra kg or part thereof	\$5

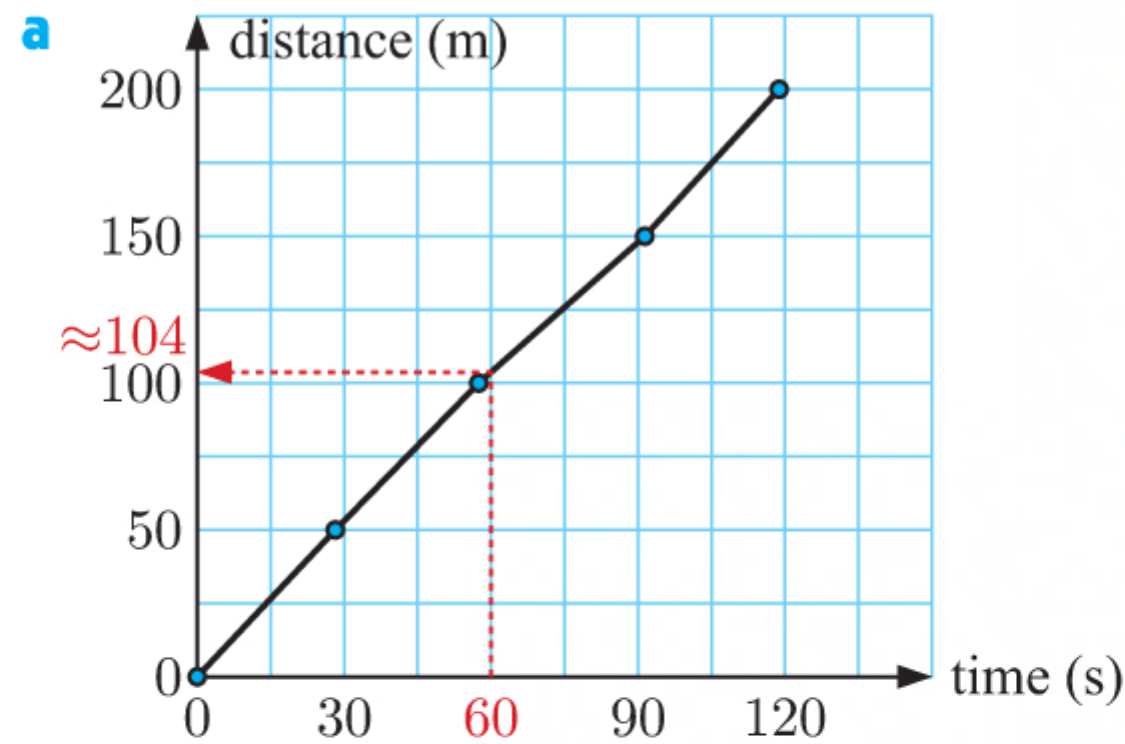


- b**
- i** It costs \$23 to send a parcel weighing 750 g.
 - ii** It costs \$32 to send a parcel weighing 1.6 kg.
 - iii** It costs \$45 to send a parcel weighing 3.4 kg.

- c** It costs \$32 to send a parcel weighing 1.7 kg and \$40 to send a parcel weighing 2.8 kg.
 \therefore it costs a total of $\$32 + \$40 = \$72$ to send the parcels separately.
 If Heather puts the two parcels together as a single package, it will weigh $1.7 \text{ kg} + 2.8 \text{ kg} = 4.5 \text{ kg}$.
 It costs \$50 to send a parcel weighing 4.5 kg.
 Heather will save $\$72 - \$50 = \$22$ if she sends the parcels together as a single package.

8

Stroke	Time (s)
Butterfly	28.20
Backstroke	29.28
Breaststroke	33.91
Freestyle	27.43



- b** We assume Laura swims at a constant rate in each 50 m leg.
- c** Using our graph from **a**, we estimate that Laura had swum approximately 104 m after 1 minute.

9

Taxable income (per year)	Tax paid
Up to £11 850	Nil
£11 851 - £46 350	£0.20 for each £1 above £11 850
£46 351 - £150 000	£6900 + £0.40 for each £1 above £46 350
Over £150 000	£48 360 + £0.45 for each £1 above £150 000

- a** Taxable income = £7.83 per hour \times 36 hours per week \times 50 weeks per year
 $=$ £14 094 per year
 Now £14 094 = £11 850 + £2244
 \therefore tax paid = £0.20 \times 2244
 $=$ £448.80

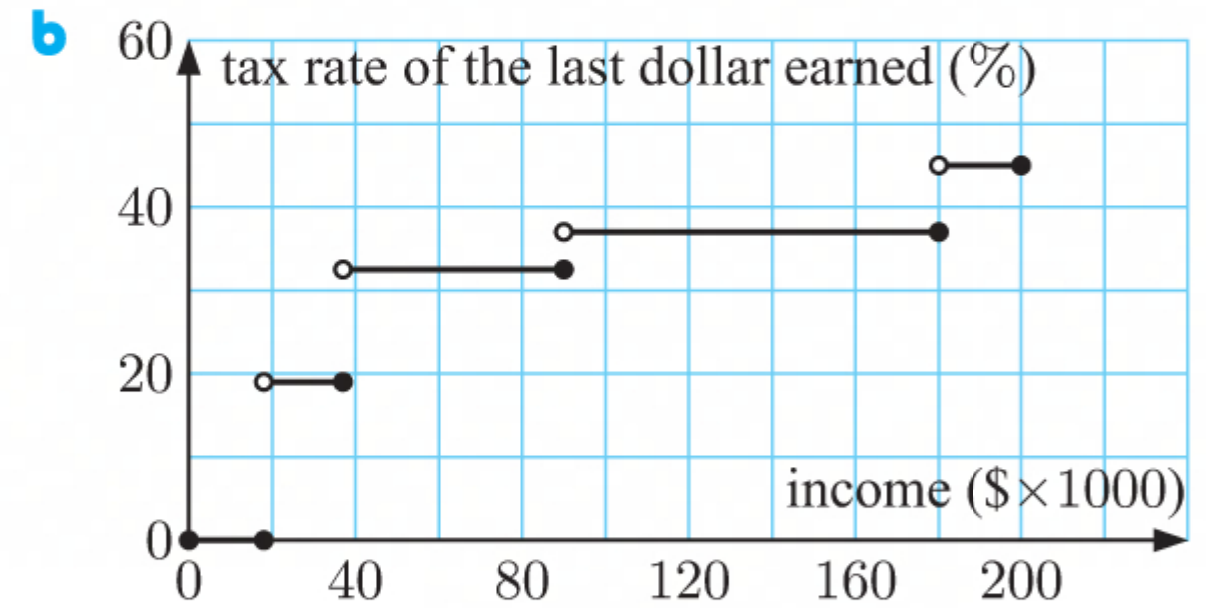
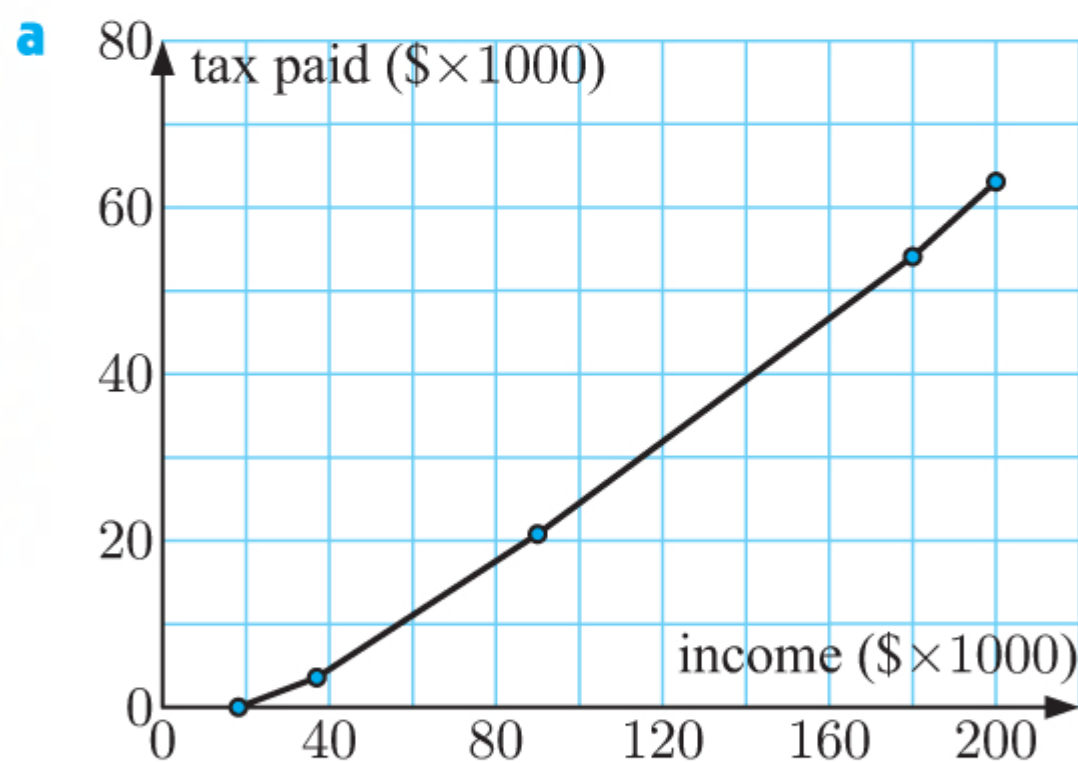
- b** **i** £53 172 = £46 350 + £6822
 \therefore tax paid = £6900 + £0.40 \times 6822
 $=$ £9628.80
- ii** After-tax income = £53 172 - £9628.80
 $=$ £43 543.20

- c** **i** £153 907 = £150 000 + £3907
 \therefore tax paid = £48 360 + £0.45 \times 3907
 $=$ £50 118.15

- ii** Percentage of income paid as income tax = $\frac{£50\,118.15}{£153\,907} \times 100\%$
 $\approx 32.6\%$

10

<i>Taxable income (per year)</i>	<i>Tax paid</i>
Up to \$18 200	Nil
\$18 201 - \$37 000	\$0.19 for each \$1 over \$18 200
\$37 001 - \$90 000	\$3572 plus \$0.325 for each \$1 over \$37 000
\$90 001 - \$180 000	\$20 797 plus \$0.37 for each \$1 over \$90 000
\$180 001 and over	\$54 097 plus \$0.45 for each \$1 over \$180 000



c i $\$28\,000 = \$18\,200 + \$9800$
 \therefore income tax = $\$0.19 \times 9800$
 $= \$1862$

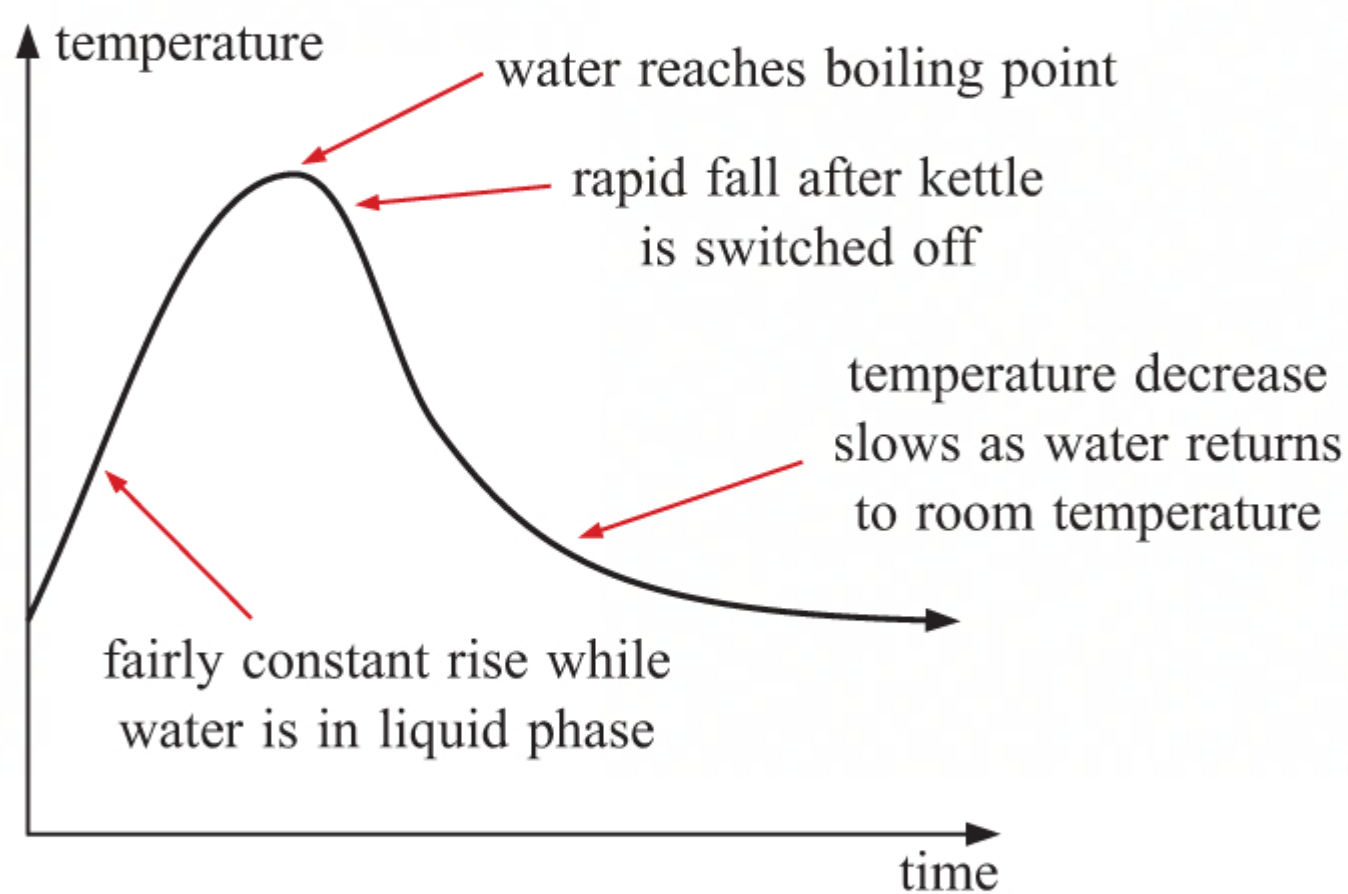
ii $\$48\,300 = \$37\,000 + \$11\,300$
 \therefore income tax = $\$3572 + \$0.325 \times 11\,300$
 $= \$7244.50$

iii $\$96\,150 = \$90\,000 + \$6150$
 \therefore income tax = $\$20\,797 + \0.37×6150
 $= \$23\,072.50$

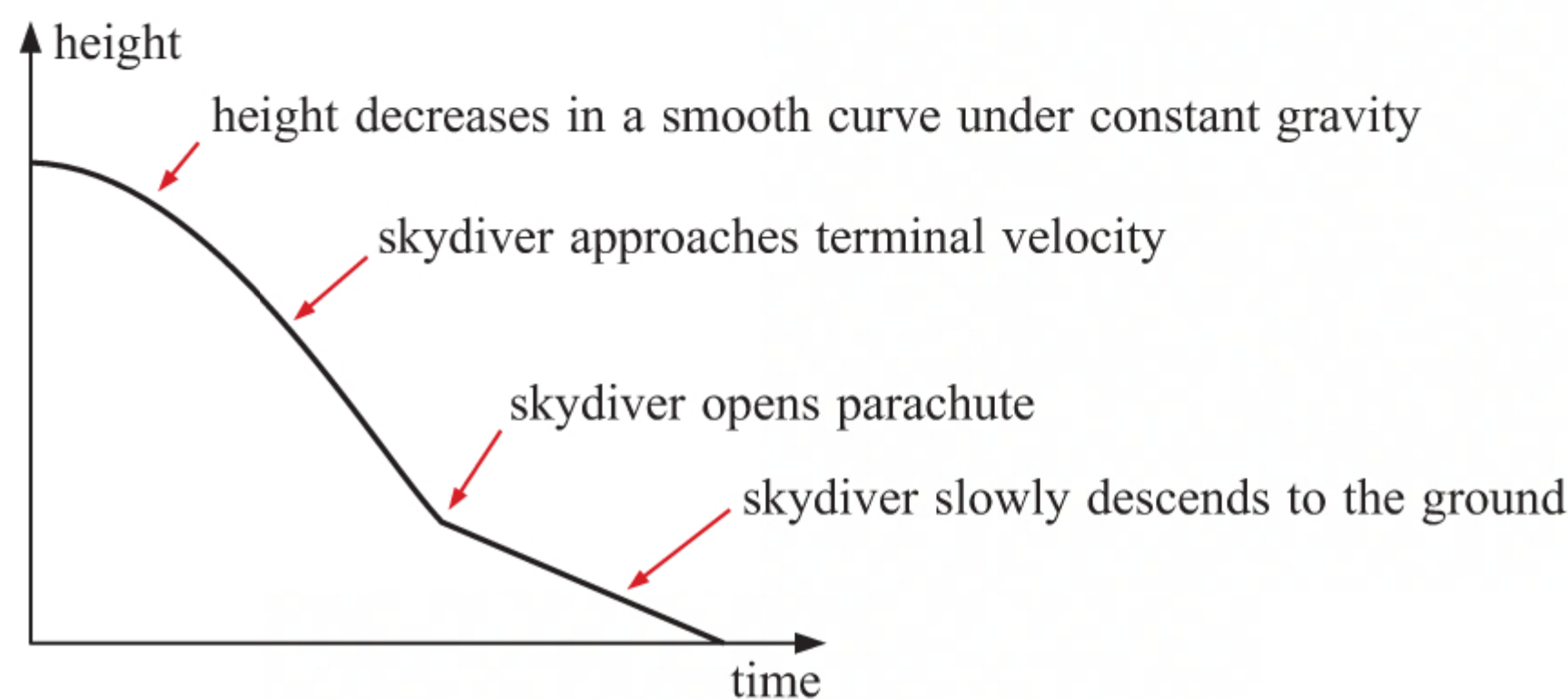
ACTIVITY 2

NON-LINEAR PIECEWISE MODELS

1 The most accurate graph is **B**.



2 The most accurate graph is **C**.



EXERCISE 4D

1 a Substituting $(4, 3)$ into the model gives

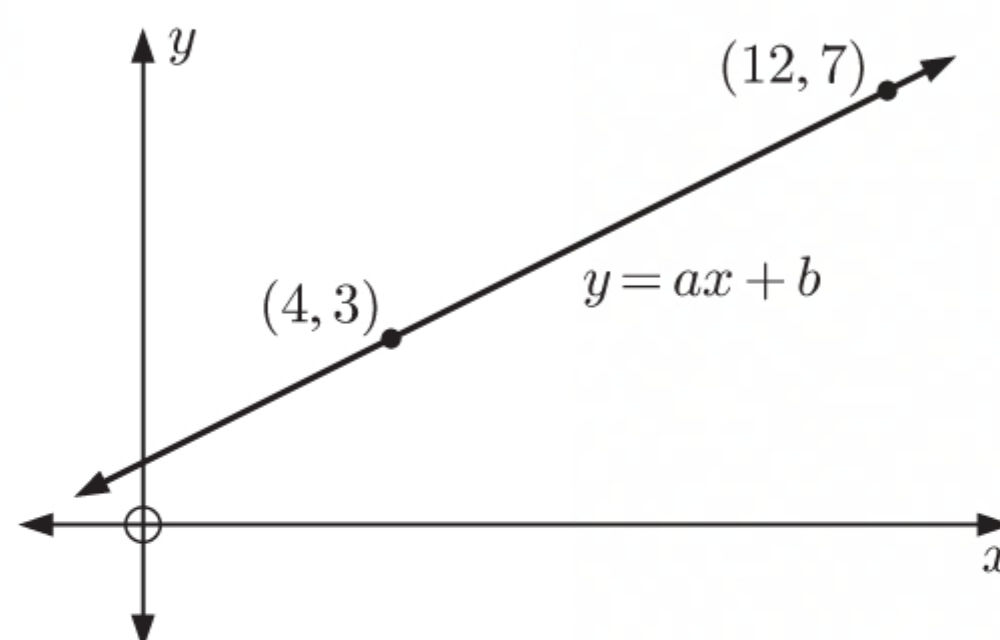
$$3 = a(4) + b$$

$$\therefore 4a + b = 3$$

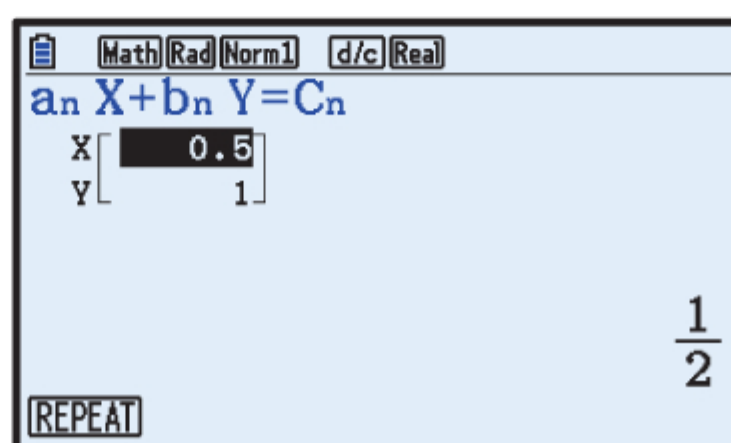
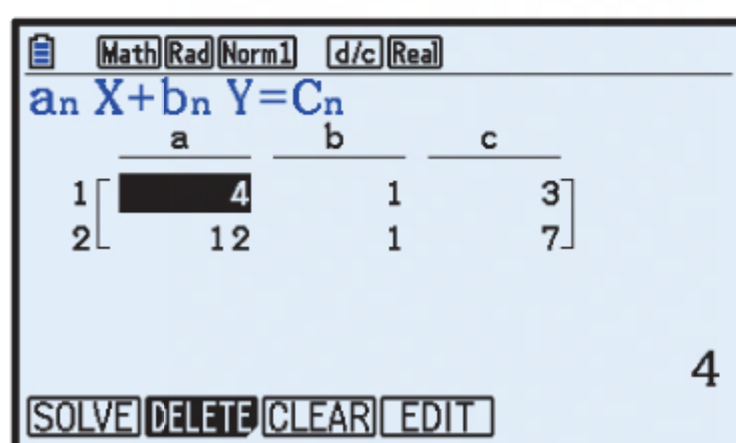
Substituting $(12, 7)$ into the model gives

$$7 = a(12) + b$$

$$\therefore 12a + b = 7$$



So, we have the system of equations
$$\begin{cases} 4a + b = 3 \\ 12a + b = 7 \end{cases}$$



Solving these equations simultaneously using technology, we find that $a = \frac{1}{2}$ and $b = 1$.

So, the model is $y = \frac{1}{2}x + 1$.

b When $x = 4$, $y = \frac{1}{2}(4) + 1$
 $= 2 + 1$
 $= 3$ ✓

When $x = 12$, $y = \frac{1}{2}(12) + 1$
 $= 6 + 1$
 $= 7$ ✓

- 2 a** Substituting $(1, 7)$ into the model gives $7 = a(1)^2 + b(1) + c$
 $\therefore a + b + c = 7$

Substituting $(2, 10)$ into the model gives $10 = a(2)^2 + b(2) + c$
 $\therefore 4a + 2b + c = 10$

Substituting $(3, 11)$ into the model gives $11 = a(3)^2 + b(3) + c$
 $\therefore 9a + 3b + c = 11$

So, we have the system of equations
$$\begin{cases} a + b + c = 7 \\ 4a + 2b + c = 10 \\ 9a + 3b + c = 11 \end{cases}$$

b

	a	b	c	d
1	1	1	1	7
2	4	2	1	10
3	9	3	1	11

1

SOLVE DELETE CLEAR EDIT

	X	Y	Z
1	-1	6	2

-1

REPEAT

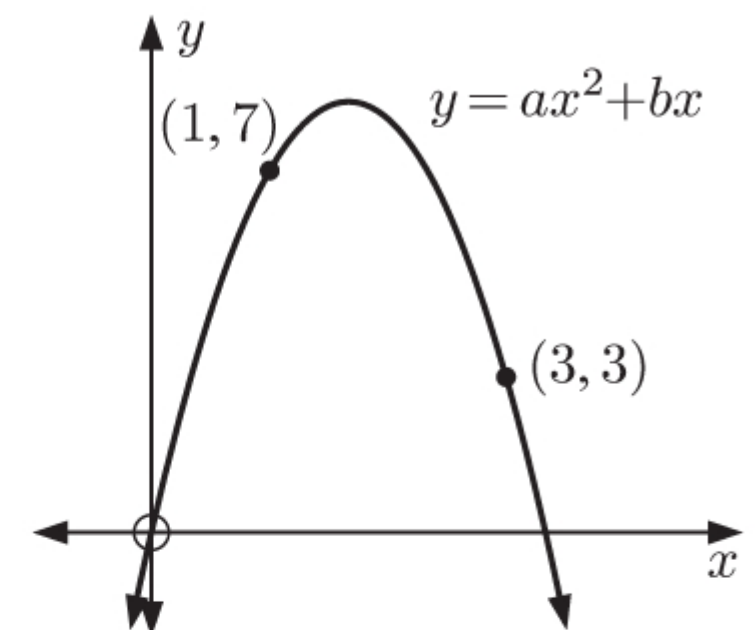
Solving the system of equations in **a** using technology, we find that $a = -1$, $b = 6$, and $c = 2$.

- c** The model is $y = -x^2 + 6x + 2$.

- 3 a** Substituting $(1, 7)$ into the model gives $7 = a(1)^2 + b(1)$
 $\therefore a + b = 7$

Substituting $(3, 3)$ into the model gives $3 = a(3)^2 + b(3)$
 $\therefore 9a + 3b = 3$

So, we have the system of equations
$$\begin{cases} a + b = 7 \\ 9a + 3b = 3 \end{cases}$$



	a	b	c
1	1	1	7
2	9	3	3

3

SOLVE DELETE CLEAR EDIT

	X	Y
1	-3	10

-3

REPEAT

Solving these equations simultaneously using technology, we find that $a = -3$ and $b = 10$.

The model is $y = -3x^2 + 10x$.

- b** Substituting (2, 2) into the model gives

$$2 = a(2)^3 + b(2)^2 + c(2)$$

$$\therefore 8a + 4b + 2c = 2$$

- Substituting (3, 6) into the model gives

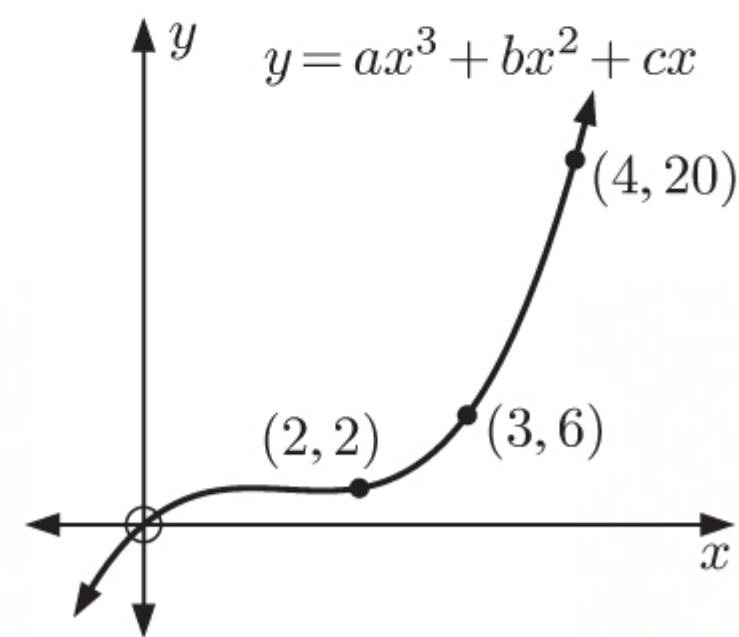
$$6 = a(3)^3 + b(3)^2 + c(3)$$

$$\therefore 27a + 9b + 3c = 6$$

- Substituting (4, 20) into the model gives

$$20 = a(4)^3 + b(4)^2 + c(4)$$

$$\therefore 64a + 16b + 4c = 20$$



So, we have the system of equations

$$\begin{cases} 8a + 4b + 2c = 2 \\ 27a + 9b + 3c = 6 \\ 64a + 16b + 4c = 20 \end{cases}.$$

	a	b	c	d
1	8	4	2	2
2	27	9	3	6
3	64	16	4	20

20

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	X	Y	Z
1	1	-4	5

1

REPEAT

Solving these equations simultaneously using technology, we find that $a = 1$, $b = -4$, and $c = 5$.

The model is $y = x^3 - 4x^2 + 5x$.

- c** Substituting (1, 5) into the model gives

$$5 = a(1) + \frac{b}{1} + c$$

$$\therefore a + b + c = 5$$

- Substituting (2, 4) into the model gives

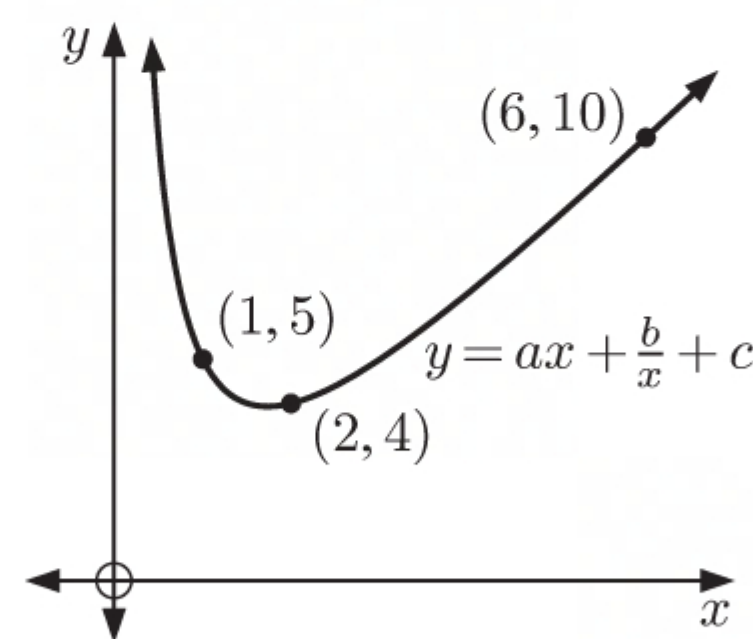
$$4 = a(2) + \frac{b}{2} + c$$

$$\therefore 2a + \frac{1}{2}b + c = 4$$

- Substituting (6, 10) into the model gives

$$10 = a(6) + \frac{b}{6} + c$$

$$\therefore 6a + \frac{1}{6}b + c = 10$$



So, we have the system of equations

$$\begin{cases} a + b + c = 5 \\ 2a + \frac{1}{2}b + c = 4 \\ 6a + \frac{1}{6}b + c = 10 \end{cases}.$$

	a	b	c	d
1	1	1	1	5
2	2	0.5	1	4
3	6	0.1666	1	10

10

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	X	Y	Z
1	2	6	-3

2

REPEAT

Solving these equations simultaneously using technology, we find that $a = 2$, $b = 6$, and $c = -3$.

The model is $y = 2x + \frac{6}{x} - 3$.

4 When $t = 1$, $D = 1.7$

$$\therefore 1.7 = a(1)^2 + b(1) + c\sqrt{1}$$

$$\therefore a + b + c = 1.7$$

When $t = 4$, $D = 7.2$

$$\therefore 7.2 = a(4)^2 + b(4) + c\sqrt{4}$$

$$\therefore 16a + 4b + 2c = 7.2$$

When $t = 9$, $D = 23.7$

$$\therefore 23.7 = a(9)^2 + b(9) + c\sqrt{9}$$

$$\therefore 81a + 9b + 3c = 23.7$$

So, we have the system of equations

$$\begin{cases} a + b + c = 1.7 \\ 16a + 4b + 2c = 7.2 \\ 81a + 9b + 3c = 23.7 \end{cases}$$

t (seconds)	1	4	9
D (metres)	1.7	7.2	23.7

	a	b	c	d
1	1	1	1	1.7
2	16	4	2	7.2
3	81	9	3	23.7

23.7

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	X	Y	Z
1	0.2	0.5	1

0.2

REPEAT

Solving these equations simultaneously using technology, we find that $a = 0.2$, $b = 0.5$, and $c = 1$.

5 a Substituting $(2, 16)$ into the model gives

$$16 = \frac{a}{2} + b(2)$$

$$\therefore \frac{1}{2}a + 2b = 16$$

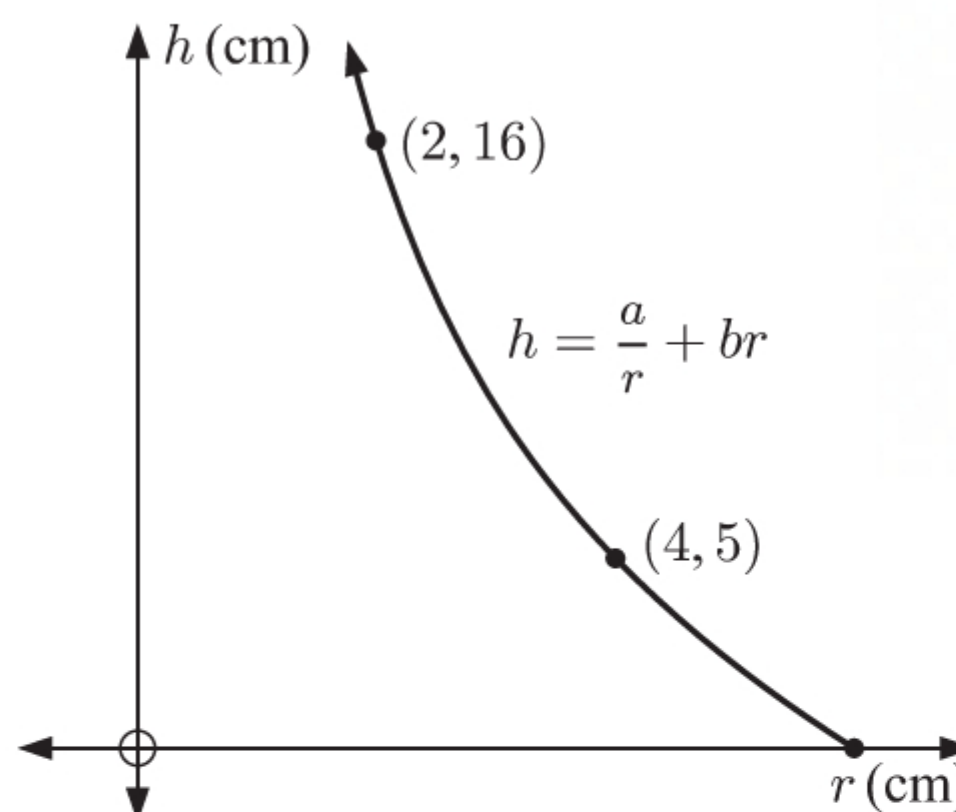
Substituting $(4, 5)$ into the model gives

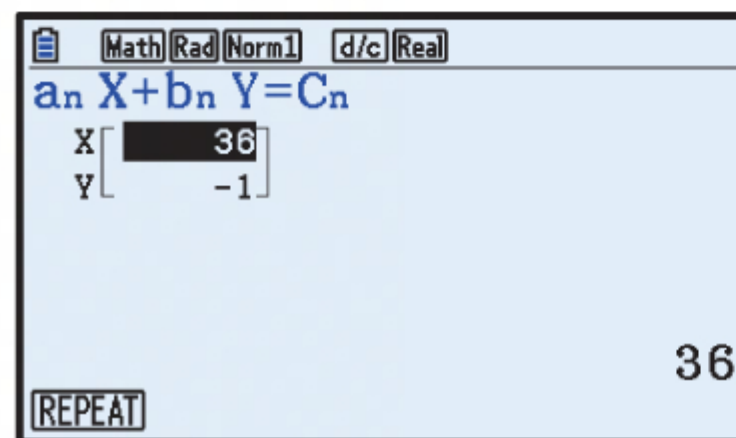
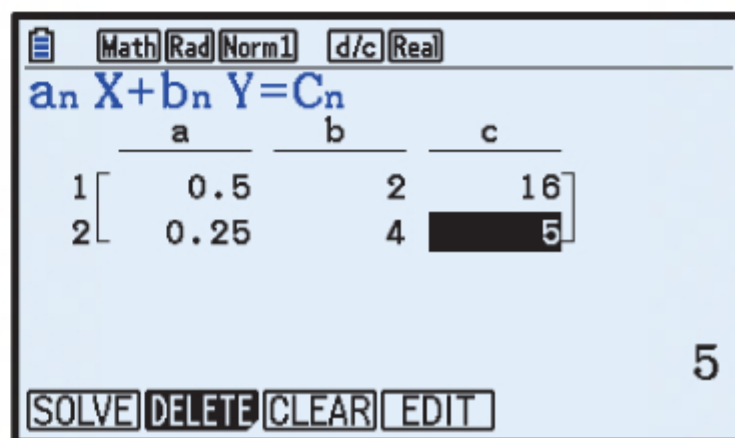
$$5 = \frac{a}{4} + b(4)$$

$$\therefore \frac{1}{4}a + 4b = 5$$

So, we have the system of equations

$$\begin{cases} \frac{1}{2}a + 2b = 16 \\ \frac{1}{4}a + 4b = 5 \end{cases}$$





Solving these equations simultaneously using technology, we find that $a = 36$, $b = -1$.

The model is $h = \frac{36}{r} - r$.

- b** The surface area of a cylinder is $A = 2\pi rh + 2\pi r^2$

$$\therefore 2\pi rh = A - 2\pi r^2$$

$$\therefore h = \frac{A}{2\pi r} - r$$

$$\therefore h = \frac{\frac{A}{2\pi}}{r} - r$$

which has the same form as the model in **a**. The model in **a** therefore seems reasonable.

$$\begin{aligned} \text{c } A &= 2\pi rh + 2\pi r^2 \\ &= 2\pi r \left(\frac{36}{r} - r \right) + 2\pi r^2 \quad \{\text{using a}\} \\ &= 72\pi - \cancel{2\pi r^2} + \cancel{2\pi r^2} \\ &= 72\pi \end{aligned}$$

\therefore the constant surface area of these particular cylinders is $72\pi \text{ cm}^2$.

$$\begin{aligned} \text{d } \text{When } r = 9, \quad h &= \frac{36}{9} - 9 \\ &= 4 - 9 \\ &= -5 \text{ cm} \end{aligned}$$

This implies that the cylinder has negative height, which is not reasonable.

- e** We require the radius and height to be positive.

$$\therefore r > 0 \quad \text{and} \quad h > 0$$

$$\frac{36}{r} - r > 0$$

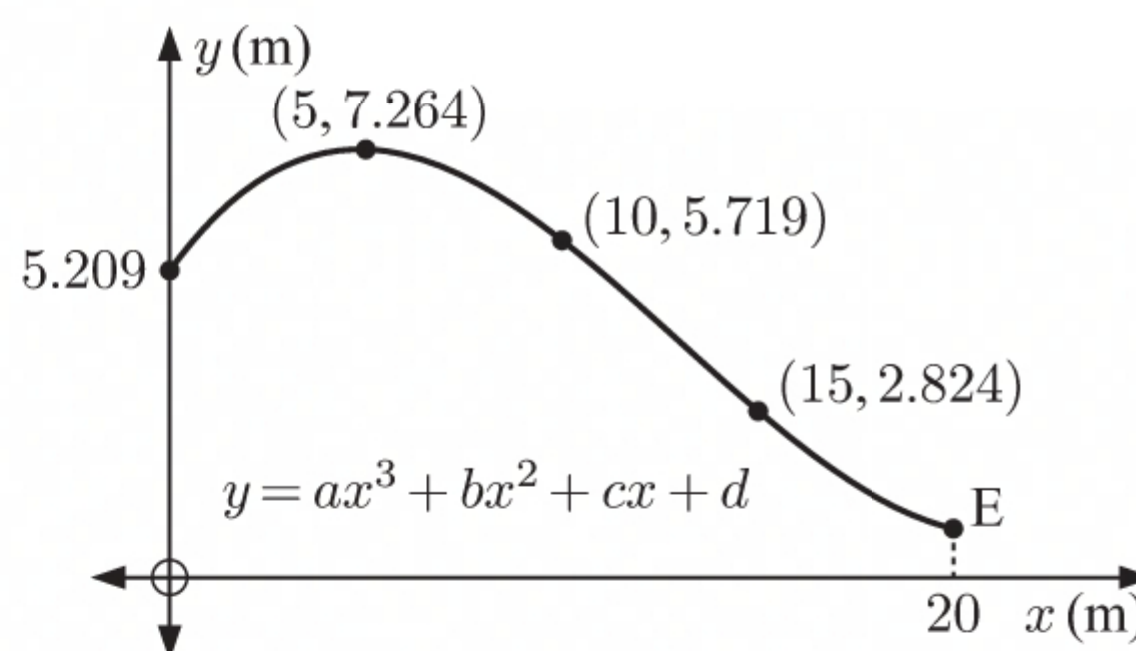
$$\therefore \frac{36}{r} > r$$

$$\therefore 36 > r^2 \quad \{r > 0\}$$

$$\therefore r < 6 \quad \{r > 0\}$$

\therefore it is reasonable to apply this model for $0 < r < 6$.

- 6 a When $x = 0$, $y = 5.209$.
 $\therefore 5.209 = a(0)^3 + b(0)^2 + c(0) + d$
 $\therefore d = 5.209$



- b Substituting $(5, 7.264)$ into the model gives $7.264 = a(5)^3 + b(5)^2 + c(5) + 5.209$
 $\therefore 125a + 25b + 5c = 2.055$

Substituting $(10, 5.719)$ into the model gives $5.719 = a(10)^3 + b(10)^2 + c(10) + 5.209$
 $\therefore 1000a + 100b + 10c = 0.51$

Substituting $(15, 2.824)$ into the model gives $2.824 = a(15)^3 + b(15)^2 + c(15) + 5.209$
 $\therefore 3375a + 225b + 15c = -2.385$

So, we have the system of equations
$$\begin{cases} 125a + 25b + 5c = 2.055 \\ 1000a + 100b + 10c = 0.51 \\ 3375a + 225b + 15c = -2.385 \end{cases}$$

	a	b	c	d
1	125	25	5	2.055
2	1000	100	10	0.51
3	3375	225	15	-2.385
				-2.385

	X	Y	Z
	$3E-3$	-0.117	0.921
			3×10^{-3}

Solving these equations simultaneously using technology, we find that $a = 0.003$, $b = -0.117$, and $c = 0.921$.

- c The model is $y = 0.003x^3 - 0.117x^2 + 0.921x + 5.209$
 When $x = 20$, $y = 0.003(20)^3 - 0.117(20)^2 + 0.921(20) + 5.209$
 $= 0.829$
 \therefore the height of the mound at point E is 0.829 m.

REVIEW SET 4A

- 1 a Ben can paddle 100 metres in 40 seconds.

We assume that Ben paddles at a constant speed of $\frac{100}{40} = \frac{5}{2} \text{ m s}^{-1}$.

Let t be the time in seconds it takes Ben to kayak D metres.

$$\text{Now speed} = \frac{\text{distance}}{\text{time}}$$

$$\therefore \text{distance} = \text{speed} \times \text{time}$$

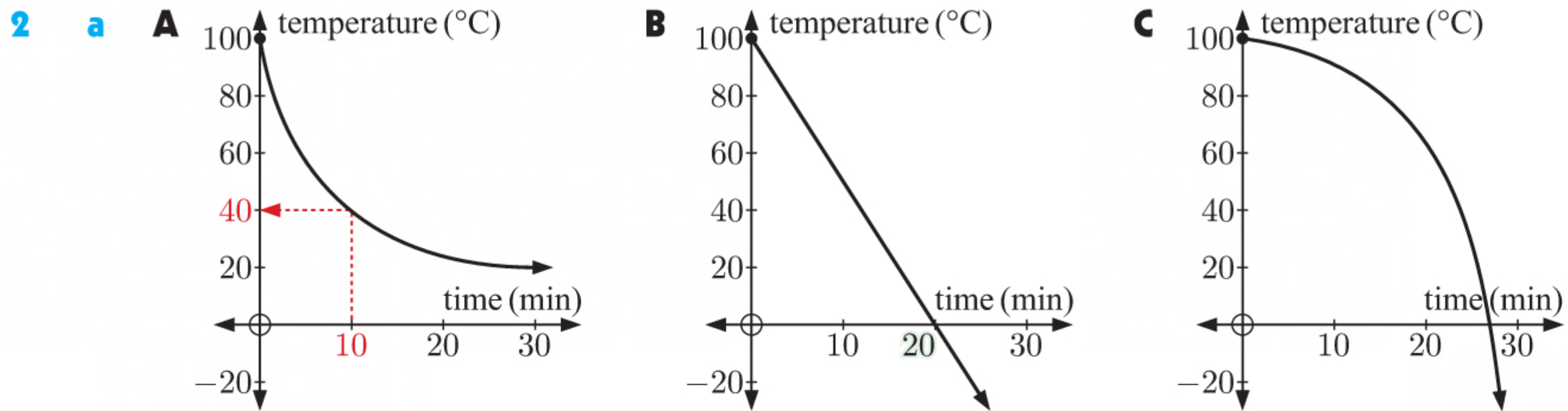
$$\therefore D = \frac{5}{2}t \text{ m}$$

- b** 10 minutes = $60 \times 10 = 600$ seconds

$$\begin{aligned}\text{When } t = 600, \quad D &= \frac{5}{2}(600) \\ &= 1500\end{aligned}$$

\therefore we predict that Ben can kayak 1500 m in 10 minutes.

- c** The actual distance will be less than our prediction. Ben will not be able to kayak at the same speed for 10 minutes as he can for 40 seconds.



The most appropriate model for the temperature of the water is **A**. Initially, the temperature will decrease quickly, then the rate of temperature decrease will slow as it approaches room temperature.

- b** Using model **A**, we predict that the temperature of the water after 10 minutes will be about 40°C .

- 3 a** The line passes through $(0, 160)$ and $(4, 140)$, so the gradient is $\frac{140 - 160}{4 - 0} = -5$. This means that oil is leaking out of the barrel at a rate of 5 L per minute. The A -intercept is 160. This means that the barrel initially contained 160 litres of oil.

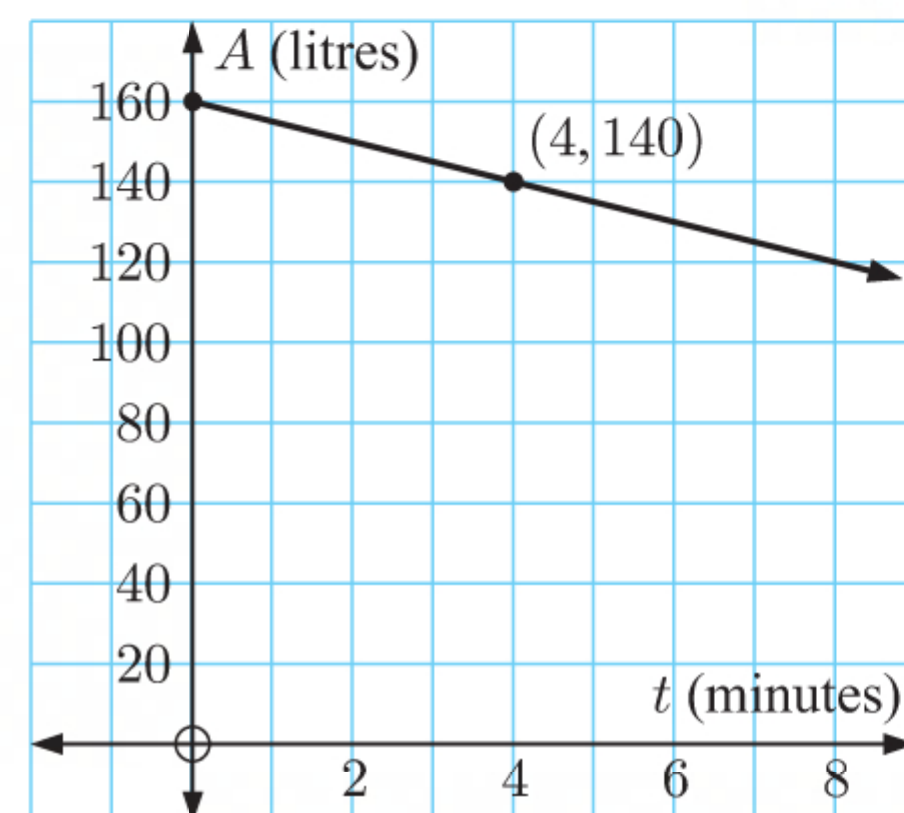
- b** Using **a**, the equation connecting A and t is $A = 160 - 5t$.

- c** When $t = 15$, $A = 160 - 5(15)$
 $= 160 - 75$
 $= 85$

There are 85 litres of oil left after 15 minutes.

- d** We require $t \geq 0$ and $A \geq 0$
 $\therefore 160 - 5t \geq 0$
 $\therefore 5t \leq 160$
 $\therefore t \leq 32$

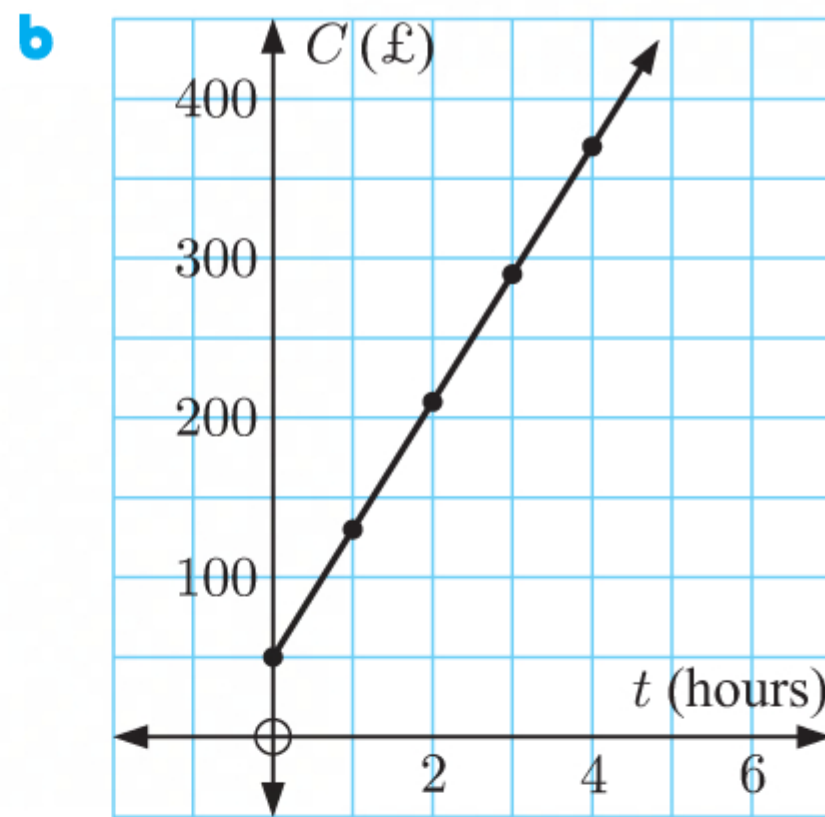
\therefore it is reasonable to apply this model for $0 \leq t \leq 32$.



4 a

Time (t hours)	0	1	2	3	4
Cost (£ C)	50	130	210	290	370

$\overset{\curvearrowright}{+80}$ $\overset{\curvearrowright}{+80}$ $\overset{\curvearrowright}{+80}$ $\overset{\curvearrowright}{+80}$



- c** The line passes through $(0, 50)$ and $(1, 130)$, so the gradient is $\frac{130 - 50}{1 - 0} = 80$.
 The C -intercept is 50.
 $\therefore C = 80t + 50$

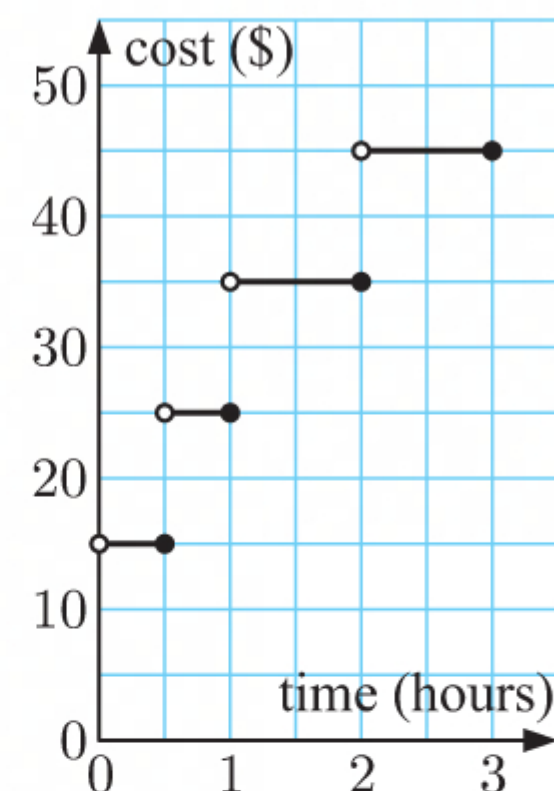
d When $t = 6$, $C = 80(6) + 50$
 $= 480 + 50$
 $= 530$

\therefore the cost of a job which takes 6 hours is £530.

- 5 a** Amy can wire $\frac{1}{3}$ of the house in one day, and Bernard can wire $\frac{1}{3}$ of the house in one day. Assuming they can work together without getting in each other's way, Amy and Bernard could wire $\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$ of the house in one day.
 \therefore it would take them $\frac{3}{2} = 1\frac{1}{2}$ days to wire the house if they work together.

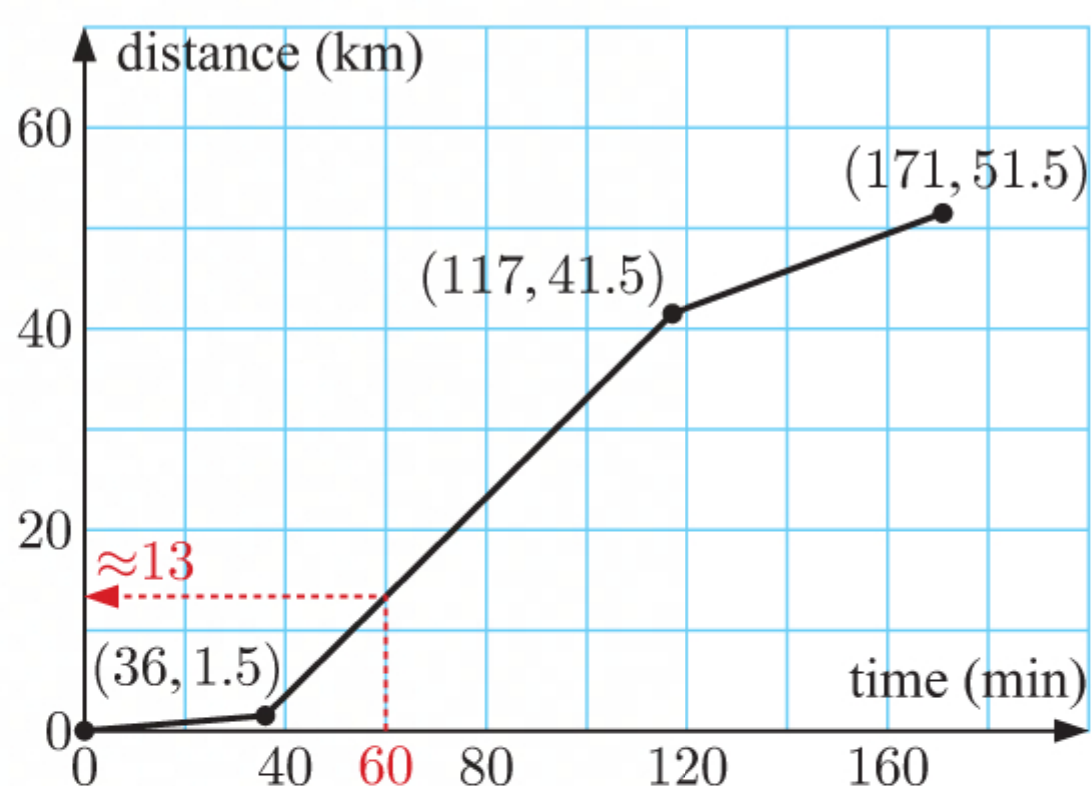
- b** The assumption is not reasonable. The work that Amy or Bernard does is likely to affect the other person, given the complexity of the task. The job will take more time than our prediction in **a**.

- 6 a i** From the graph, the cost of hiring the court for 45 minutes is \$25.
ii From the graph, the cost of hiring the court for 2 hours is \$35.
b Kate and Peggy can spend no more than \$15 each, or \$30 in total.
 The longest time Kate and Peggy can hire the court for is 1 hour.



7 a

Leg	Time (min)
Swim	36
Bicycle ride	81
Run	54



- b** We have assumed that Alana travels at a constant speed during each leg of the race.
c Using the graph in **a**, we estimate that Alana had travelled about 13 km after 1 hour.

- 8 a** Substituting $(1, 7)$ into the model gives

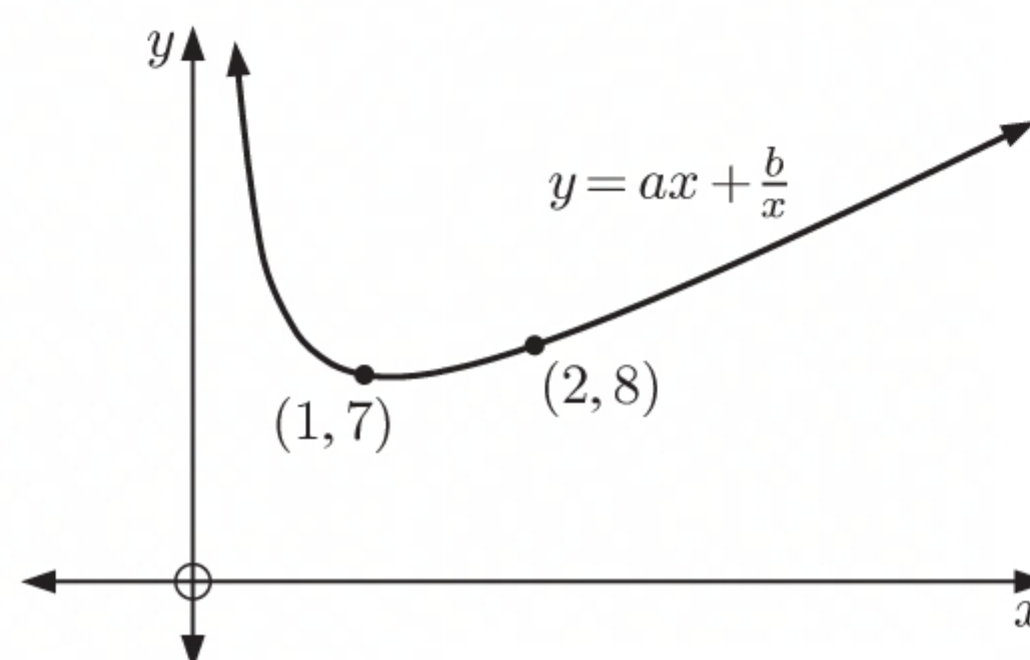
$$7 = a(1) + \frac{b}{1}$$

$$\therefore a + b = 7$$

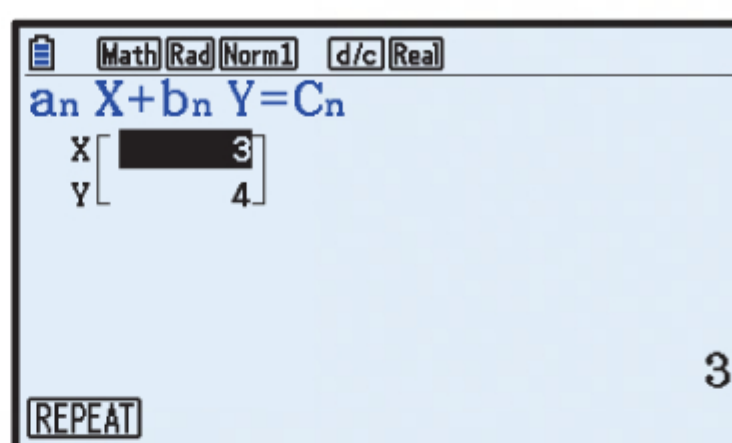
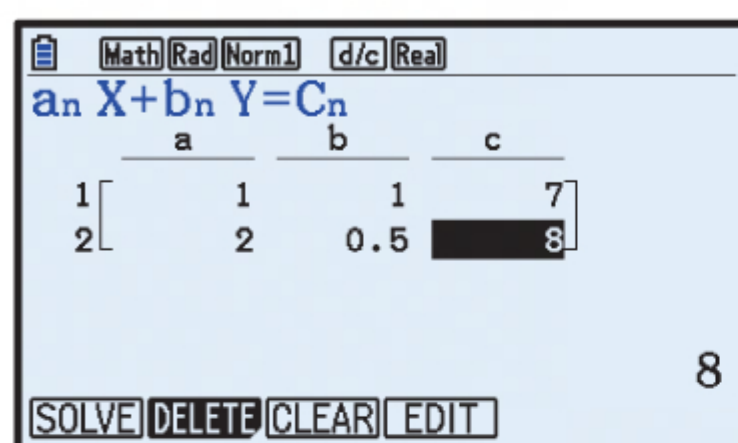
Substituting $(2, 8)$ into the model gives

$$8 = a(2) + \frac{b}{2}$$

$$\therefore 2a + \frac{1}{2}b = 8$$



So, we have the system of equations
$$\begin{cases} a + b = 7 \\ 2a + \frac{1}{2}b = 8 \end{cases}$$



Solving these equations simultaneously using technology, we find that $a = 3$ and $b = 4$.

The model is $y = 3x + \frac{4}{x}$.

- b** Substituting $(1, 1)$ into the model gives

$$1 = a(1)^3 + b(1)^2 + c(1)$$

$$\therefore a + b + c = 1$$

Substituting $(2, 6)$ into the model gives

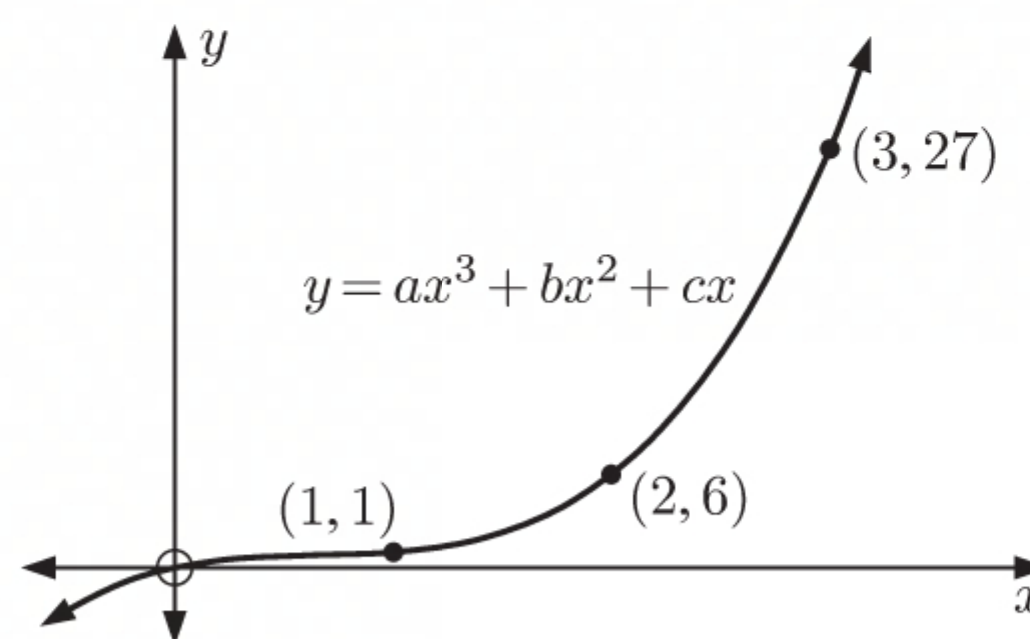
$$6 = a(2)^3 + b(2)^2 + c(2)$$

$$\therefore 8a + 4b + 2c = 6$$

Substituting $(3, 27)$ into the model gives

$$27 = a(3)^3 + b(3)^2 + c(3)$$

$$\therefore 27a + 9b + 3c = 27$$



So, we have the system of equations
$$\begin{cases} a + b + c = 1 \\ 8a + 4b + 2c = 6 \\ 27a + 9b + 3c = 27 \end{cases}.$$

	a	b	c	d
1	1	1	1	1
2	8	4	2	6
3	27	9	3	27

27

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	X	Y	Z
	2	-4	3

2

REPEAT

Solving these equations simultaneously using technology, we find that $a = 2$, $b = -4$, and $c = 3$.

The model is $y = 2x^3 - 4x^2 + 3x$.

- 9 a Substituting $(10, 19)$ into the model gives

$$19 = a(10)^2 + b(10)$$

$$\therefore 100a + 10b = 19$$

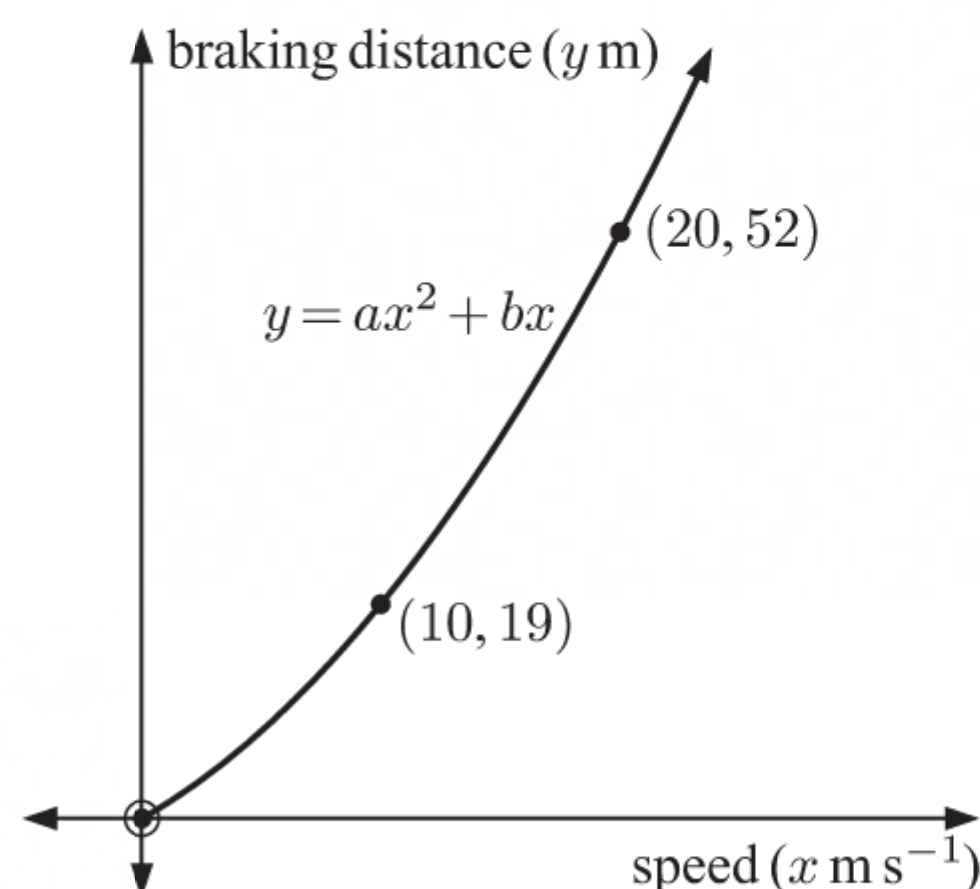
Substituting $(20, 52)$ into the model gives

$$52 = a(20)^2 + b(20)$$

$$\therefore 400a + 20b = 52$$

So, we have the system of equations

$$\begin{cases} 100a + 10b = 19 \\ 400a + 20b = 52 \end{cases}.$$



	a	b	c
1	100	10	19
2	400	20	52

52

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	X	Y
	0.07	1.2

$\frac{7}{100}$

REPEAT

Solving these equations simultaneously using technology, we find that $a = \frac{7}{100}$ and $b = \frac{6}{5}$.

- b The model is $y = \frac{7}{100}x^2 + \frac{6}{5}x$

$$\begin{aligned} \text{When } x = 30, \quad y &= \frac{7}{100}(30)^2 + \frac{6}{5}(30) \\ &= 99 \end{aligned}$$

\therefore we predict that Daniel's braking distance when he is travelling at 30 m s^{-1} is 99 metres.

- c No, we cannot use this model to predict the braking distance for a different person. Braking distance is dependent on the car and the reaction time of the driver.

REVIEW SET 4B

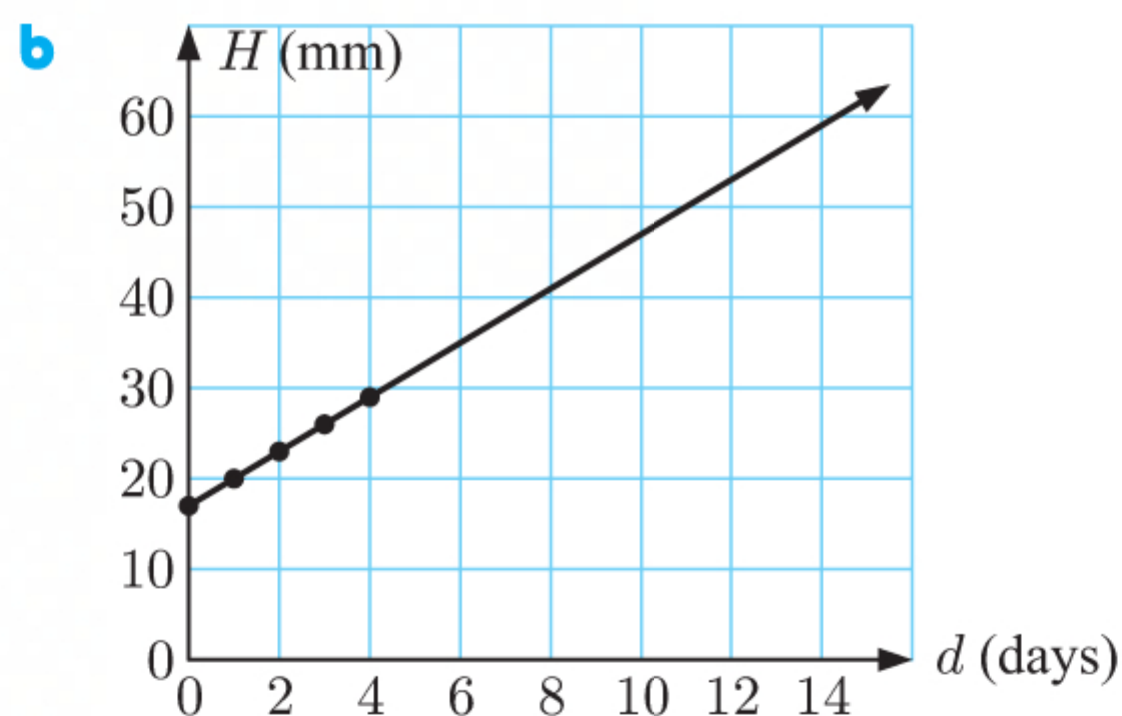
- 1 Todd can weed $\frac{1}{3}$ of the garden each hour, and Sophie can weed $\frac{1}{3.5} = \frac{2}{7}$ of the garden each hour. We assume Todd and Sophie can work together without getting in each other's way. So, working together they will weed $\frac{1}{3} + \frac{2}{7} = \frac{13}{21}$ of the garden each hour.

\therefore it would take them $\frac{21}{13} = 1\frac{8}{13}$ hours ≈ 1 hour 37 minutes to weed the garden together.

2 a

d (days)	0	1	2	3	4
H (mm)	17	20	23	26	29

$\overset{\text{red}}{\curvearrowright} \quad \overset{\text{red}}{\curvearrowright} \quad \overset{\text{red}}{\curvearrowright} \quad \overset{\text{red}}{\curvearrowright}$
 $\text{+3} \quad \text{+3} \quad \text{+3} \quad \text{+3}$



- c The line passes through $(0, 17)$ and $(1, 20)$, so the gradient is $\frac{20 - 17}{1 - 0} = 3$.

The H -intercept is 17.

$$\therefore H = 3d + 17$$

- d When $d = 12$, $H = 3(12) + 17$
 $= 36 + 17$
 $= 53$

\therefore after 12 days, the height of the lawn is 53 mm.

- e When the lawn is 8 cm high, $H = 80$
 $\therefore 3d + 17 = 80$
 $\therefore 3d = 63$
 $\therefore d = 21$

\therefore Rohan mows the lawn every 21 days (or 3 weeks).

- 3 a Yes, these are reasonable assumptions. Not all of the wood in a tree is usable, and assuming that the trees are cylindrical makes it easier to perform calculations.

b $V = 80\% \times \pi r^2 h$
 $= 0.8 \times \pi \left(\frac{0.45}{2}\right)^2 \times h \text{ m}^3$
 $= 0.8 \times \pi (0.225)^2 h \text{ m}^3$
 $= \frac{81\pi h}{2000} \text{ m}^3$

- c When $h = 15$, $V = \frac{81\pi(15)}{2000}$
 $\approx 1.91 \text{ m}^3$

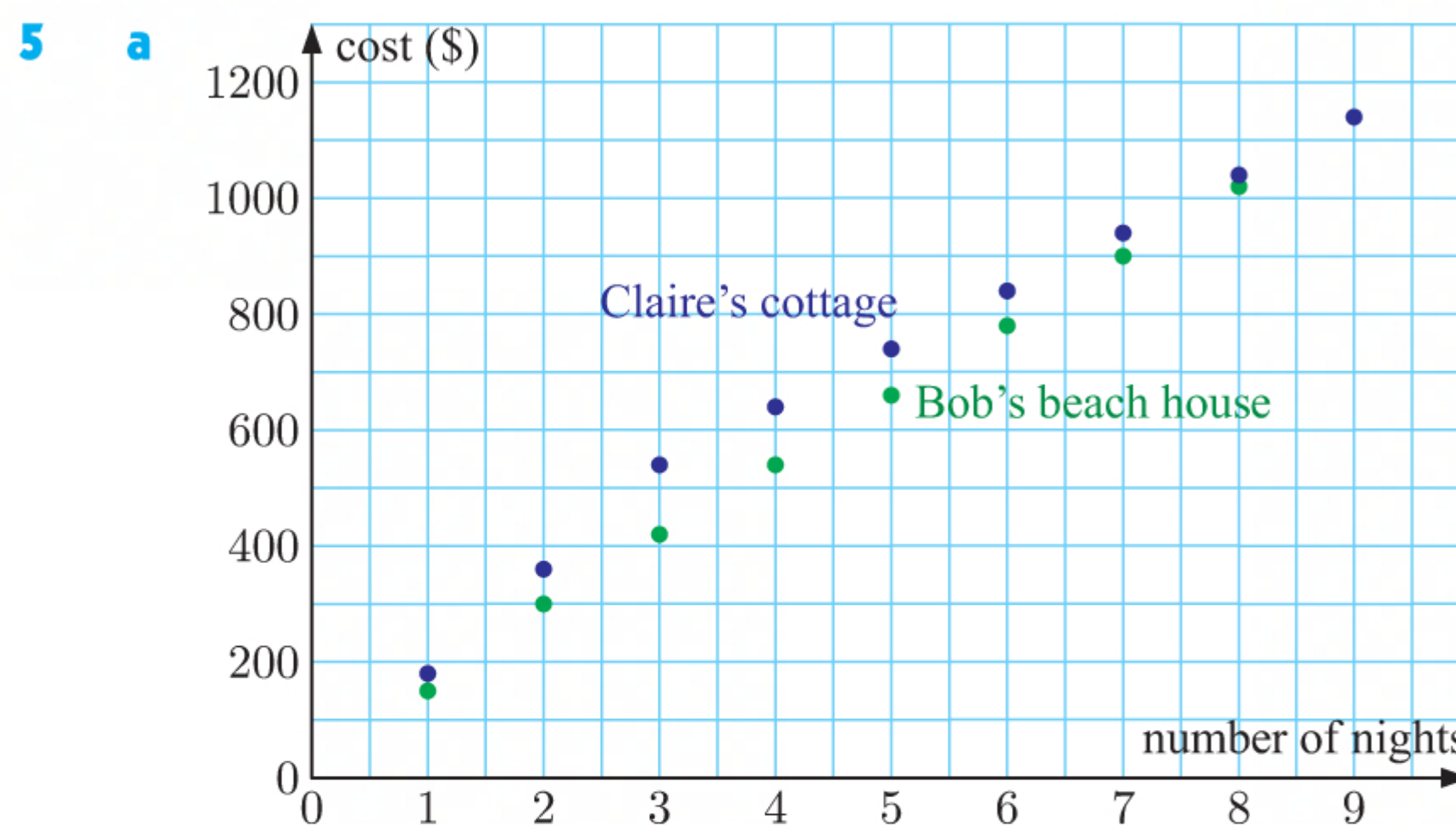
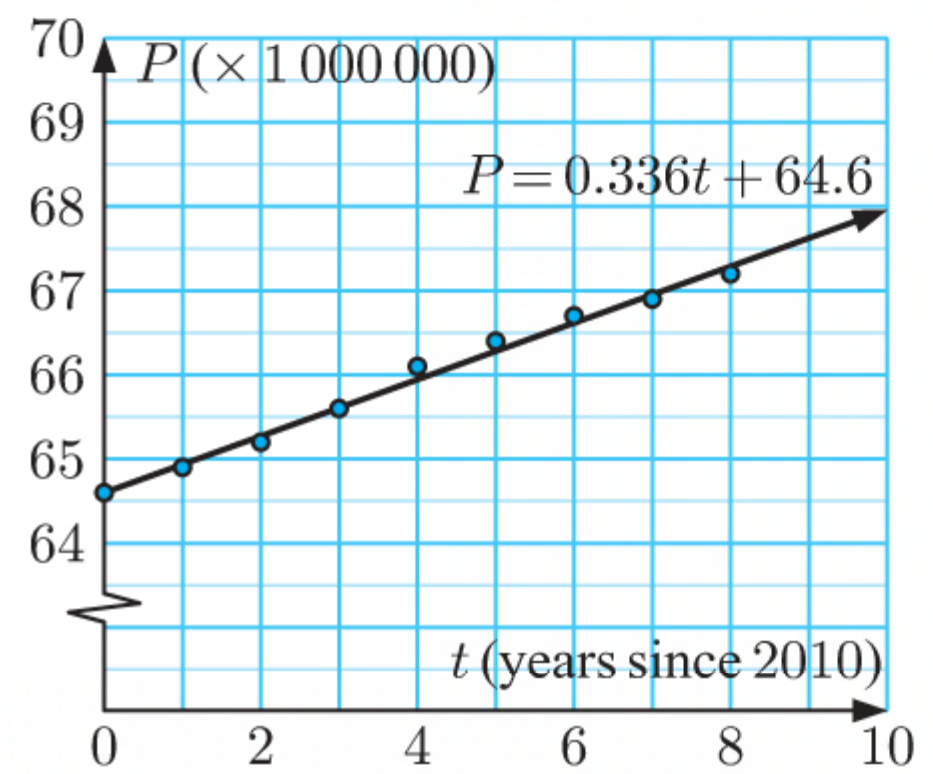
\therefore we predict that there is about 1.91 m^3 of usable wood in a 15 m high tree.

- 4 a** The points do not lie exactly on the line.
 \therefore the model is approximate.

b When $P = 75$, $75 = 0.336t + 64.6$
 $\therefore 0.336t = 10.4$
 $\therefore t \approx 30.96$

\therefore the model predicts that France's population will reach 75 million in the 31st year after the start of 2010, which corresponds to the year 2040.

This is an extrapolation, so this prediction is not likely to be reasonable.



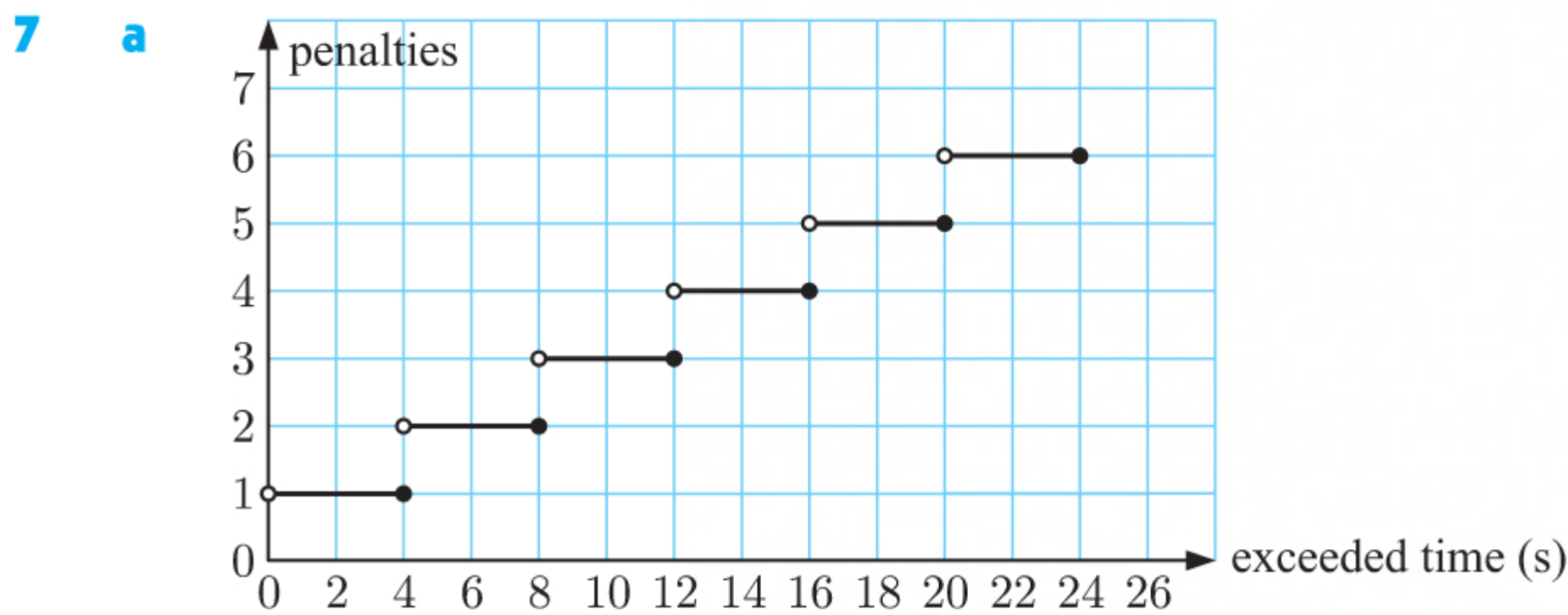
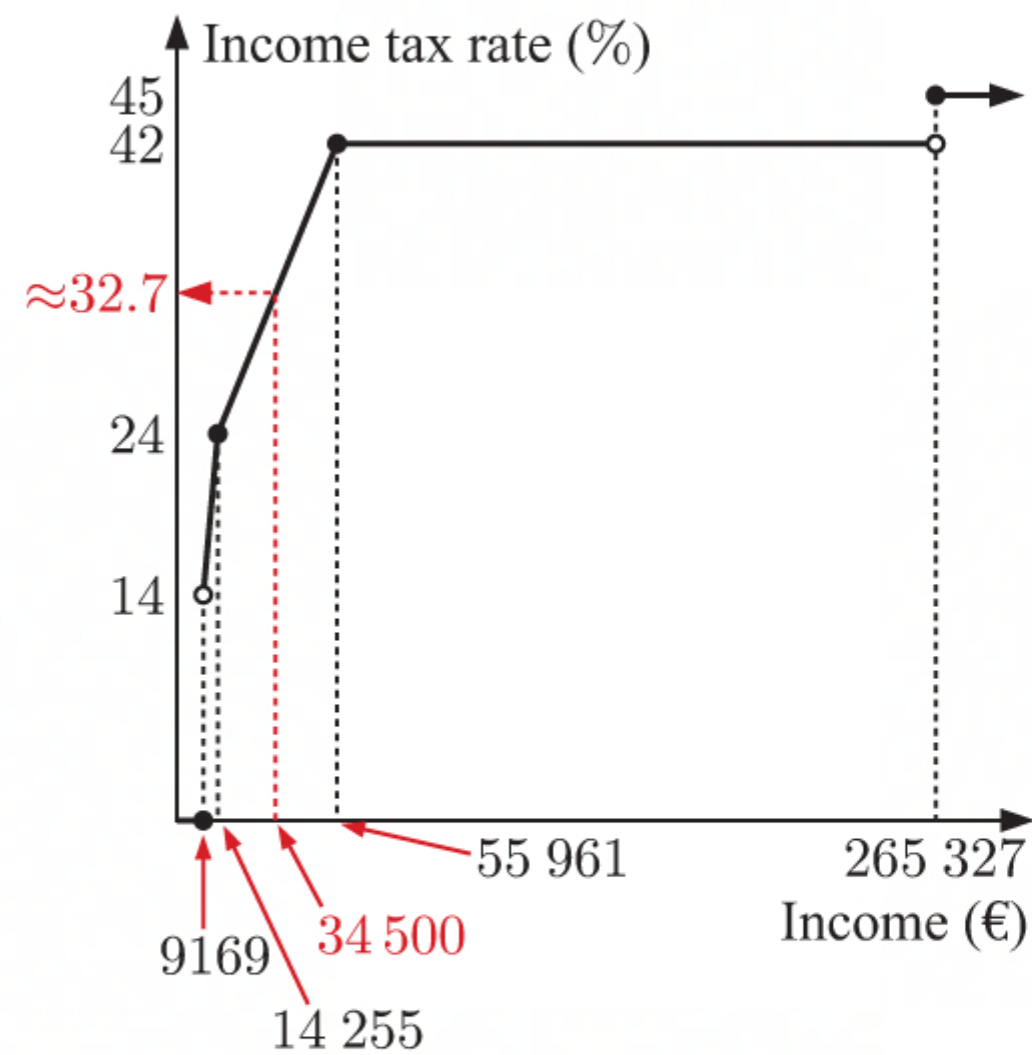
- b i** Cost of staying for 4 nights at Bob's Beach House $= 2 \times \$150 + 2 \times \120
 $= \$540$
- ii** Cost of staying for 4 nights at Claire's Cottage $= 3 \times \$180 + 1 \times \100
 $= \$640$
- c** From the graph, the cost is the same at each place when staying for 9 nights.
- d** Cost of staying for 8 nights at Bob's Beach House $= 2 \times \$150 + 6 \times \120
 $= \$1020$

$$\text{Cost of staying for 8 nights at Claire's Cottage} = 3 \times \$180 + 5 \times \$100 \\ = \$1040$$

It is $\$1040 - \$1020 = \$20$ cheaper to stay 8 nights at Bob's Beach House.

- 6 a** This is a piecewise linear model.
- b** Using the graph, we find that the rate of income tax payable for Gert-Jan, who has an annual income of €34 500, is about 32.7%.

$$\begin{aligned} \text{€}108\,609 &= \text{€}55\,961 + \text{€}52\,648 \\ \therefore \text{tax payable} &= \text{€}14\,729.32 + 0.42 \times \text{€}52\,648 \\ &= \text{€}14\,729.32 + \text{€}22\,112.16 \\ &= \text{€}36\,841.48 \end{aligned}$$



- b i** 83.1 seconds is $83.1 - 82 = 1.1$ seconds over the time allowed. The rider would be given 1 time penalty.
- ii** 87.9 seconds is $87.9 - 82 = 5.9$ seconds over the time allowed. The rider would be given 2 time penalties.
- iii** 81.5 seconds is less than the time allowed, so the rider would not be given any time penalties.
- iv** 96.3 seconds is $96.3 - 82 = 14.3$ seconds over the time allowed. The rider would be given 4 time penalties.

- 8 a** Substituting (1, 3) into the model gives

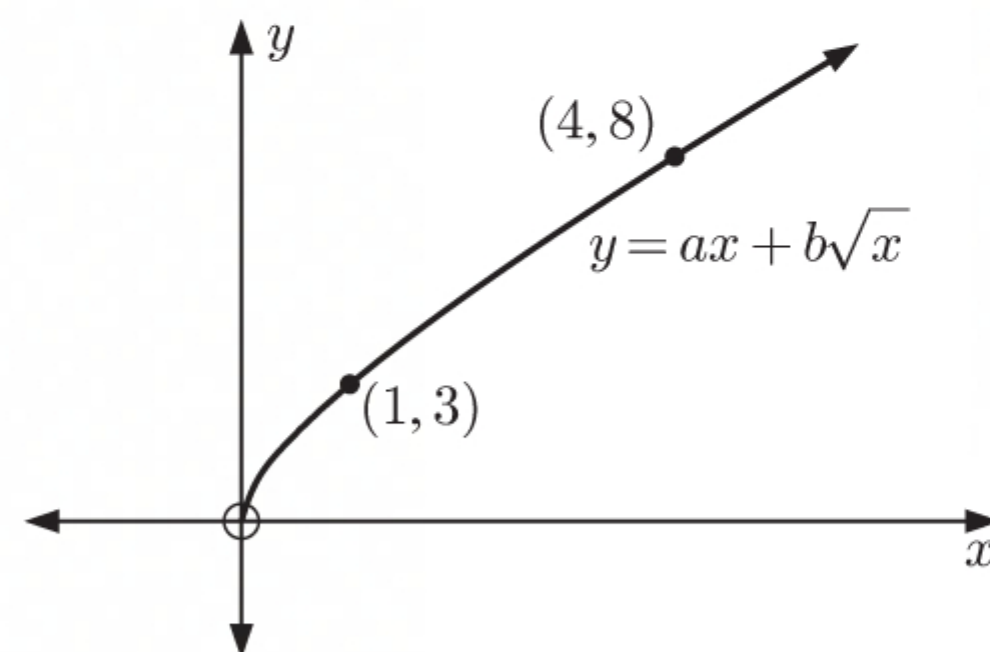
$$\begin{aligned} 3 &= a(1) + b\sqrt{1} \\ \therefore a + b &= 3 \end{aligned}$$

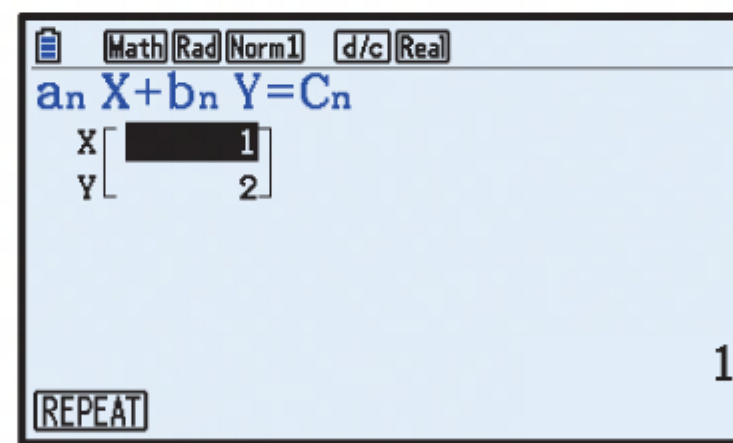
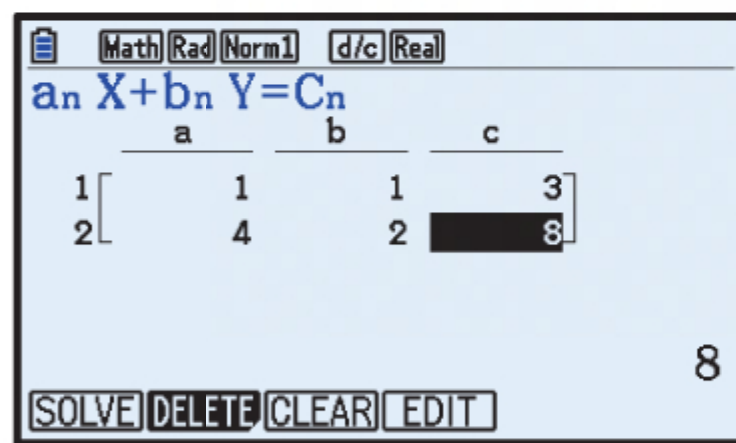
Substituting (4, 8) into the model gives

$$\begin{aligned} 8 &= a(4) + b\sqrt{4} \\ \therefore 4a + 2b &= 8 \end{aligned}$$

So, we have the system of equations

$$\begin{cases} a + b = 3 \\ 4a + 2b = 8 \end{cases}$$





Solving these equations simultaneously using technology, we find that $a = 1$ and $b = 2$.

The model is $y = x + 2\sqrt{x}$.

- b** Substituting $(1, 12)$ into the model gives

$$12 = a(1)^2 + b(1) + \frac{c}{1}$$

$$\therefore a + b + c = 12$$

Substituting $(2, 10)$ into the model gives

$$10 = a(2)^2 + b(2) + \frac{c}{2}$$

$$\therefore 4a + 2b + \frac{1}{2}c = 10$$

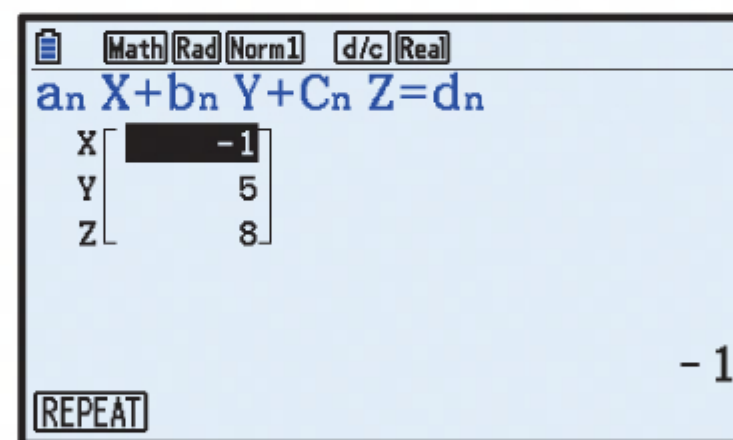
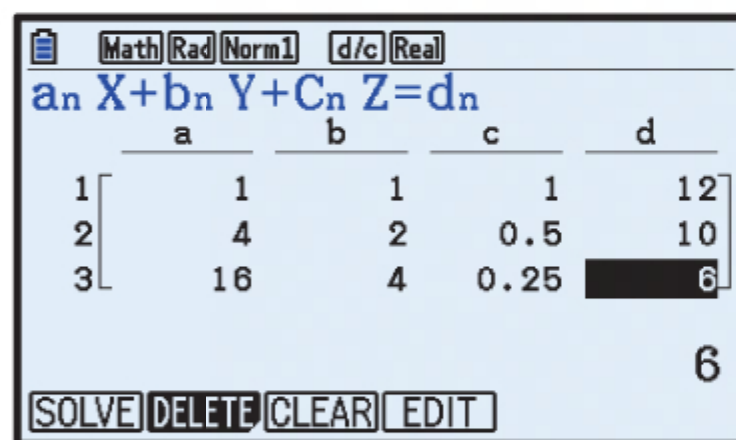
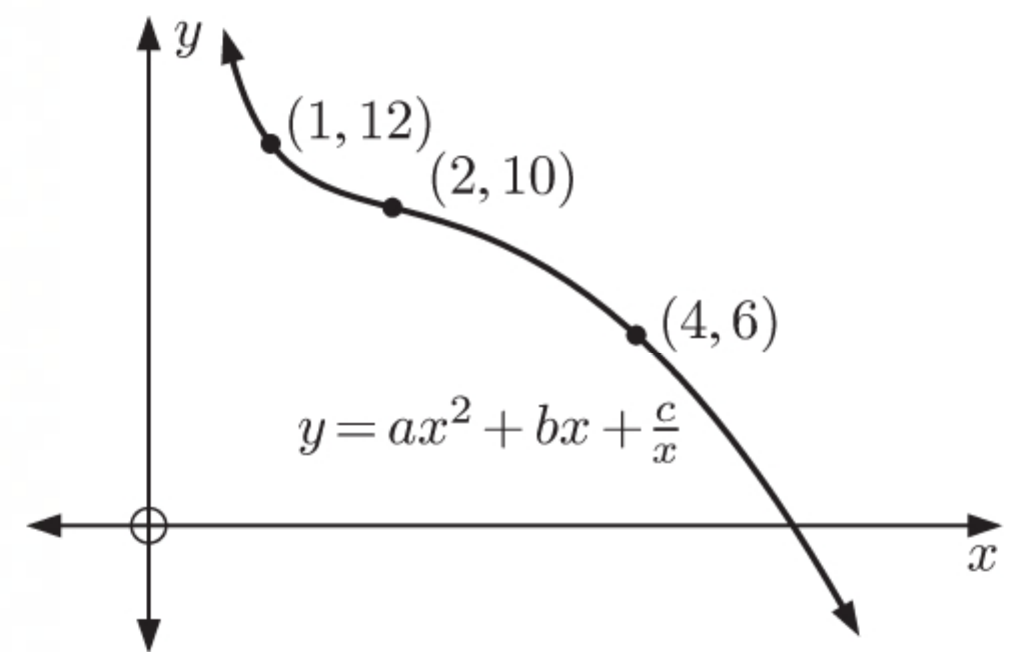
Substituting $(4, 6)$ into the model gives

$$6 = a(4)^2 + b(4) + \frac{c}{4}$$

$$\therefore 16a + 4b + \frac{1}{4}c = 6$$

So, we have the system of equations

$$\begin{cases} a + b + c = 12 \\ 4a + 2b + \frac{1}{2}c = 10 \\ 16a + 4b + \frac{1}{4}c = 6 \end{cases}$$

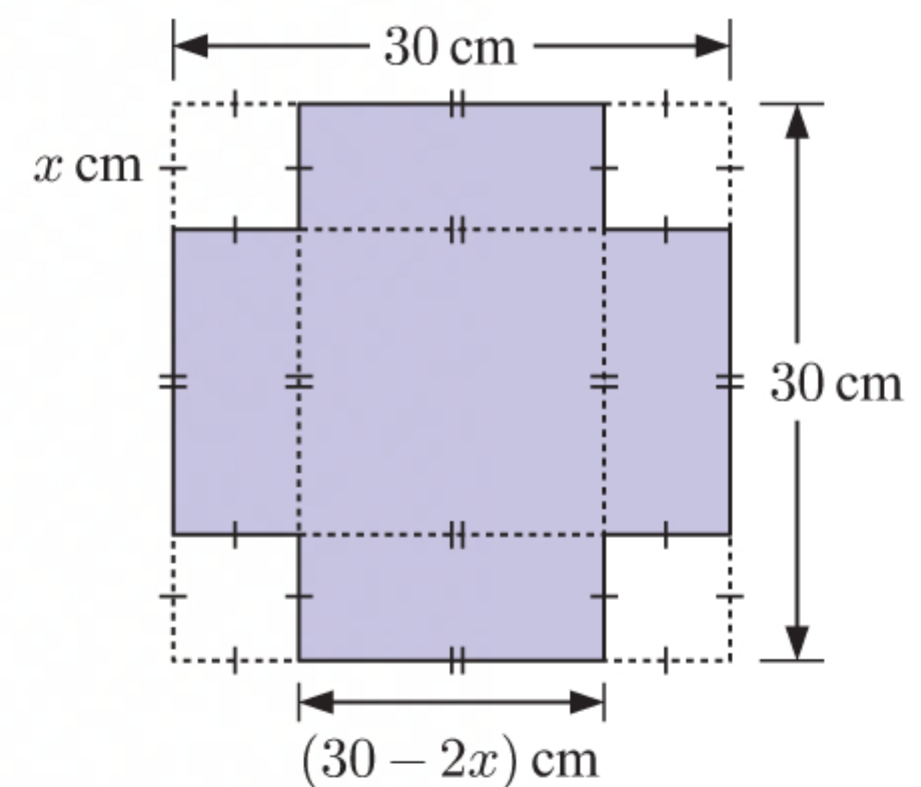
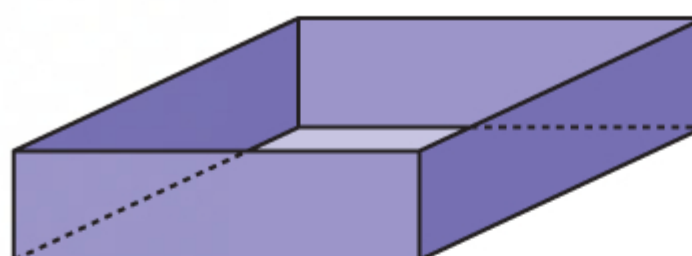


Solving these equations simultaneously using technology, we find that $a = -1$, $b = 5$, and $c = 8$.

The model is $y = -x^2 + 5x + \frac{8}{x}$.

- 9 a** If $x = 0$, then we are not cutting out squares from each corner, and the sheet of metal cannot be folded into a tray.

$$\begin{aligned} \text{The volume } V &= a(0)^3 + b(0)^2 + c(0) + d = 0 \\ \therefore d &= 0 \end{aligned}$$



b When $x = 2$, $V = 1352$
 $\therefore 1352 = a(2)^3 + b(2)^2 + c(2)$
 $\therefore 8a + 4b + 2c = 1352$

When $x = 5$, $V = 2000$
 $\therefore 2000 = a(5)^3 + b(5)^2 + c(5)$
 $\therefore 125a + 25b + 5c = 2000$

When $x = 10$, $V = 1000$
 $\therefore 1000 = a(10)^3 + b(10)^2 + c(10)$
 $\therefore 1000a + 100b + 10c = 1000$

So, we have the system of equations
$$\begin{cases} 8a + 4b + 2c = 1352 \\ 125a + 25b + 5c = 2000 \\ 1000a + 100b + 10c = 1000 \end{cases}$$

x (cm)	2	5	10
V (cm ³)	1352	2000	1000

	a	b	c	d
1	8	4	2	1352
2	125	25	5	2000
3	1000	100	10	1000
				1000

	X	Y	Z
	4	-120	900

Solving these equations simultaneously using technology, we find that $a = 4$, $b = -120$, and $c = 900$.

c The volume of the rectangular prism = length \times width \times height
 $= (30 - 2x) \text{ cm} \times (30 - 2x) \text{ cm} \times x \text{ cm}$
 $= (900 - 120x + 4x^2) \times x \text{ cm}^3$
 $= 4x^3 - 120x^2 + 900x \text{ cm}^3$

which is in the form $V = ax^3 + bx^2 + cx + d \text{ cm}^3$, where $d = 0$.

So a model of this form is reasonable.

d We must make a cut greater than 0 cm otherwise the sheet cannot be folded.

We also require $30 - 2x > 0$ as the side lengths must be positive

$$\therefore 2x < 30$$

$$\therefore x < 15$$

\therefore it is reasonable to apply this model for $0 < x < 15$.

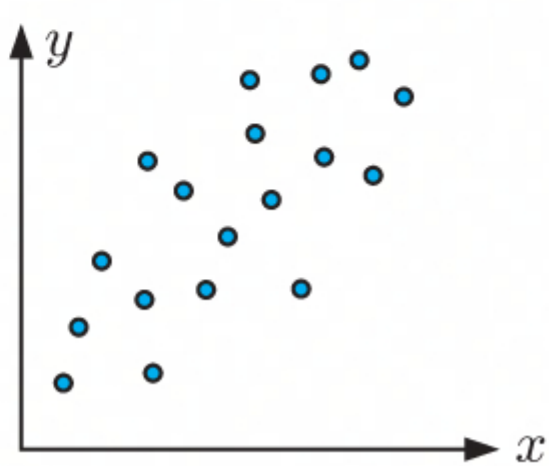
Chapter 5

BIVARIATE STATISTICS

EXERCISE 5A

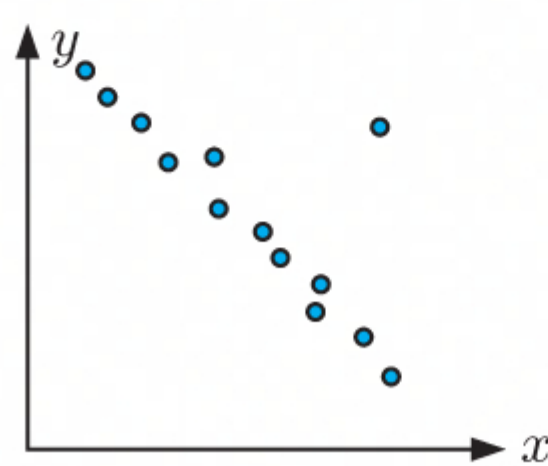
1

a



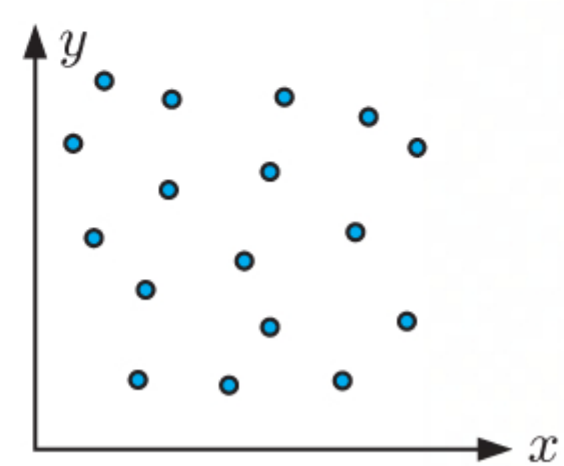
There is a weak, positive, linear correlation with no outliers.

b



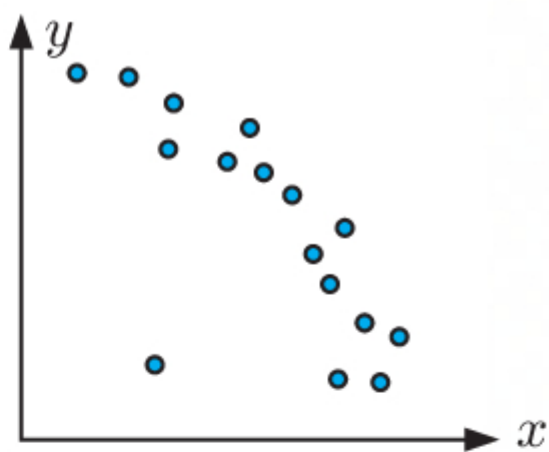
There is a strong, negative, linear correlation with one outlier.

c



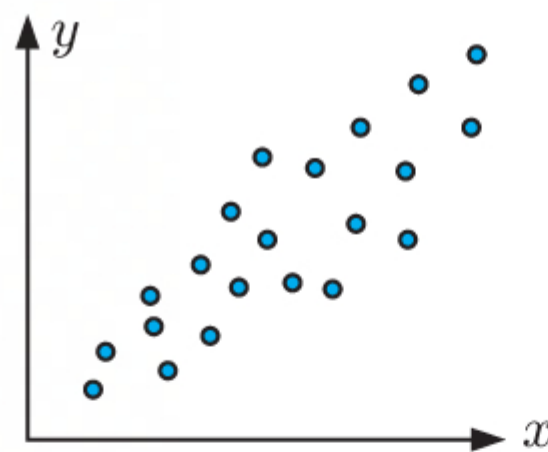
There is no correlation.

d



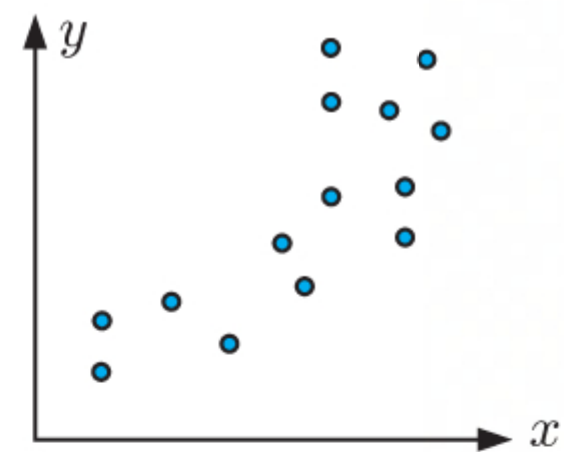
There is a strong, negative, non-linear correlation with one outlier.

e



There is a moderate, positive, linear correlation with no outliers.

f



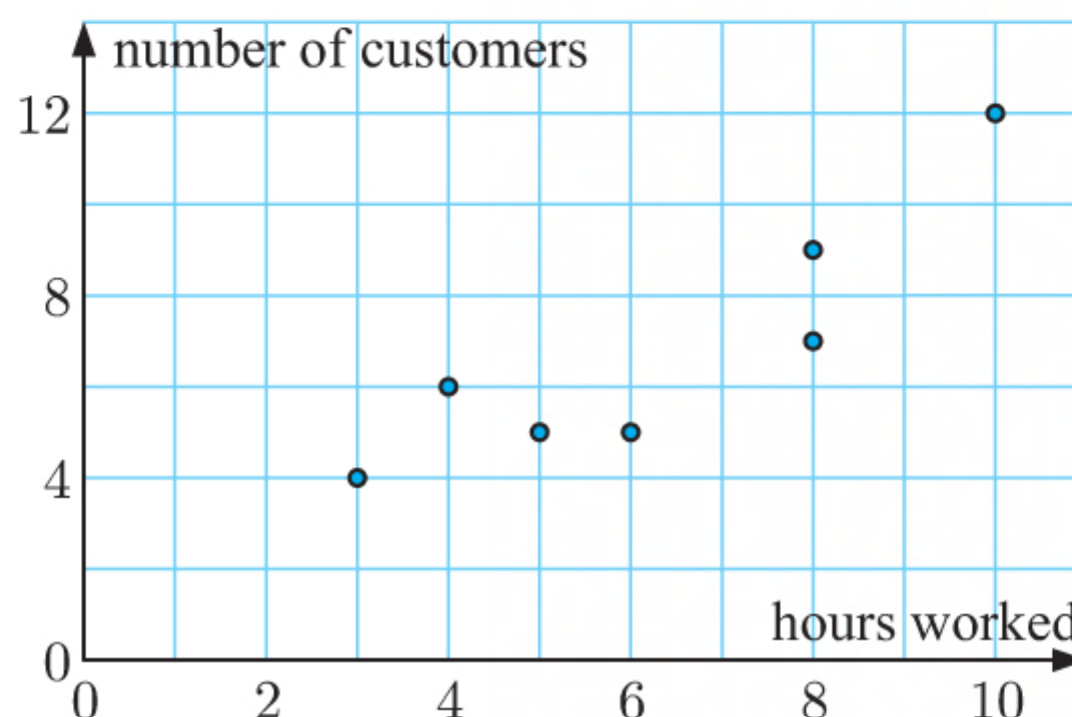
There is a weak, positive, non-linear correlation with no outliers.

2

Day	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Hours worked	8	4	5	10	8	3	6
Number of customers	9	6	5	12	7	4	5

- a *Hours worked* is the explanatory variable.
Number of customers is the response variable.

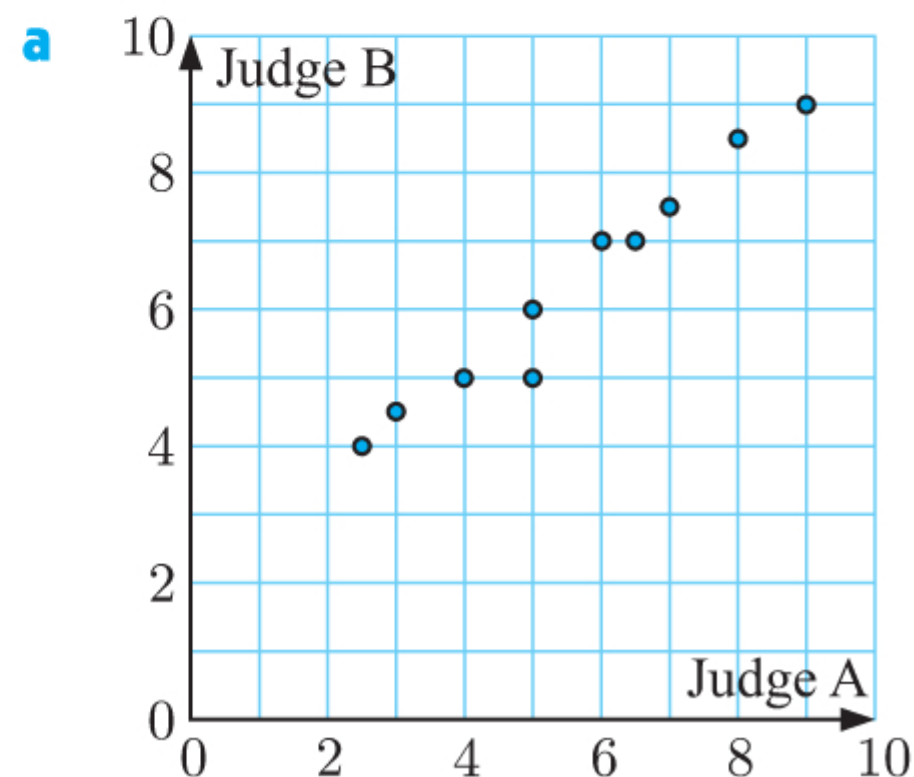
b



- c
- i Tiffany worked the same number of hours (8 hours) on Monday and Friday.
 - ii Tiffany had the same number of customers (5 customers) on Wednesday and Sunday.
- d The more hours that Tiffany works, the more customers she is likely to have, so we would expect a positive correlation between the variables.

3

Competitor	P	Q	R	S	T	U	V	W	X	Y
Judge A	5	6.5	8	9	4	2.5	7	5	6	3
Judge B	6	7	8.5	9	5	4	7.5	5	7	4.5



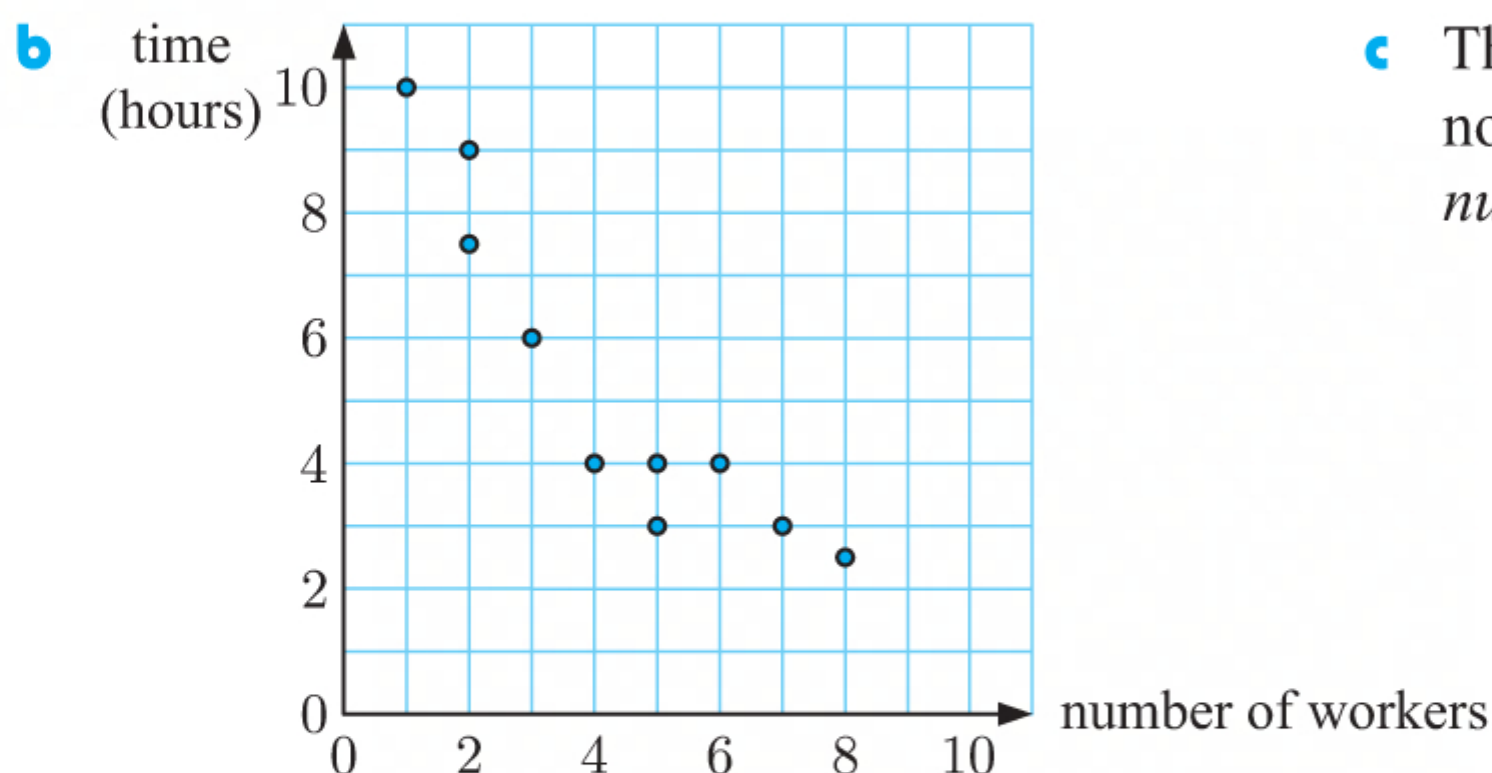
b There appears to be **strong, positive, linear** correlation between Judge A's scores and Judge B's scores. This means that as Judge A's scores increase, Judge B's scores **increase**.

c No, an increase in Judge A's scores are not likely to cause an increase in Judge B's scores. It is much more likely that both scores are related to the quality of the ice skaters' performances.

4

Job	A	B	C	D	E	F	G	H	I	J
Number of workers	5	3	8	2	5	6	1	4	2	7
Time (hours)	4	6	2.5	9	3	4	10	4	7.5	3

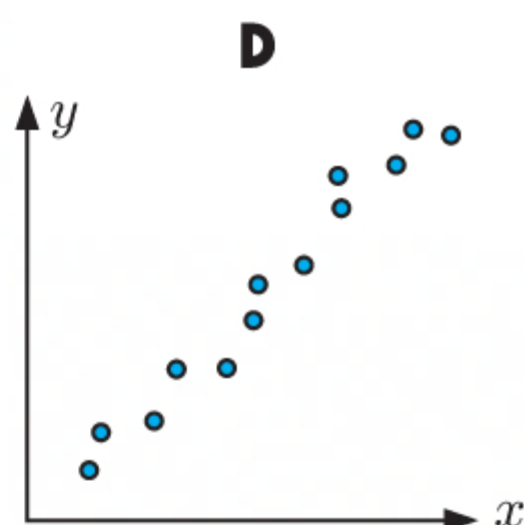
- a**
- i** Job G took the longest to complete (10 hours).
 - ii** Job C involved the most workers (8 workers).



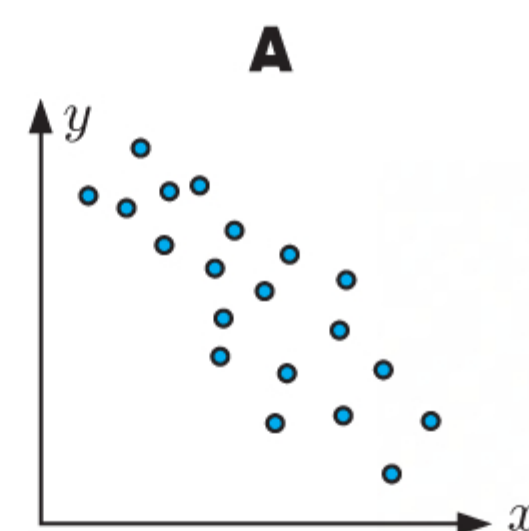
c There is a strong, negative, non-linear correlation between the *number of workers* and *time*.

5

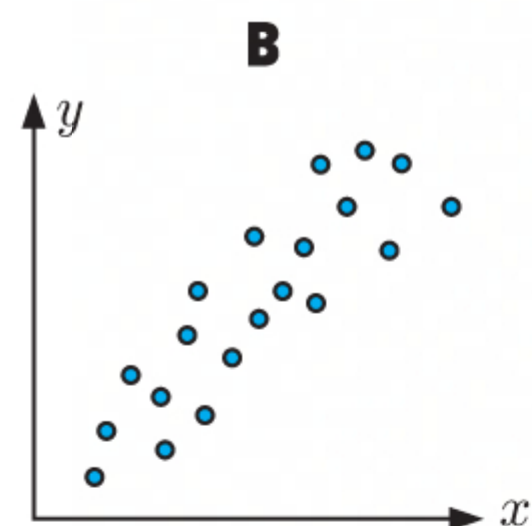
a x = the number of apples bought by customers
 y = the total cost of apples bought
 We expect strong, positive, linear correlation. This corresponds to **D**.



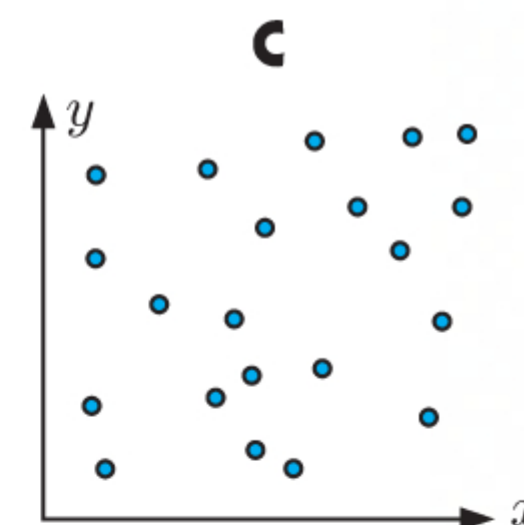
b x = the number of pushups a student can perform in one minute
 y = the time taken for a student to run 100 metres
 We expect moderate, negative, linear correlation. This corresponds to **A**.



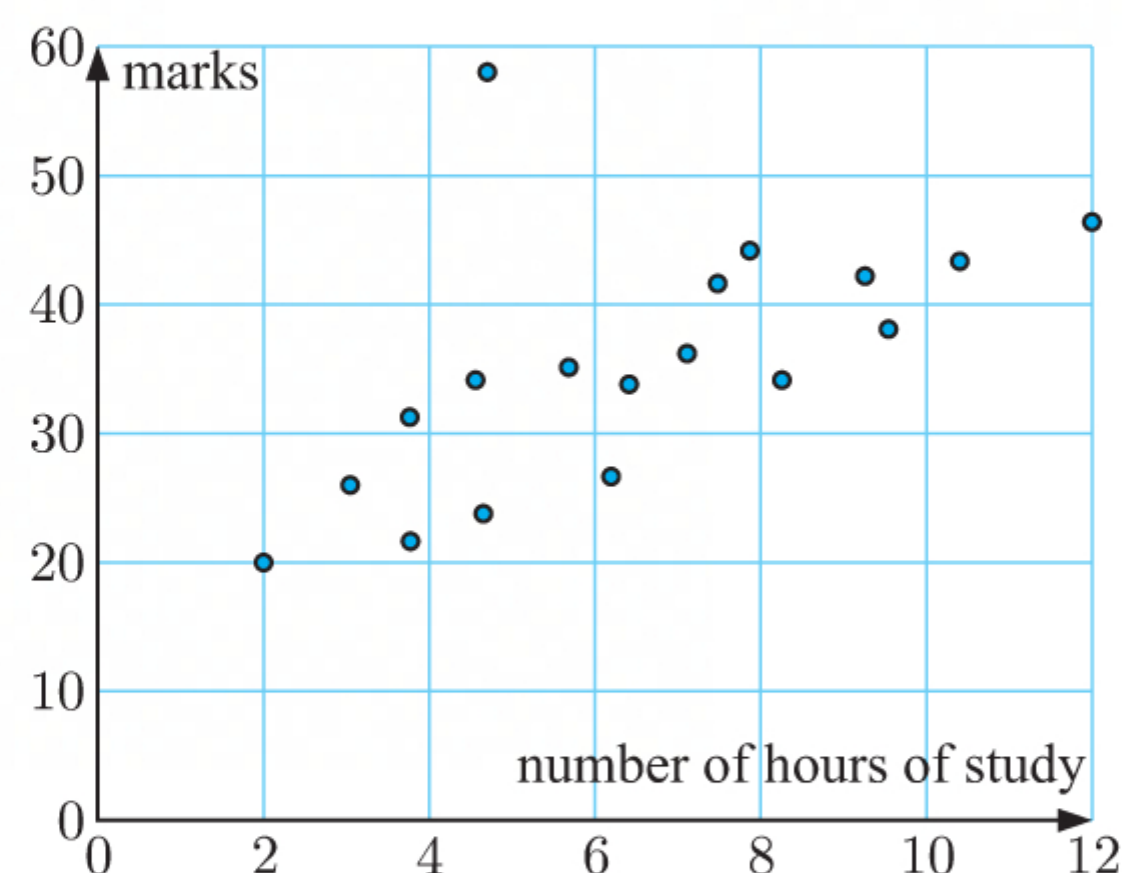
- c** x = the height of a person
 y = the weight of the person
 We expect moderate, positive, linear correlation. This corresponds to **B**.



- d** x = the distance a student travels to school
 y = the height of the student's uncle
 We expect no correlation. This corresponds to **C**.



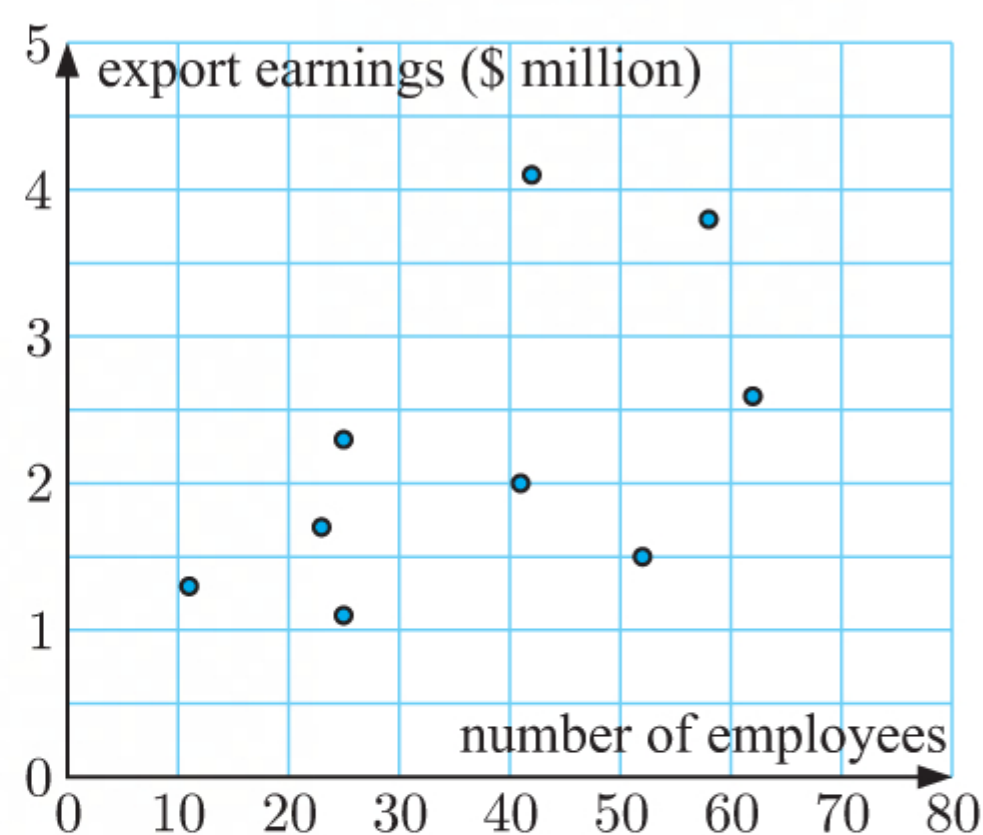
- 6**
- a** There is a moderate, positive, linear correlation between the *number of hours of study* and the *marks obtained*.
 - b** As the test is out of 50 marks and the outlier is greater than 50, we can assume it is an error and discard it.
 - c** Yes, this is a causal relationship as spending more time studying for the test is likely to cause a higher mark.

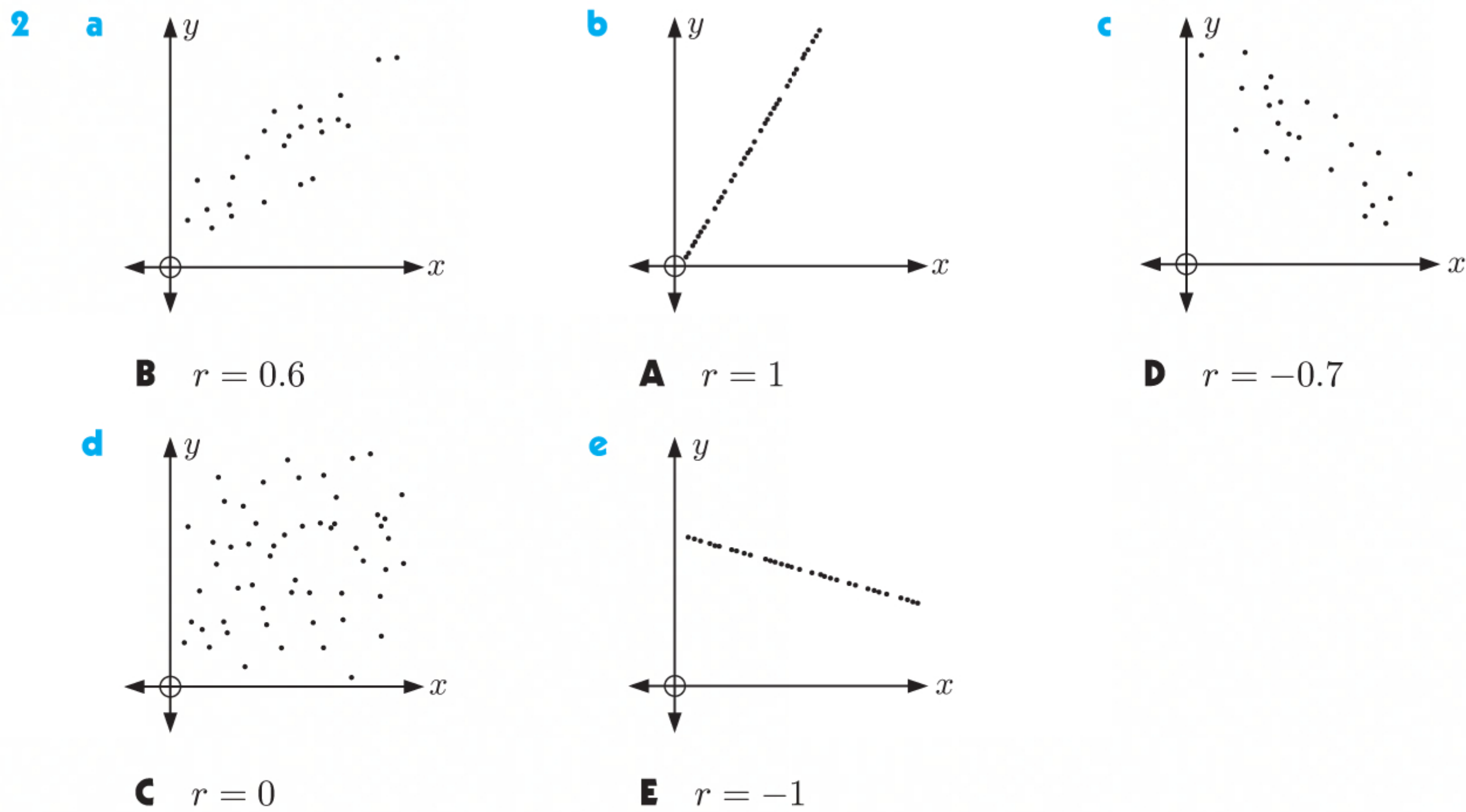


- 7**
- a** Not causal, dependent on genetics and/or age.
 - b** Not causal, dependent on the size/location of the fire.
 - c** Causal, an increase in advertising is likely to cause an increase in sales.
 - d** Causal, the childrens' adult height is determined by the genetics inherited from their parents to a great extent.
 - e** Not causal, dependent on the population of the towns.

EXERCISE 5B

- 1** $r = 0.556$
 There is a weak, positive correlation between the *number of employees of a company* and its *export earnings*.



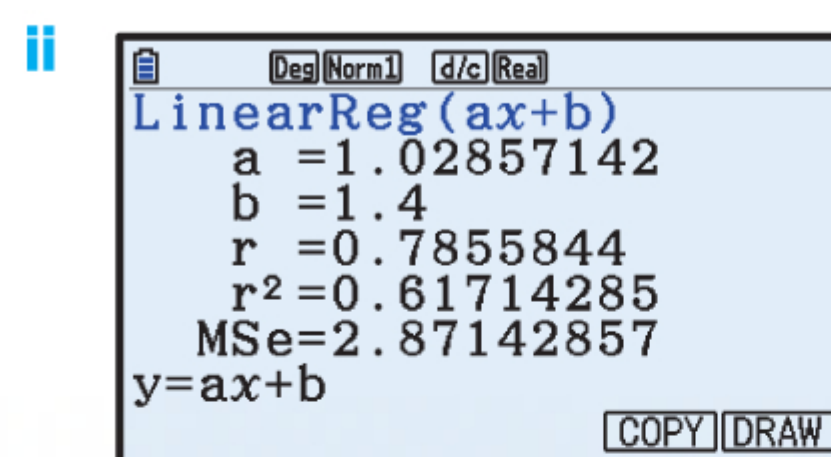
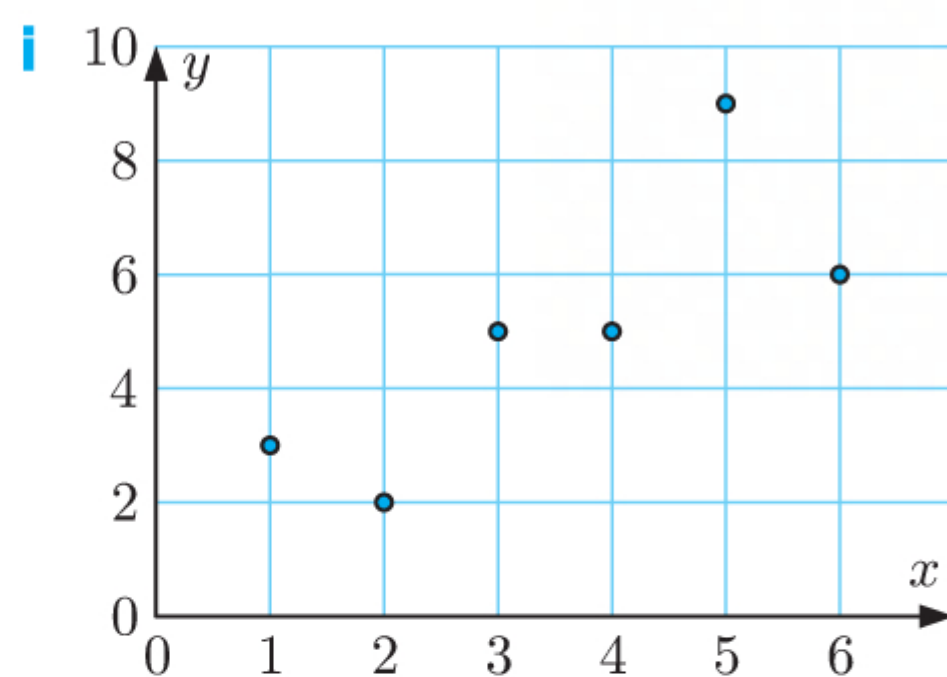
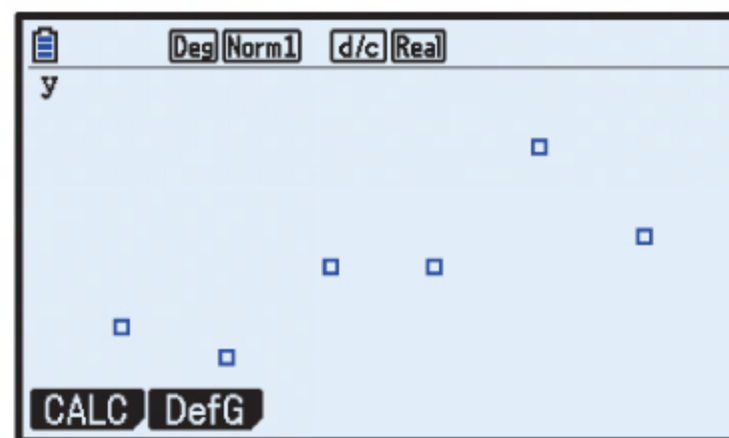


3

a

x	1	2	3	4	5	6
y	3	2	5	5	9	6

	List 1	List 2	List 3	List 4
SUB				
1	1	3		
2	2	2		
3	3	5		
4	4	5		



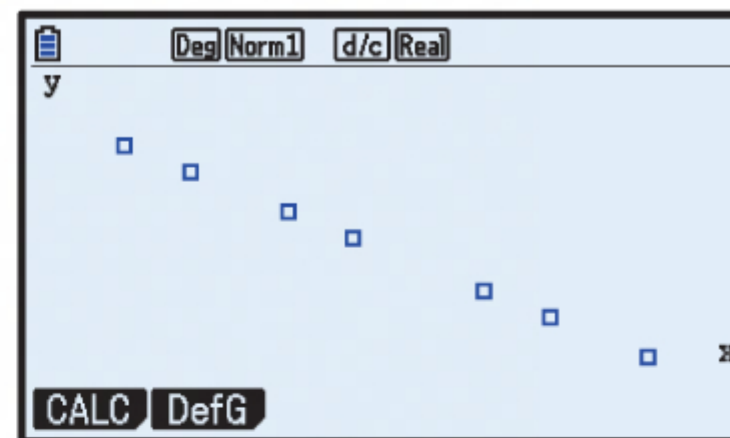
So, $r \approx 0.786$.

iii There is a moderate, positive correlation between x and y .

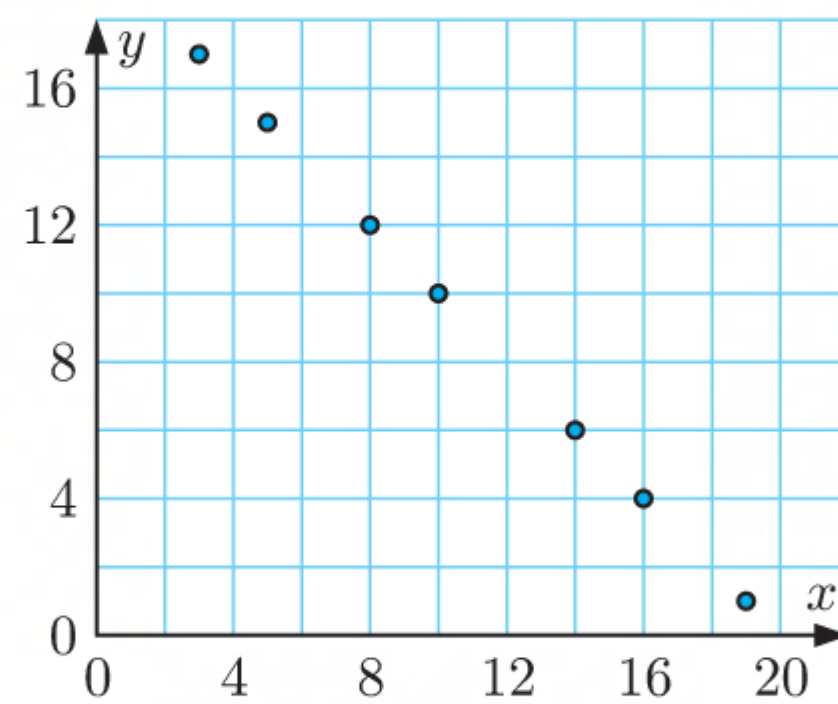
b

x	3	8	5	14	19	10	16
y	17	12	15	6	1	10	4

	List 1	List 2	List 3	List 4
SUB				
1	3	17		
2	8	12		
3	5	15		
4	14	6		



i



ii

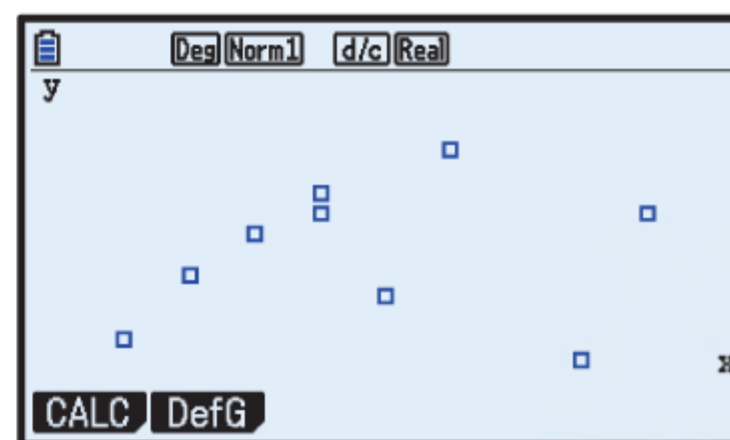
	List 1	List 2	List 3	List 4
SUB				
1	3	17		
2	8	12		
3	5	15		
4	14	6		

So, $r = -1$.iii There is a perfect, negative correlation between x and y .

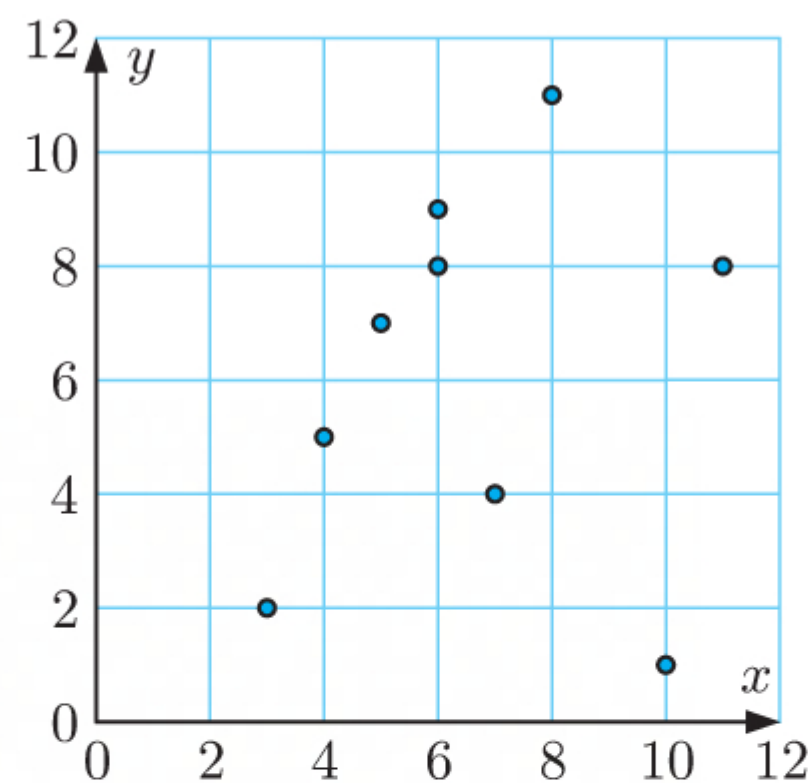
c

x	3	6	11	7	5	6	8	10	4
y	2	8	8	4	7	9	11	1	5

	List 1	List 2	List 3	List 4
SUB				
1	3	2		
2	6	8		
3	11	8		
4	7	4		



i



ii

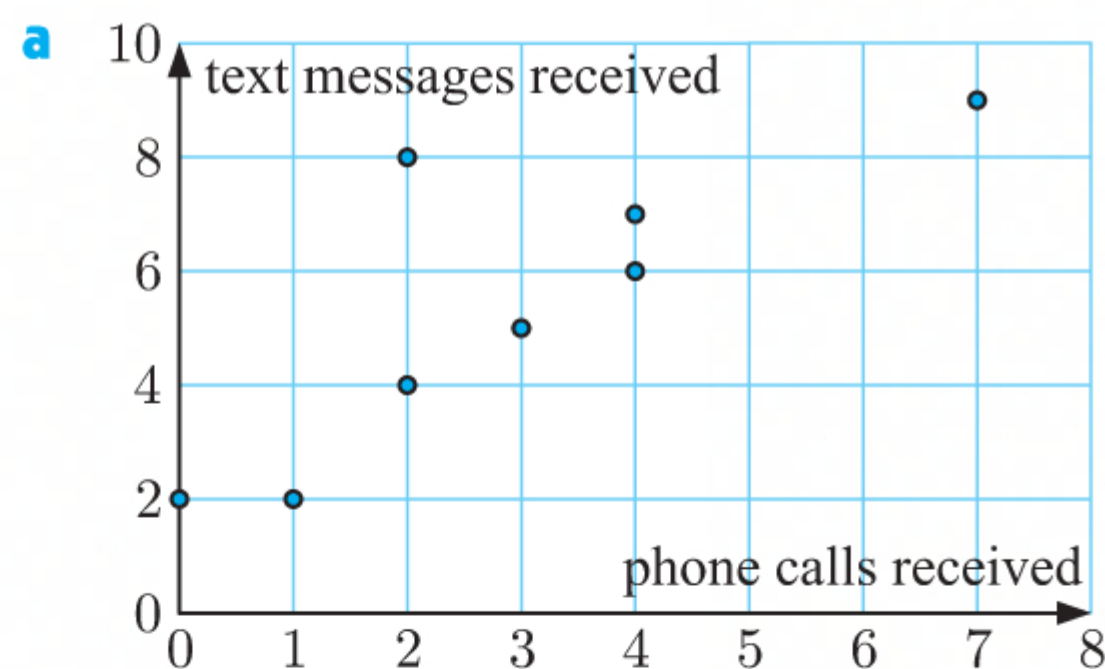
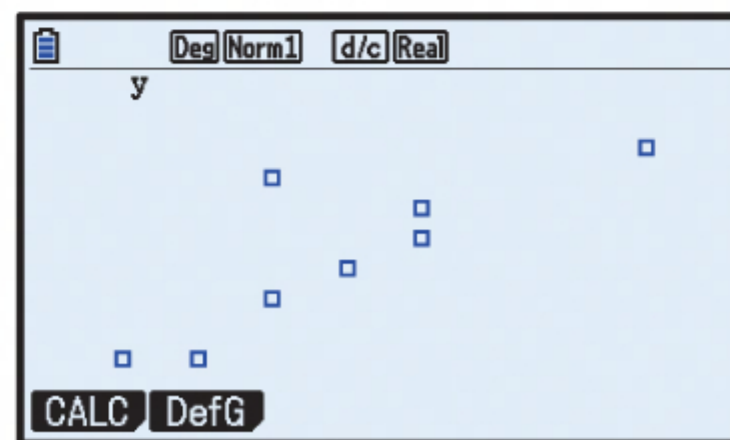
	List 1	List 2	List 3	List 4
SUB				
1	3	2		
2	6	8		
3	11	8		
4	7	4		

So, $r \approx 0.146$.iii There is a very weak, positive correlation between x and y .

4

Student	A	B	C	D	E	F	G	H
Phone calls received	4	7	1	0	3	2	2	4
Text messages received	6	9	2	2	5	8	4	7

	List 1	List 2	List 3	List 4
SUB				
1	4	6		
2	7	9		
3	1	2		
4	0	2		



b

LinearReg(ax+b)
a = 0.98479087
b = 2.54372623
r = 0.81606077
r ² = 0.66595518
MSe = 2.66539923
y = ax + b

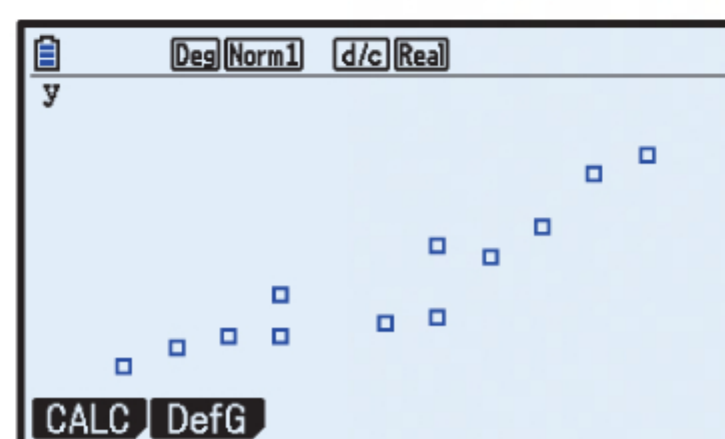
So, $r \approx 0.816$.

- c There is a moderate, positive correlation between *phone calls received* and *text messages received*.
- d Those students who receive several phone calls are also likely to receive several text messages and vice versa.

5

Athlete	A	B	C	D	E	F	G	H	I	J	K	L
Age (years)	12	16	16	18	13	19	11	10	20	17	15	13
Distance thrown (m)	20	35	23	38	27	47	18	15	50	33	22	20

	List 1	List 2	List 3	List 4
SUB				
1	12	20		
2	16	35		
3	16	23		
4	18	38		



LinearReg(ax+b)
a = 3.28947368
b = -20.342105
r = 0.91730097
r ² = 0.84144108
MSe = 23.2447368
y = ax + b

So, $r \approx 0.917$.

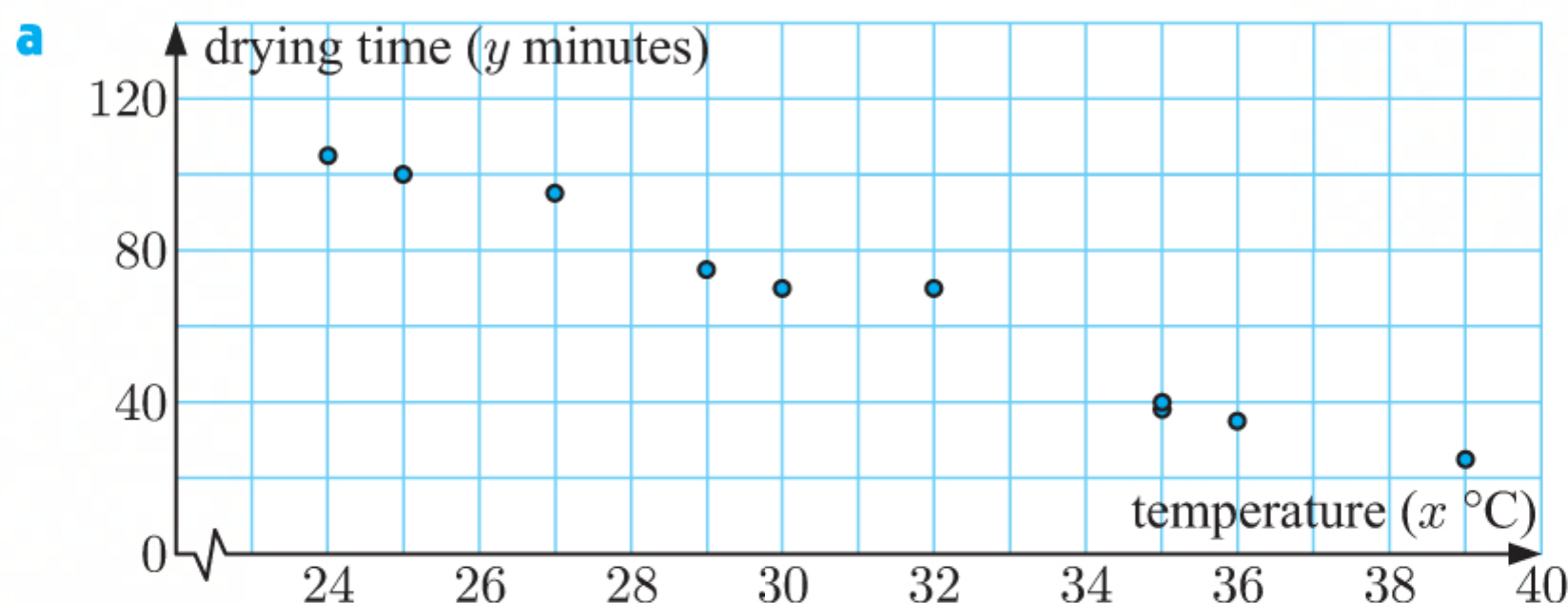
- b There is a strong, positive correlation between the *age* of the young athlete and the *distance thrown*. In general, the higher the young athlete's age, the further they can throw a discus.

6

<i>Temperature (x °C)</i>	25	32	27	39	35	24	30	36	29	35
<i>Drying time (y minutes)</i>	100	70	95	25	38	105	70	35	75	40

	List 1	List 2	List 3	List 4
SUB				
1	25	100		
2	32	70		
3	27	95		
4	39	25		
				25

GRAPH CALC TEST INTR DIST



b

LinearReg(ax+b)
a = -5.7495606
b = 244.686291
r = -0.9866983
r ² = 0.97357371
MSe = 25.5281195
y = ax + b

COPY DRAW

- c** There is a very strong, negative correlation between *temperature* and *drying time*. In general, the higher the temperature, the lower the drying time.

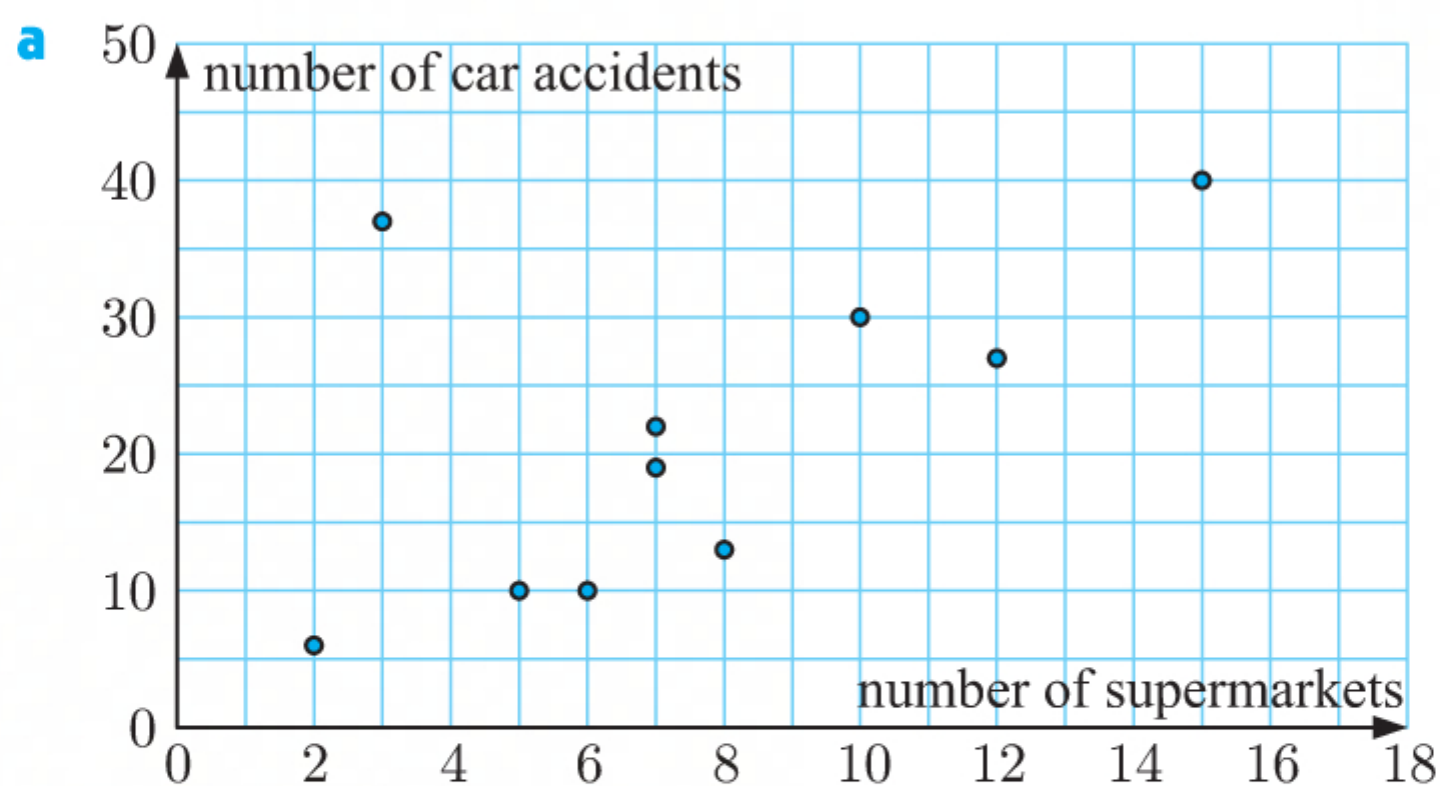
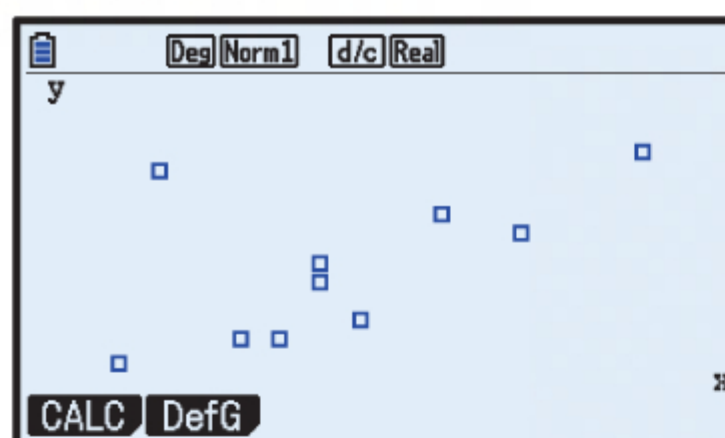
So, $r \approx -0.987$.

7

<i>Number of supermarkets</i>	5	8	12	7	6	2	15	10	7	3
<i>Number of car accidents</i>	10	13	27	19	10	6	40	30	22	37

	List 1	List 2	List 3	List 4
SUB				
1	5	10		
2	8	13		
3	12	27		
4	7	19		
				19

GRAPH CALC TEST INTR DIST



b

```

Deg Norm1 d/c Real
LinearReg(ax+b)
a =1.70526315
b =8.61052631
r =0.5715717
r²=0.32669421
MSe=106.752631
y=ax+b
COPY DRAW

```

So, $r \approx 0.572$.

- c The point (3, 37), which represents 37 car accidents in a town with 3 supermarkets, is an outlier.

d i

```

Deg Norm1 d/c Real
LinearReg(ax+b)
a =2.675
b =-1.7333333
r =0.92757522
r²=0.86039579
MSe=19.9035714
y=ax+b
COPY DRAW

```

So, $r \approx 0.928$.

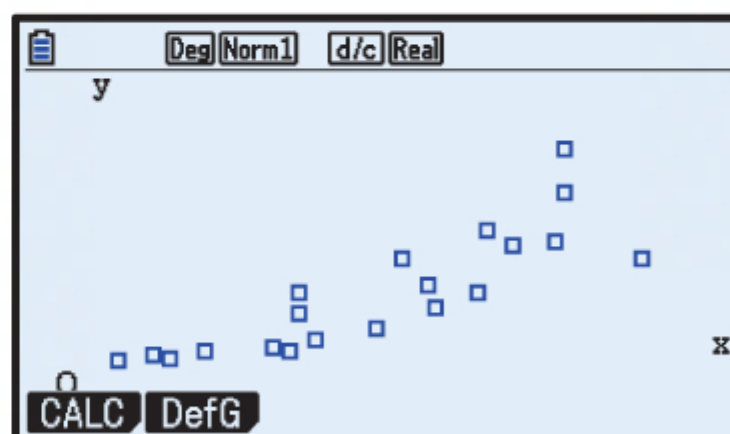
- ii There is a strong, positive correlation between the *number of supermarkets* and the *number of car accidents*.
- iii By removing the outlier, the value of r increased significantly.
- e No, it is not a causal relationship. Both variables depend on the number of people in each town, not on each other.

8

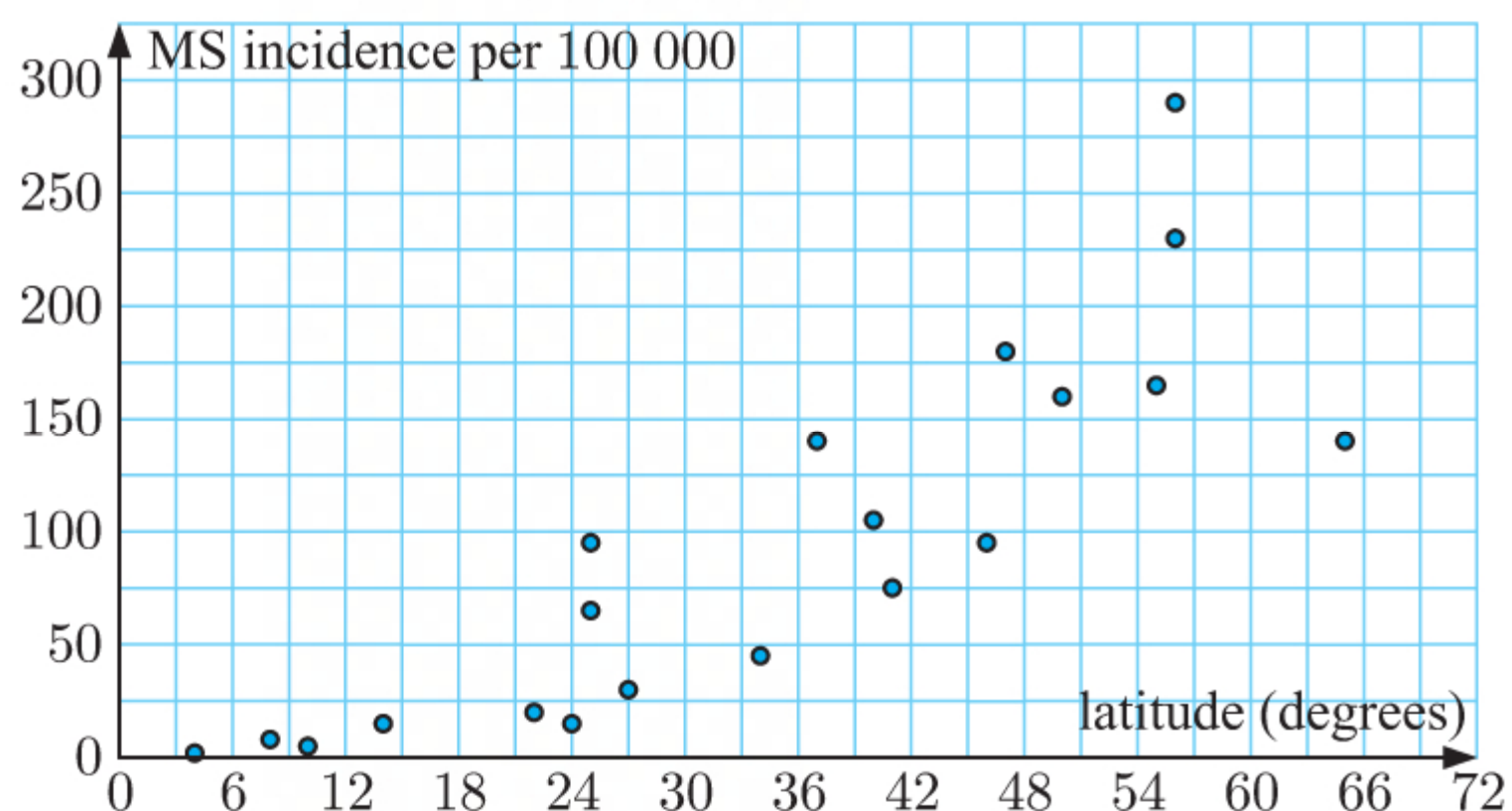
<i>Latitude (degrees)</i>	55	25	41	22	47	37	56	14	34	25
<i>MS incidence per 100 000</i>	165	95	75	20	180	140	230	15	45	65

<i>Latitude (degrees)</i>	27	65	10	24	4	56	46	8	50	40
<i>MS incidence per 100 000</i>	30	140	5	15	2	290	95	8	160	105

	List 1	List 2	List 3	List 4
SUB				
1	55	165		
2	25	95		
3	41	75		
4	22	20		
				20



a



b

Deg Norm1 d/c Real	
LinearReg(ax+b)	
a	=3.90314994
b	=-39.878043
r	=0.84940985
r ²	=0.7214971
MSe	=1972.69789
y=ax+b	
COPY DRAW	

So, $r \approx 0.849$.

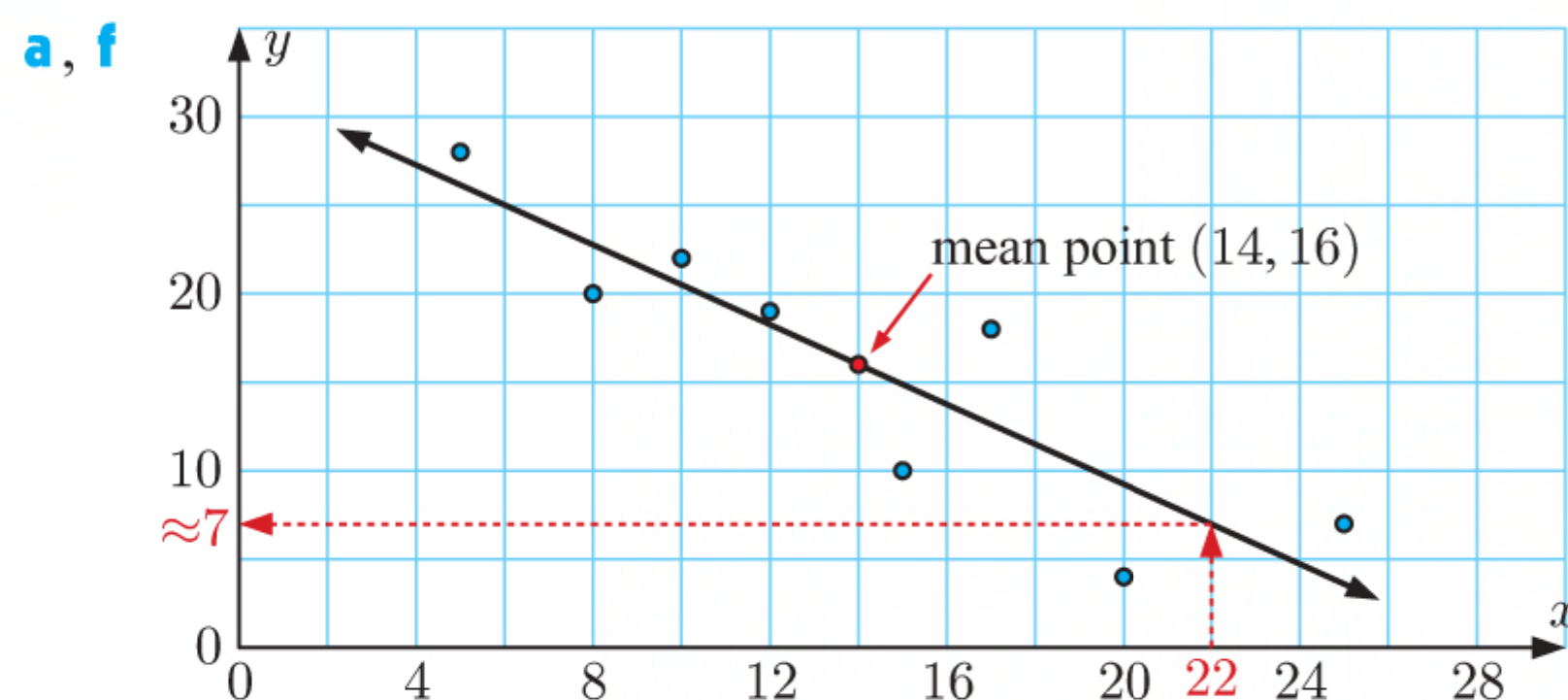
- d** The incidence of MS is higher near the poles.

- c** There is a moderate, positive correlation between *latitude* and *MS incidence*.

EXERCISE 5C

1

x	5	12	20	17	10	8	25	15
y	28	19	4	18	22	20	7	10



- b** The data appears to be negatively correlated.

c

Deg Norm1 d/c Real	
LinearReg(ax+b)	
a	=-1.0953947
b	=31.3355263
r	=-0.8809647
r ²	=0.77609882
MSe	=17.5389254
y=ax+b	
COPY DRAW	

So, $r \approx -0.881$.

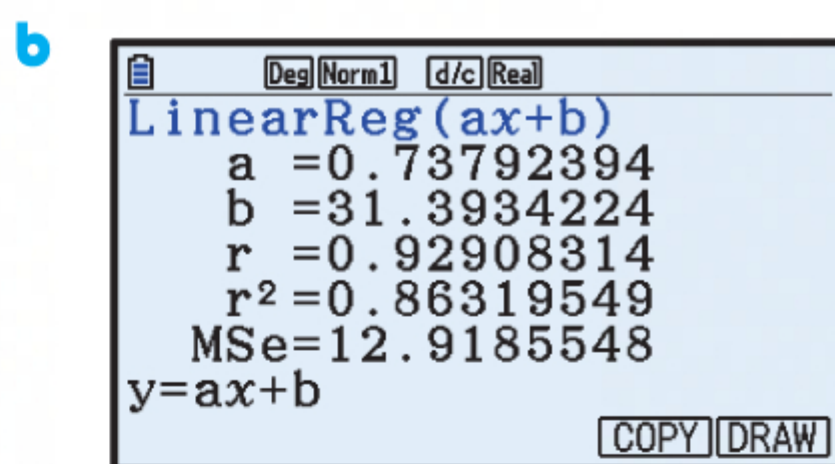
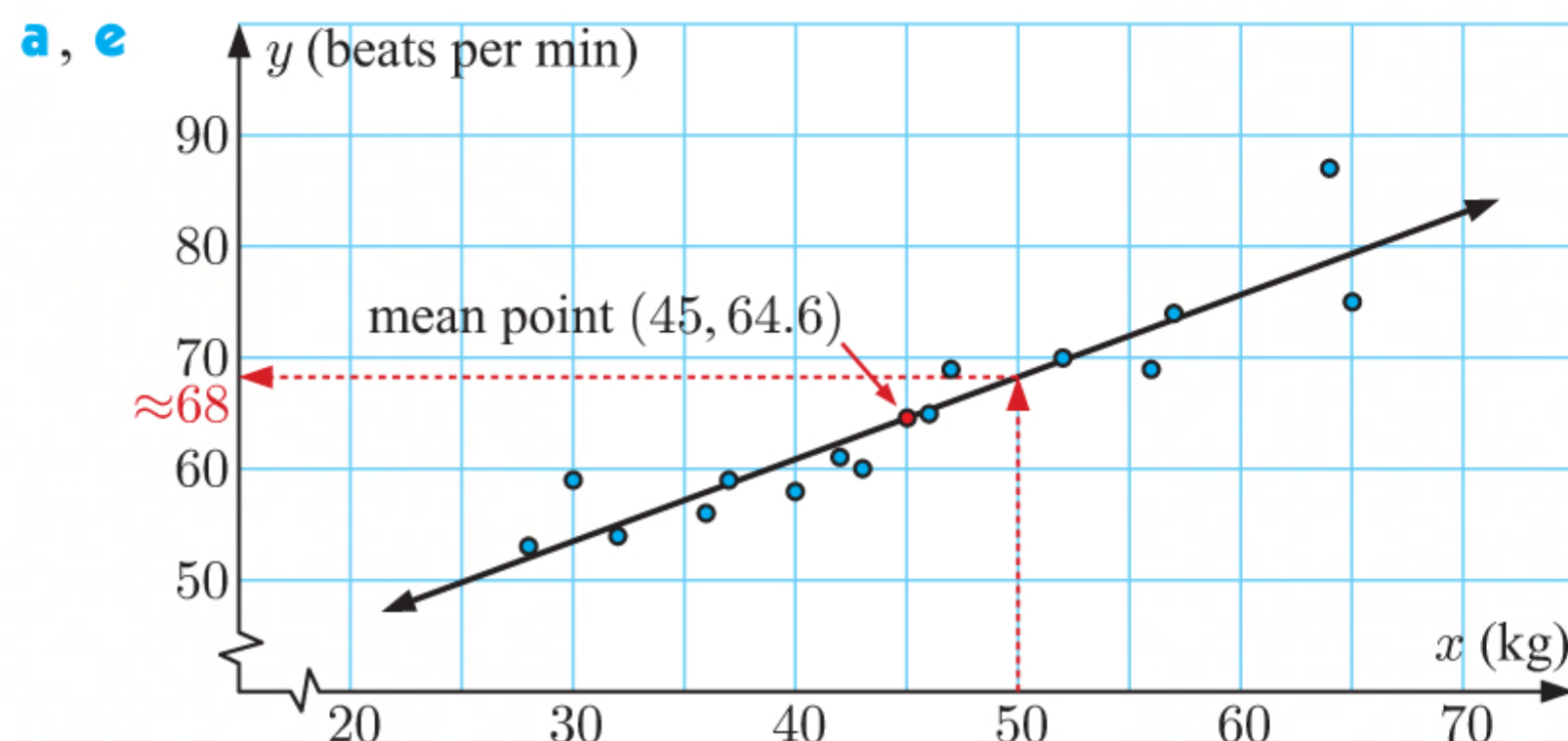
- d** There is a strong, negative correlation between x and y .

e $\bar{x} = \frac{5 + 12 + 20 + 17 + 10 + 8 + 25 + 15}{8}, \quad \bar{y} = \frac{28 + 19 + 4 + 18 + 22 + 20 + 7 + 10}{8}$
 $= 14 \quad \quad \quad = 16$

So the mean point is (14, 16).

- g** When $x = 22$, $y \approx 7$.

2	<i>Weight (x kg)</i>	46	37	32	57	47	64	42	30	52	56	65	43	36	28	40
	<i>Pulse rate (y beats per min)</i>	65	59	54	74	69	87	61	59	70	69	75	60	56	53	58



So, $r \approx 0.929$.

c There is a strong, positive correlation between the *weight* of a student and their *pulse rate*.

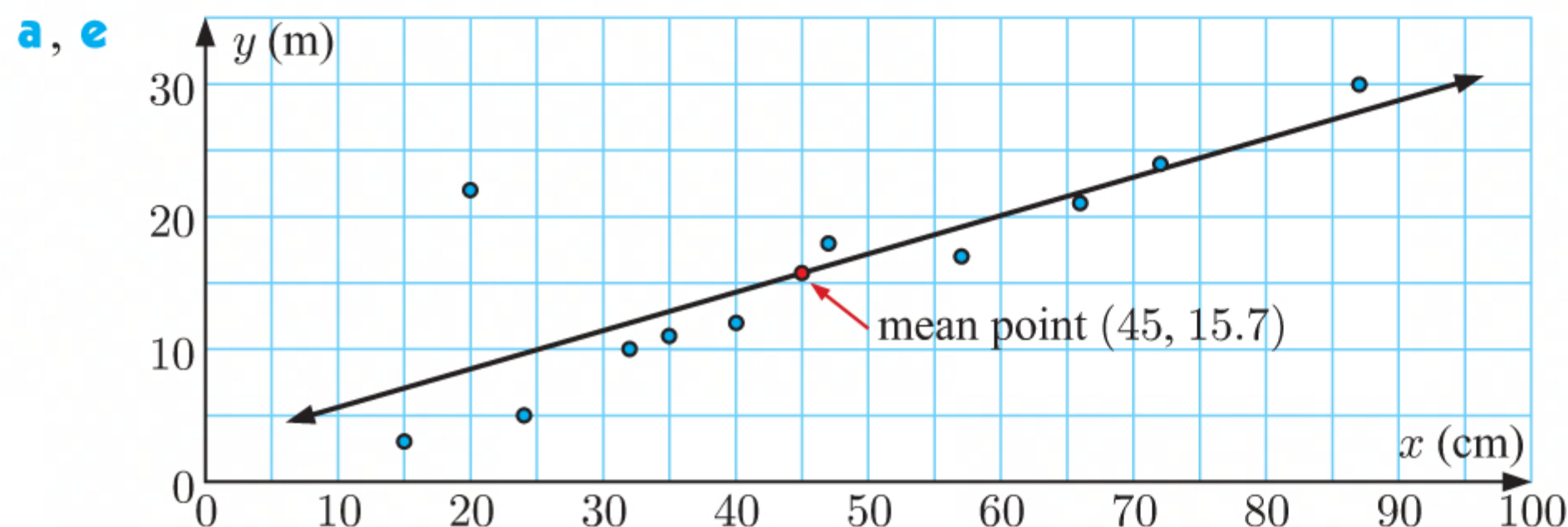
d $\bar{x} = \frac{46 + 37 + \dots + 28 + 40}{15}, \quad \bar{y} = \frac{65 + 59 + \dots + 53 + 58}{15}$
 $= 45 \qquad \qquad \qquad = 64.6$

So the mean point is $(45, 64.6)$.

f When $x = 50$, $y \approx 68$.

A student who weighs 50 kg will have a pulse rate of approximately 68 beats per minute. This is an interpolation, so the estimate is reliable.

3	<i>Trunk width (x cm)</i>	35	47	72	40	15	87	20	66	57	24	32
	<i>Height (y m)</i>	11	18	24	12	3	30	22	21	17	5	10



b $(20, 22)$ is an outlier as it appears separated from the rest of the data.

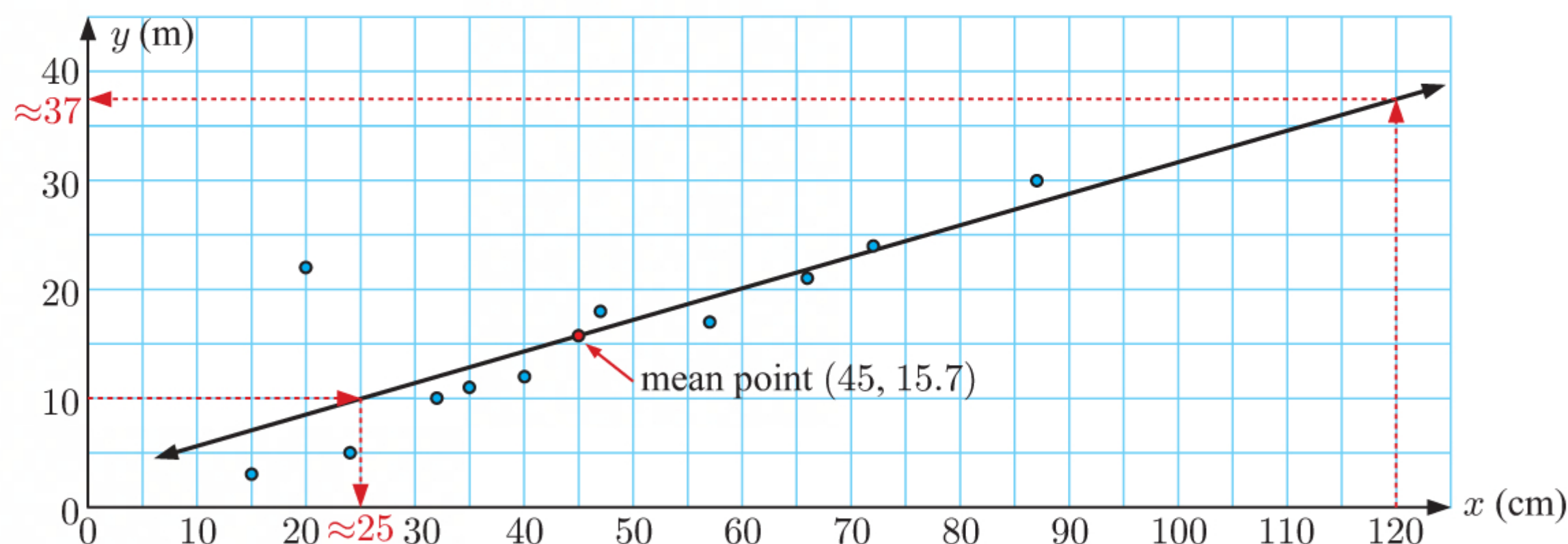
c The tree represented by the outlier would be very tall and thin.

$$\text{d } \bar{x} = \frac{35 + 47 + \dots + 24 + 32}{11}, \quad \bar{y} = \frac{11 + 18 + \dots + 5 + 10}{11}$$

$$= 45 \quad \approx 15.7$$

So the mean point is (45, 15.7).

f We extend the scatter diagram from a to include the value $x = 120$:



When $x = 120$, $y \approx 37$.

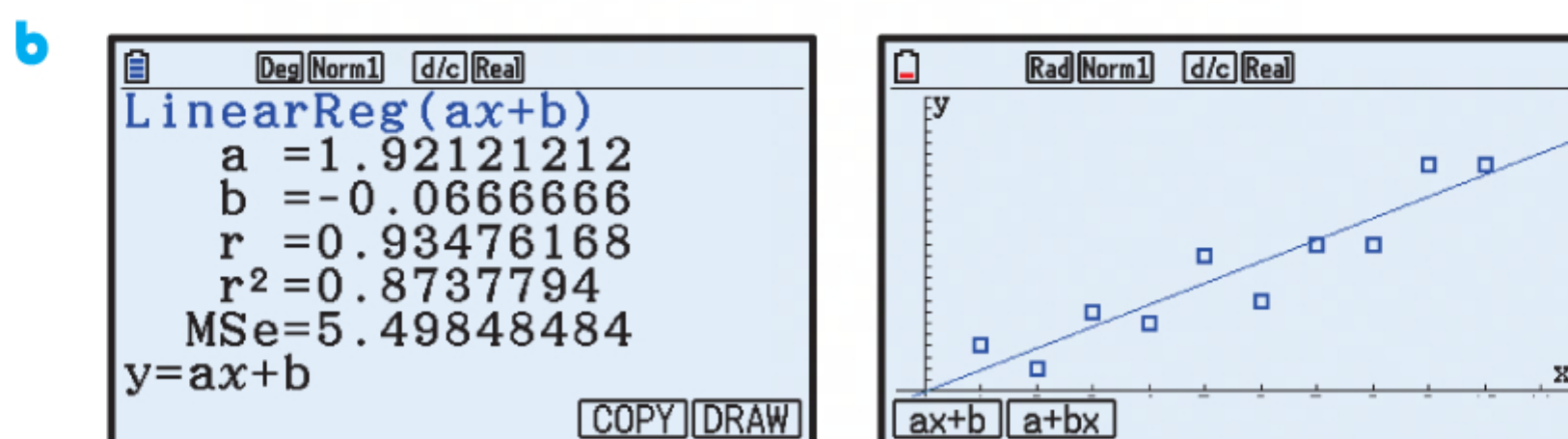
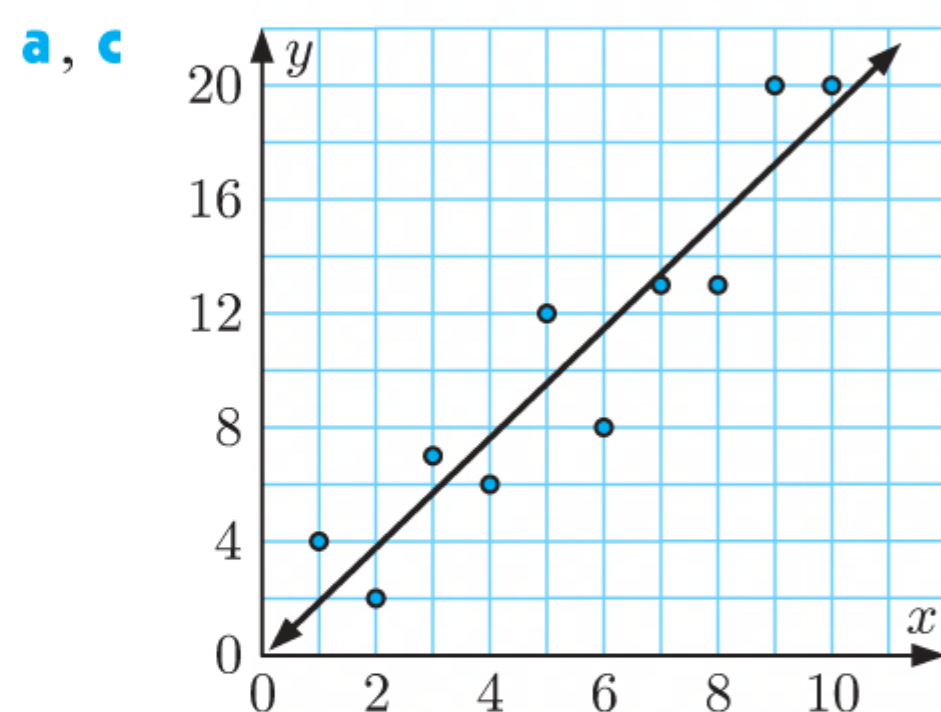
A tree with trunk width 120 cm will have a height of approximately 37 m. This is an extrapolation, so the prediction may not be reliable.

g When $y = 10$, $x \approx 25$.

A tree with height 10 m will have a trunk width of approximately 25 cm. This is an interpolation, so the estimate is reliable.

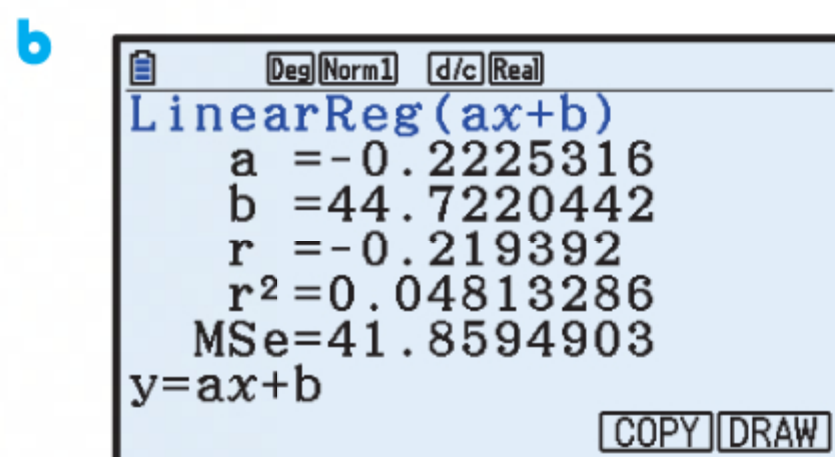
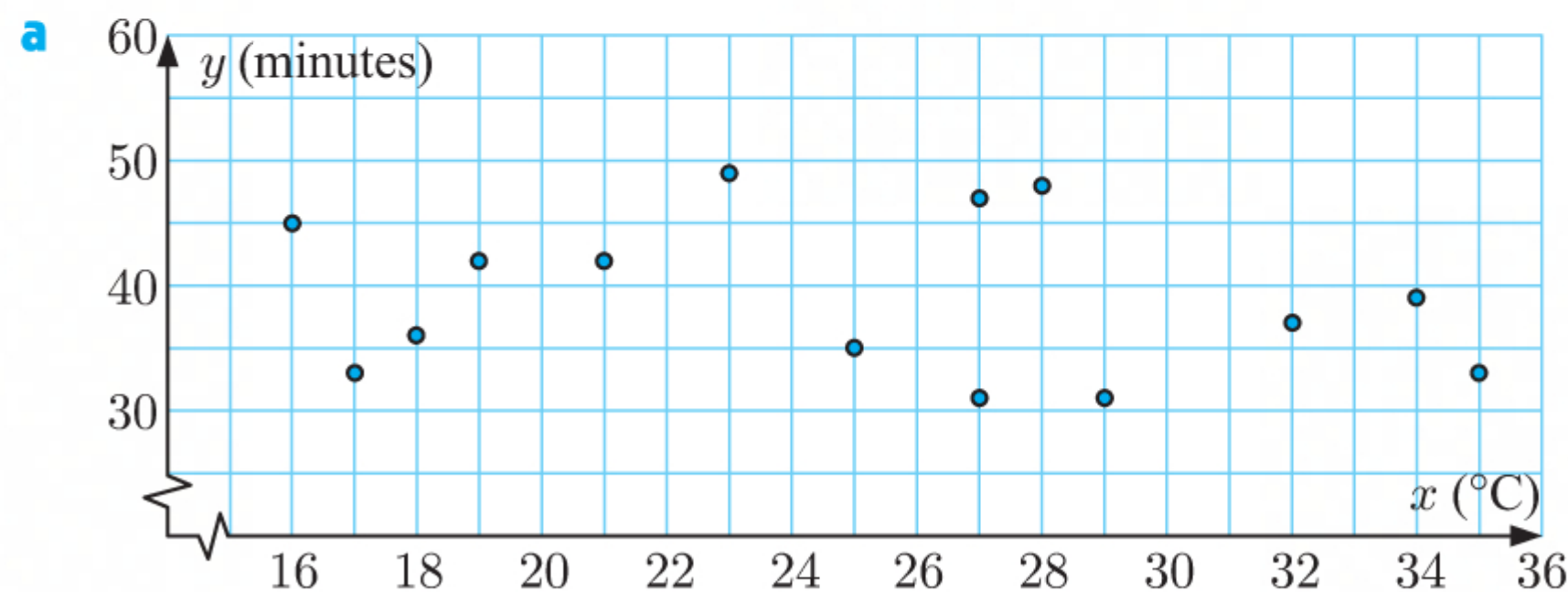
EXERCISE 5D

1	x	10	4	6	8	9	5	7	1	2	3
	y	20	6	8	13	20	12	13	4	2	7



Using technology, the least squares regression line is $y \approx 1.92x - 0.0667$.

2	Temperature (x °C)	25	19	23	27	32	35	29	27	21	18	16	17	28	34
	Time (y minutes)	35	42	49	31	37	33	31	47	42	36	45	33	48	39

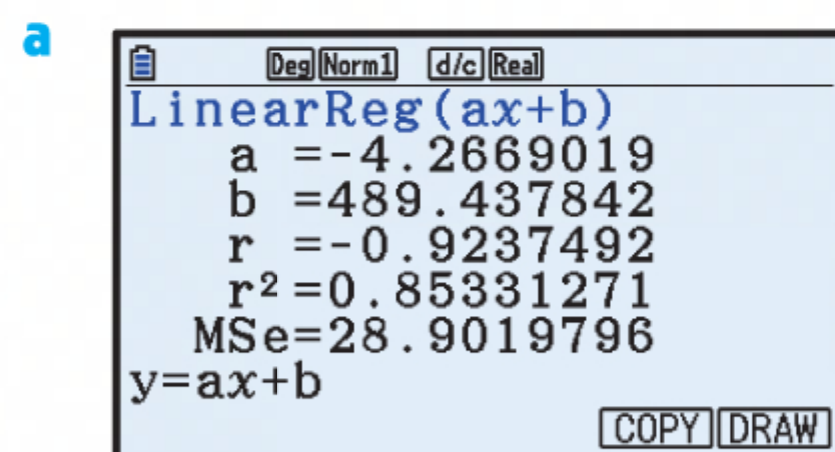


So, $r \approx -0.219$.

- c** There is a very weak, negative correlation between *temperature* and *time*.
d No, it is not reasonable to find a line of best fit for this data as there is almost no correlation.

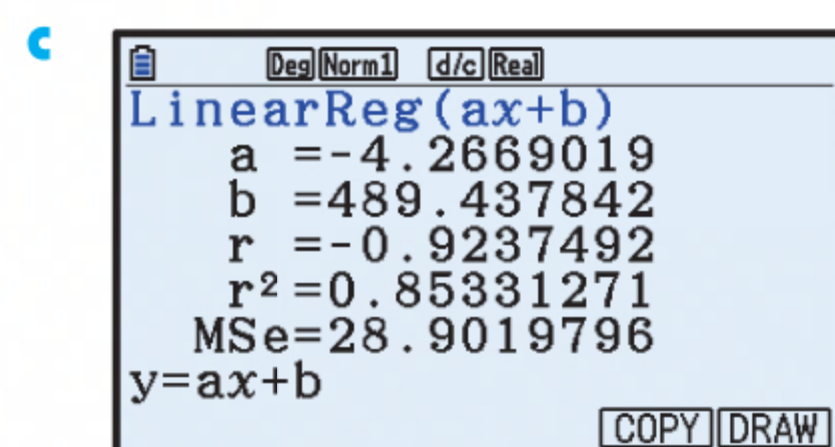
3	Petrol price (x cents per litre)	105.9	106.9	109.9	104.5	104.9	111.9	110.5	112.9
	Number of customers (y)	45	42	25	48	43	15	19	10

	Petrol price (x cents per litre)	107.5	108.0	104.9	102.9	110.9	106.9	105.5	109.5
	Number of customers (y)	30	23	42	50	12	24	32	17



So, $r \approx -0.924$.

- b** There is a strong, negative correlation between *petrol price* and the *number of customers*.



Using technology, the least squares regression line is $y \approx -4.27x + 489$.

d The gradient of the least squares regression line ≈ -4.27 . This means that for every cent per litre the petrol price increases by, the number of customers will decrease by approximately 4.27.

e When $x = 115.9$, $y \approx -4.27(115.9) + 489$
 ≈ -5.10

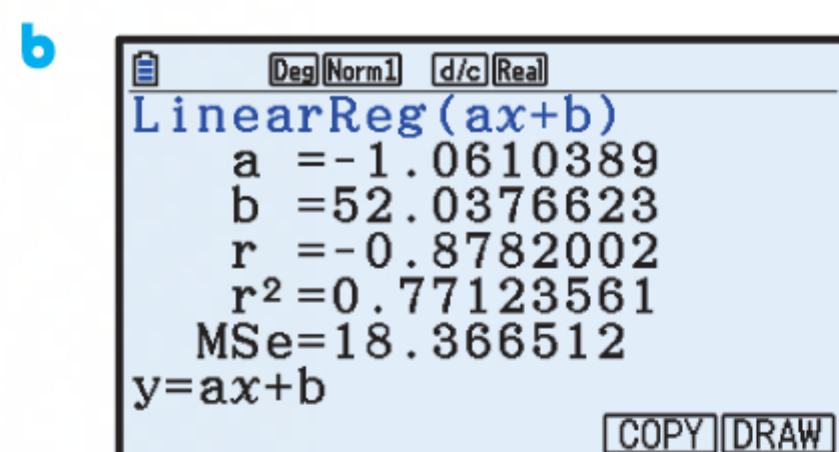
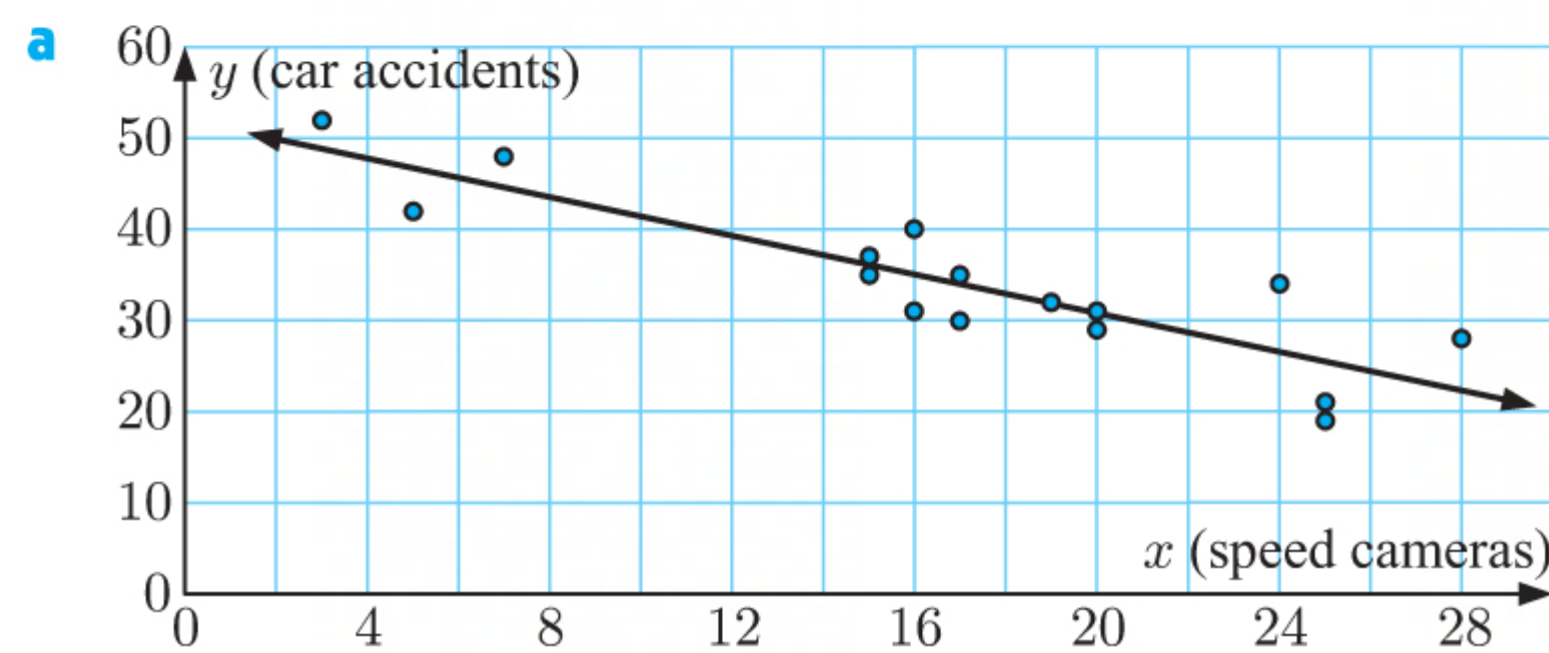
So, when petrol is 115.9 cents per litre, we would expect about -5.10 customers per hour.

f When $y = 40$, $40 \approx -4.27x + 489$
 $\therefore -449 \approx -4.27x$
 $\therefore x \approx 105.3$

So, a petrol station which has 40 customers per hour would sell petrol at approximately 105.3 cents per litre.

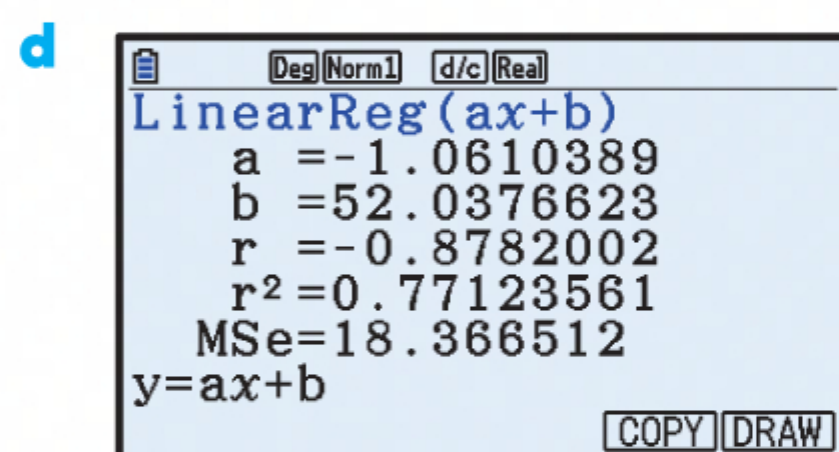
g In **e**, it is impossible to have a negative number of customers. This extrapolation is not valid. In **f**, this is an interpolation, so this estimate is likely to be reliable.

4	Number of speed cameras (x)	7	15	20	3	16	17	28	17	24	25	20	5	16	25	15	19
	Number of car accidents (y)	48	35	31	52	40	35	28	30	34	19	29	42	31	21	37	32



So, $r \approx -0.878$.

c There is a strong, negative correlation between the *number of speed cameras* and the *number of car accidents*.



Using technology, the least squares regression line is $y \approx -1.06x + 52.0$.

- e The gradient of the least squares regression line ≈ -1.06 . This indicates that for every additional speed camera, the number of car accidents per week decreases by an average of 1.06.

The y -intercept of the least squares regression line ≈ 52.0 . This indicates that if there were no speed cameras in a city, an average of 52.0 car accidents would occur each week.

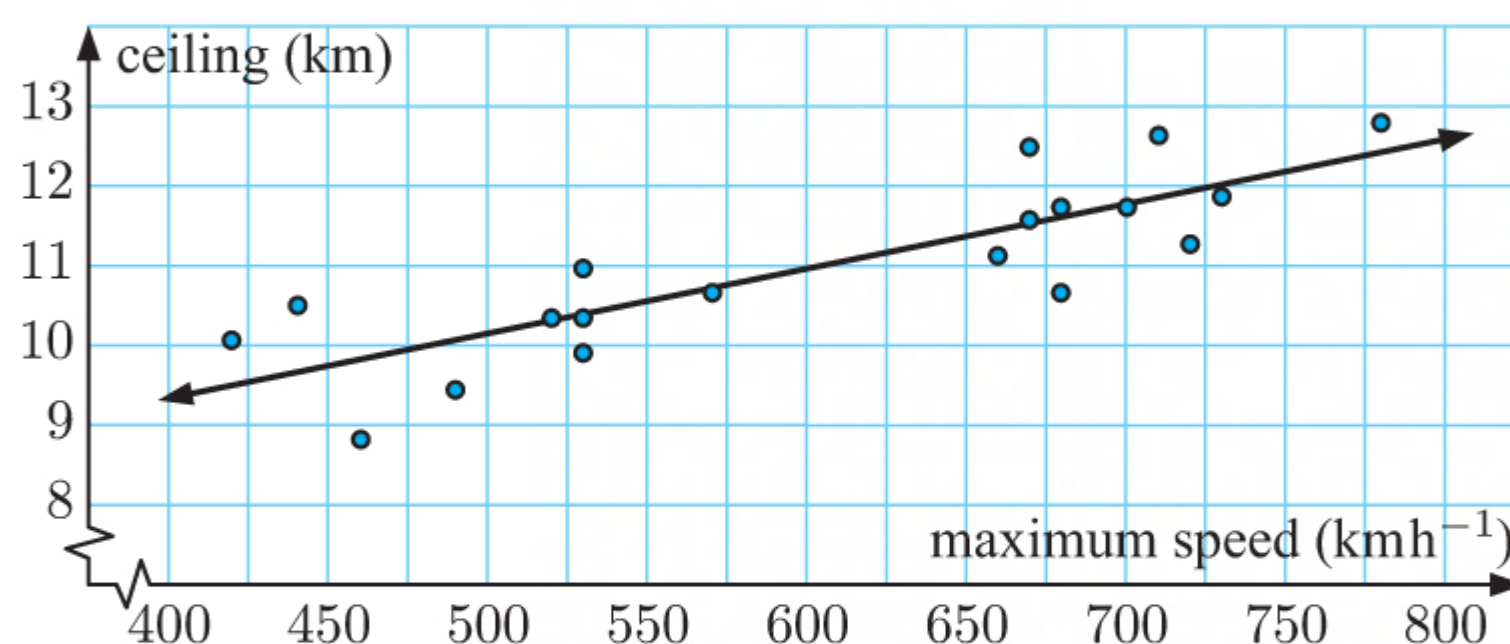
- f When $x = 10$, $y \approx -1.06(10) + 52.0$
 ≈ 41.4

So, there will be approximately 41.4 car accidents per week in a city with 10 speed cameras.

5

Maximum speed	Ceiling	Maximum speed	Ceiling	Maximum speed	Ceiling
460	8.84	680	10.66	670	12.49
420	10.06	720	11.27	570	10.66
530	10.97	710	12.64	440	10.51
530	9.906	660	11.12	670	11.58
490	9.448	780	12.80	700	11.73
530	10.36	730	11.88	520	10.36
680	11.73				

a, d



b

```

LinearReg(ax+b)
a = 8.1202E-03
b = 6.09013455
r = 0.84010344
r² = 0.70577379
MSe = 0.36102817
y = ax + b
  
```

So, $r \approx 0.840$.

- c There is a moderate, positive, linear correlation between *maximum speed* and *ceiling*.

d

```

LinearReg(ax+b)
a = 8.1202E-03
b = 6.09013455
r = 0.84010344
r² = 0.70577379
MSe = 0.36102817
y = ax + b
  
```

Using technology, the least squares regression line is $y \approx 0.00812x + 6.09$.

- e The gradient of the least squares regression line ≈ 0.00812 . This indicates that for each additional km h^{-1} , the ceiling increases by an average of approximately 0.00812 km or 8.12 m.

f When $x = 600$, $y \approx 0.008\,12(600) + 6.09$
 ≈ 11.0

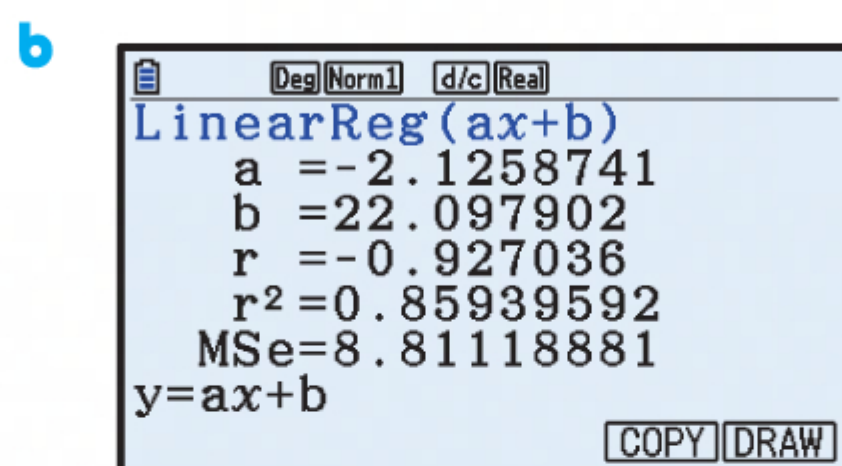
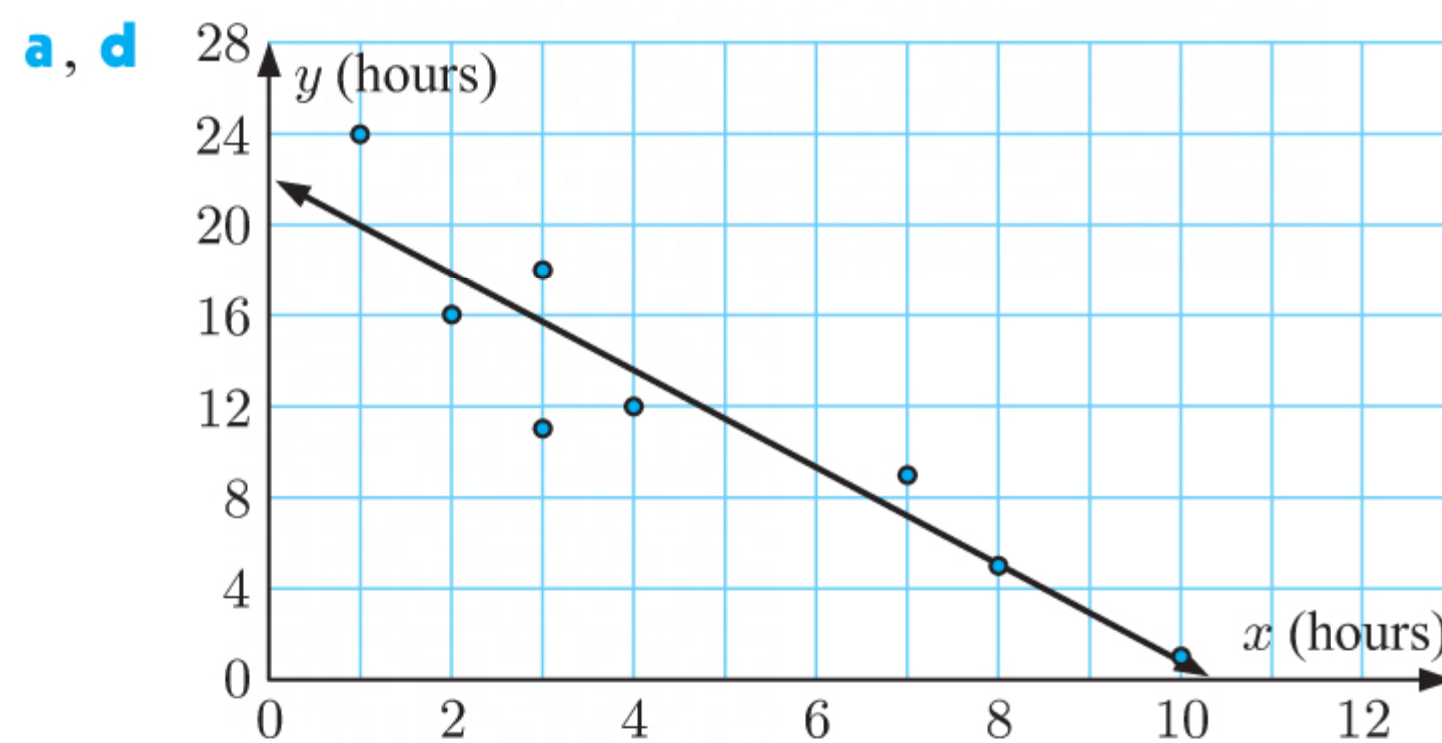
So, a fighter plane with maximum speed 600 km h^{-1} would have a ceiling of approximately 11.0 km.

g When $y = 11$, $11 \approx 0.008\,12x + 6.09$
 $\therefore 4.91 \approx 0.008\,12x$
 $\therefore x \approx 605$

So, a fighter plane with a ceiling of 11 km would have maximum speed of approximately 605 km h^{-1} .

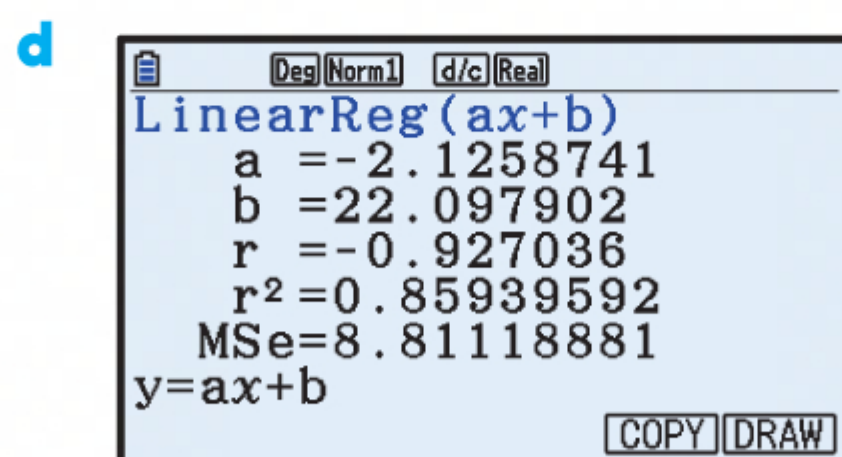
6

<i>Exercise</i> (x hours per week)	4	1	8	7	10	3	3	2
<i>Television</i> (y hours per week)	12	24	5	9	1	18	11	16



So, $r \approx -0.927$.

c There is a strong, negative, linear correlation between *time exercising* and *time watching television*.



Using technology, the least squares regression line is $y \approx -2.13x + 22.1$.

e The gradient of the least squares regression line ≈ -2.13 . This indicates that for each additional hour a child exercises each week, the number of hours they spend watching television each week decreases by 2.13.

The y -intercept of the least squares regression line ≈ 22.1 . This indicates that for children who do not spend time exercising, they would watch television for an average of about 22.1 hours per week.

- f** **i** From the table, the student who exercised for 7 hours each week watched 9 hours of television each week.

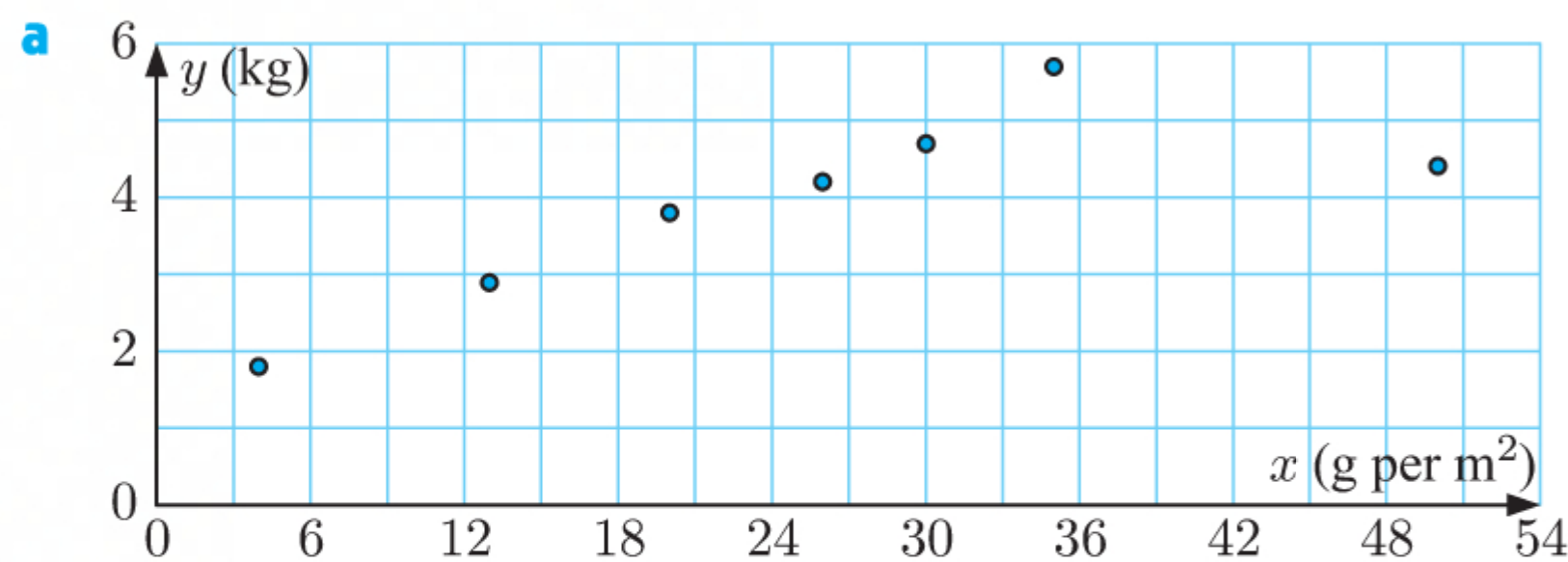
ii When $x = 7$, $y \approx -2.126(7) + 22.1$
 ≈ 7.22

Using the least squares regression line, a child who exercises for 7 hours each week watches approximately 7.22 hours of television each week.

- iii** This particular child spent more time watching television than predicted.

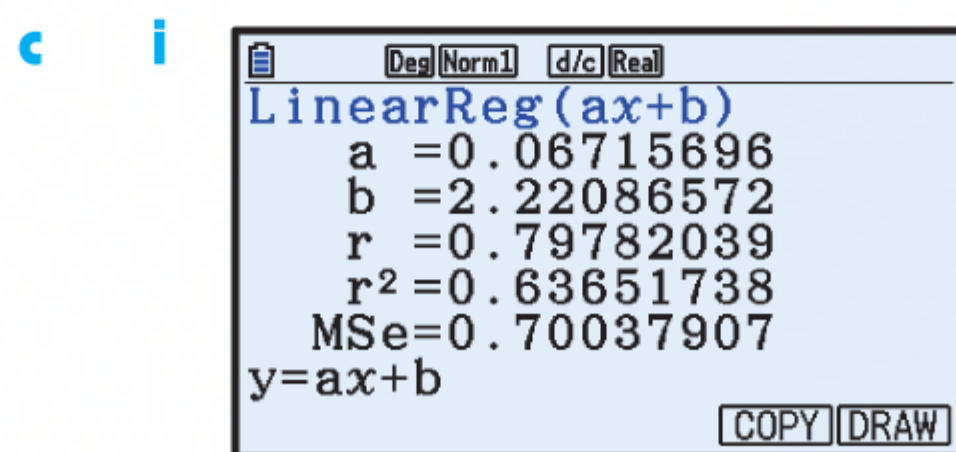
7

Fertiliser (x g per m^2)	4	13	20	26	30	35	50
Yield (y kg)	1.8	2.9	3.8	4.2	4.7	5.7	4.4

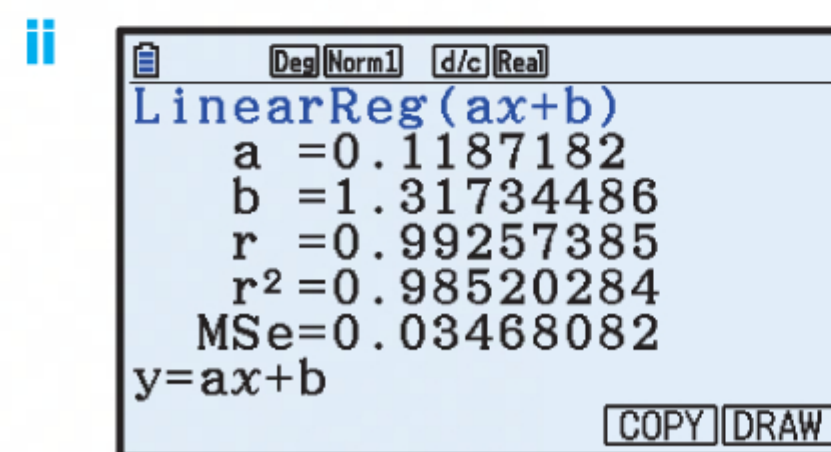


(50, 4.4) is the outlier.

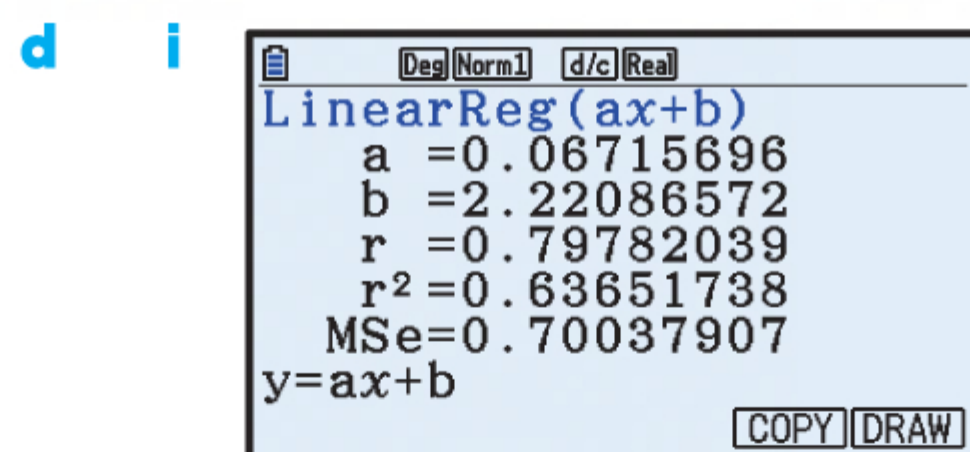
- b** **i** The outlier reduces the strength of correlation of the data.
ii The outlier decreases the gradient of the least squares regression line.



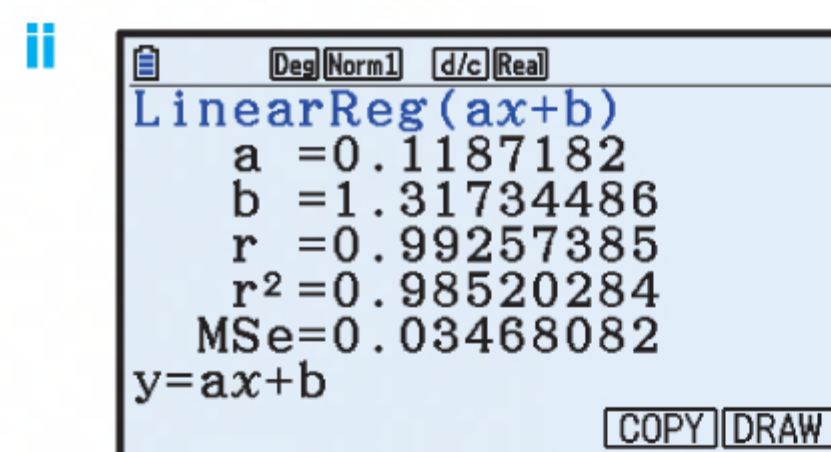
So, $r \approx 0.798$.



So, $r \approx 0.993$.

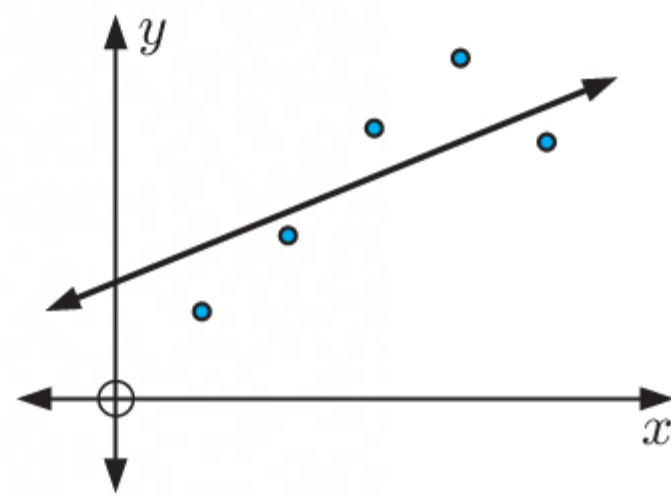


Using technology, the least squares regression line is $y \approx 0.0672x + 2.22$.

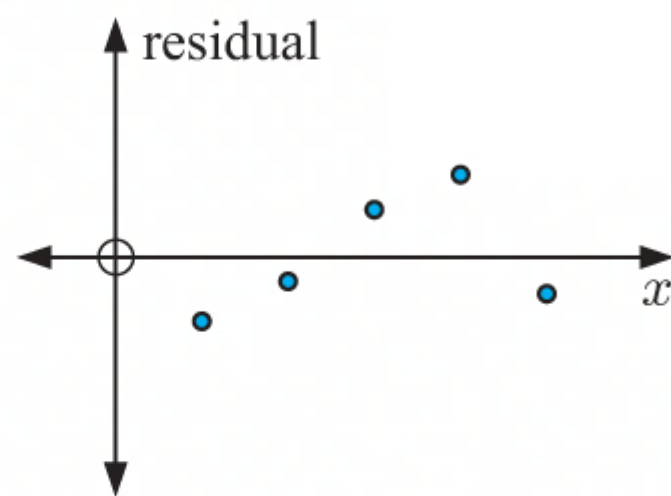
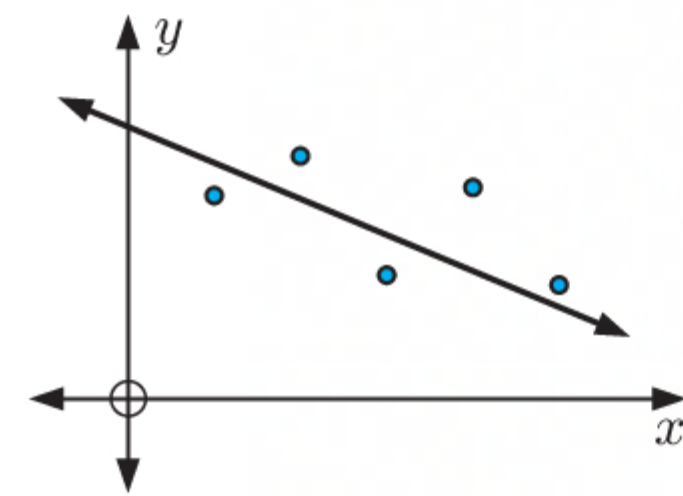


Using technology, the least squares regression line is $y \approx 0.119x + 1.32$.

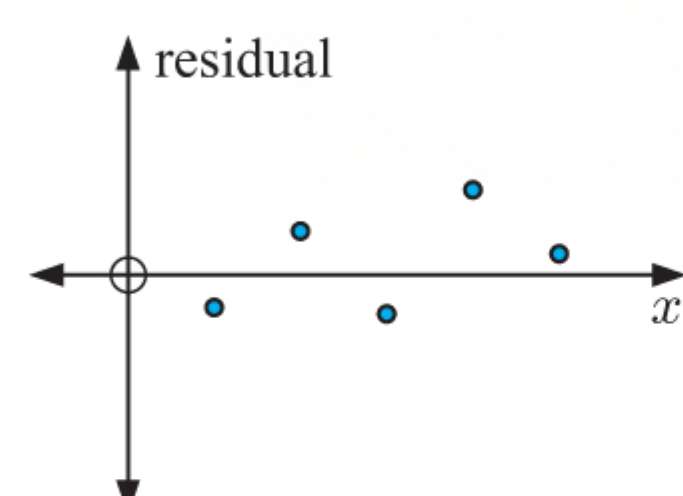
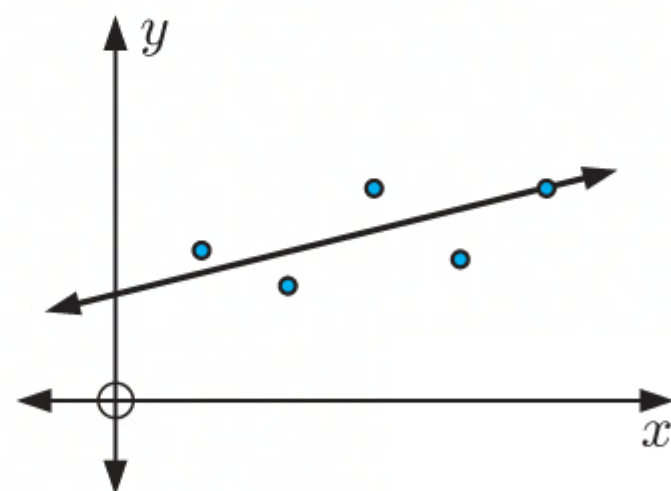
- e** The regression line which excludes the outlier should be used to estimate the yield when 15 g per m^2 of fertiliser is used. This will be more accurate for an interpolation.
- f** Too much fertiliser often kills the plants. In this case, the outlier should be kept when analysing the data as it is a valid data value. If the outlier is a recording error caused by bad measurement or recording skills, it should be removed before analysing data.

ACTIVITY 2**RESIDUAL PLOTS****PART 1: CONSTRUCTING A RESIDUAL PLOT****1 a**

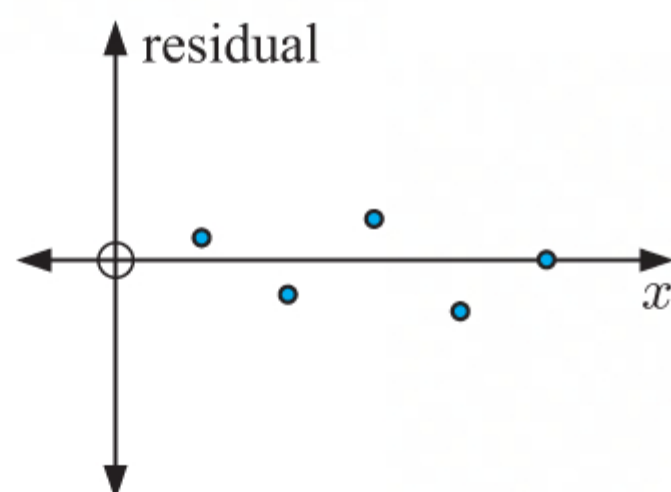
This scatter diagram has 2 data points above the least squares regression line and 3 points below the regression line. This pattern is shown in the residual plot in **B**.

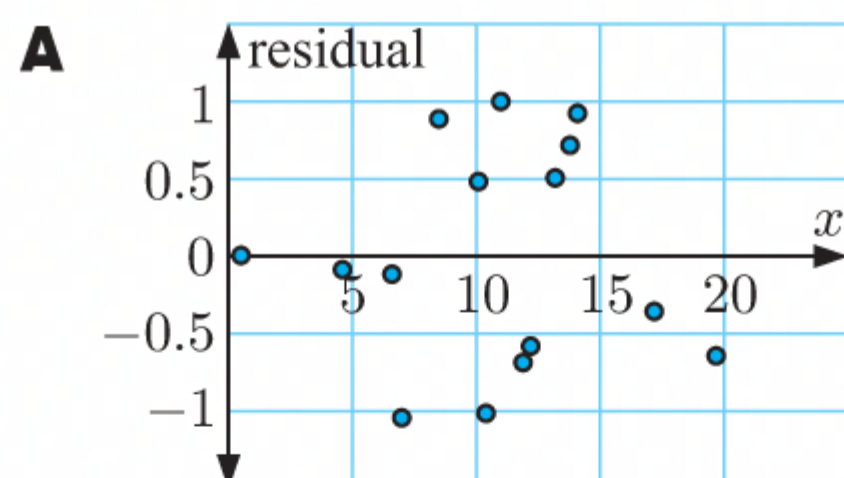
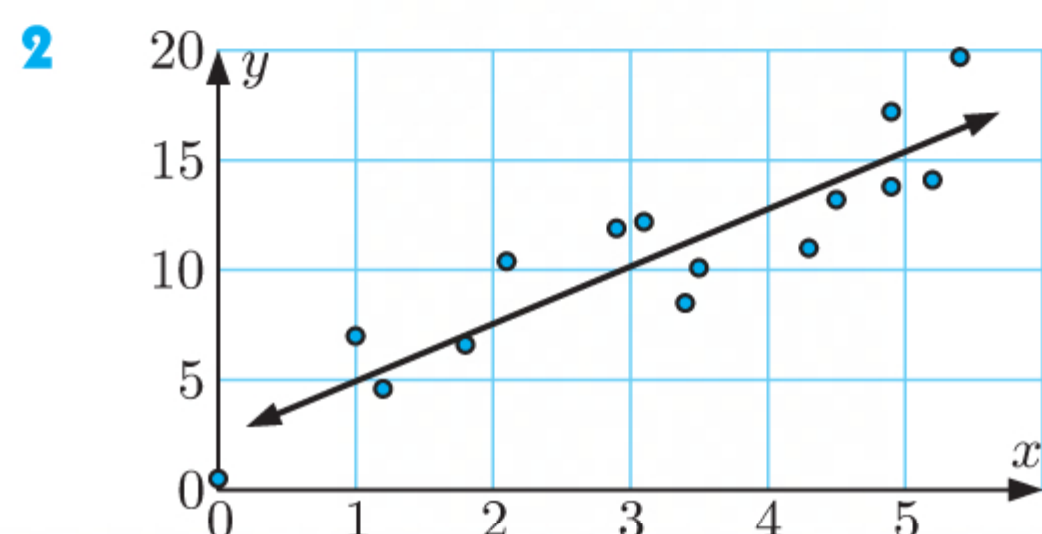
**b**

This scatter diagram has 3 data points above the least squares regression line and 2 points below the regression line. This pattern is shown in the residual plot in **C**.

**c**

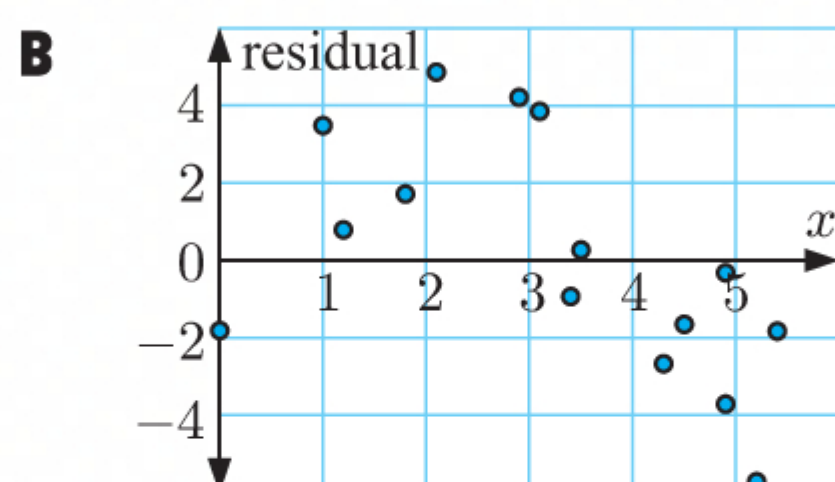
This scatter diagram has 2 data points above the least squares regression line, 2 points below the regression line, and 1 point on the regression line. This pattern is shown in the residual plot in **A**.





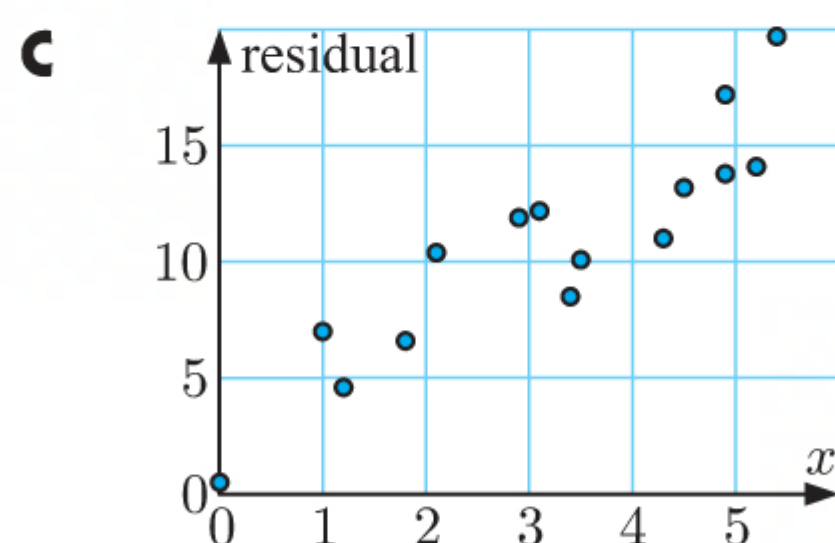
The values on the x -axis do not correspond to those on the scatter diagram.

So **A** is not the correct residual plot.



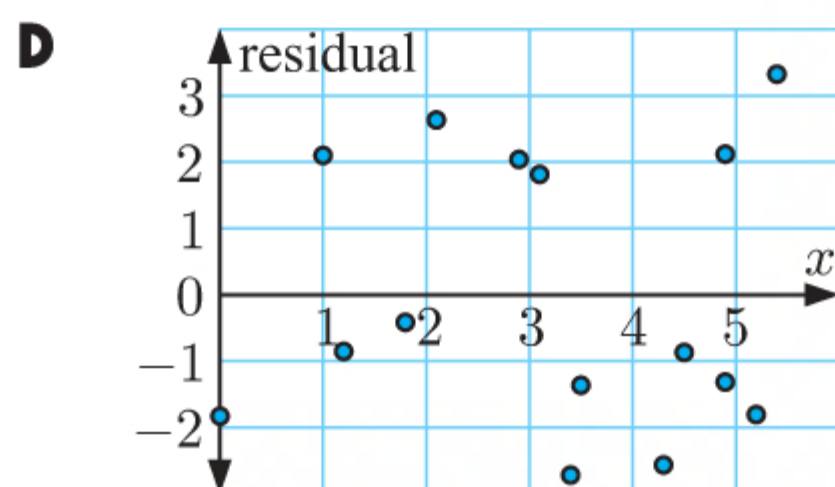
The residuals in this plot indicate that there are values which are more than 4 units from the regression line. The scatter diagram however shows that the values are within about ± 3 of the regression line.

So **B** is not the correct residual plot.



The residuals in this plot indicate that all values are above the regression line. The scatter diagram however shows that there are values below the regression line.

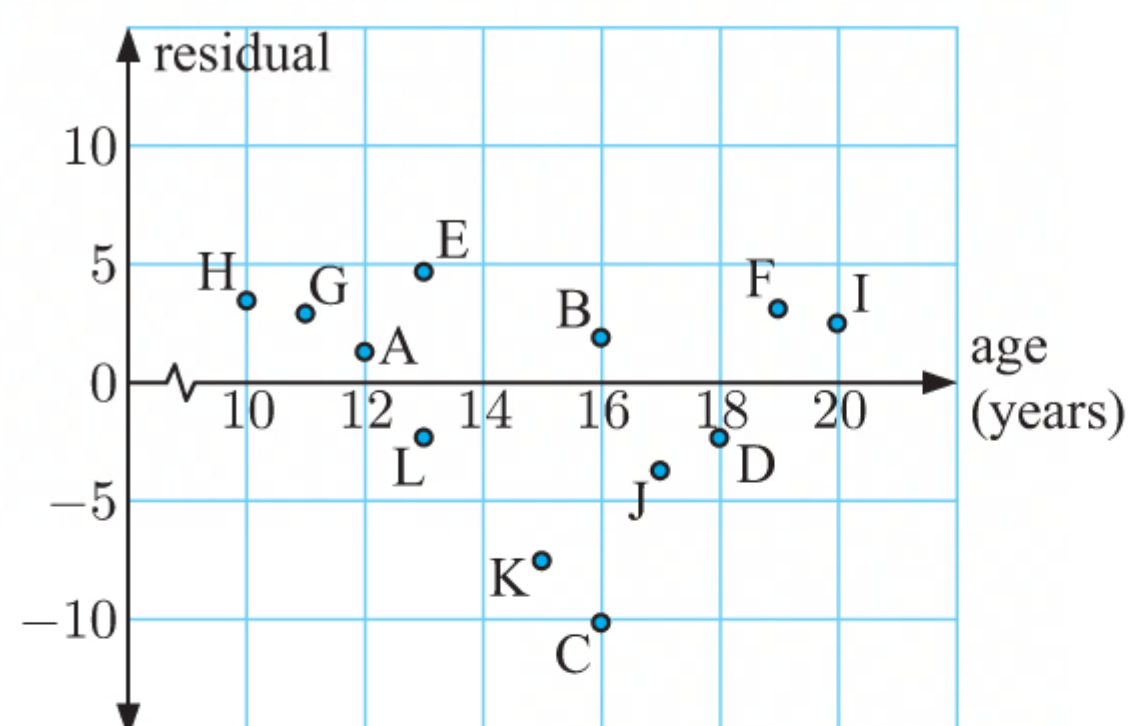
So **C** is not the correct residual plot.



The residuals in this plot indicate that all values are within about ± 3 of the regression line. The scatter diagram shows that the values are within about ± 3 of the regression line.

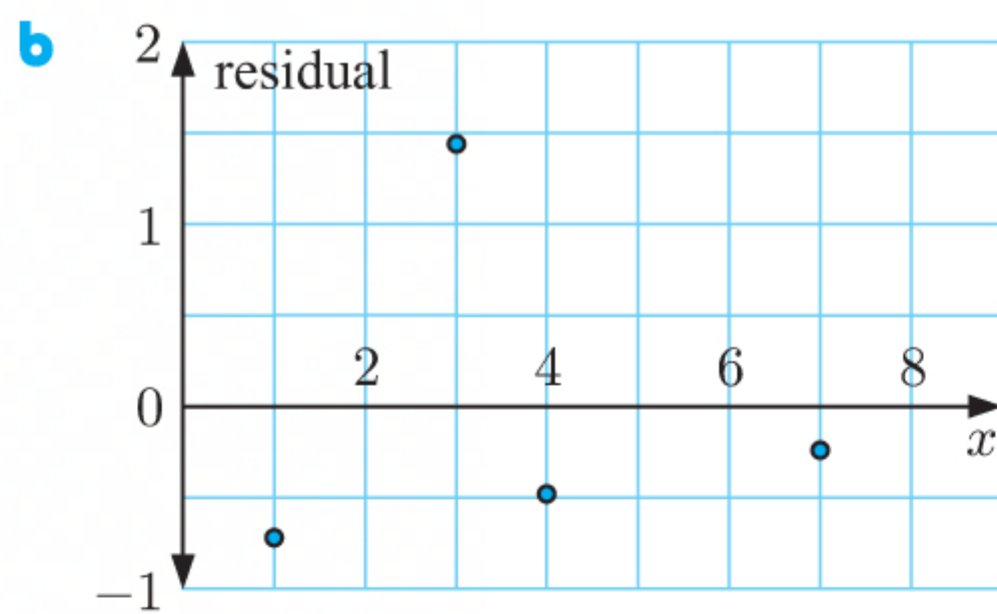
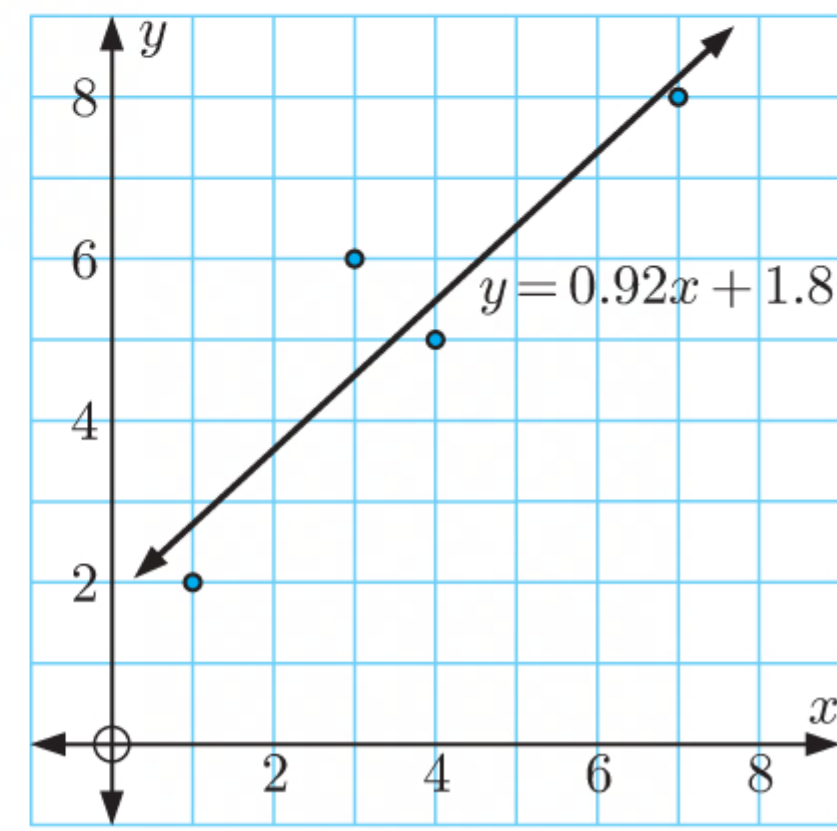
So **D** is the correct residual plot.

- 3**
- a** The athletes corresponding to the points above the x -axis threw the discus further than expected. These were athletes H, G, A, E, B, F, and I.
 - b** The athlete closest to the x -axis is A. So athlete A performed closest to what the linear model predicted.
 - c** No, it is not possible to determine which athlete threw the discus furthest. The residual plot only shows the difference between the actual distance and the predicted distance.



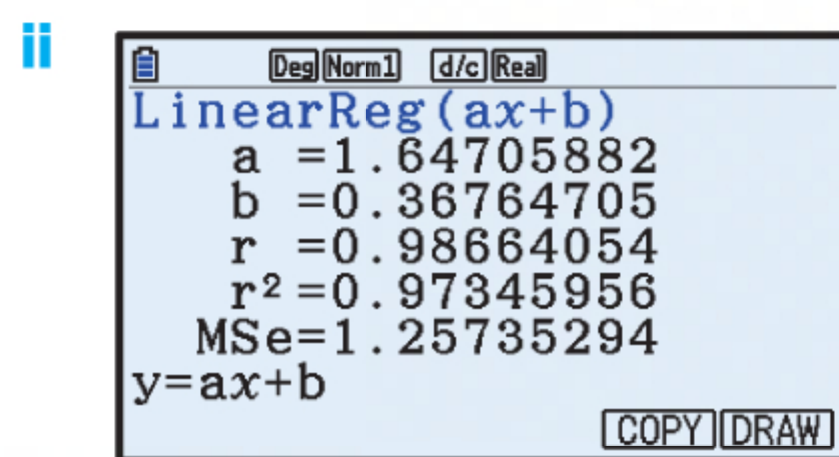
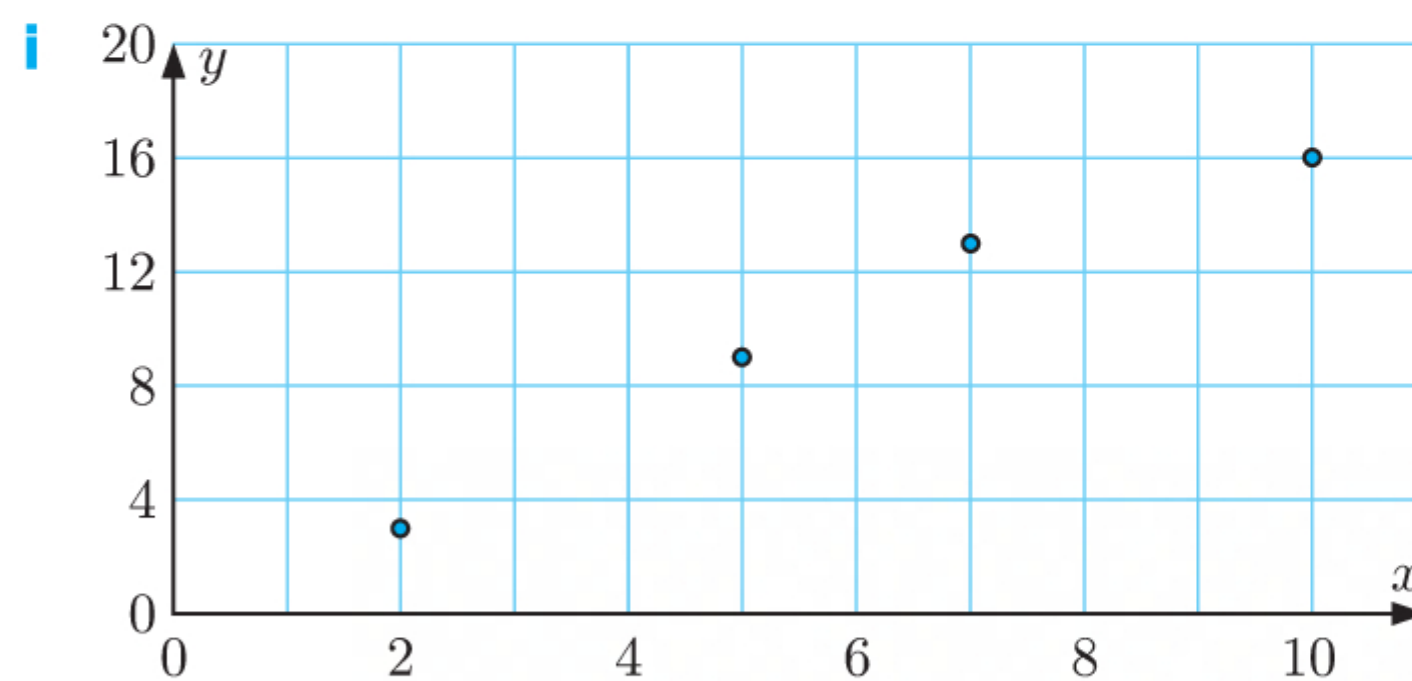
- 4 a** When $x = 3$, $y = 0.92(3) + 1.8 = 4.56$
 When $x = 4$, $y = 0.92(4) + 1.8 = 5.48$
 When $x = 7$, $y = 0.92(7) + 1.8 = 8.24$
 So, the table is:

x	y_{obs}	y_{pred}	residual = $y_{\text{obs}} - y_{\text{pred}}$
1	2	2.72	-0.72
3	6	4.56	1.44
4	5	5.48	-0.48
7	8	8.24	-0.24



5 a

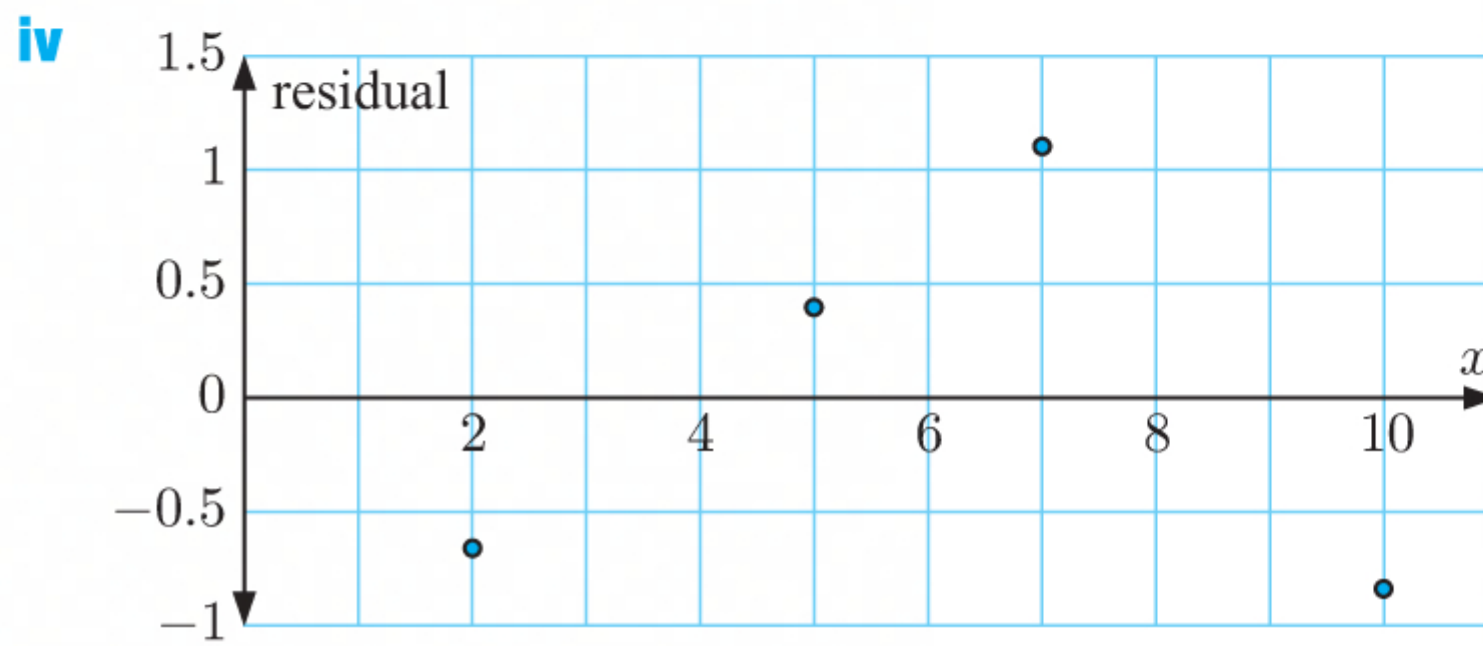
x	2	5	7	10
y	3	9	13	16



Using technology, the least squares regression line is $y \approx 1.65x + 0.368$.

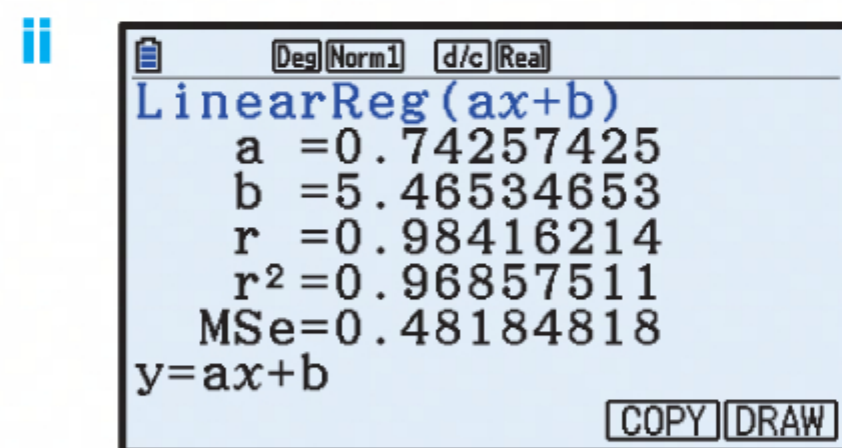
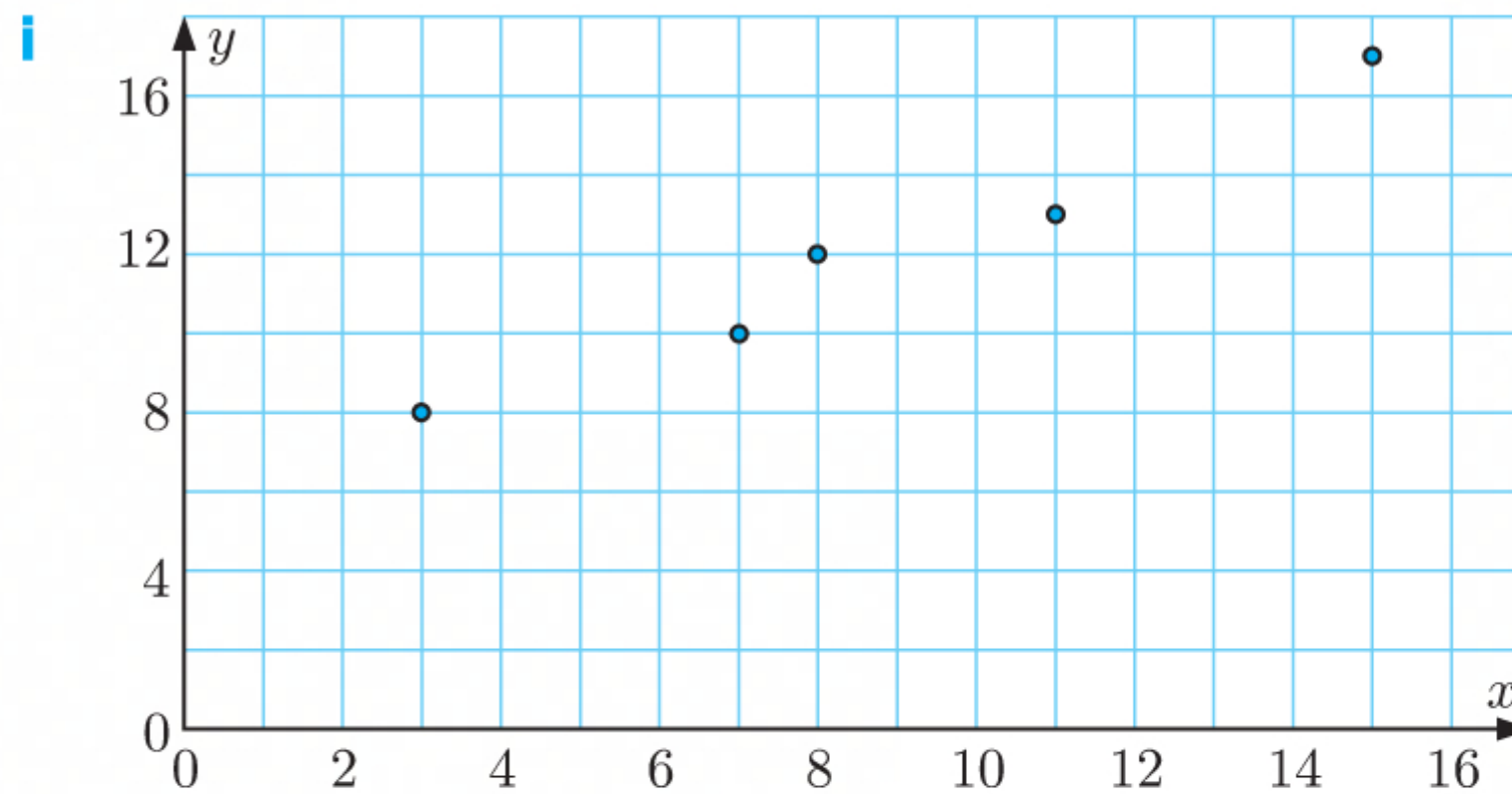
- iii** We find y_{pred} for each data point by evaluating $y \approx 1.65x + 0.368$ for each of the x -values.

x	y_{obs}	y_{pred}	residual = $y_{\text{obs}} - y_{\text{pred}}$
2	3	3.66	-0.66
5	9	8.60	0.40
7	13	11.90	1.10
10	16	16.84	-0.84



b

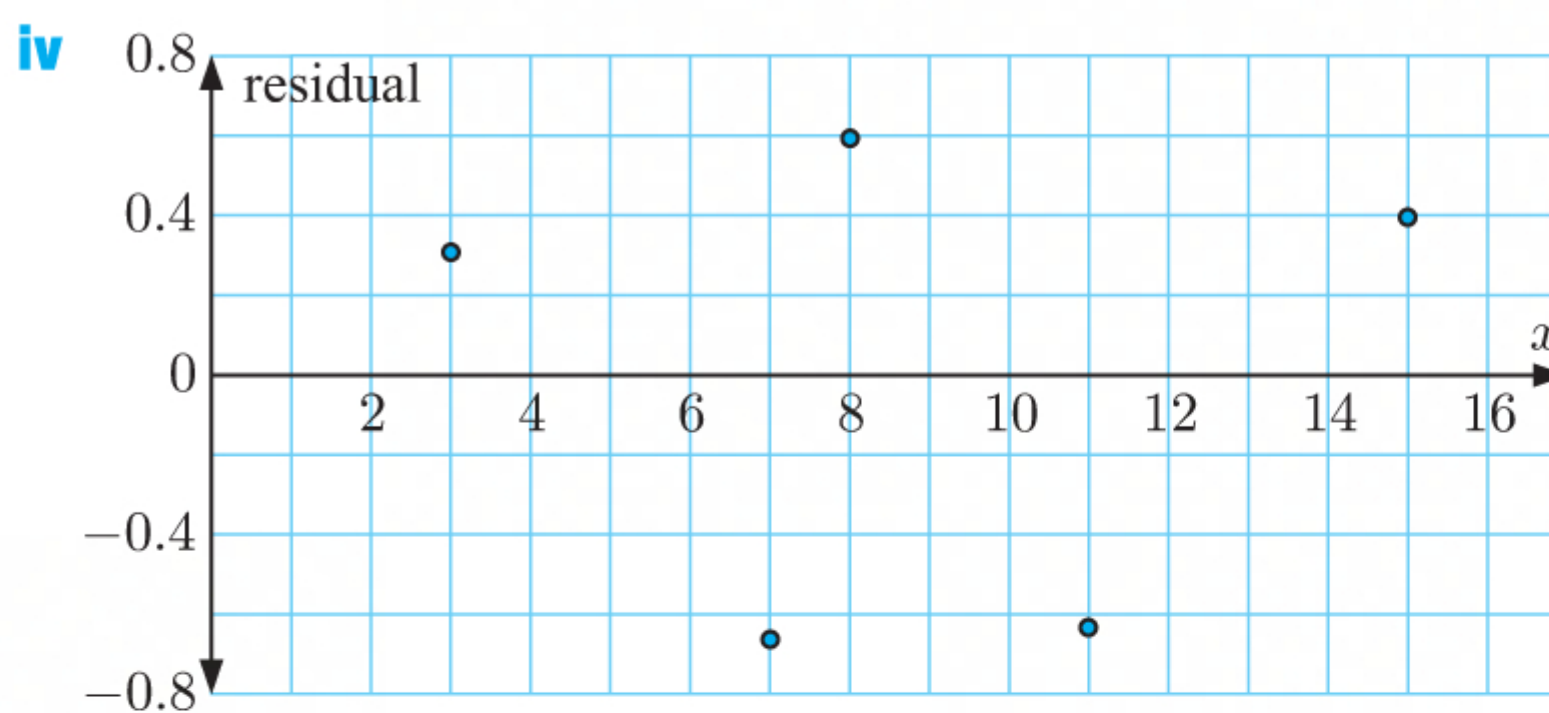
x	3	7	8	11	15
y	8	10	12	13	17



Using technology, the least squares regression line is $y \approx 0.743x + 5.47$.

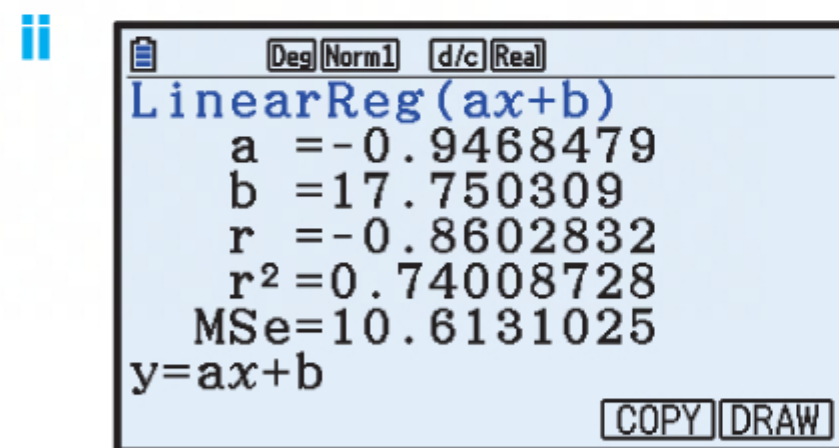
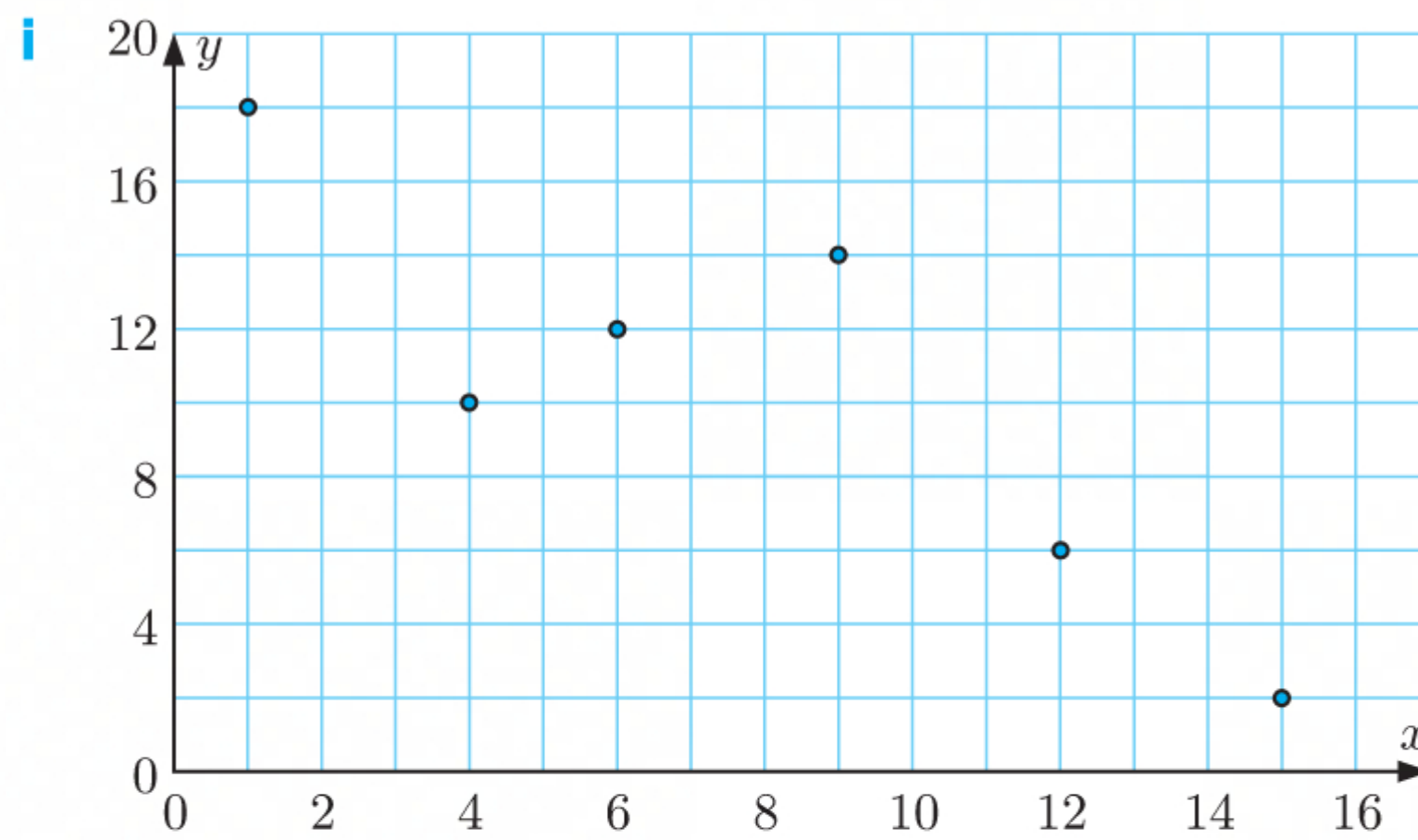
- iii** We find y_{pred} for each data point by evaluating $y \approx 0.743x + 5.47$ for each of the x -values.

x	y_{obs}	y_{pred}	residual = $y_{\text{obs}} - y_{\text{pred}}$
3	8	7.69	0.31
7	10	10.66	-0.66
8	12	11.41	0.59
11	13	13.63	-0.63
15	17	16.60	0.40



c

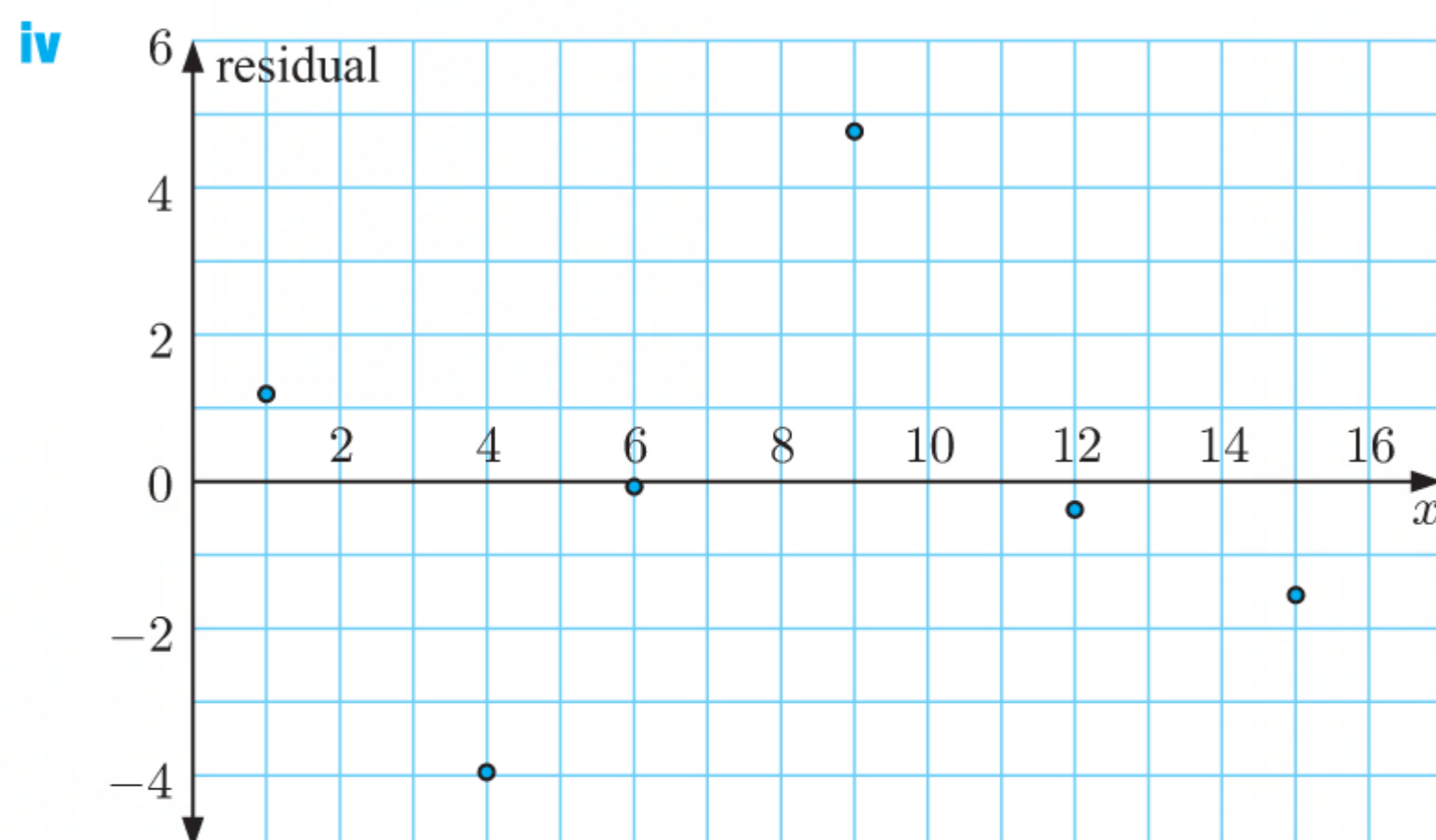
x	1	9	6	15	4	12
y	18	14	12	2	10	6

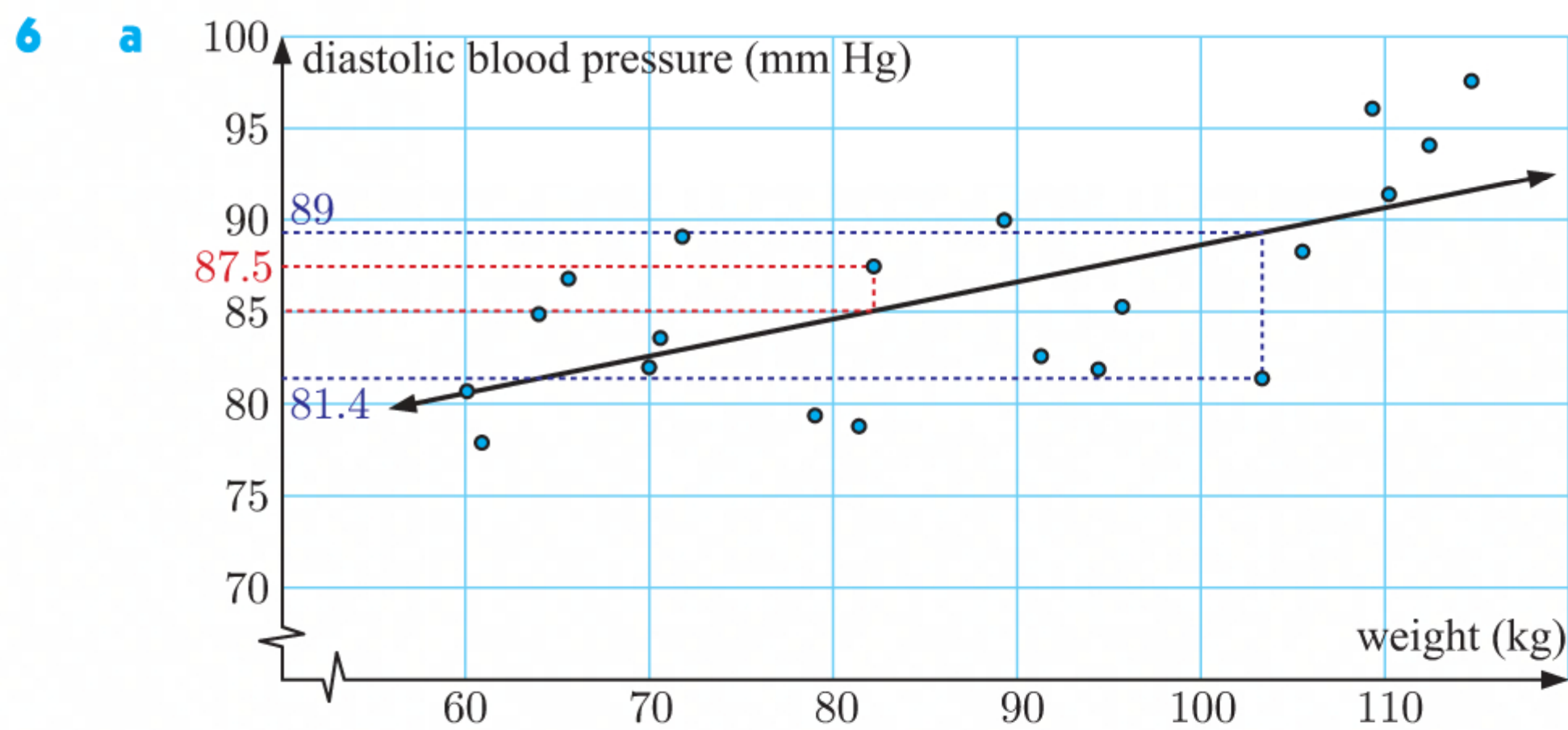


Using technology, the least squares regression line is $y \approx -0.947x + 17.8$.

- iii** We find y_{pred} for each data point by evaluating $y \approx -0.947x + 17.8$ for each of the x -values.

x	y_{obs}	y_{pred}	residual = $y_{\text{obs}} - y_{\text{pred}}$
1	18	16.80	1.20
9	14	9.23	4.77
6	12	12.07	-0.07
15	2	3.55	-1.55
4	10	13.96	-3.96
12	6	6.39	-0.39





- b i** From the graph, the residual for the point (82, 87.5) is about 2.5.

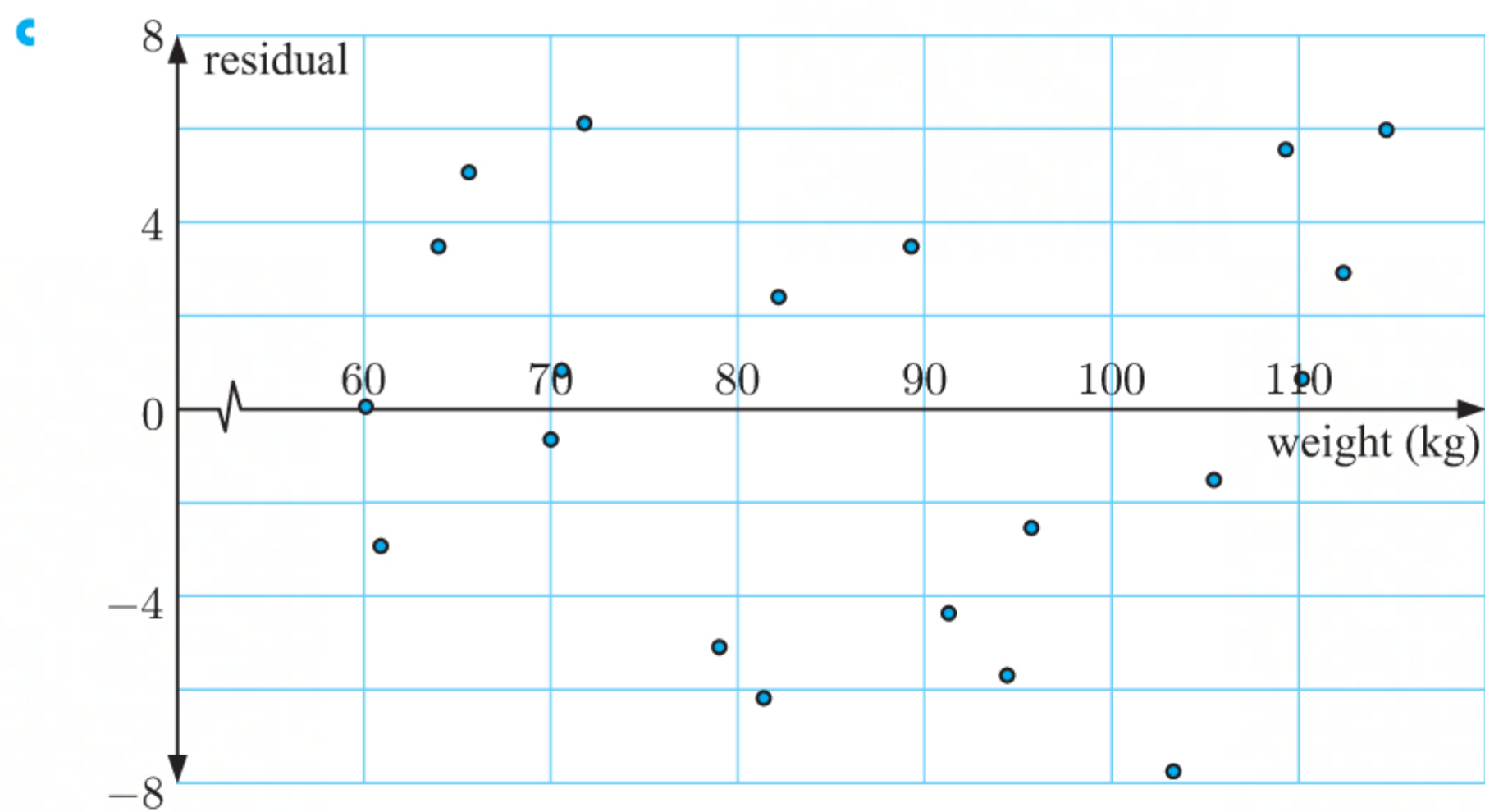
$$\begin{aligned} \text{From the equation, when } x = 82, \quad y_{\text{pred}} &= 68.5 + 0.2 \times 82 \\ &= 84.9 \end{aligned}$$

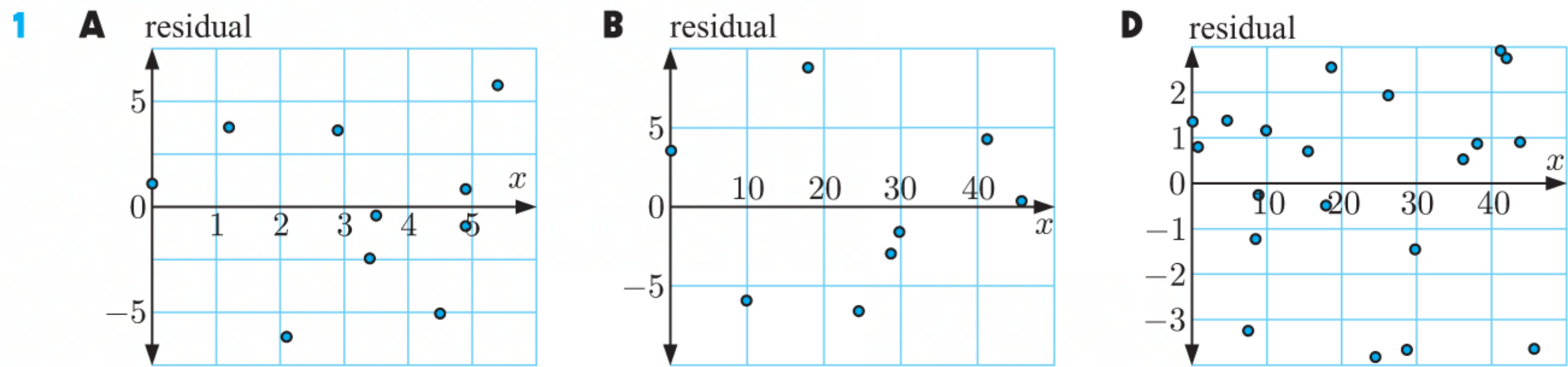
$$\begin{aligned} \therefore \text{ the residual} &= y_{\text{obs}} - y_{\text{pred}} \\ &= 87.5 - 84.9 \\ &= 2.6 \end{aligned}$$

- ii** From the graph, the residual for the point (103.3, 81.4) is about -8.

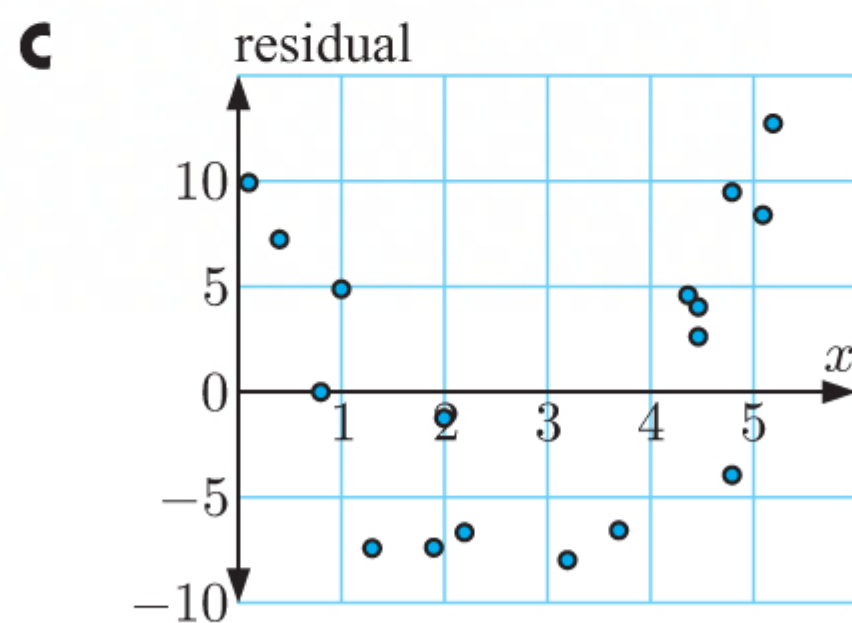
$$\begin{aligned} \text{From the equation, when } x = 103.3, \quad y_{\text{pred}} &= 68.5 + 0.2 \times 103.3 \\ &= 89.16 \end{aligned}$$

$$\begin{aligned} \therefore \text{ the residual} &= y_{\text{obs}} - y_{\text{pred}} \\ &= 81.4 - 89.16 \\ &= -7.76 \end{aligned}$$



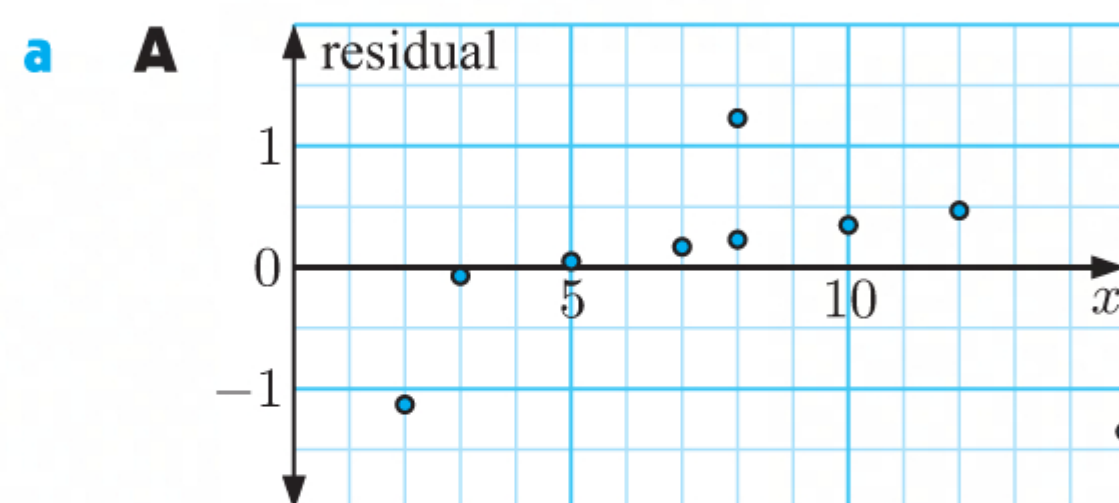
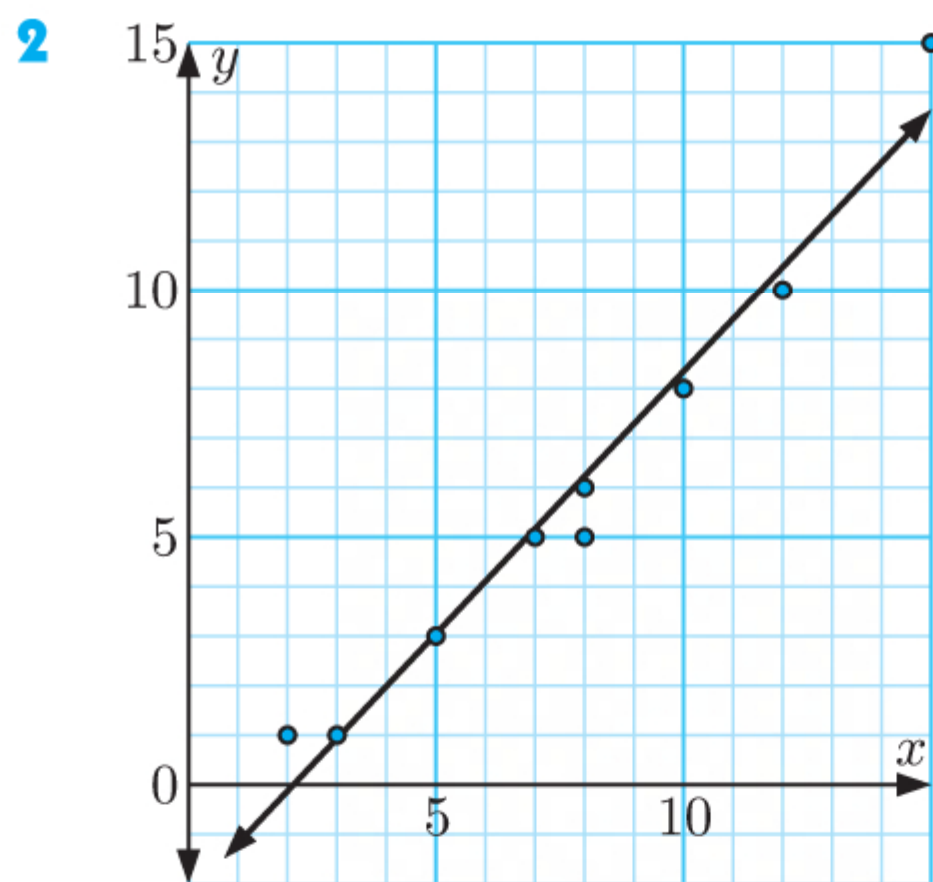
PART 2: ANALYSING RESIDUAL PLOTS

The residual plots for **A**, **B**, and **D** show points randomly scattered about the x -axis, with no obvious pattern.



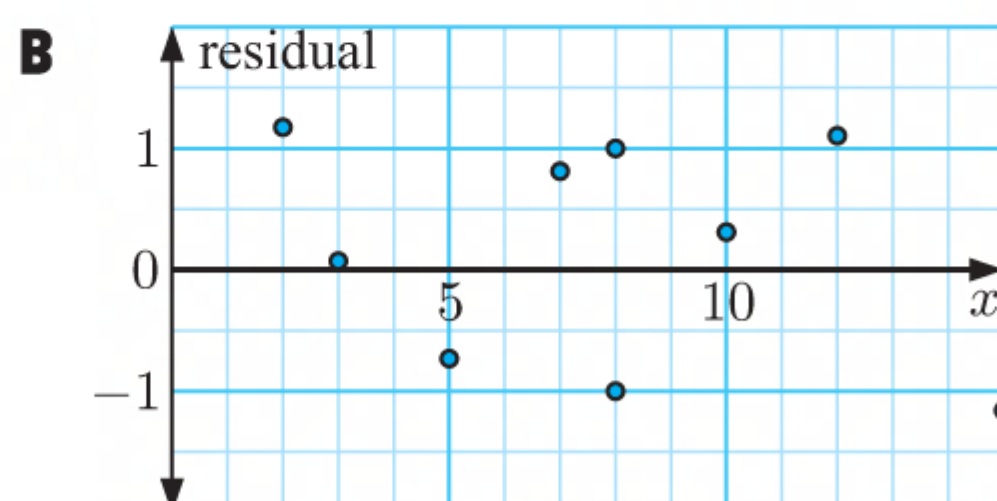
The residual plot for **C** however shows a clear, non-random pattern.

So the residual plot for **C** shows a regression line which is not a good fit for the data.



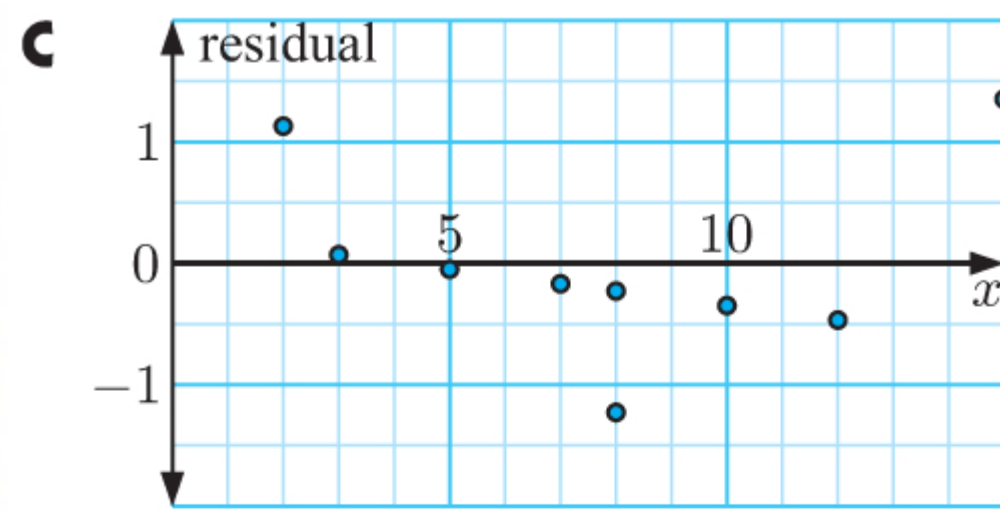
Most of the residuals in this plot are above the x -axis. The scatter diagram however shows that only three data values are above the regression line.

So **A** is not the correct residual plot.



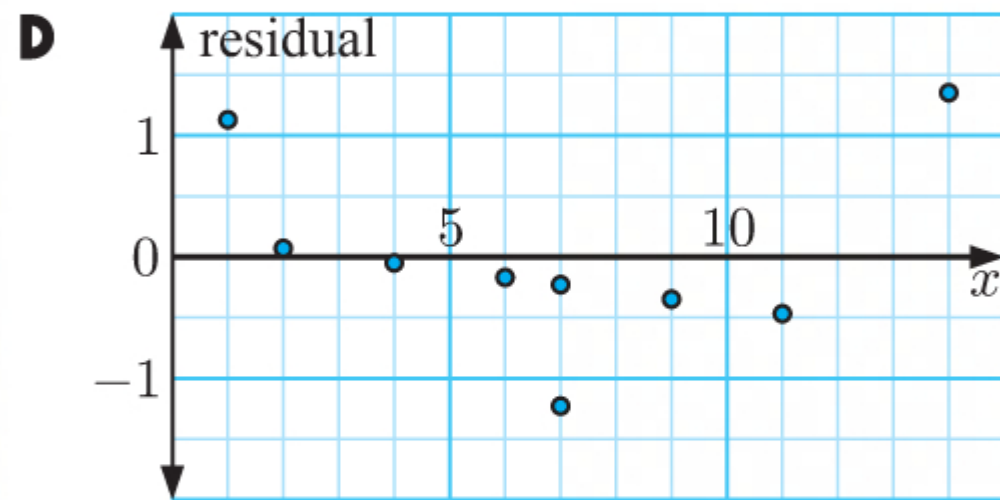
Most of the residuals in this plot are above the x -axis. The scatter diagram however shows that only three data values are above the regression line.

So **B** is not the correct residual plot.



Most of the residuals in this plot are below the x -axis, with only three residuals above it. The scatter diagram shows that only three data values are above the regression line, and the values on the x -axis correspond to those on the scatter diagram.

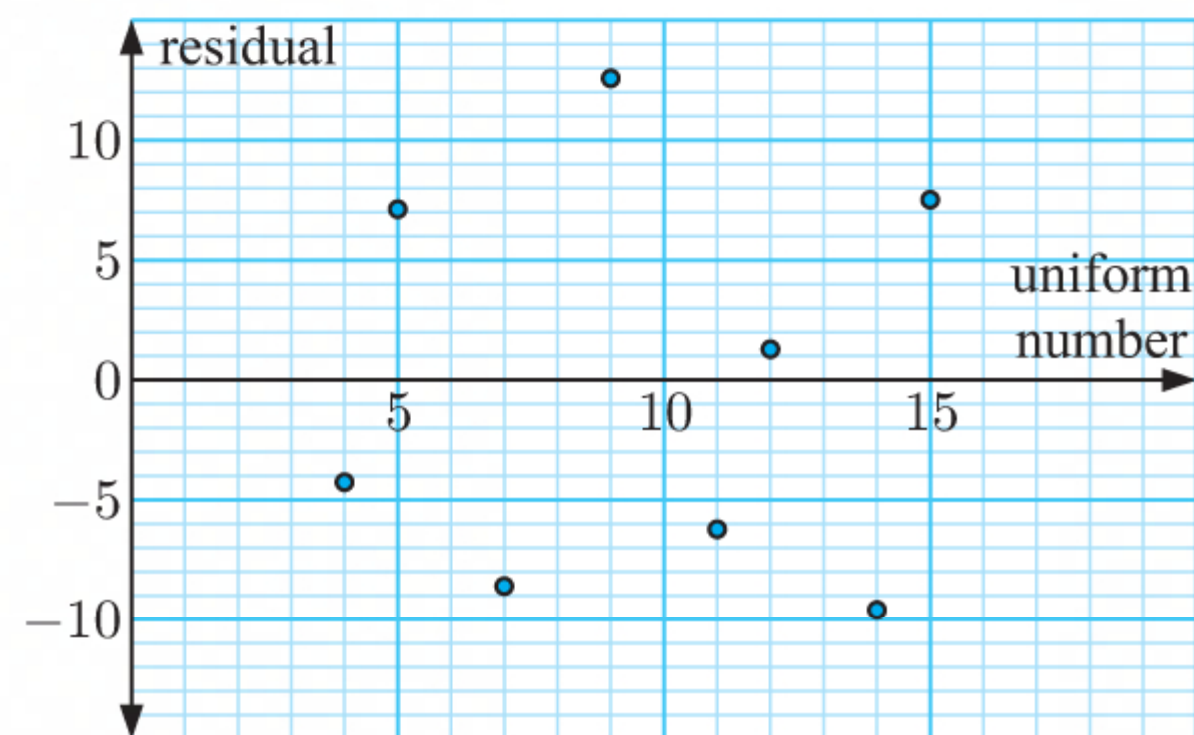
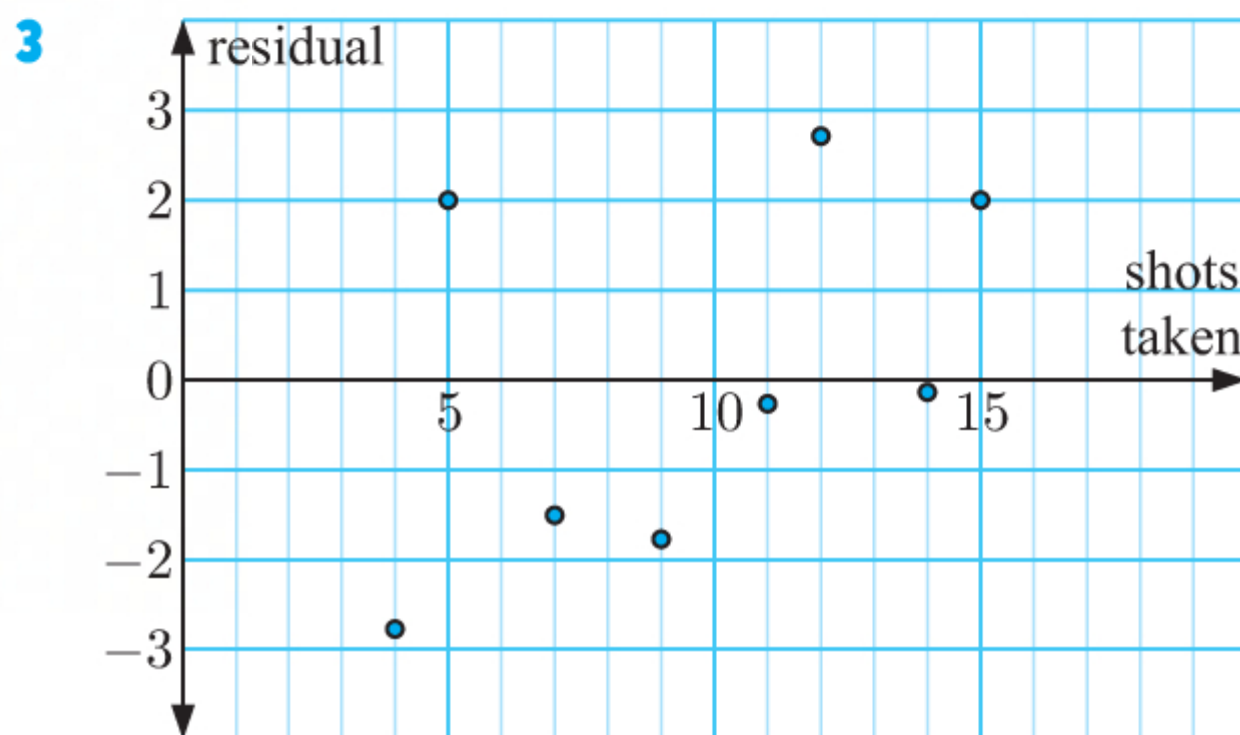
So **C** is the correct residual plot.



This is very similar to residual plot **C**, except the values on the x -axis do not correspond to those on the scatter diagram.

So **D** is not the correct residual plot.

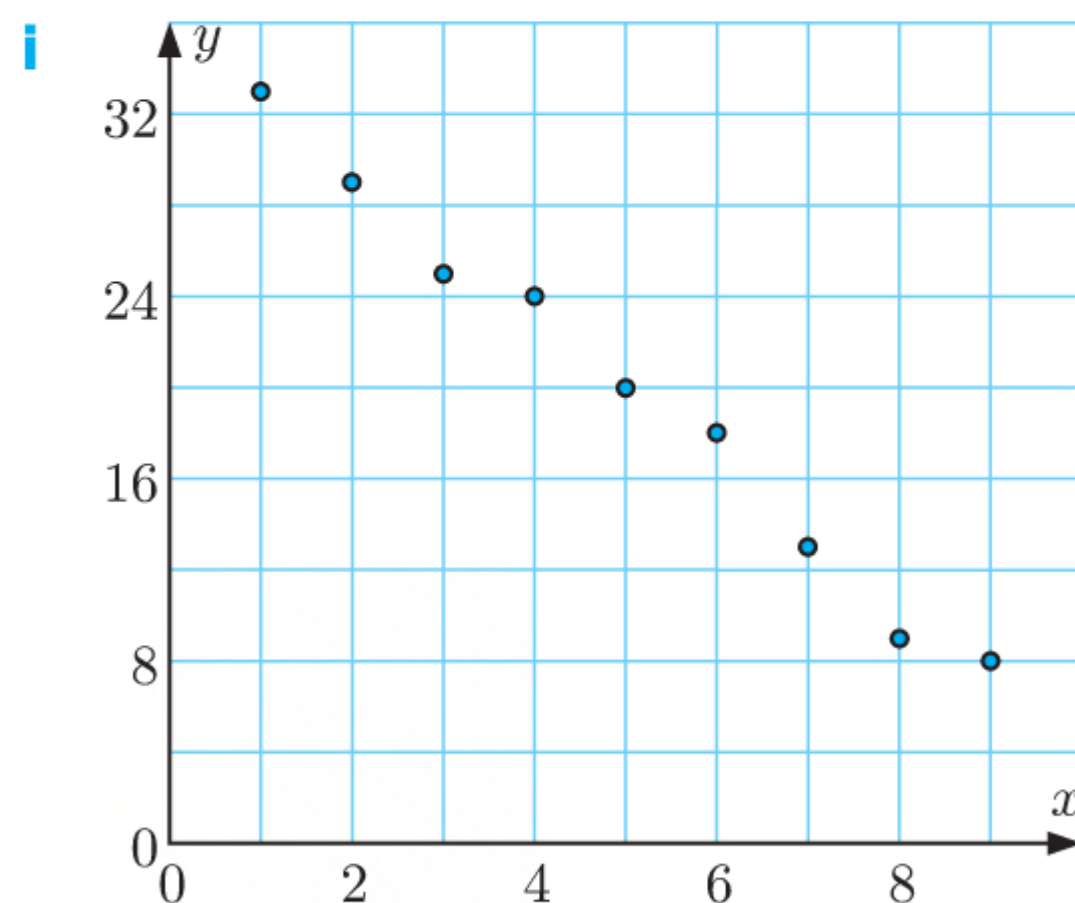
- b** The residual plot does not appear to be random, so a linear model may not be appropriate for the data.

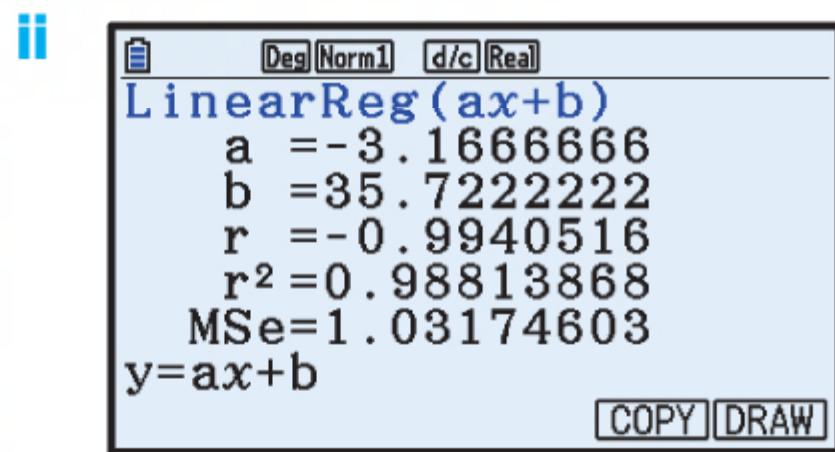


- a** Yes, the points in both plots appear to be randomly scattered.
- b** A linear model is most appropriate for the *points scored vs shots taken* data set. The residuals in this plot are generally smaller, which means that the points are generally closer to the least squares regression line.

4 a

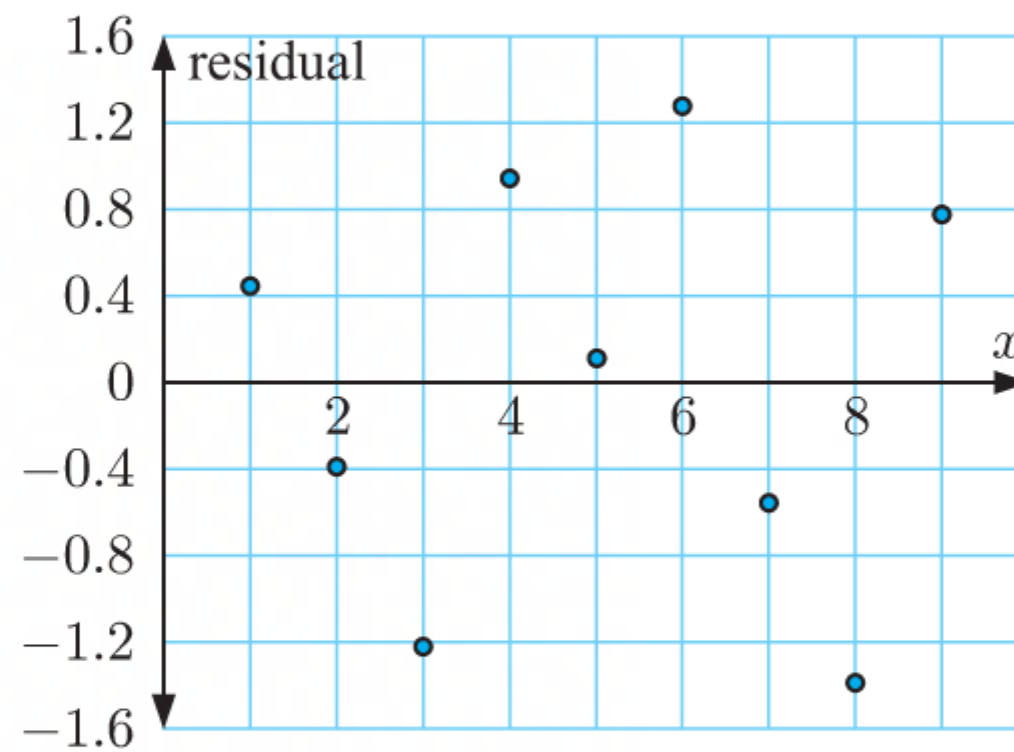
x	1	2	3	4	5	6	7	8	9
y	33	29	25	24	20	18	13	9	8





Using technology, the least squares regression line is $y \approx -3.17x + 35.7$, and $r \approx -0.994$.

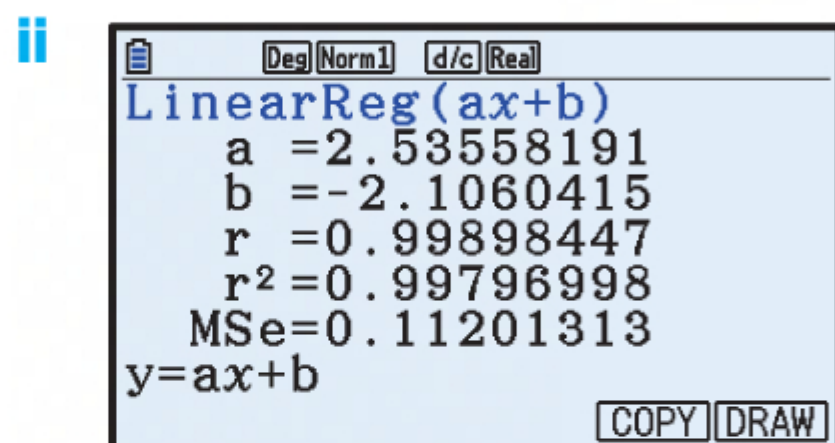
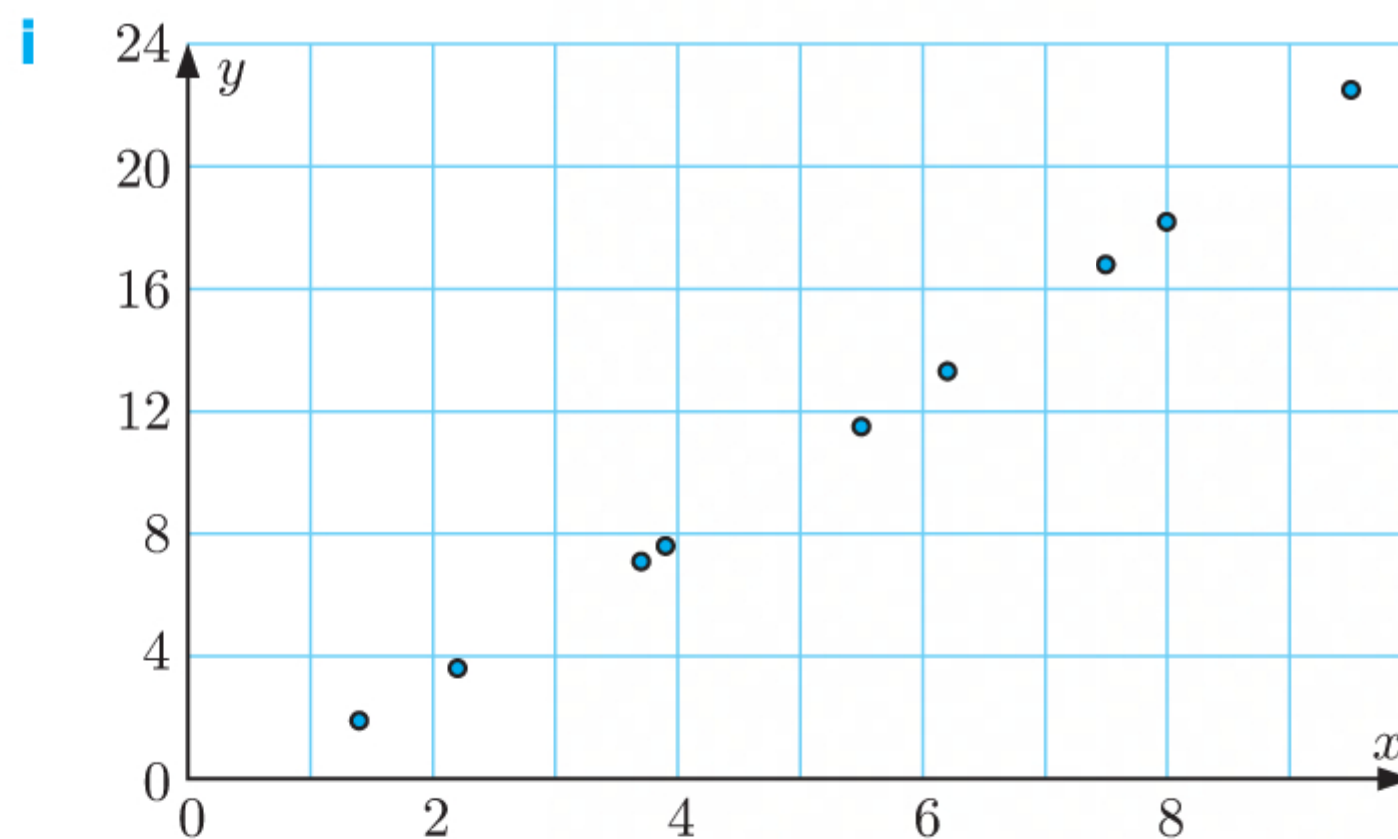
iii Using technology, the residual plot is:



iv There is very strong correlation and the residuals are randomly scattered, so the least squares regression line is appropriate to model the data.

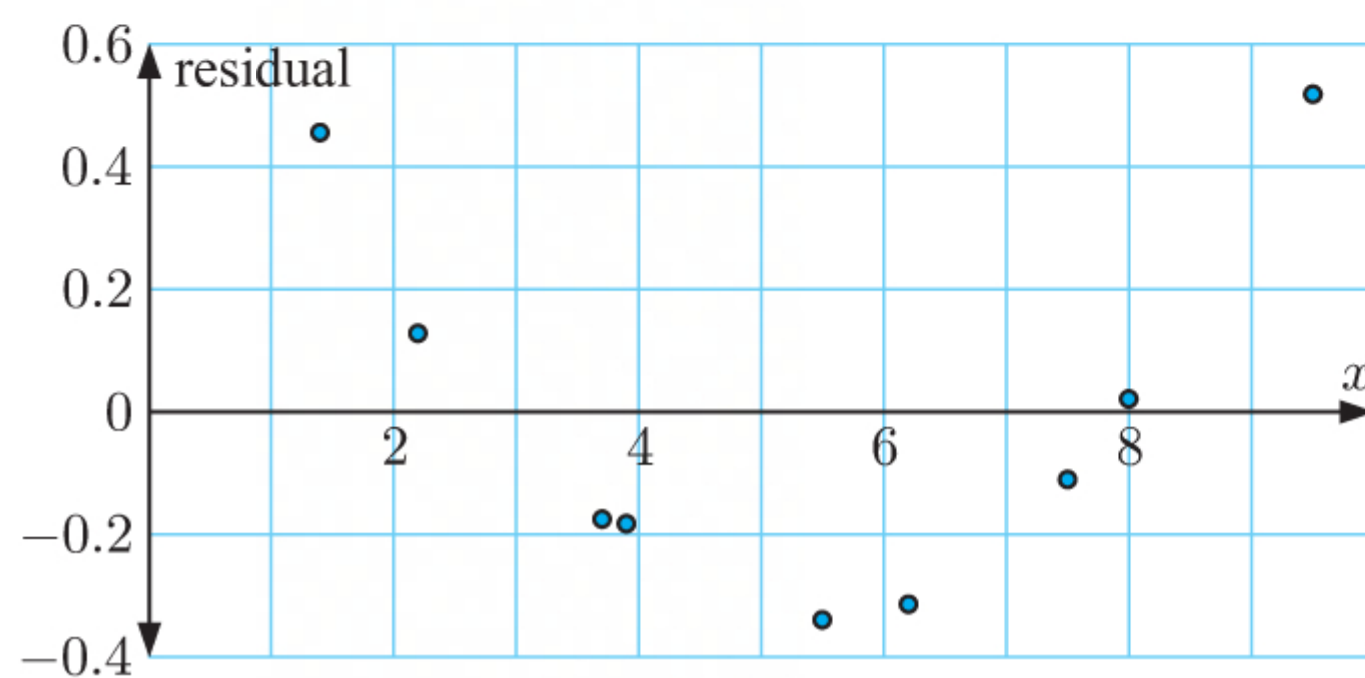
b

x	2.2	3.7	9.5	6.2	1.4	3.9	7.5	8	5.5
y	3.6	7.1	22.5	13.3	1.9	7.6	16.8	18.2	11.5



Using technology, the least squares regression line is $y \approx 2.54x - 2.11$, and $r \approx 0.999$.

iii Using technology, the residual plot is:

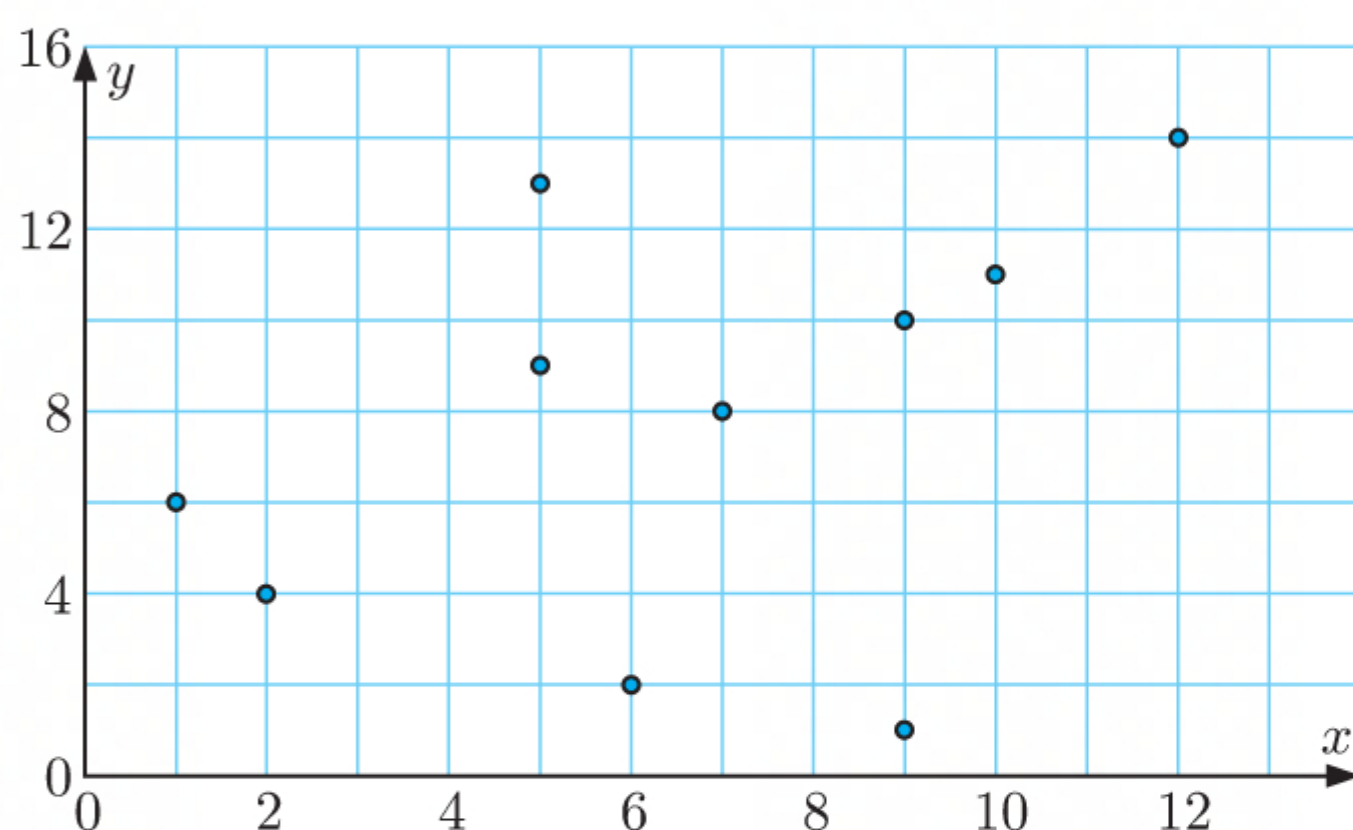


iv The residual plot shows a clear pattern and does not appear random. So the least squares regression line is not appropriate to model the data.

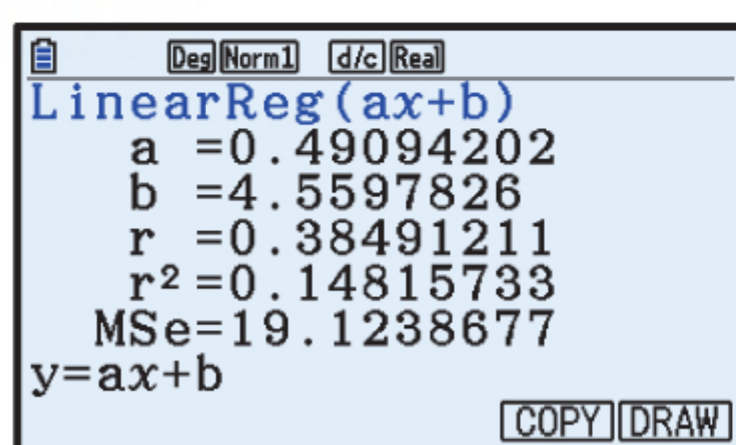
c

x	5	9	1	12	6	5	9	7	2	10
y	13	1	6	14	2	9	10	8	4	11

i

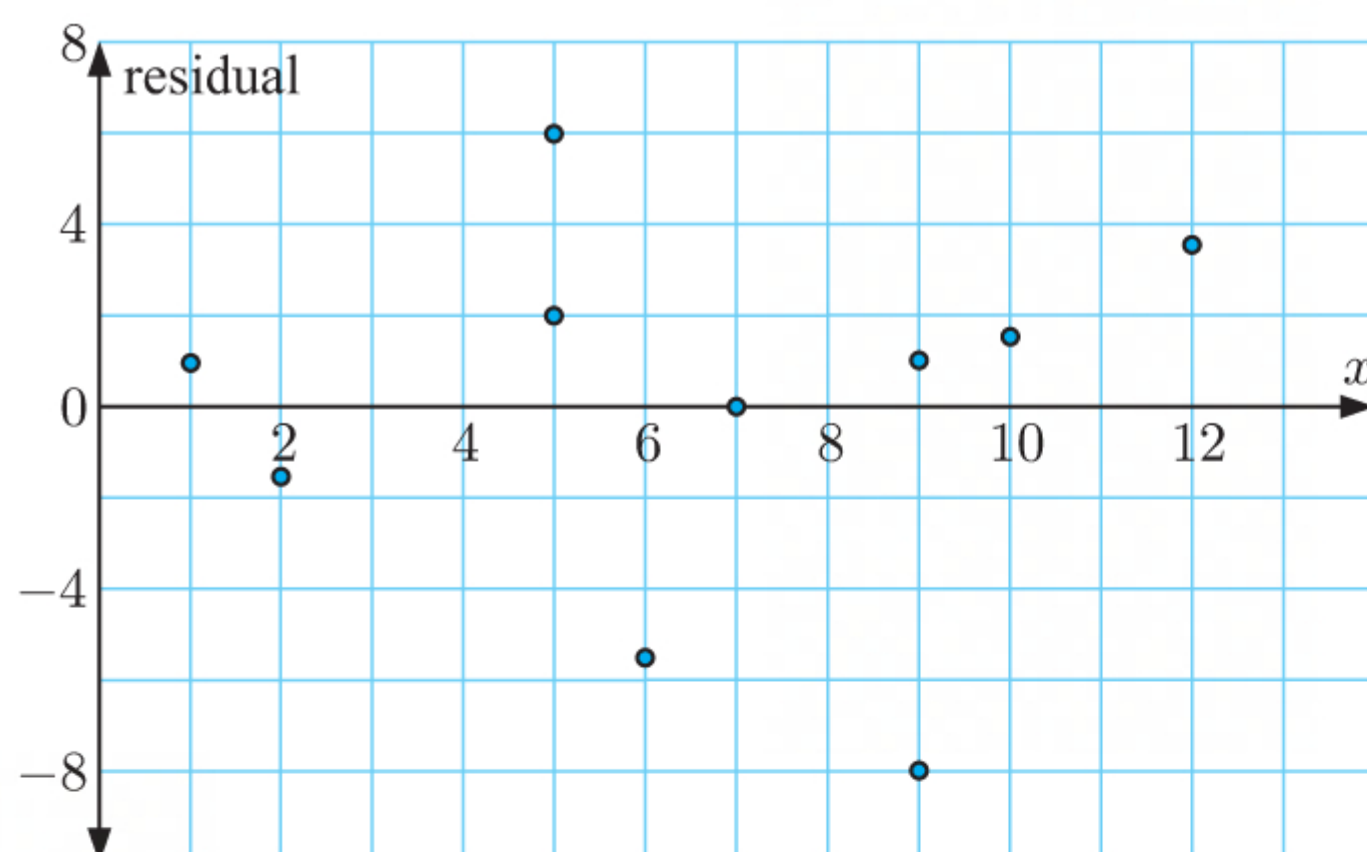


ii



Using technology, the least squares regression line is $y \approx 0.491x + 4.56$, and $r \approx 0.385$.

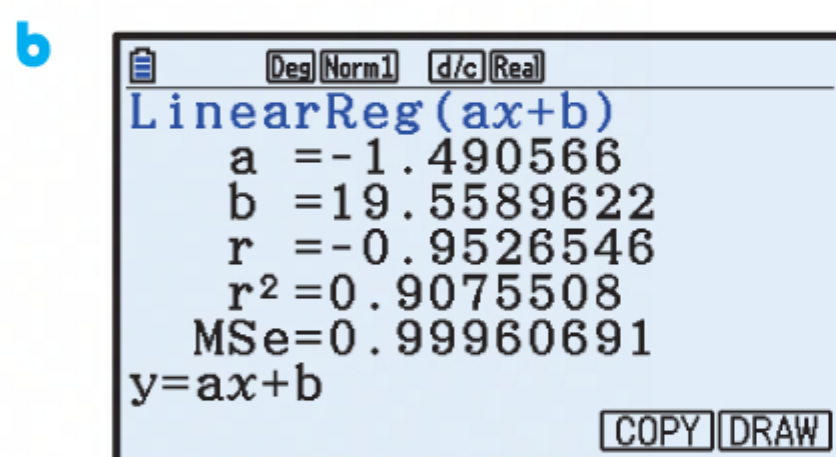
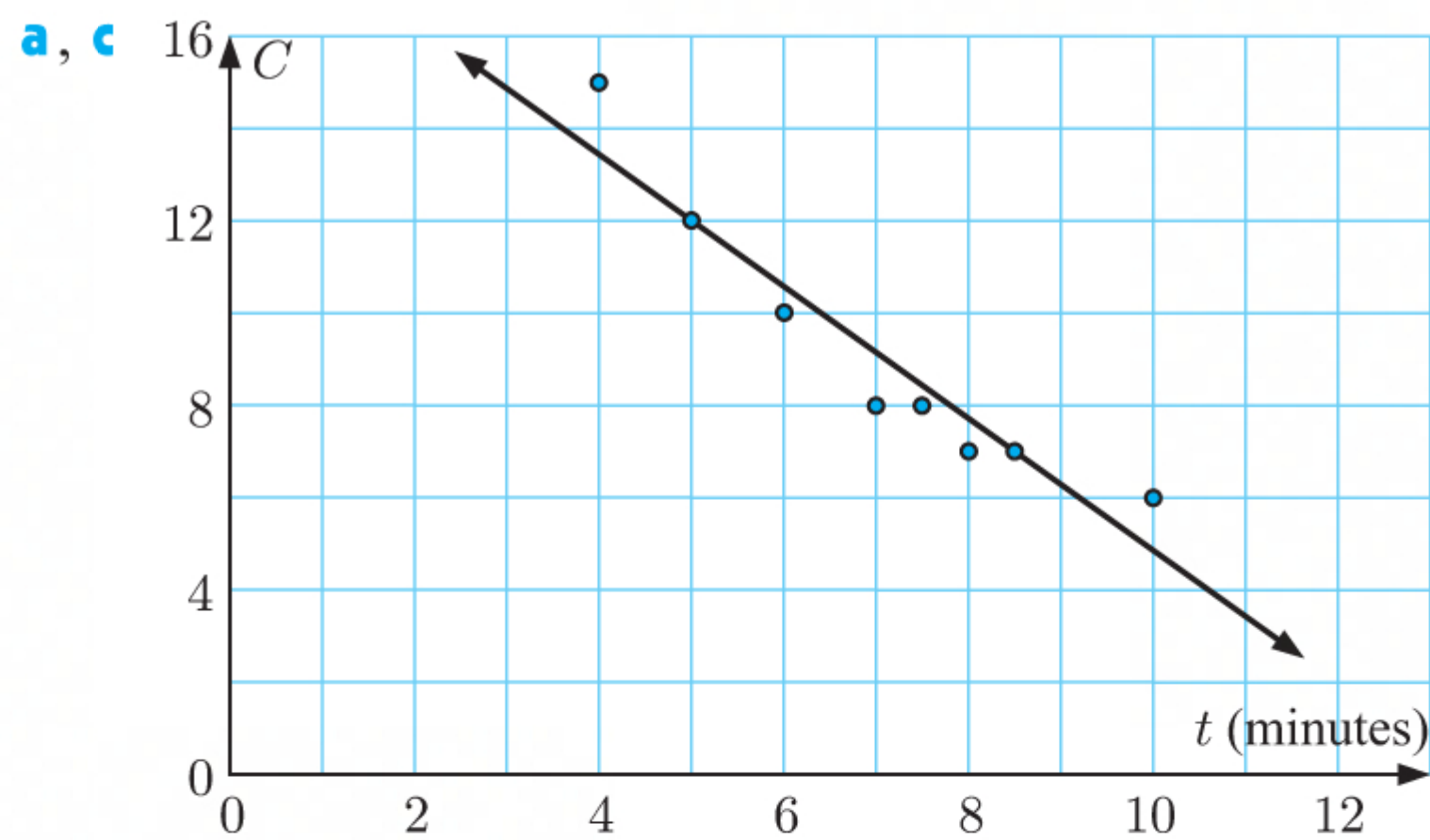
iii Using technology, the residual plot is:



- iv The value of r is very small which indicates there is very weak correlation. So the least squares regression line is not appropriate to model the data.

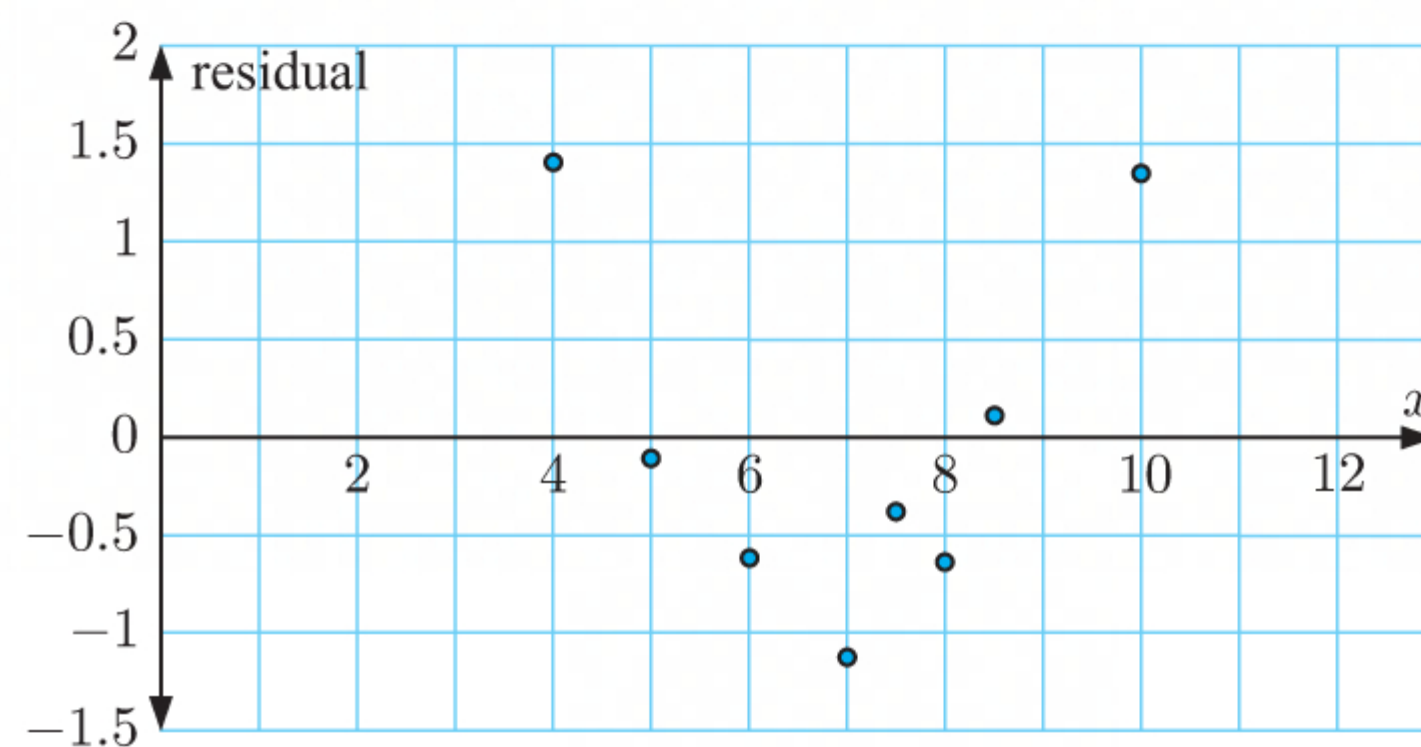
5

Time taken (t minutes)	6	8.5	4	5	8	7.5	10	7
Cranes made (C)	10	7	15	12	7	8	6	8



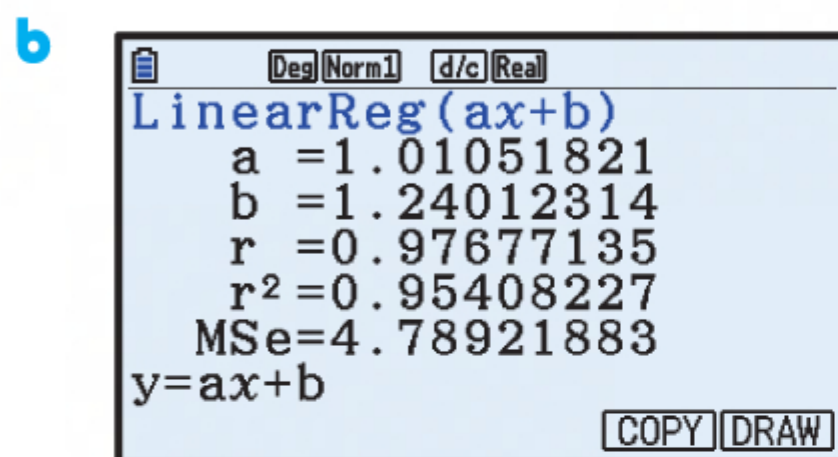
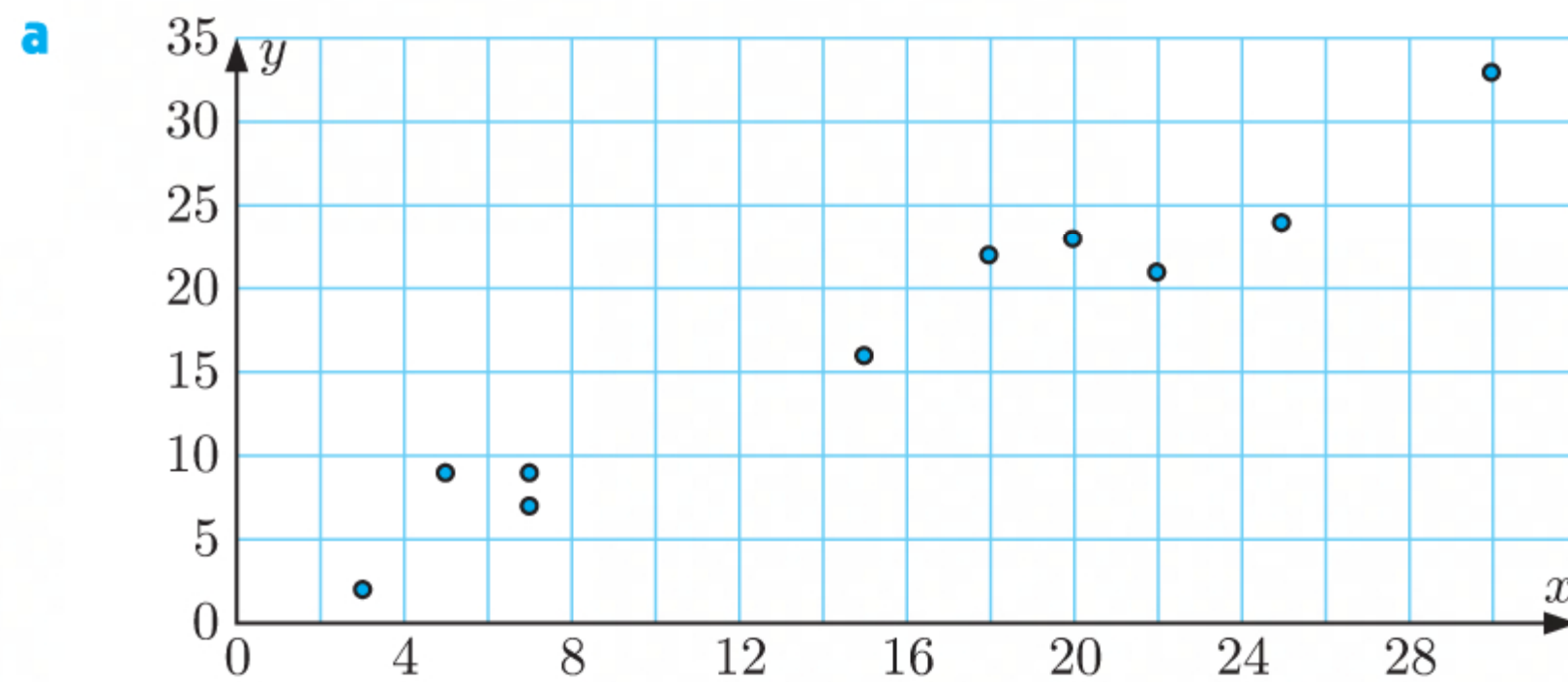
Using technology, the least squares regression line is $y \approx -1.49x + 19.6$, and $r \approx -0.953$.

- d Using technology, the residual plot is:



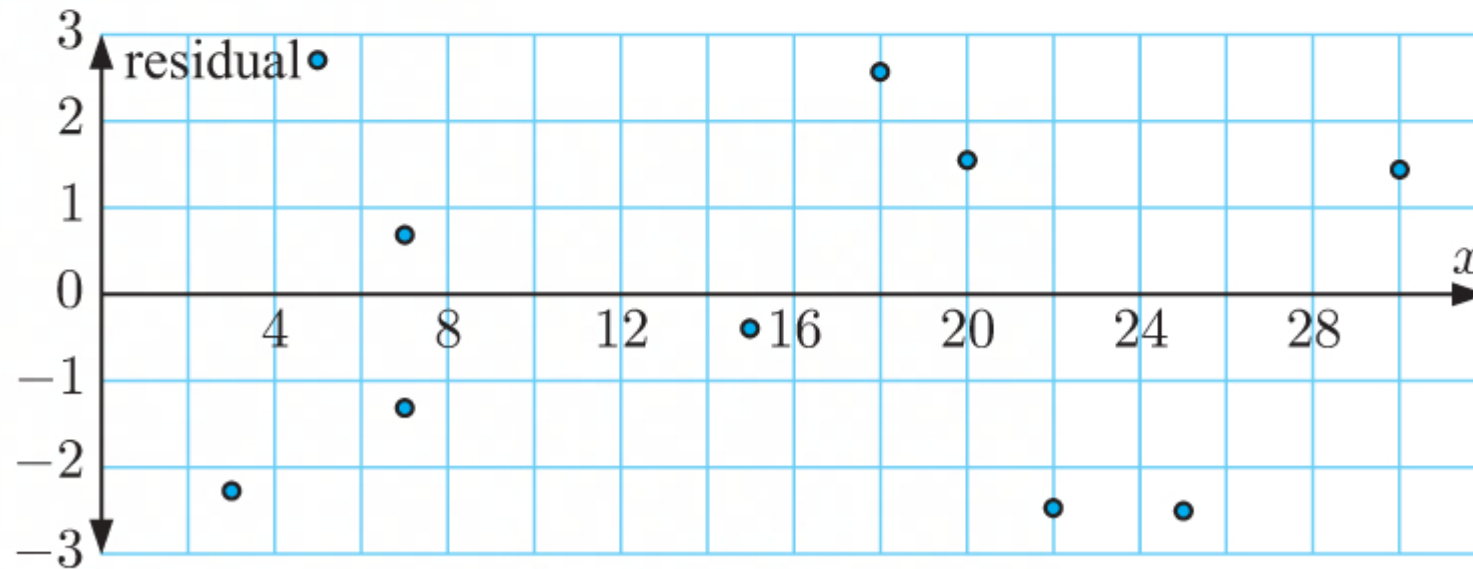
- e The residual plot shows a clear pattern, and does not appear random. So the least squares regression line is not appropriate to model the data.

6	<i>Text messages sent (x)</i>	18	3	7	22	15	5	20	30	7	25
	<i>Text messages received (y)</i>	22	2	9	21	16	9	23	33	7	24



Using technology, the least squares regression line is $y \approx 1.01x + 1.24$, and $r \approx 0.977$.

- c** There is very strong, positive correlation between *text messages sent* and *text messages received*.
- d** Using technology, the residual plot is:

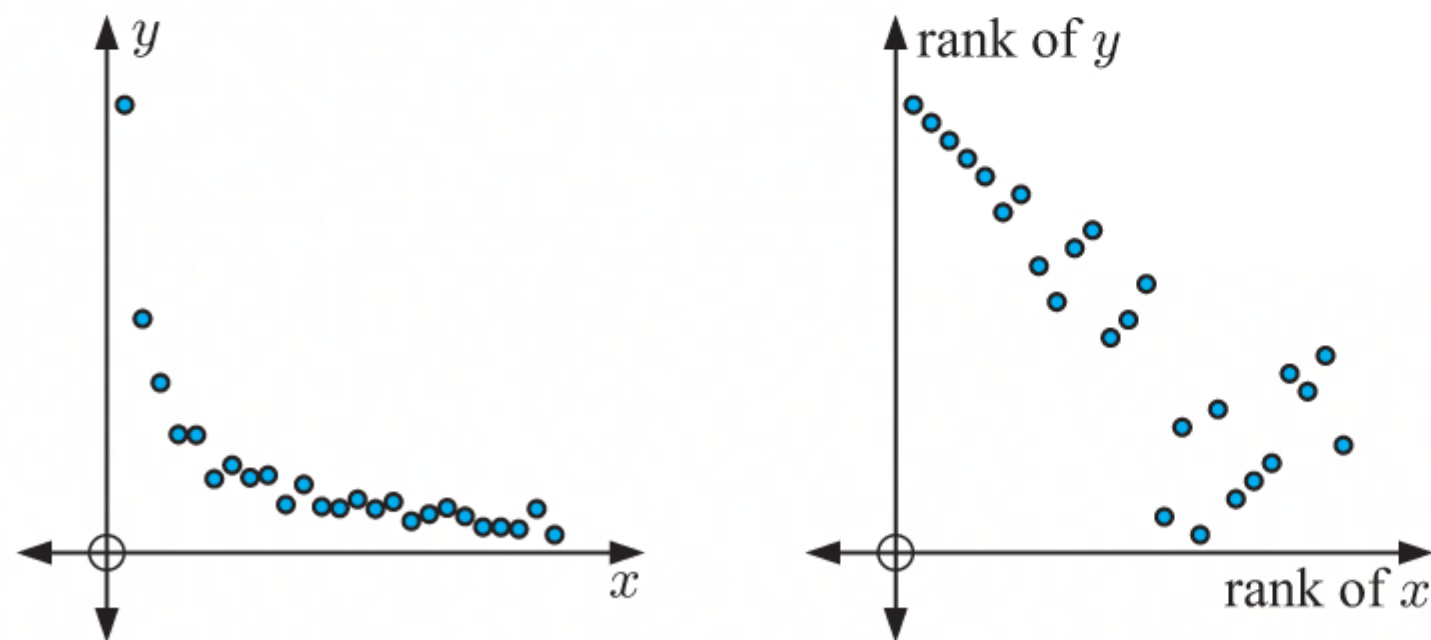


- e** There is very strong, positive correlation and the residuals are randomly scattered. So the least squares regression line is appropriate to model the data.

f **i** When $y = 10$, $10 \approx 1.01x + 1.24$
 $\therefore 8.76 \approx 1.01x$
 $\therefore x \approx 8.67$
 ≈ 9 {rounded to nearest integer}

So, we estimate Ted sent about 9 text messages.

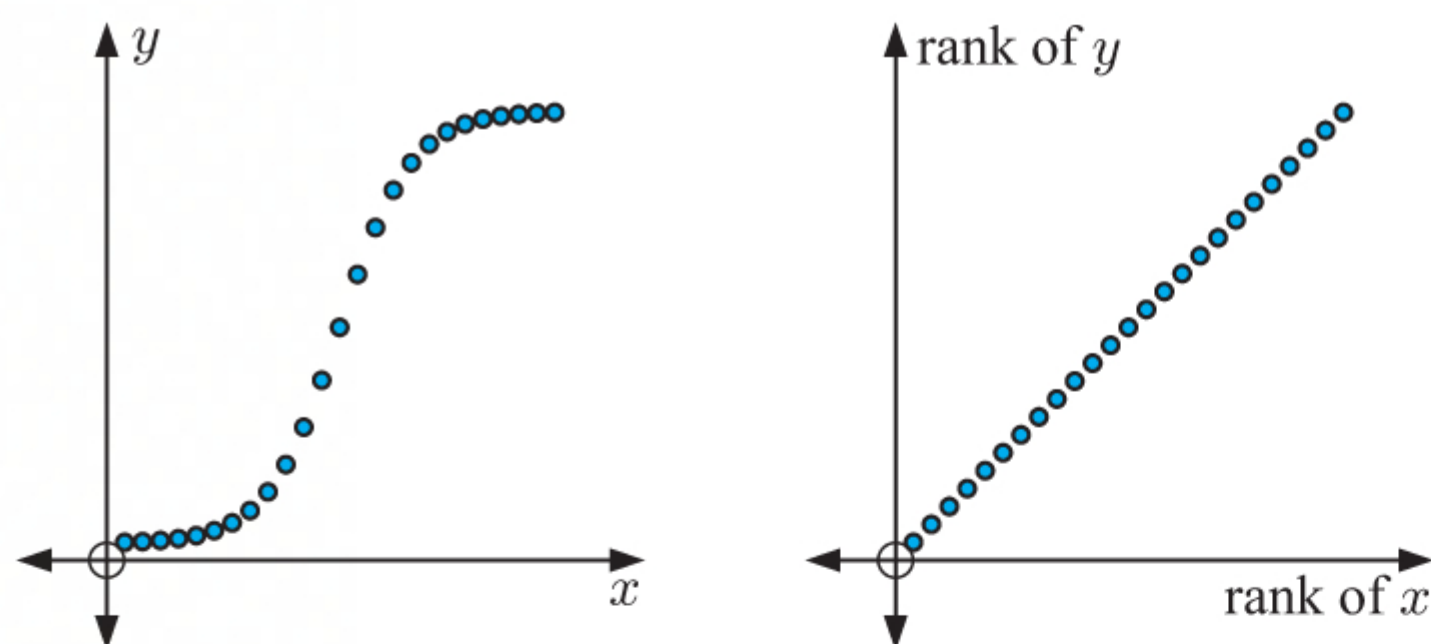
- ii** As the estimate is an interpolation with strongly correlated data, it is fairly reliable.

EXERCISE 5E**1 a**

In general, as x increases, y decreases.

So a higher rank of x generally corresponds to a lower rank of y .

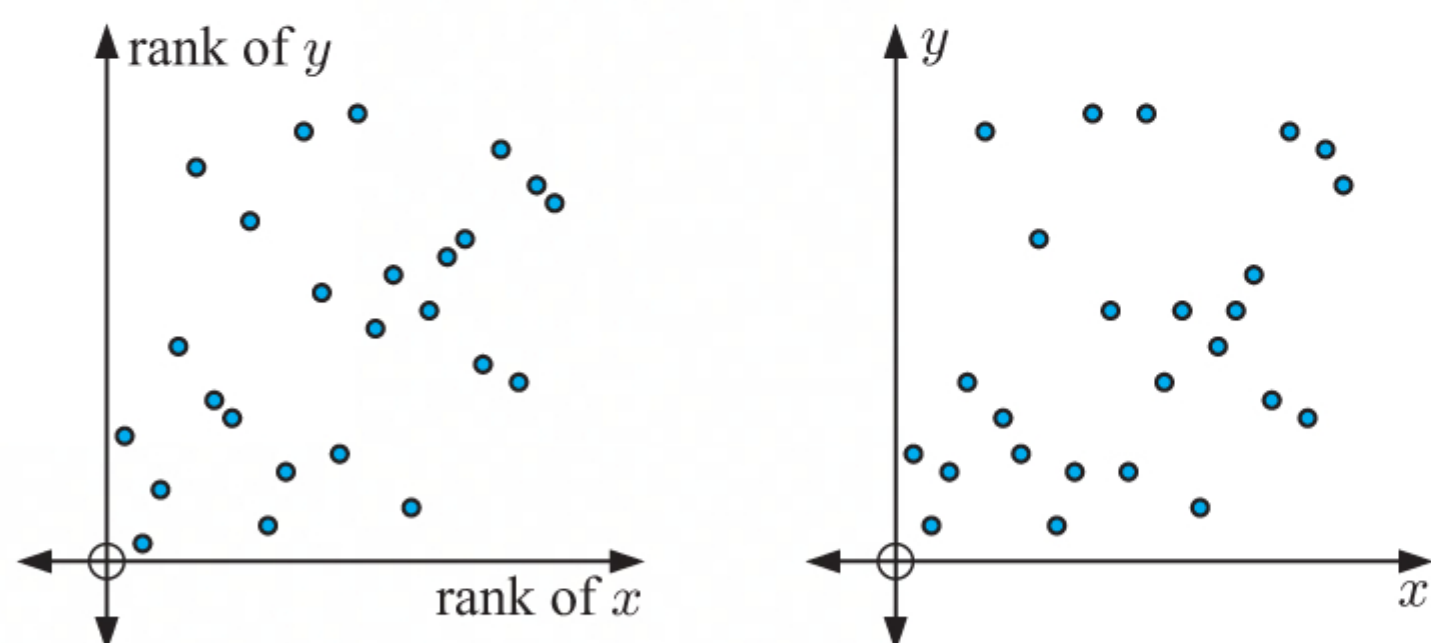
The correct scatter rank diagram is **B**.

b

As x increases, y always increases.

So for each data point (x, y) , $\text{rank of } x = \text{rank of } y$.

The correct scatter rank diagram is **C**.

c

There appears to be no correlation in the scatter diagram. So we expect to see no correlation in the corresponding rank scatter diagram.

The correct scatter rank diagram is **A**.

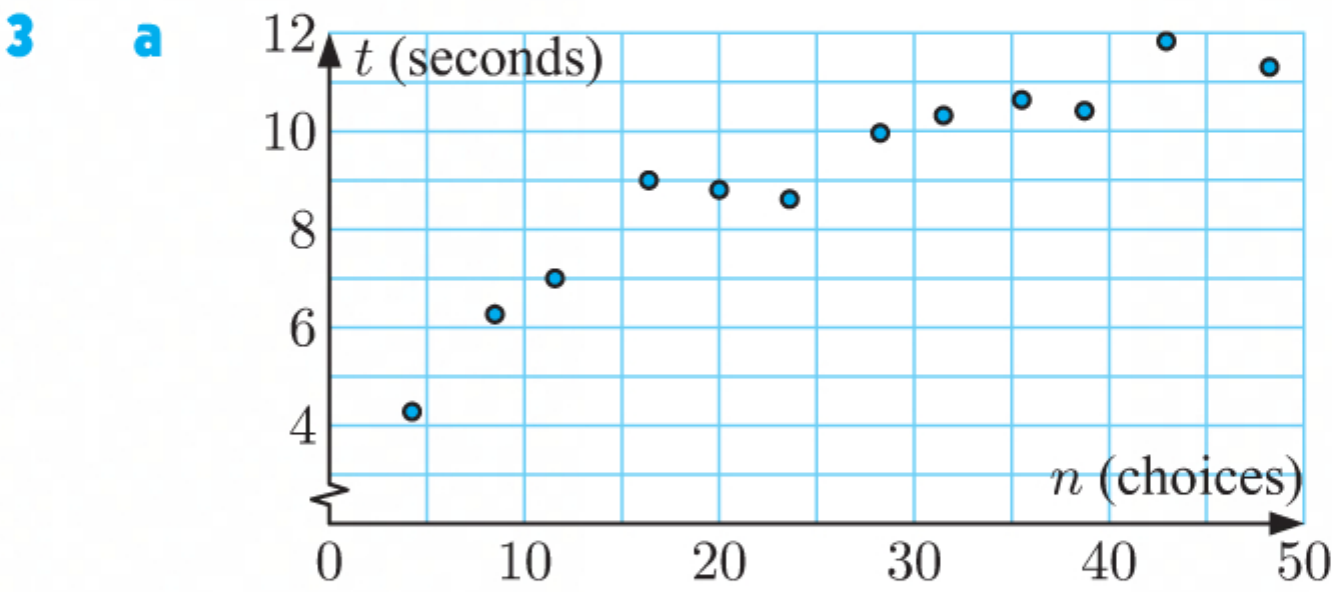
2

a Since $r_s \approx -0.7$, the trend in the ranks is negative.

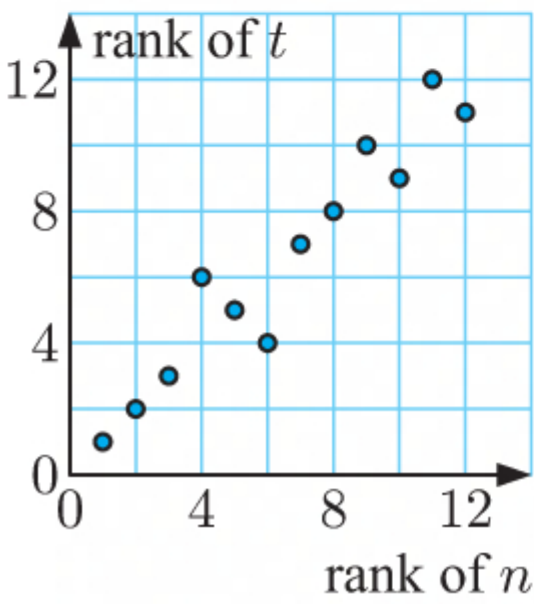
\therefore as x increases, y decreases.

\therefore the trend in the data is negative.

b No, Spearman's rank correlation coefficient indicates a positive or negative trend in the data but does not tell us about the linearity of the data set.



As the number of choices n increases, the time taken t generally increases. So as the rank of n increases, the rank of t generally increases.

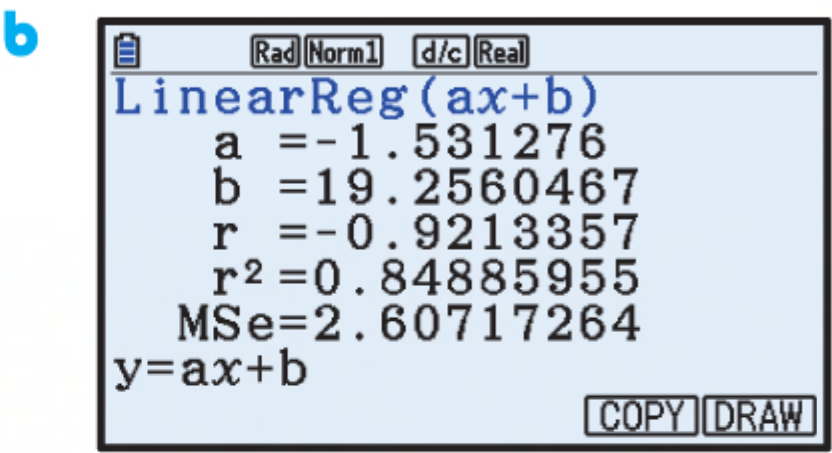
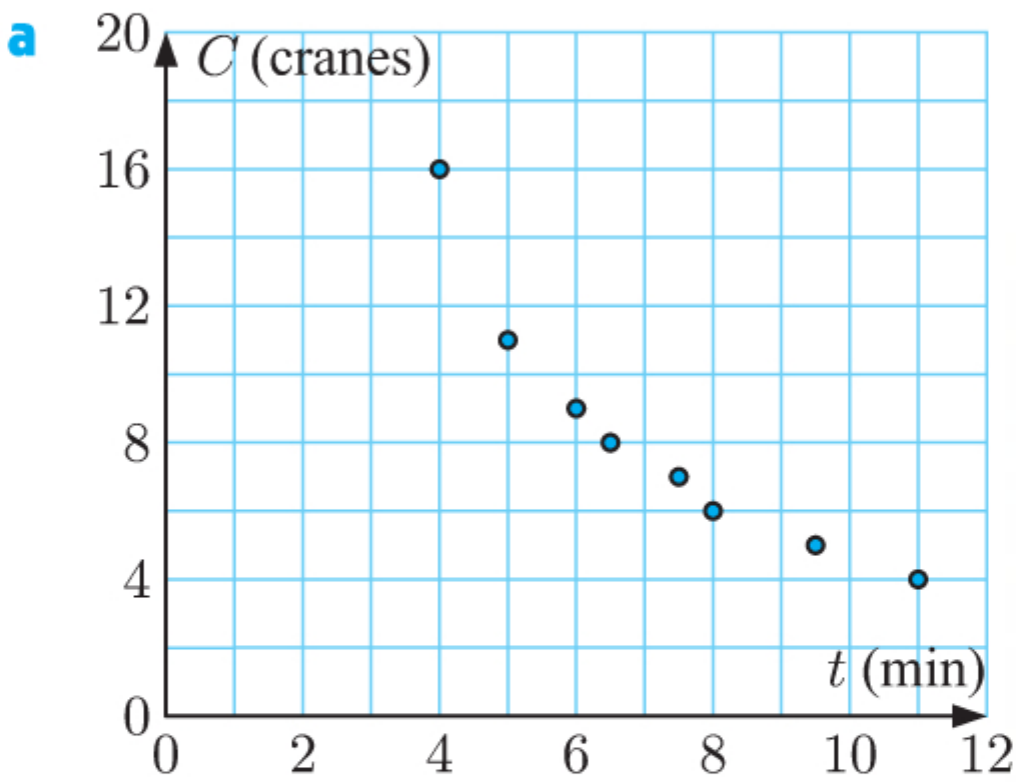


Looking at the first few points, we see that **A** is the correct scatter rank diagram.

b The scatter rank diagram has a strong, positive linear correlation, so the correct value of Spearman's rank correlation coefficient is $r_s \approx 0.958$ (**A**).

4

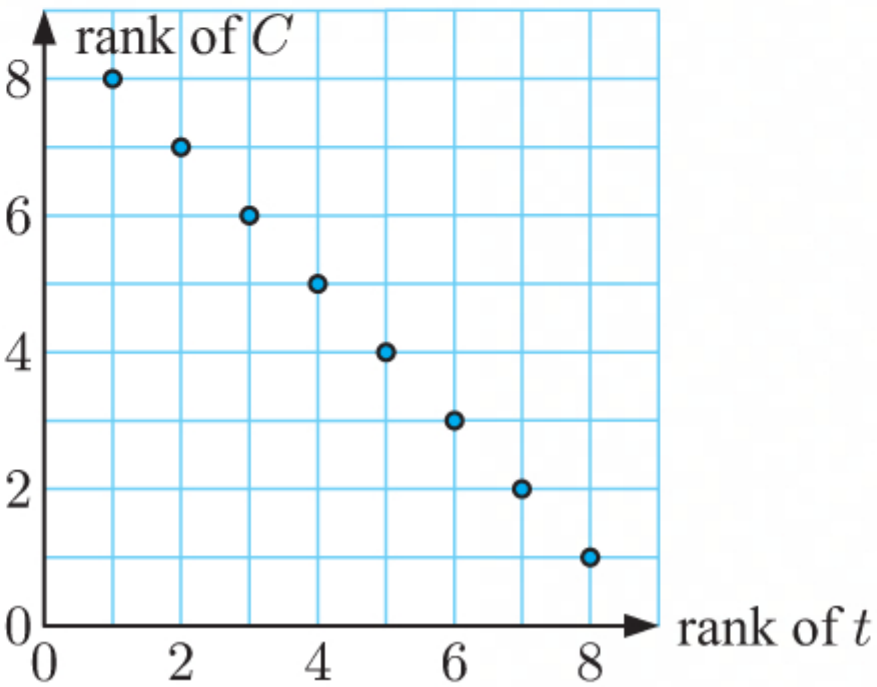
Time taken (t min)	6	9.5	4	5	8	7.5	11	6.5
Cranes made (C)	9	5	16	11	6	7	4	8



Using technology, $r_p \approx -0.921$.

c

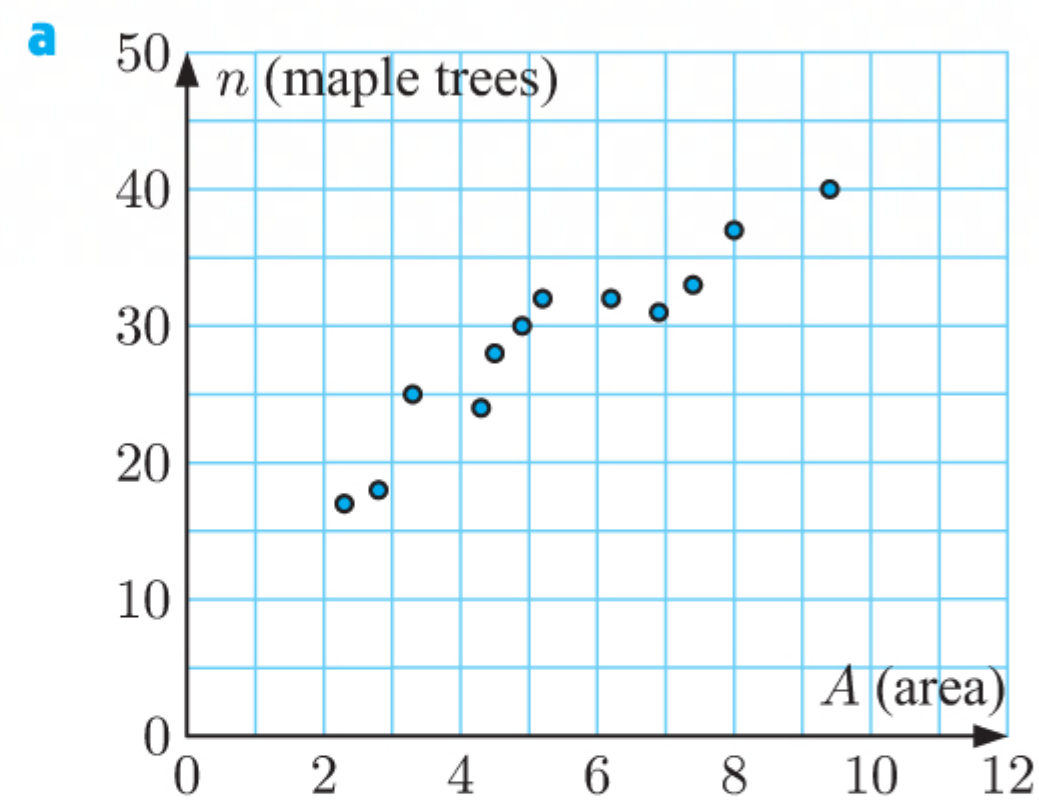
t	4	5	6	6.5	7.5	8	9.5	11
rank of t	1	2	3	4	5	6	7	8
C	16	11	9	8	7	6	5	4
rank of C	8	7	6	5	4	3	2	1



- d** We can see that the scatter rank diagram shows a perfect negative linear correlation.
 $\therefore r_s = -1$
- e** The scatter diagram of the raw data shows a non-linear trend. Spearman's rank correlation coefficient is therefore more appropriate.
- f** Using **d** and **e**, there is a very strong, negative, non-linear correlation between *time taken* and *number of cranes made*.

5

Area (A hectares)	2.8	6.9	7.4	4.3	2.3	9.4	5.2	8.0	4.9	6.2	3.3	4.5
Number of maple trees (n)	18	31	33	24	17	40	32	37	30	32	25	28



- b** There is a positive, linear correlation in the scatter diagram of the raw data. We expect to see the same in the rank scatter diagram. r_p and r_s will be very similar.

c

Area (A hectares)	2.8	6.9	7.4	4.3	2.3	9.4	5.2	8.0	4.9	6.2	3.3	4.5
rank of A	2	9	10	4	1	12	7	11	6	8	3	5
Number of maple trees (n)	18	31	33	24	17	40	32	37	30	32	25	28
rank of n	2	7	10	3	1	12	8.5	11	6	8.5	4	5

Original data:

Rad	Norm1	d/c	Real
LinearReg(ax+b)			
a = 2.99152863			
b = 12.6626943			
r = 0.94274696			
r ² = 0.88877184			
MSe = 5.90528827			
y = ax + b			
COPY		DRAW	

So, $r_p \approx 0.943$

Ranked data:

Rad	Norm1	d/c	Real
LinearReg(ax+b)			
a = 0.96853146			
b = 0.20454545			
r = 0.97022915			
r ² = 0.94134462			
MSe = 0.83583916			
y = ax + b			
COPY		DRAW	

So, $r_s \approx 0.970$

6 a

Number of words (x)	40	53	20	65	35	60	85	49	35	76
rank of x	4	6	1	8	2.5	7	10	5	2.5	9
Number of errors (y)	11	15	2	20	4	22	30	16	27	25
rank of y	3	4	1	6	2	7	10	5	9	8

Original data:

Rad	Norm1	d/c	Real
LinearReg(ax+b)			
a = 0.35701773			
b = -1.2935187			
r = 0.76220437			
r ² = 0.5809555			
MSe = 41.9882581			
y = ax + b			
COPY		DRAW	

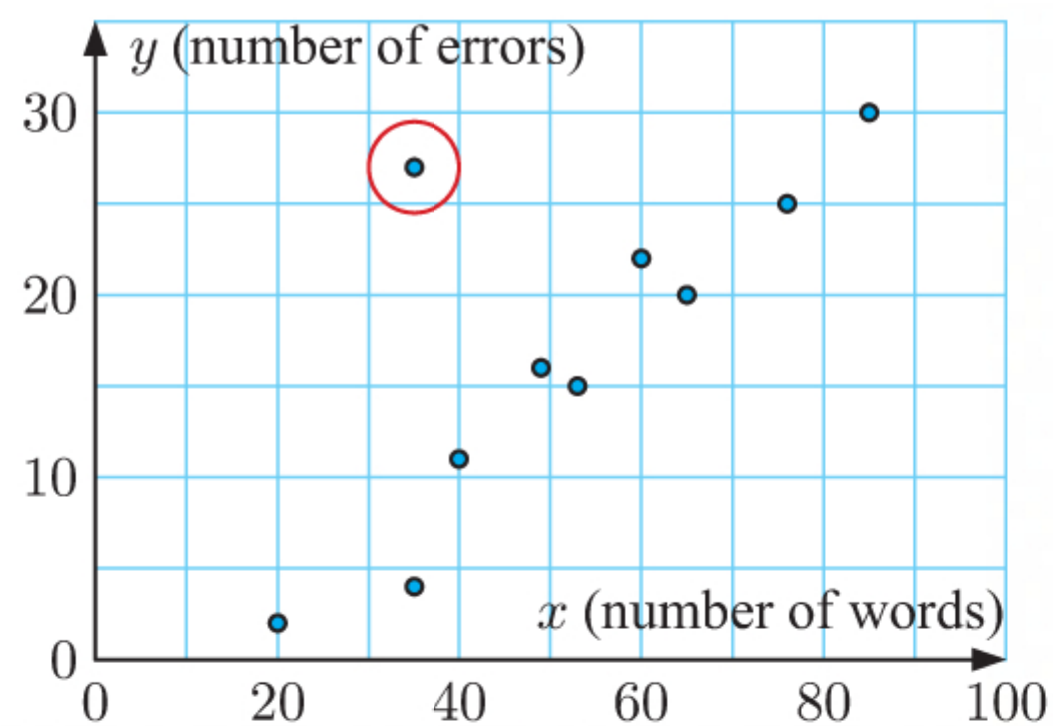
So, $r_p \approx 0.762$

Ranked data:

Rad	Norm1	d/c	Real
LinearReg(ax+b)			
a = 0.68292682			
b = 1.74390243			
r = 0.6808542			
r ² = 0.46356245			
MSe = 5.53201219			
y = ax + b			
COPY		DRAW	

So, $r_s \approx 0.681$

b



The outlier (35, 27) is circled.

c

Number of words (x)	40	53	20	65	35	60	85	49	76
rank of x	3	5	1	7	2	6	9	4	8
Number of errors (y)	11	15	2	20	4	22	30	16	25
rank of y	3	4	1	6	2	7	9	5	8

Original data (without outlier):

Rad	Norm1	d/c	Real
LinearReg(ax+b)			
a = 0.44530938			
b = -7.787159			
r = 0.97628674			
r ² = 0.95313581			
MSe = 4.65220036			
y = ax + b			
COPY		DRAW	

So, $r_p \approx 0.976$

Ranked data (without outlier):

Rad	Norm1	d/c	Real
LinearReg(ax+b)			
a = 0.96666666			
b = 0.16666666			
r = 0.96666666			
r ² = 0.93444444			
MSe = 0.56190476			
y = ax + b			
COPY		DRAW	

So, $r_s \approx 0.967$ d r_s was more affected by the presence of the outlier.

7 a

Longest jump (y m)	5.29	5.22	4.64	4.62	4.58
Placing	1	2	3	4	5

Longest jump (y m)	4.38	4.31	4.28	3.94	3.89
Placing	6	7	8	9	10

- b *Placing* is the **dependent** variable since it depends on the length of the longest jump.
 \therefore *longest jump* is the independent variable.
- c i The variable *placing* has the values 1 to 10 which act as a ranking.
 ii Spearman's correlation coefficient for the variables *longest jump* and *placing* must be exactly -1 since the longest jump always decreases as the placing increases.

ACTIVITY 3

ANSCOMBE'S QUARTET

1 Data set A:

x	10	8	13	9	11	14	6	4	12	7	5
y	8.04	6.95	7.58	8.81	8.33	9.96	7.24	4.26	10.84	4.82	5.68

\bar{x}	=9
Σx	=99
Σx^2	=1001
σx	=3.16227766
sx	=3.31662479
n	=11

$$\mu_x = 9, \sigma_x \approx 3.16$$

\bar{x}	=7.50090909
Σx	=82.51
Σx^2	=660.1763
σx	=1.93710869
sx	=2.03165673
n	=11

$$\mu_y \approx 7.50, \sigma_y \approx 1.94$$

Data set B:

x	10	8	13	9	11	14	6	4	12	7	5
y	9.14	8.14	8.74	8.77	9.26	8.1	6.13	3.1	9.13	7.26	4.74

\bar{x}	=9
Σx	=99
Σx^2	=1001
σx	=3.16227766
sx	=3.31662479
n	=11

$$\mu_x = 9, \sigma_x \approx 3.16$$

\bar{x}	=7.50090909
Σx	=82.51
Σx^2	=660.1763
σx	=1.93710869
sx	=2.03165673
n	=11

$$\mu_y \approx 7.50, \sigma_y \approx 1.94$$

Data set C:

x	10	8	13	9	11	14	6	4	12	7	5
y	7.46	6.77	12.74	7.11	7.81	8.84	6.08	5.39	8.15	6.42	5.73

1-Variable	
\bar{x}	=9
Σx	=99
Σx^2	=1001
σx	=3.16227766
sx	=3.31662479
n	=11

1-Variable	
\bar{x}	=7.5
Σx	=82.5
Σx^2	=659.9762
σx	=1.93593294
sx	=2.0304236
n	=11

$$\mu_x = 9, \sigma_x \approx 3.16$$

$$\mu_y = 7.5, \sigma_y \approx 1.94$$

Data set D:

x	8	8	8	8	8	8	8	19	8	8	8
y	6.58	5.76	7.71	8.84	8.47	7.04	5.25	12.5	5.56	7.91	6.89

1-Variable	
\bar{x}	=9
Σx	=99
Σx^2	=1001
σx	=3.16227766
sx	=3.31662479
n	=11

1-Variable	
\bar{x}	=7.50090909
Σx	=82.51
Σx^2	=660.1325
σx	=1.93608064
sx	=2.03057851
n	=11

$$\mu_x = 9, \sigma_x \approx 3.16$$

$$\mu_y \approx 7.50, \sigma_y \approx 1.94$$

- a** In each data set: The mean of x is 9.
The mean of y is 7.5 (or very close to 7.5).
- b** In each data set: The population standard deviation of $x \approx 3.16$.
The population standard deviation of $y \approx 1.94$.

2 Data set A:

LinearReg(ax+b)	
a	=0.5000909
b	=3.0000909
r	=0.81642051
r^2	=0.66654245
MSe	=1.52918777
$y=ax+b$	

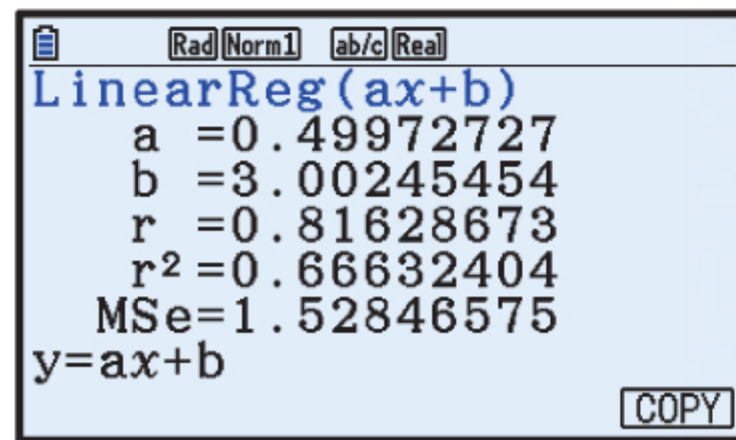
The regression line is $y \approx 0.500x + 3.00$.
 $r_p \approx 0.816$

Data set B:

LinearReg(ax+b)	
a	=0.5
b	=3.00090909
r	=0.8162365
r^2	=0.66624203
MSe	=1.53069898
$y=ax+b$	

The regression line is $y \approx 0.5x + 3.00$.
 $r_p \approx 0.816$

Data set C:

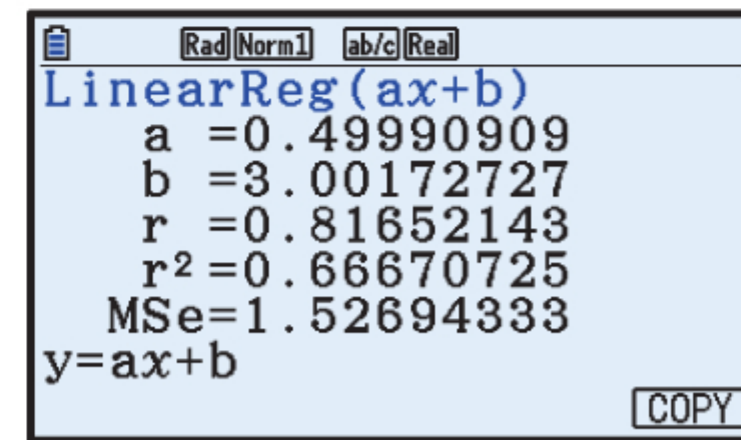


The regression line is $y \approx 0.500x + 3.00$.

$$r_p \approx 0.816$$

The regression lines and the values of Pearson's product-moment correlation coefficient are almost identical for each data set.

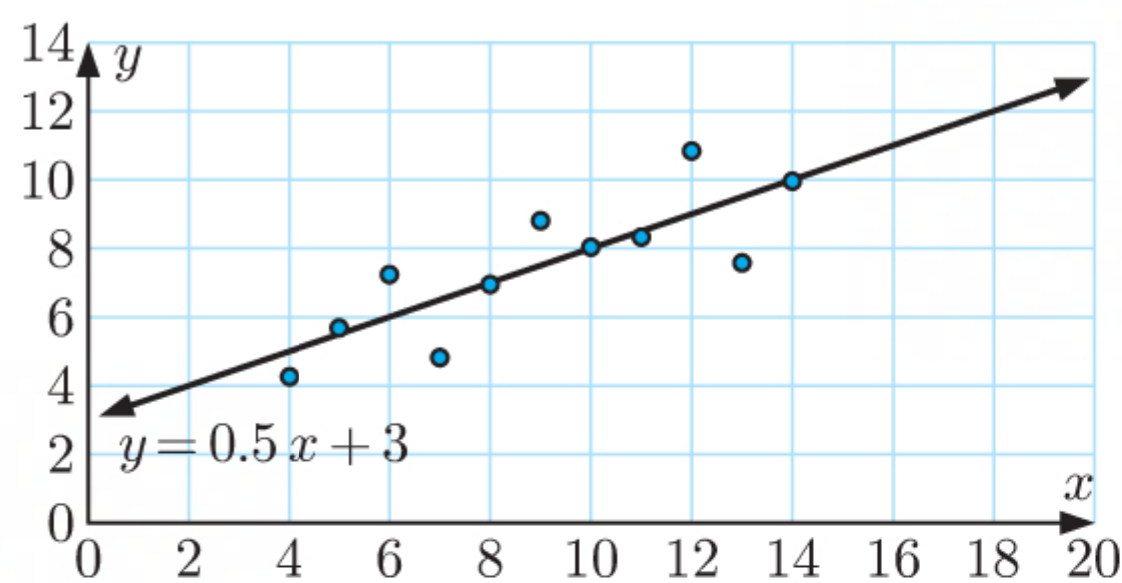
Data set D:



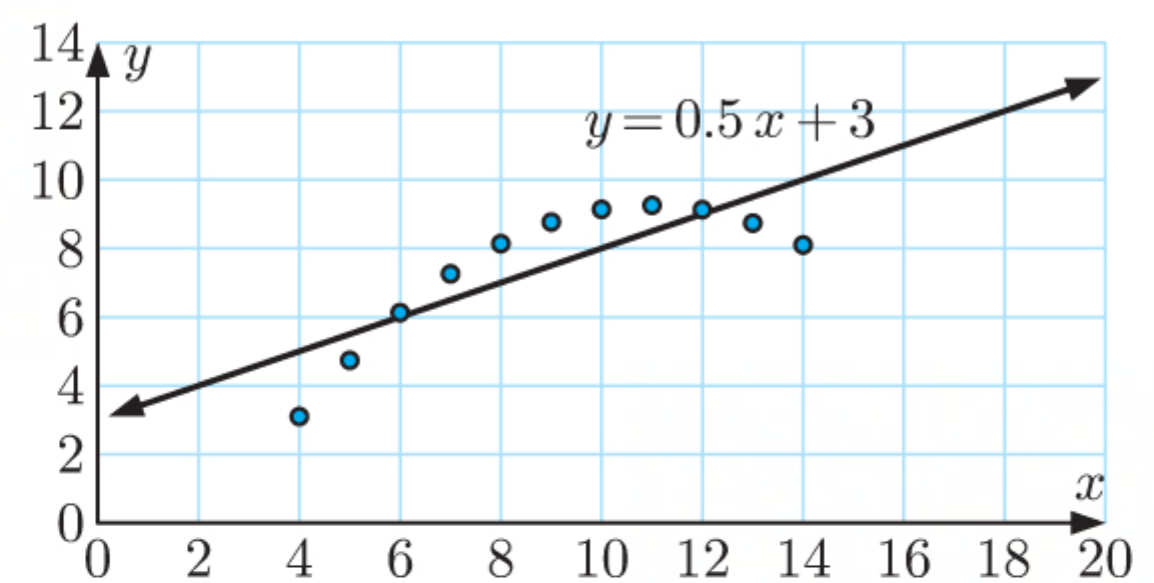
The regression line is $y \approx 0.500x + 3.00$.

$$r_p \approx 0.817$$

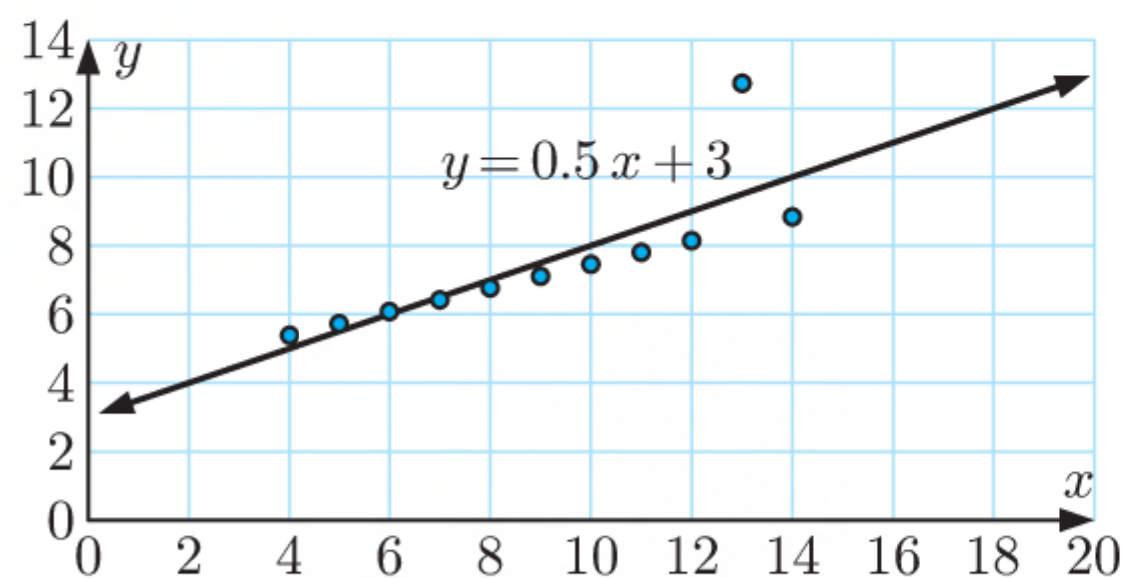
3 Data set A:



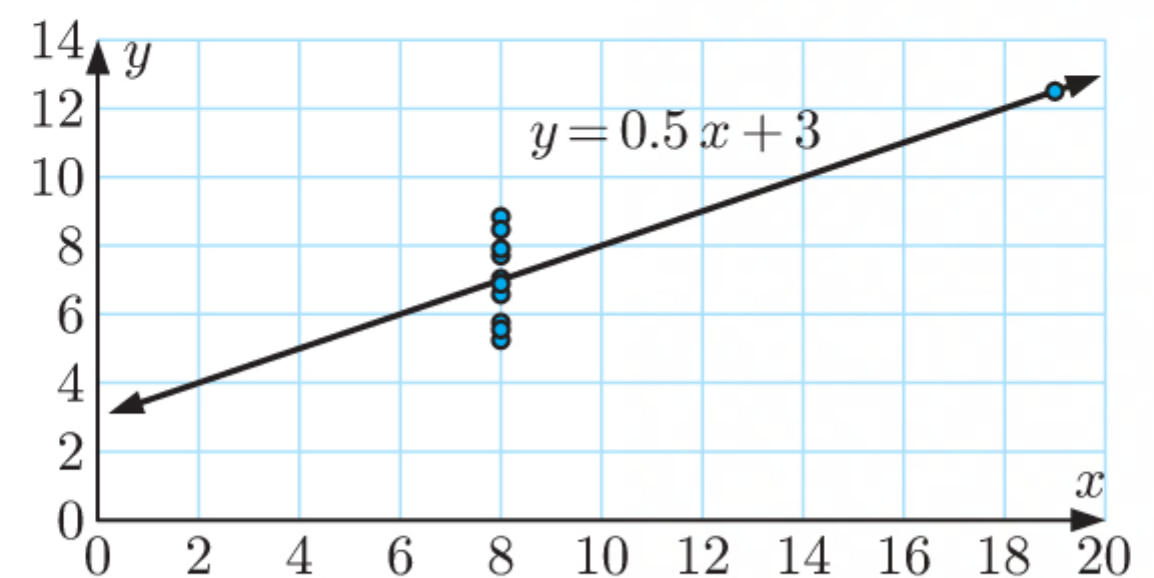
Data set B:



Data set C:



Data set D:



- 4 Each data set has the same mean and standard deviation for both variables, and the same regression line. However, we see that the scatter diagrams for each data set are wildly different from each other. A linear model is not necessarily appropriate for each data set.

5 Data set A:

x	10	8	13	9	11	14	6	4	12	7	5
rank of x	7	5	10	6	8	11	3	1	9	4	2
y	8.04	6.95	7.58	8.81	8.33	9.96	7.24	4.26	10.84	4.82	5.68
rank of y	7	4	6	9	8	10	5	1	11	2	3

```

Rad Norm1 d/c Real
LinearReg(ax+b)
a =0.81818181
b =1.09090909
r =0.81818181
r²=0.66942148
MSe=4.04040404
y=ax+b
COPY DRAW

```

$$r_s \approx 0.818$$

Data set B:

x	10	8	13	9	11	14	6	4	12	7	5
rank of x	7	5	10	6	8	11	3	1	9	4	2
y	9.14	8.14	8.74	8.77	9.26	8.1	6.13	3.1	9.13	7.26	4.74
rank of y	10	6	7	8	11	5	3	1	9	4	2

```

Rad Norm1 d/c Real
LinearReg(ax+b)
a =0.69090909
b =1.85454545
r =0.69090909
r²=0.47735537
MSe=6.38787878
y=ax+b
COPY DRAW

```

$$r_s \approx 0.691$$

Data set C:

x	10	8	13	9	11	14	6	4	12	7	5
rank of x	7	5	10	6	8	11	3	1	9	4	2
y	7.46	6.77	12.74	7.11	7.81	8.84	6.08	5.39	8.15	6.42	5.73
rank of y	7	5	11	6	8	10	3	1	9	4	2

```

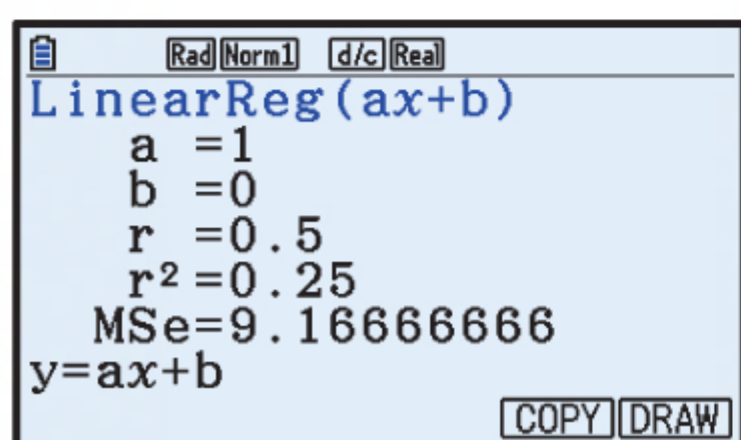
Rad Norm1 d/c Real
LinearReg(ax+b)
a =0.99090909
b =0.05454545
r =0.99090909
r²=0.98190082
MSe=0.22121212
y=ax+b
COPY DRAW

```

$$r_s \approx 0.991$$

Data set D:

x	8	8	8	8	8	8	8	19	8	8	8
rank of x	5.5	5.5	5.5	5.5	5.5	5.5	5.5	11	5.5	5.5	5.5
y	6.58	5.76	7.71	8.84	8.47	7.04	5.25	12.5	5.56	7.91	6.89
rank of y	4	3	7	10	9	6	1	11	2	8	5

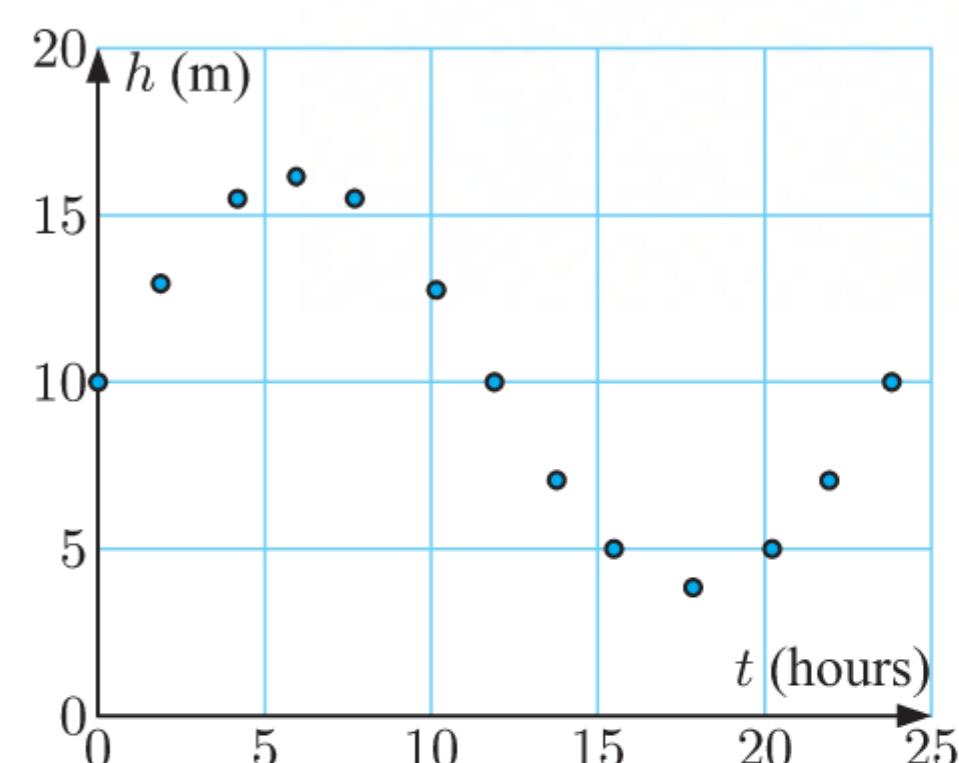


$$r_s = 0.5$$

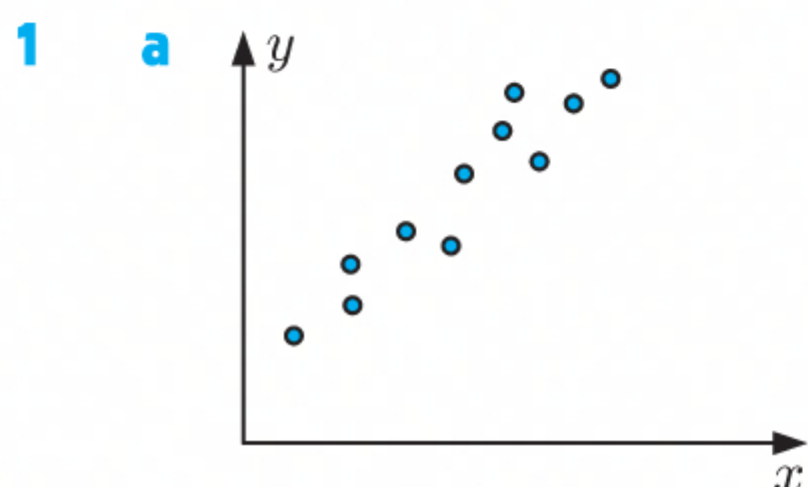
The values of Spearman's rank correlation coefficient vary greatly between each data set. The values alone cannot predict the trend in the original data sets.

- 6 A scatter diagram allows us to see patterns in data that cannot be conveyed with descriptive statistics alone.

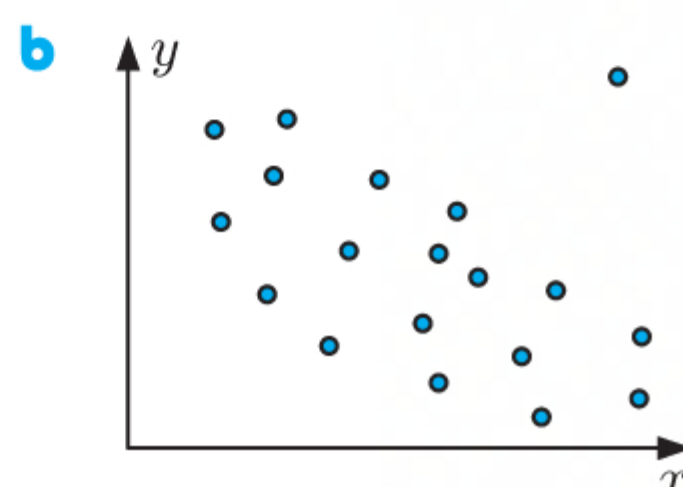
In the scatter diagram alongside, neither Pearson's product-moment correlation coefficient nor Spearman's rank correlation coefficient will be indicative of the periodic trend in the data.



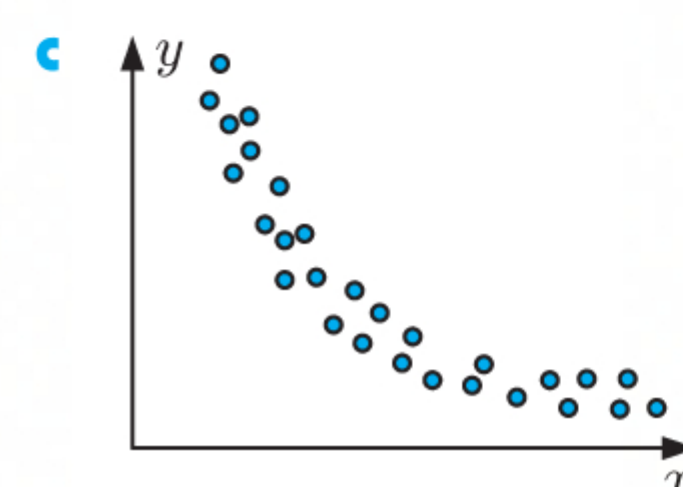
REVIEW SET 5A



There is a strong, positive, linear correlation, with no outliers.



There is a weak, negative, linear correlation, with one outlier.

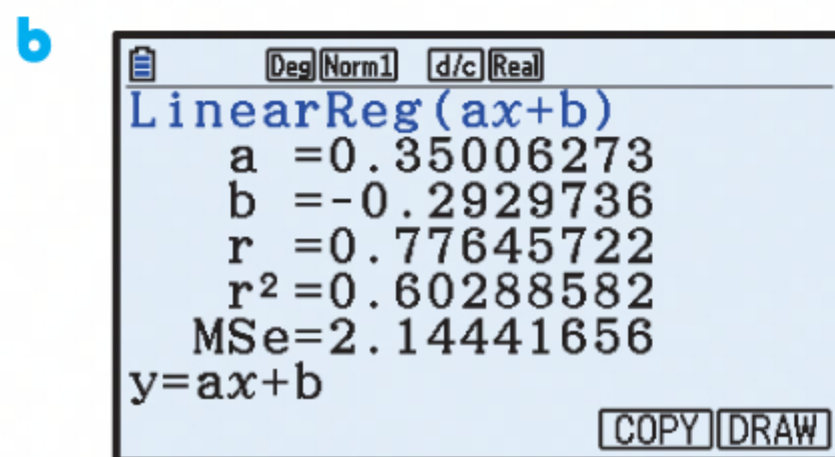
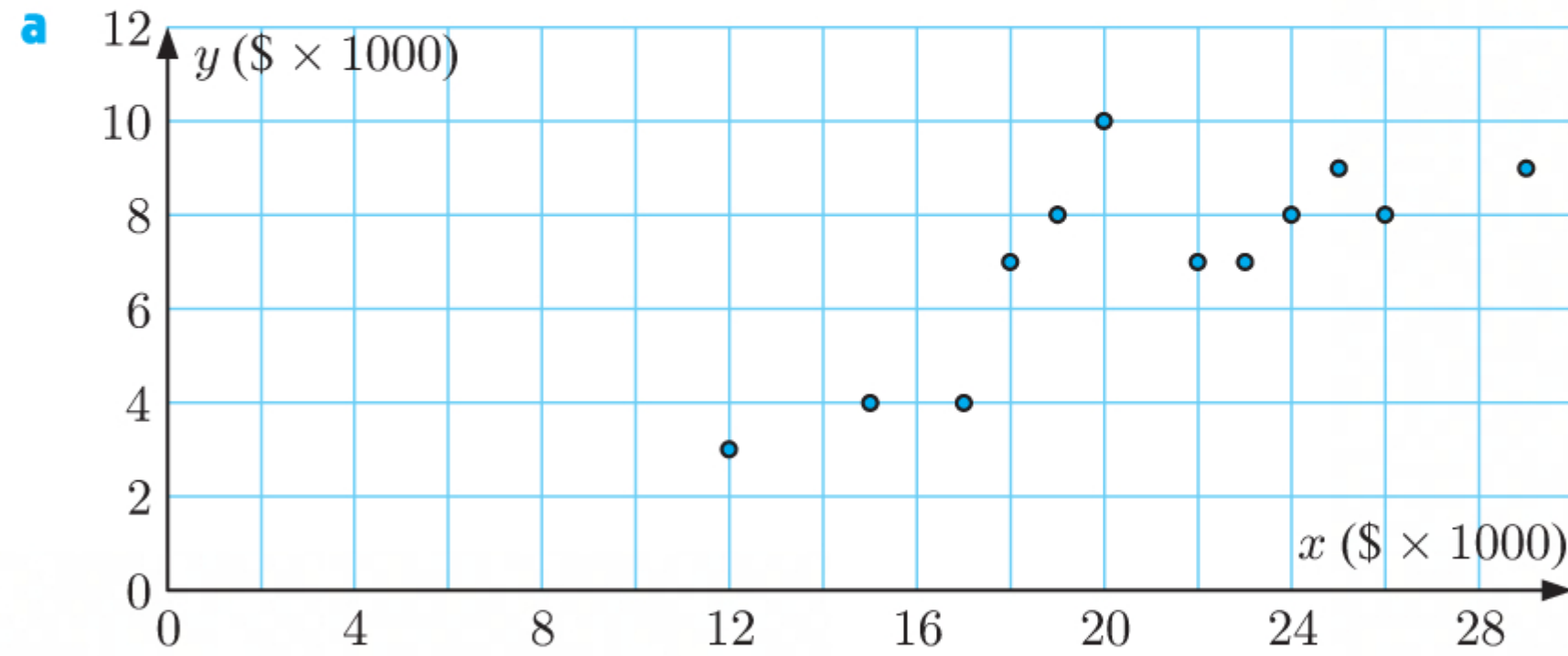


There is a strong, negative, non-linear correlation, with no outliers.

- 2 a The correlation between water bills and electricity bills is likely to be positive, as a household with a high water bill is also likely to have a high electricity bill, and vice versa.

- b** No, there is not a causal relationship. Both variables mainly depend on the number of occupants in each house.

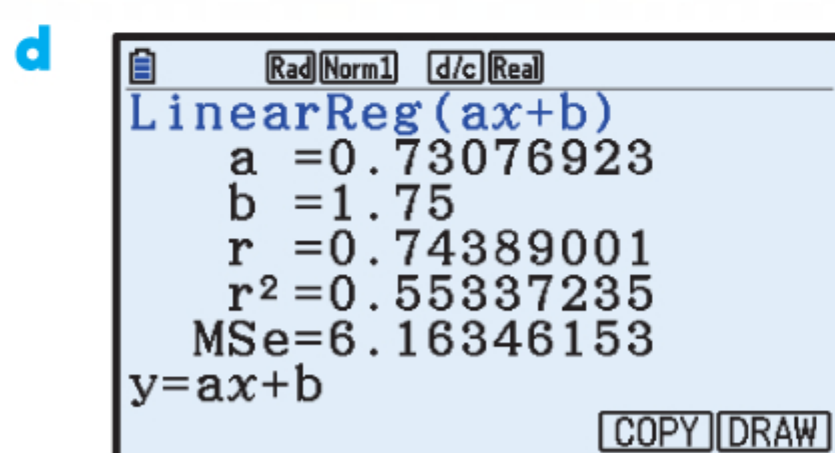
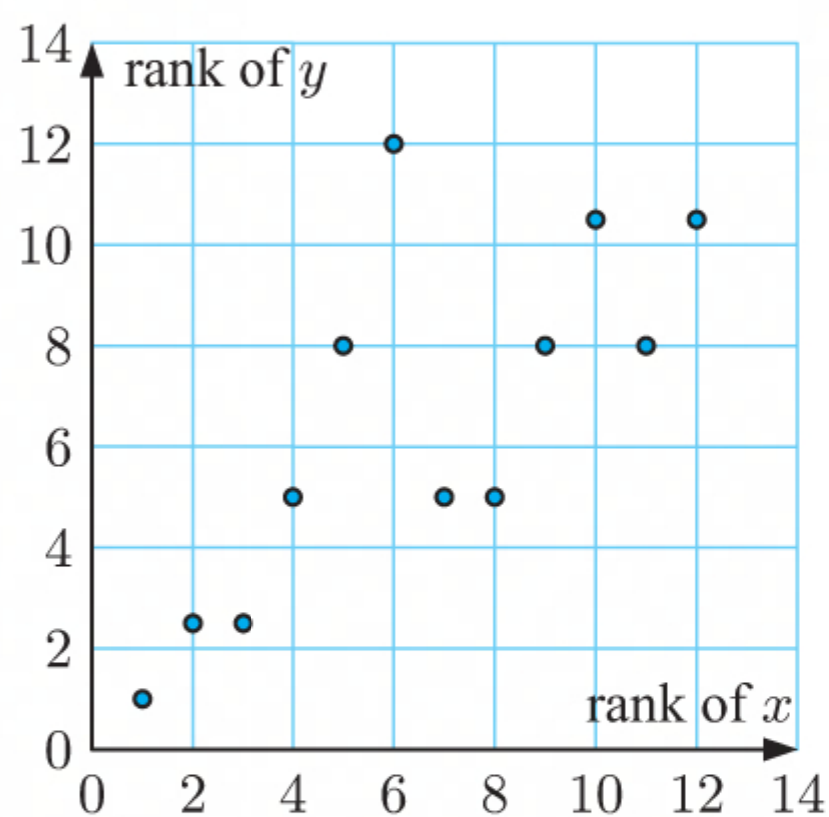
3	<i>Ticket sales</i> ($\$x \times 1000$)	25	22	15	19	12	17	24	20	18	23	29	26
	<i>Beverage sales</i> ($\$y \times 1000$)	9	7	4	8	3	4	8	10	7	7	9	8



So, $r_p \approx 0.776$.

c

<i>Ticket sales</i> ($\$x \times 1000$)	25	22	15	19	12	17	24	20	18	23	29	26
<i>rank of x</i>	10	7	2	5	1	3	9	6	4	8	12	11
<i>Beverage sales</i> ($\$y \times 1000$)	9	7	4	8	3	4	8	10	7	7	9	8
<i>rank of y</i>	10.5	5	2.5	8	1	2.5	8	12	5	5	10.5	8

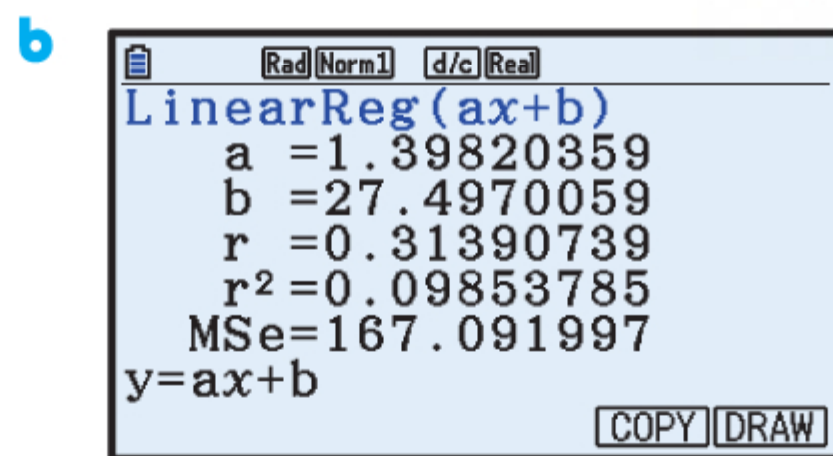
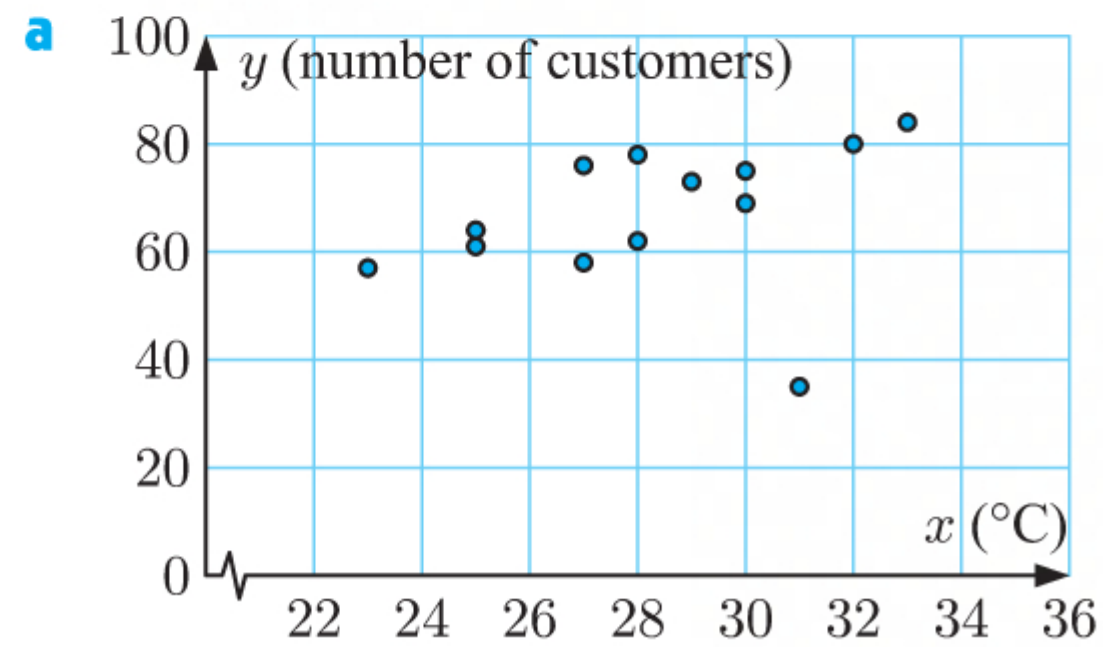


So, $r_s \approx 0.744$.

- e There is a moderate, positive correlation between *ticket sales* and *beverage sales*.

4

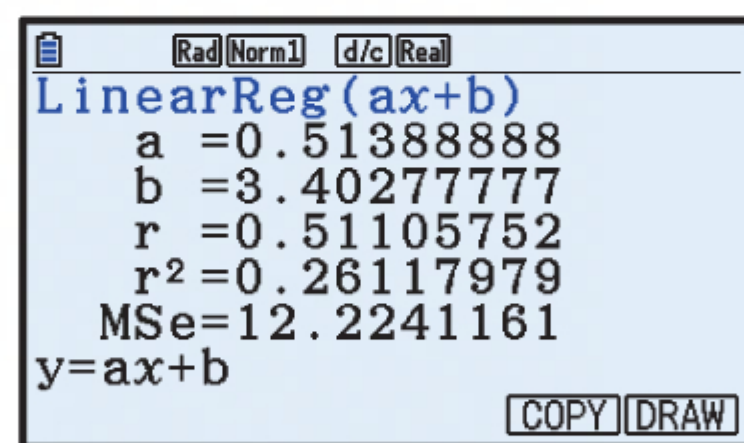
Temperature ($x^{\circ}\text{C}$)	23	25	28	30	30	27	25	28	32	31	33	29	27
Number of customers (y)	57	64	62	75	69	58	61	78	80	35	84	73	76



So, $r_p \approx 0.314$.

c

Temperature ($x^{\circ}\text{C}$)	23	25	28	30	30	27	25	28	32	31	33	29	27
rank of x	1	2.5	6.5	9.5	9.5	4.5	2.5	6.5	12	11	13	8	4.5
Number of customers (y)	57	64	62	75	69	58	61	78	80	35	84	73	76
rank of y	2	6	5	9	7	3	4	11	12	1	13	8	10



So, $r_s \approx 0.511$.

- d The outlier is (31, 35) as it is greatly separated from the rest of the data.

e

Temperature ($x^{\circ}\text{C}$)	23	25	28	30	30	27	25	28	32	33	29	27
rank of x	1	2.5	6.5	9.5	9.5	4.5	2.5	6.5	11	12	8	4.5
Number of customers (y)	57	64	62	75	69	58	61	78	80	84	73	76
rank of y	1	5	4	8	6	2	3	10	11	12	7	9

Rad Norm1 d/c Real
LinearReg(ax+b)
a = 2.49956101
b = -0.4460052
r = 0.80101426
r ² = 0.64162385
MSe = 33.1229148
y = ax + b
COPY DRAW

Rad Norm1 d/c Real
LinearReg(ax+b)
a = 0.76950354
b = 1.49822695
r = 0.76410345
r ² = 0.58385408
MSe = 5.95088652
y = ax + b
COPY DRAW

$$r_p \approx 0.801$$

$$r_s \approx 0.764$$

f r_p was more affected by the presence of the outlier.

g There is a moderate, positive, linear correlation between the *number of customers* and the *noon temperature* at the garden centre.

5

Time (min)	8	18	5	10	17	11	2	13	18	4	11	20	23	22	17
Money (€)	40	78	0	46	72	86	0	59	33	0	0	122	90	137	93

a $\bar{x} = \frac{8 + 18 + \dots + 22 + 17}{15}$

$$= \frac{199}{15}$$

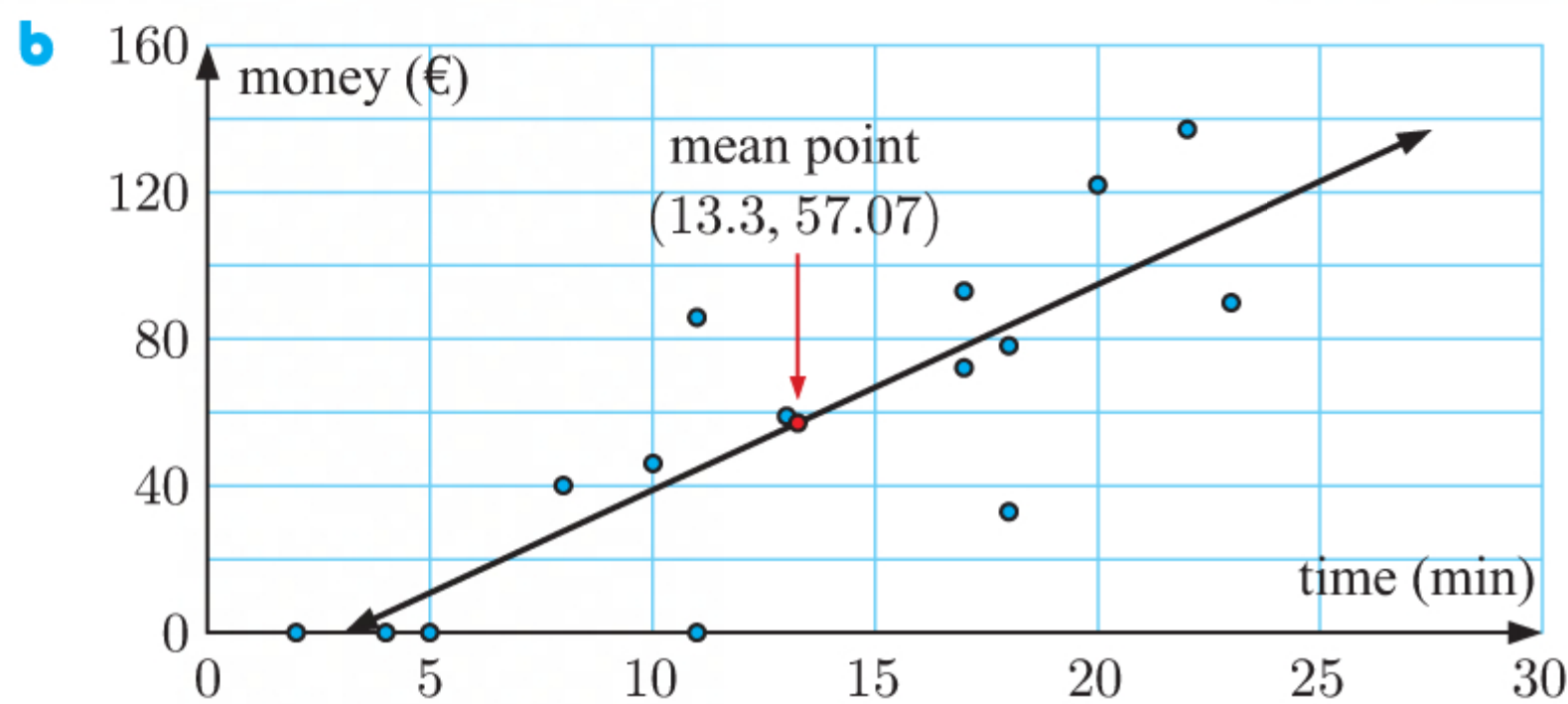
$$\approx 13.3$$

$$\bar{y} = \frac{40 + 78 + \dots + 137 + 93}{15}$$

$$= \frac{856}{15}$$

$$\approx 57.07$$

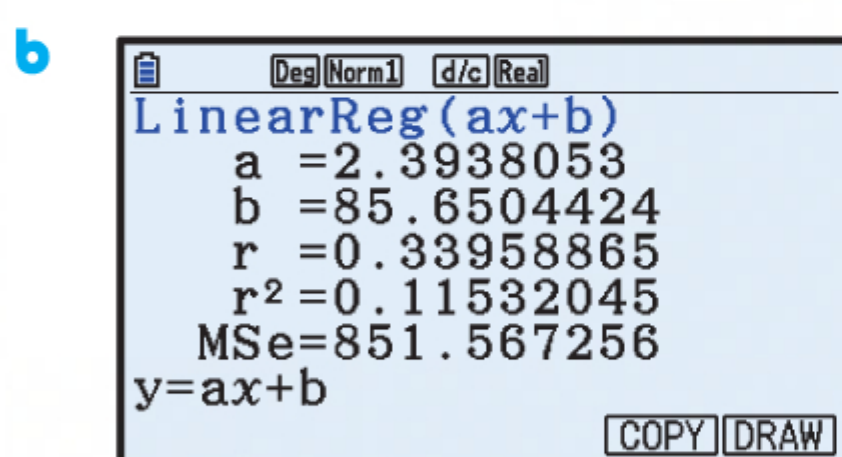
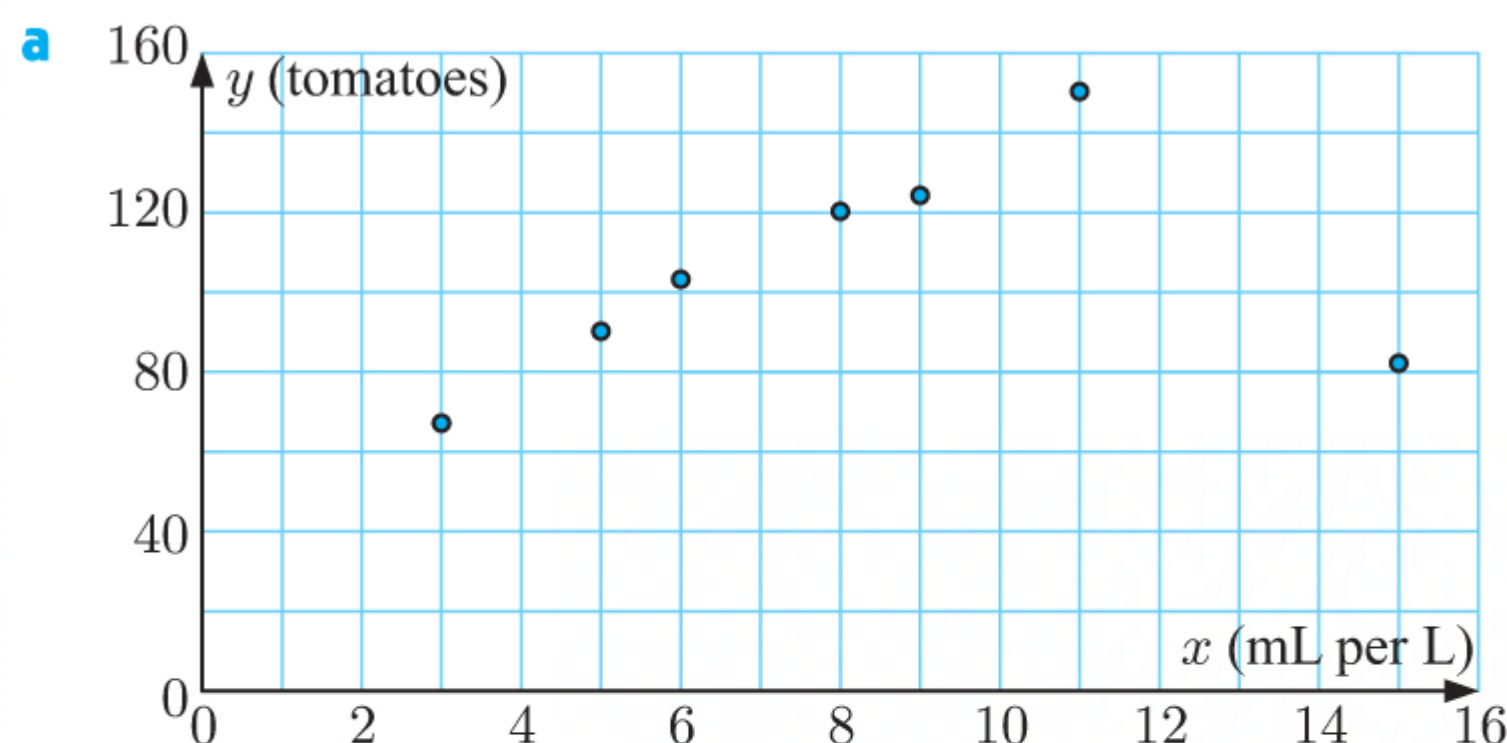
So the mean time is about 13.3 minutes, and the mean spending is about €57.07.



c There is a moderate, positive, linear correlation between *time in the store* and *money spent*.

6

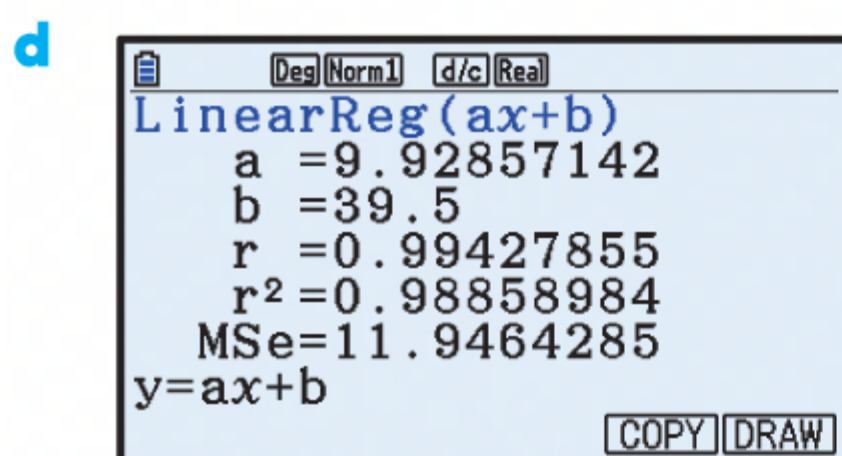
Spray concentration (x mL per L)	3	5	6	8	9	11	15
Yield of tomatoes per bush (y)	67	90	103	120	124	150	82



So, $r \approx 0.340$.

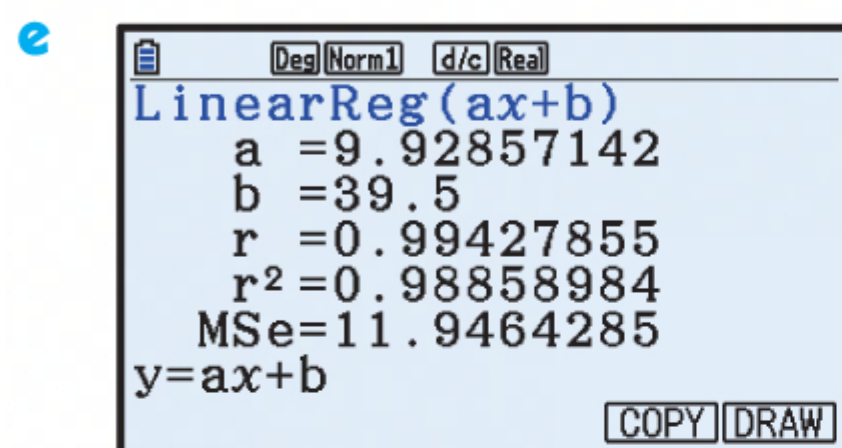
There is a very weak, positive, linear correlation between spray concentration and yield.

c Yes, $(15, 82)$ is an outlier which is affecting the correlation.



So, $r \approx 0.994$.

Yes it is now reasonable to draw a least squares regression line.



Using technology, the least squares regression line is $y \approx 9.93x + 39.5$.

f The gradient of the least squares regression line ≈ 9.93 . This indicates that for every additional mL per L the spray concentration increases, the yield of tomatoes per bush increases on average by 9.93 tomatoes.

The y -intercept of the least squares regression line ≈ 39.5 . This indicates that if the tomato bushes are not sprayed, the average yield per bush is approximately 39.5 tomatoes.

g i When $x = 7$, $y \approx 9.93(7) + 39.5$
 ≈ 109

If the spray concentration is 7 mL per L, the yield will be approximately 109 tomatoes per bush.

ii When $y = 200$, $200 \approx 9.93x + 39.5$
 $9.93x \approx 160.5$
 $x \approx 16.2$

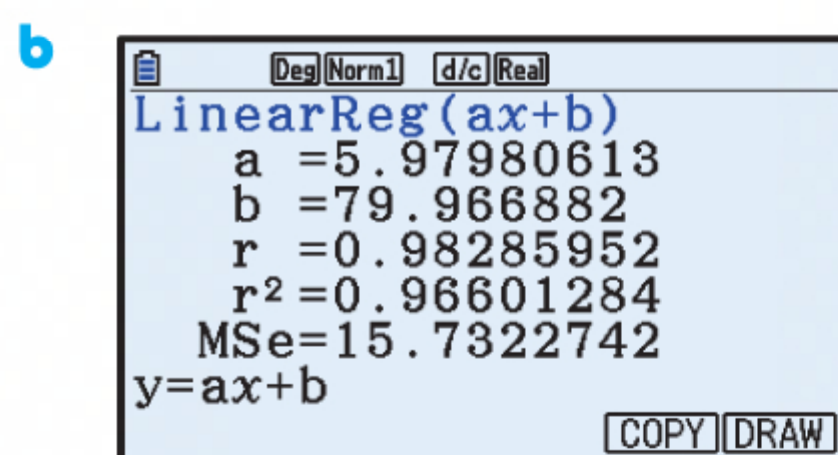
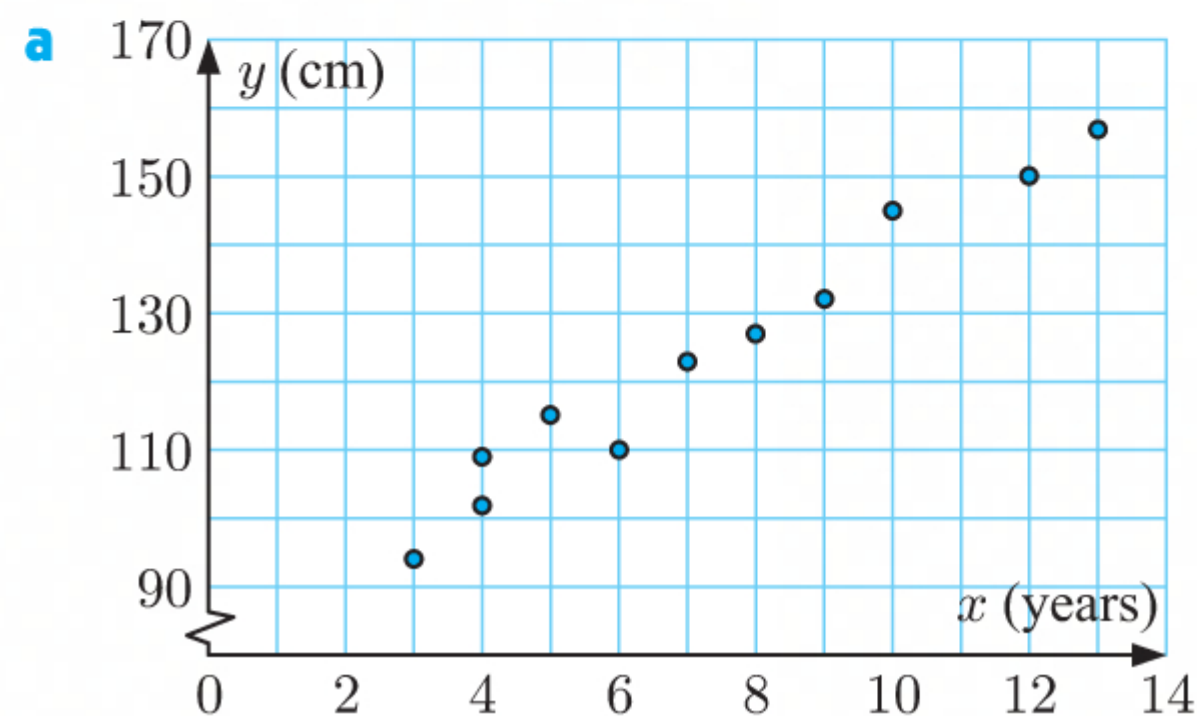
If the yield is 200 tomatoes per bush, the spray concentration would be approximately 16.2 mL per L.

h In **g i**, this is an interpolation, so this estimate is likely to be reliable.

In **g ii**, this is an extrapolation, so this estimate may not be reliable.

7

Age (x years)	3	9	7	4	4	12	8	6	5	10	13
Height (y cm)	94	132	123	102	109	150	127	110	115	145	157



Using technology, the least squares regression line is $y \approx 5.98x + 80.0$.

c The gradient of the least squares regression line ≈ 5.98 . This indicates that each year, a child grows taller by an average of 5.98 cm.

d When $x = 5$, $y \approx 5.98(5) + 80.0$
 ≈ 110

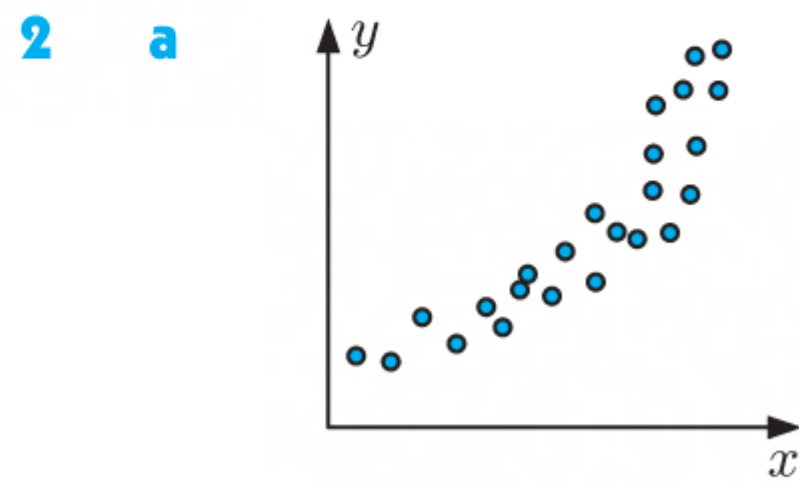
So, a 5 year old child would be approximately 110 cm tall.

e When $y = 140$, $140 \approx 5.98x + 80$
 $5.98x \approx 60$
 $x \approx 10.0$

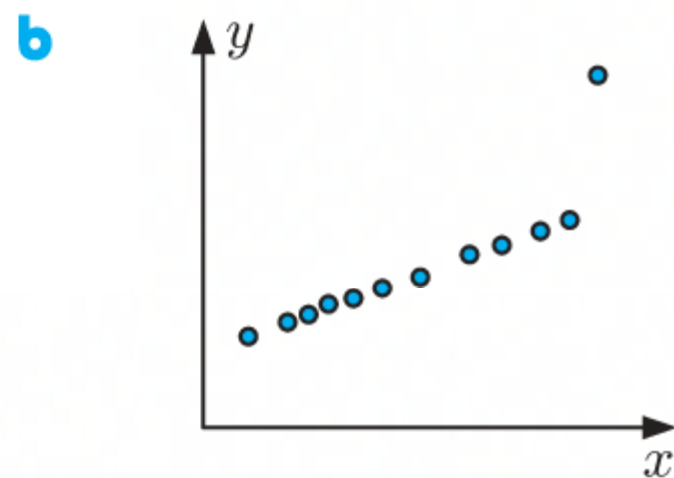
A child would be expected to reach 140 cm in height at age 10 years.

REVIEW SET 5B

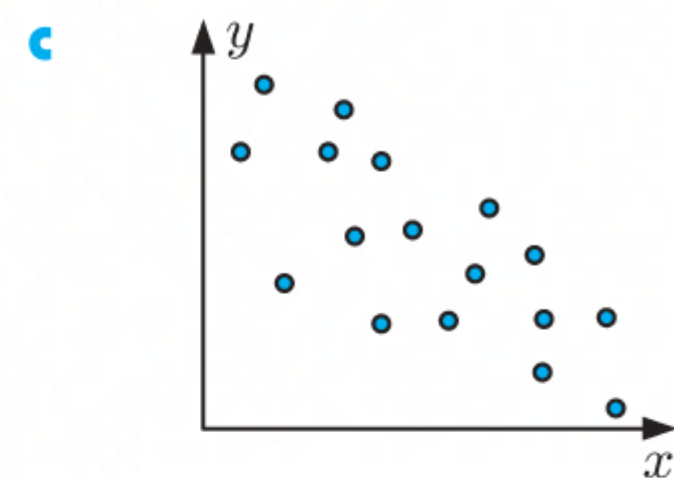
- 1 a The variables are likely to be negatively correlated, as prices increase, the number of tickets sold is likely to decrease.
This is a causal relationship as less people will be able to afford tickets as the prices increase.
- b The variables are likely to be positively correlated, as ice cream sales increase, the number of shark attacks is likely to increase.
This is not a causal relationship as both of these variables are dependent on the time of year.



In general, as x increases, y increases.
So in general, as the rank of x increases, the rank of y increases. The ranks are positively correlated.
The rank correlation coefficient is $r_s = 0.7$ (C).



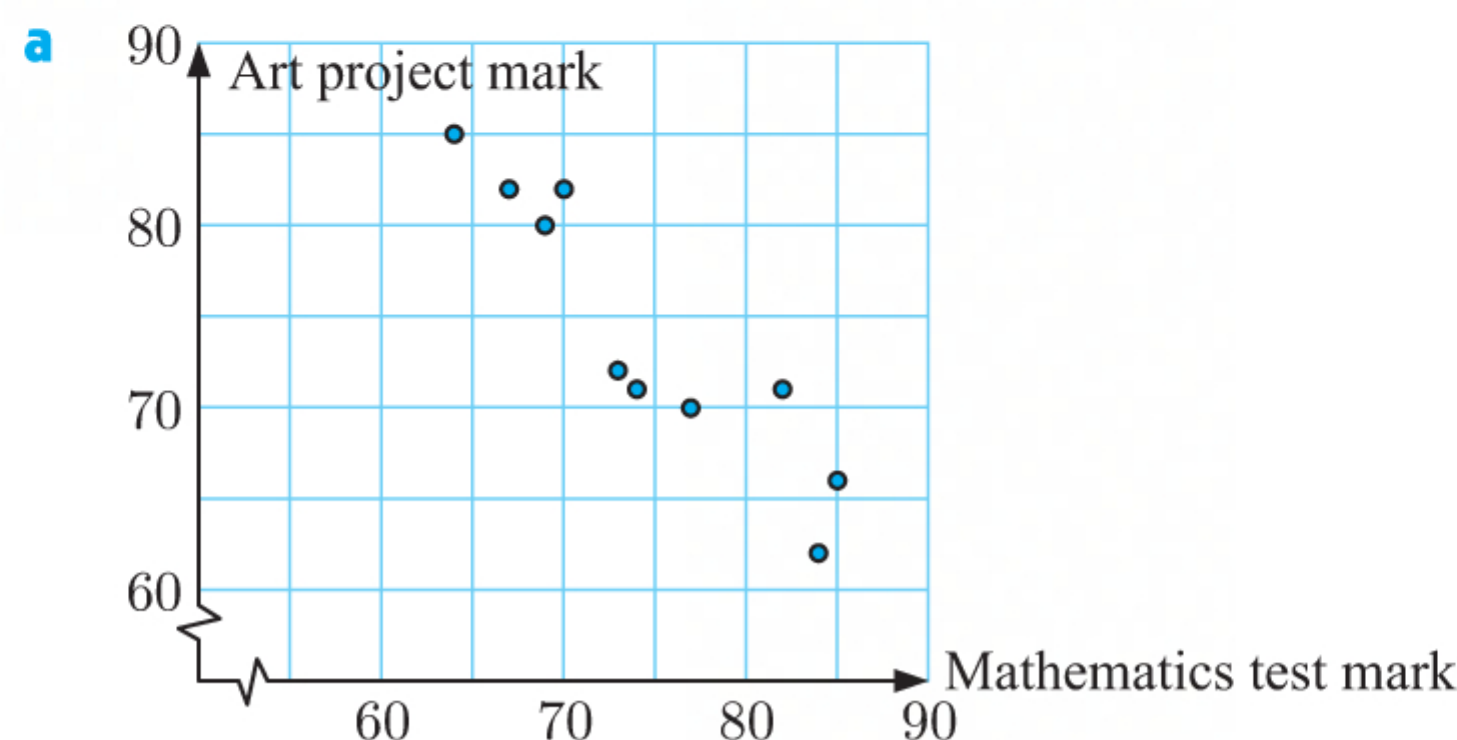
As x increases, y always increases.
So the ranks are perfectly positively correlated.
The rank correlation coefficient is $r_s = 1$ (A).



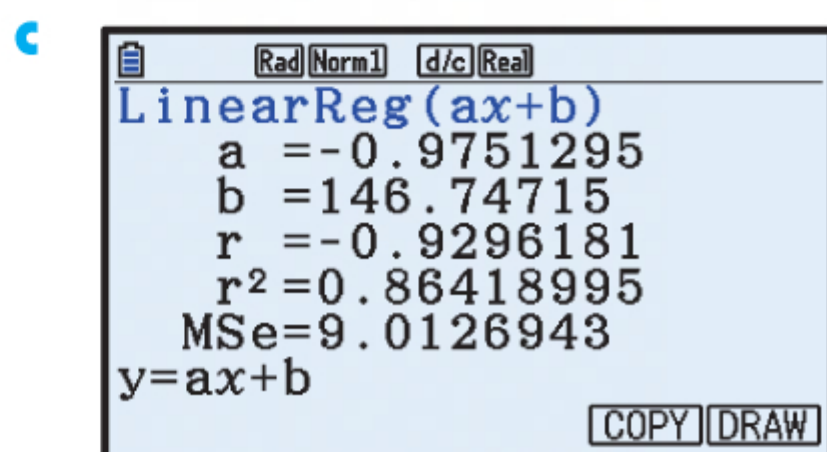
In general, as x increases, y decreases.
So in general, as the rank of x increases, the rank of y decreases. The ranks are negatively correlated.
The rank correlation coefficient is $r_s = -0.4$ (B).

3

Student	A	B	C	D	E	F	G	H	I	J
Mathematics test	64	67	69	70	73	74	77	82	84	85
Art project	85	82	80	82	72	71	70	71	62	66



- b There is a strong, negative, linear correlation between the Mathematics and Art marks.



So, $r \approx -0.930$.

4

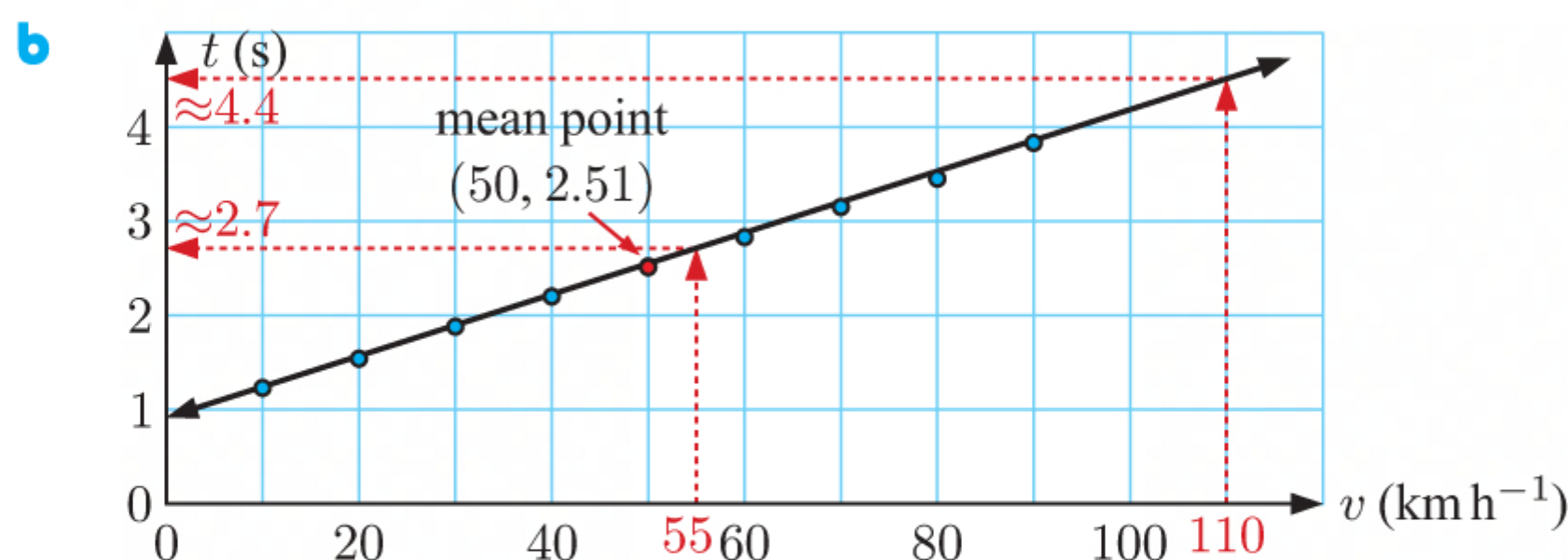
Speed (v km h ⁻¹)	10	20	30	40	50	60	70	80	90
Stopping time (t s)	1.23	1.54	1.88	2.20	2.52	2.83	3.15	3.45	3.83

a $\bar{v} = \frac{10 + 20 + 30 + \dots + 80 + 90}{9}$
 $= 50$

$$\bar{t} = \frac{1.23 + 1.54 + 1.88 + \dots + 3.45 + 3.83}{9}$$

$$\approx 2.51$$

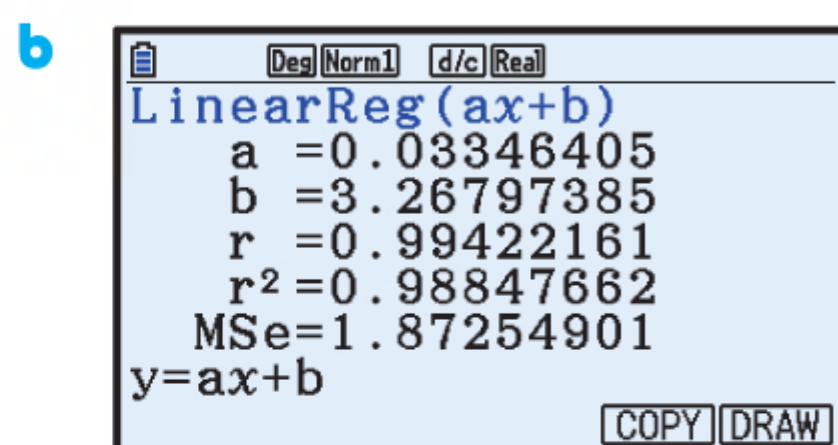
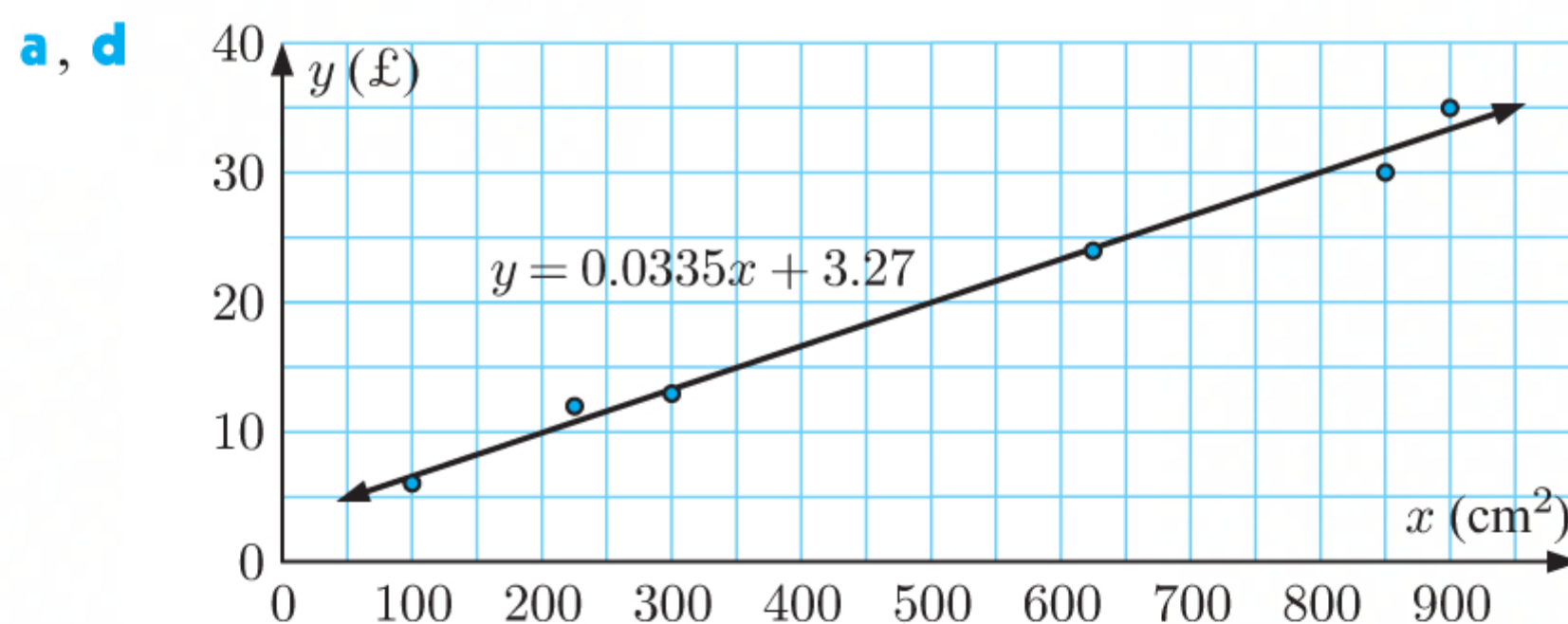
\therefore the mean point (\bar{v}, \bar{t}) is $(50, 2.51)$.



- c** **i** We estimate that the stopping time for a speed of 55 km h⁻¹ is about 2.7 seconds.
ii We estimate that the stopping time for a speed of 110 km h⁻¹ is about 4.4 seconds.
d The estimate in **c i** is more likely to be reliable, since it is an interpolation.

5

Area (x cm ²)	100	225	300	625	850	900
Price (£ y)	6	12	13	24	30	35



So, $r \approx 0.994$.

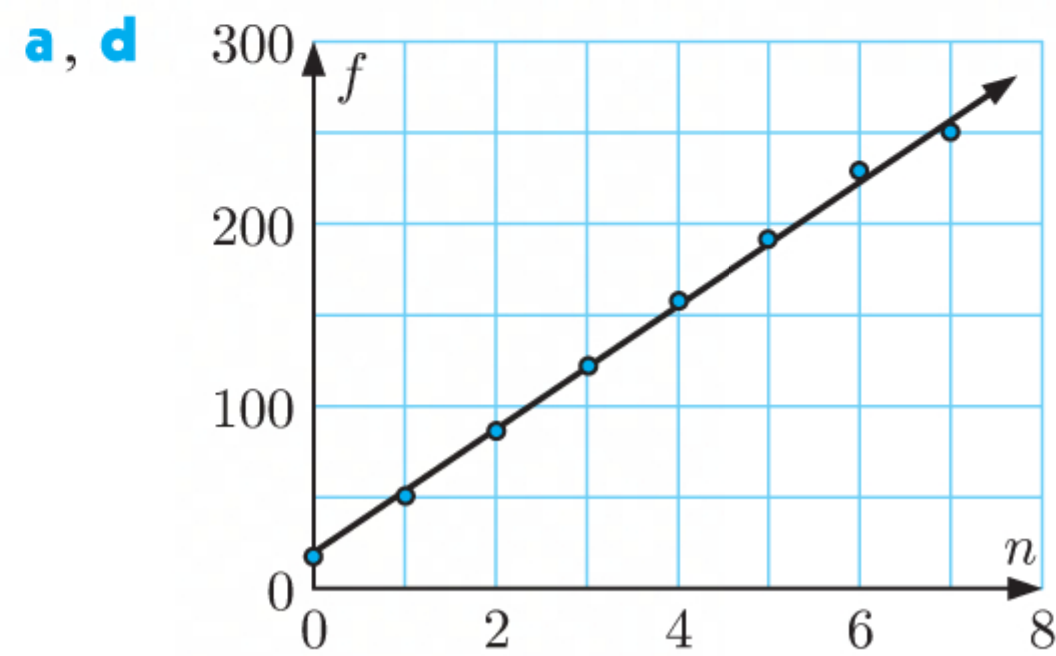
- c** There is a very strong, positive correlation between the *area* of a canvas and its *price*.
d The regression line is $y \approx 0.0335x + 3.27$.

- e When $x = 1200$, $y \approx 0.0335(1200) + 3.27$
 ≈ 43.42

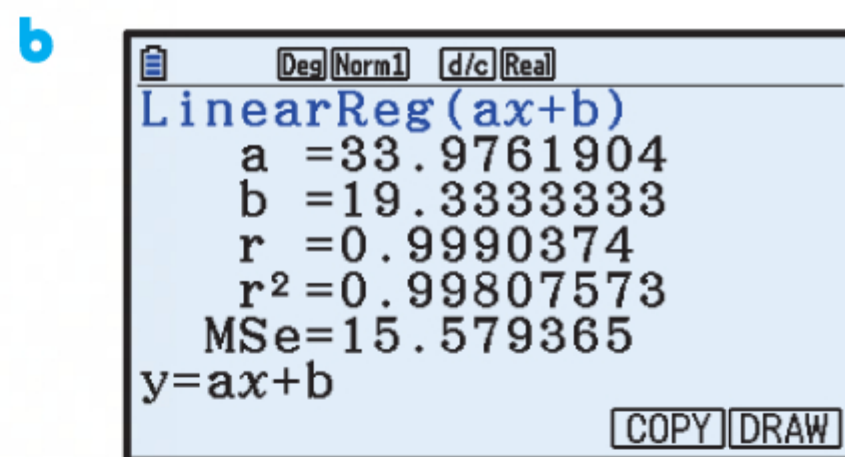
We estimate that a canvas with area 1200 cm^2 will cost about £43.42. This is an extrapolation however, so it may be unreliable.

6

Number of waterings (n)	0	1	2	3	4	5	6	7
Flowers produced (f)	18	52	86	123	158	191	228	250



There is a very strong, positive correlation between number of waterings and flowers produced.



The regression line is $f \approx 34.0n + 19.3$.

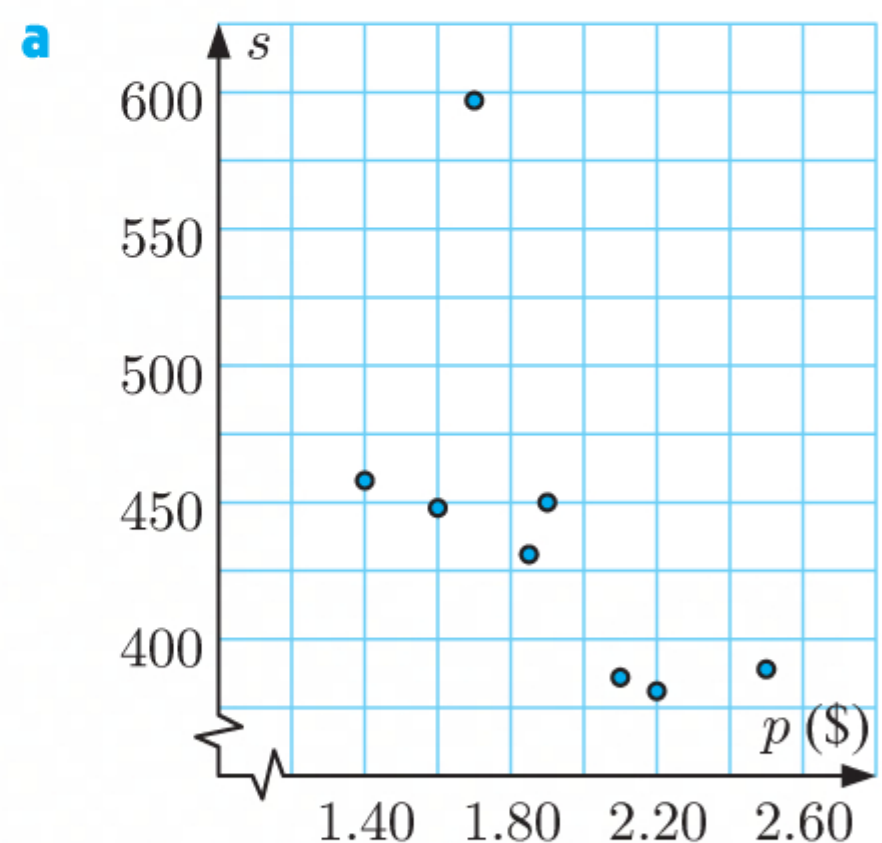
- c Yes, plants need water to grow. So it is expected that an increase in watering will result in an increase in flower production.
- e i 5 times a fortnight \equiv 2.5 times a week
- When $n = 2.5$, $f \approx 34.0(2.5) + 19.3$
 ≈ 104

Violet can expect about 104 flowers from this bed.

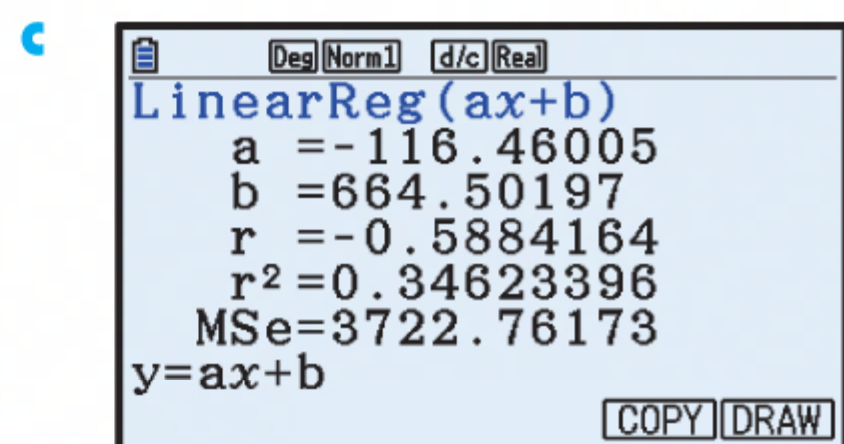
When $n = 10$, $f \approx 34.0(10) + 19.3$
 ≈ 359

Violet can expect about 359 flowers from this bed.
- ii The estimate for $n = 2.5$ is reliable as it is an interpolation.
The estimate for $n = 10$ is unreliable as it is an extrapolation and over-watering could be a problem.

7	<i>Price (\$p)</i>	2.50	1.90	1.60	2.10	2.20	1.40	1.70	1.85
	<i>Sales (s)</i>	389	450	448	386	381	458	597	431



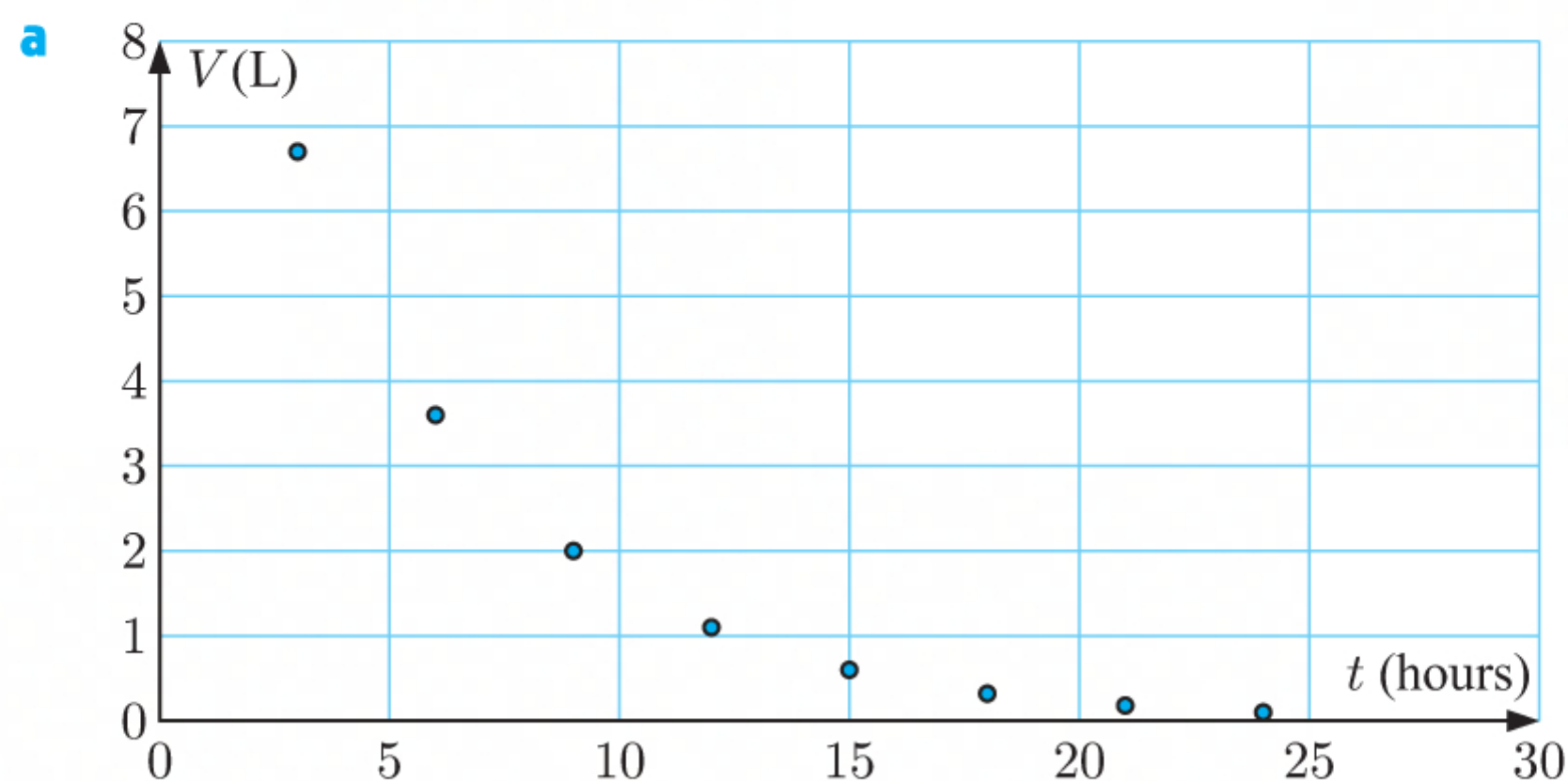
- b** Yes, the point (1.70, 597) is an outlier. It should not be deleted as there is no evidence that it is a recording error.



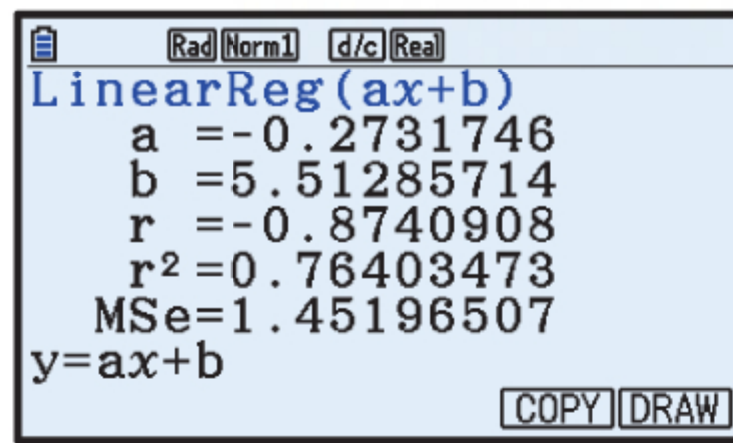
The regression line is $s \approx -116p + 665$.

- d** The gradient of the least squares regression line ≈ -116 . This indicates that for every additional dollar the price increases by, the number of sales decreases by 116.
- e** No, the prediction of sales of Supa-fizz if it was priced at 50 cents would not be accurate, as it is an extrapolation well beyond the range of data values given.

8	<i>Time (t hours)</i>	3	6	9	12	15	18	21	24
	<i>Water remaining (V litres)</i>	6.7	3.6	2	1.1	0.6	0.32	0.18	0.10



b



So, $r_p \approx -0.874$.

- c As t increases, V always decreases.

So the ranks are perfectly negatively correlated.

$$\therefore r_s = -1.$$

- d From the scatter diagram, the relationship between the variables is clearly non-linear. Spearman's rank correlation coefficient is therefore more appropriate.

Chapter 6

QUADRATIC FUNCTIONS

EXERCISE 6A

- 1 a** $y = 2x^2 - 4x + 10$ is a relationship between two variables x and y which is in the form $y = ax^2 + bx + c$ where a, b, c are constants, $a \neq 0$.
 $\therefore y = 2x^2 - 4x + 10$ is a quadratic function.
- b** $y = 8x + 3$ cannot be written in the form $y = ax^2 + bx + c$, $a \neq 0$.
 $\therefore y = 8x + 3$ is not a quadratic function.
- c** $y = -2x^2$ is a relationship between two variables x and y which is in the form $y = ax^2 + bx + c$ where a, b, c are constants, $a \neq 0$.
 $\therefore y = -2x^2$ is a quadratic function.
- d** $y = \frac{1}{3}x + 6 - x^2$ can be written as $y = -x^2 + \frac{1}{3}x + 6$.
 $\therefore y = \frac{1}{3}x + 6 - x^2$ is a relationship between two variables x and y which can be written in the form $y = ax^2 + bx + c$ where a, b, c are constants, $a \neq 0$.
 $\therefore y = \frac{1}{3}x + 6 - x^2$ is a quadratic function.
- e** $2y + x - 3 = 0$ cannot be written in the form $y = ax^2 + bx + c$, $a \neq 0$.
 $\therefore 2y + x - 3 = 0$ is not a quadratic function.
- f** $y - 2x^2 = 3x - 1$ can be written as $y = 2x^2 + 3x - 1$.
 $\therefore y - 2x^2 = 3x - 1$ is a relationship between two variables x and y which can be written in the form $y = ax^2 + bx + c$ where a, b, c are constants, $a \neq 0$.
 $\therefore y - 2x^2 = 3x - 1$ is a quadratic function.

2 a When $x = 1$,
 $y = 1^2 + 3(1) - 7$
 $= 1 + 3 - 7$
 $= -3$

c When $x = 3$,
 $y = 3(3)^2 - 2(3) - 5$
 $= 27 - 6 - 5$
 $= 16$

b When $x = -2$,
 $y = -2(-2)^2 + 5(-2) + 2$
 $= -8 - 10 + 2$
 $= -16$

d When $x = -1$,
 $y = -3(-1)^2 + 7(-1) - 2$
 $= -3 - 7 - 2$
 $= -12$

3 a $y = x^2 - 3x + 1$

x	-2	-1	0	1	2
y	11	5	1	-1	-1

b $y = -3x^2 + 2x + 4$

x	-4	-2	0	2	4
y	-52	-12	4	-4	-36

4 a $f(x) = x^2 + 3x - 7$
 $\therefore f(2) = 2^2 + 3(2) - 7$ and $f(-1) = (-1)^2 + 3(-1) - 7$
 $= 4 + 6 - 7$ $= 1 - 3 - 7$
 $= 3$ $= -9$

$$\begin{aligned}
 \text{b} \quad f(x) &= 2x^2 - x + 1 \\
 \therefore f(0) &= 2(0)^2 - 0 + 1 & \text{and} & \quad f(-3) = 2(-3)^2 - (-3) + 1 \\
 &= 0 - 0 + 1 & & \quad = 2(9) + 3 + 1 \\
 &= 1 & & \quad = 18 + 3 + 1 \\
 & & & \quad = 22
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad g(x) &= -3x^2 - 2x + 4 \\
 \therefore g(3) &= -3(3)^2 - 2(3) + 4 & \text{and} & \quad g(-2) = -3(-2)^2 - 2(-2) + 4 \\
 &= -3(9) - 6 + 4 & & \quad = -3(4) + 4 + 4 \\
 &= -27 - 6 + 4 & & \quad = -12 + 4 + 4 \\
 &= -29 & & \quad = -4
 \end{aligned}$$

$$\begin{aligned}
 \text{5 a} \quad \text{When } x &= 0, \\
 y &= 2(0)^2 + 5 \\
 &= 0 + 5 \\
 &= 5 \\
 \therefore (0, 4) &\text{ does not satisfy the function} \\
 & \quad y = 2x^2 + 5.
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad f(-1) &= -(-1)^2 + 2(-1) - 5 \\
 &= -1 - 2 - 5 \\
 &= -8 \\
 \therefore (-1, -8) &\text{ satisfies the function} \\
 & \quad f(x) = -x^2 + 2x - 5.
 \end{aligned}$$

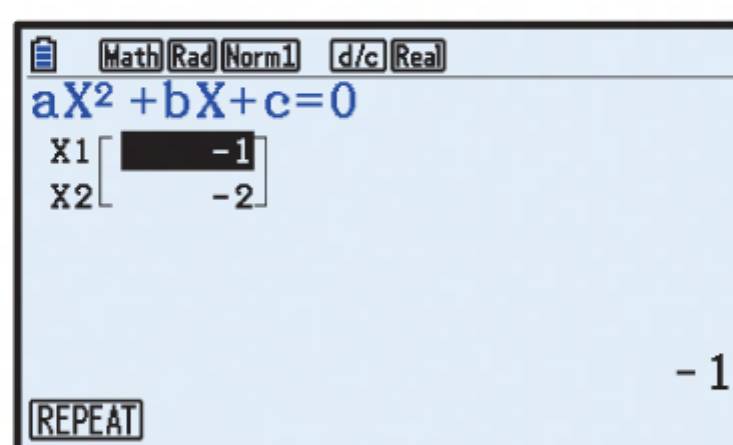
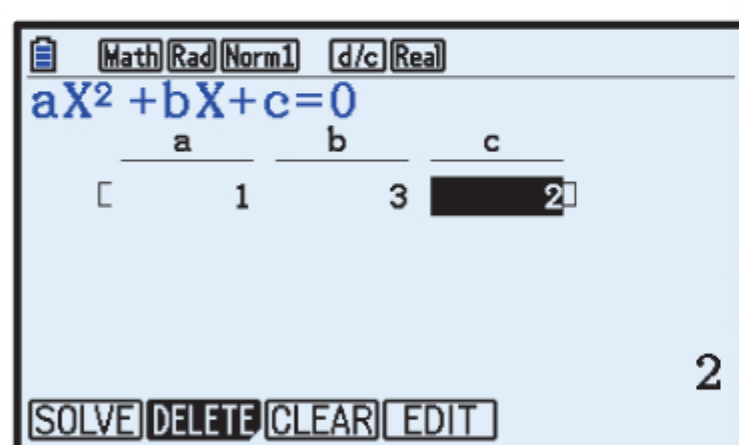
$$\begin{aligned}
 \text{e} \quad \text{When } x &= 2, \\
 y &= 3(2)^2 - 4(2) + 10 \\
 &= 12 - 8 + 10 \\
 &= 14 \\
 \therefore (2, 10) &\text{ does not satisfy the function} \\
 & \quad y = 3x^2 - 4x + 10.
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad \text{When } x &= 2, \\
 y &= (2)^2 - 3(2) + 2 \\
 &= 4 - 6 + 2 \\
 &= 0 \\
 \therefore (2, 0) &\text{ satisfies the function} \\
 & \quad y = x^2 - 3x + 2.
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad \text{When } x &= 3, \\
 y &= -2(3)^2 - 3 + 6 \\
 &= -18 - 3 + 6 \\
 &= -15 \\
 \therefore (3, -15) &\text{ satisfies the function} \\
 & \quad y = -2x^2 - x + 6.
 \end{aligned}$$

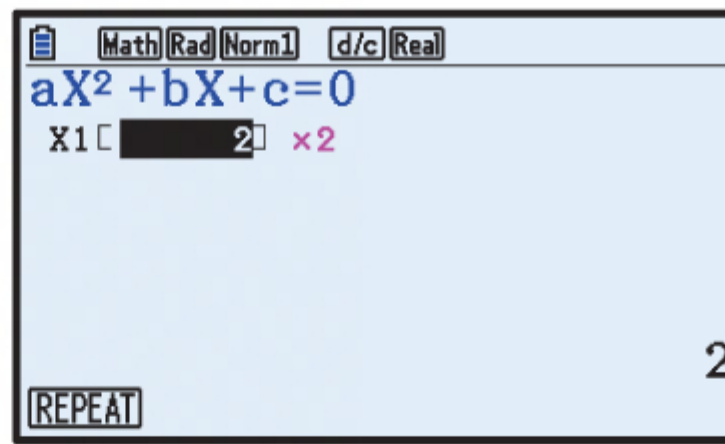
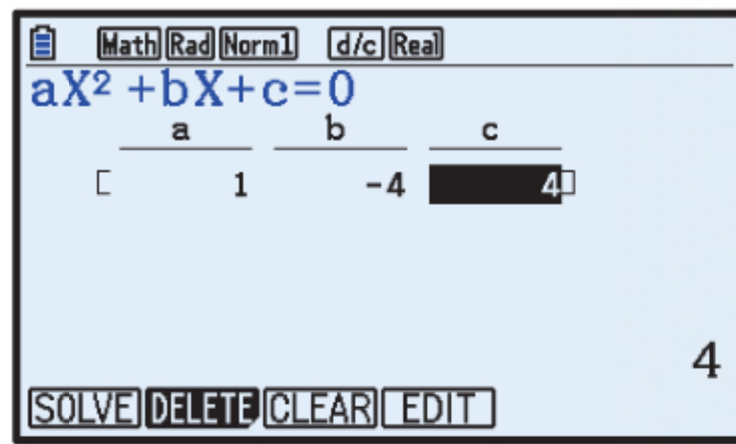
$$\begin{aligned}
 \text{f} \quad f(2) &= -\frac{1}{2}(2)^2 + 4(2) - 1 \\
 &= -2 + 8 - 1 \\
 &= 5 \\
 \therefore (2, 5) &\text{ satisfies the function} \\
 & \quad f(x) = -\frac{1}{2}x^2 + 4x - 1.
 \end{aligned}$$

$$\begin{aligned}
 \text{6 a} \quad \text{If } y &= 4 \text{ then } x^2 + 3x + 6 = 4 \\
 \therefore x^2 + 3x + 2 &= 0
 \end{aligned}$$



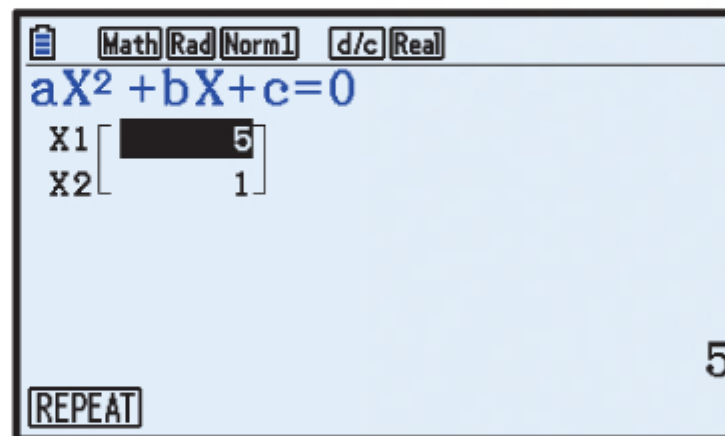
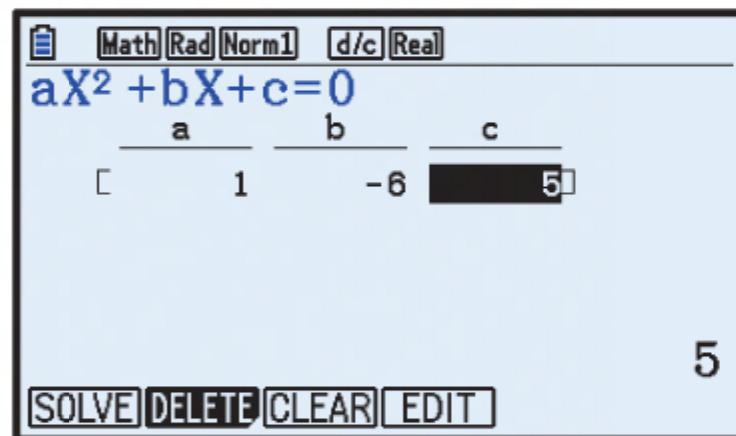
$$\therefore x = -2 \text{ or } -1$$

- b** If $y = 3$ then $x^2 - 4x + 7 = 3$
 $\therefore x^2 - 4x + 4 = 0$



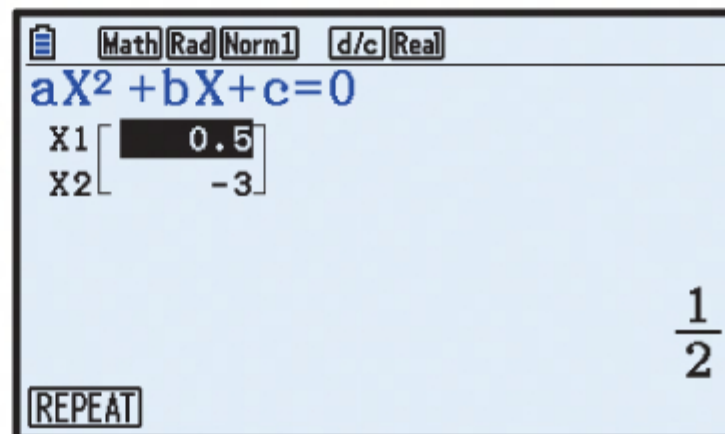
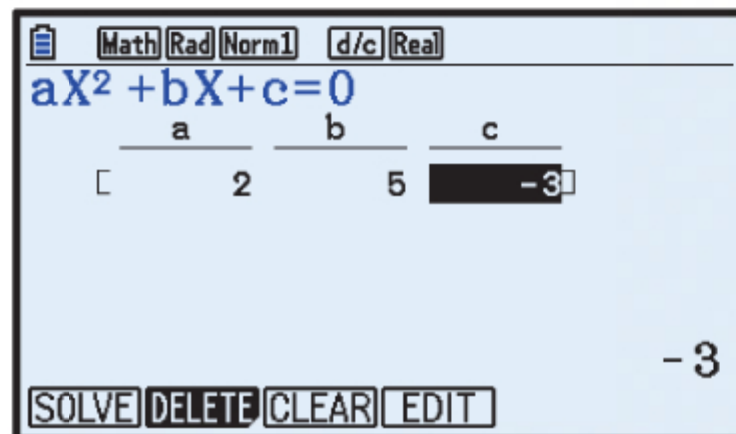
$$\therefore x = 2$$

- c** If $y = -4$ then $x^2 - 6x + 1 = -4$
 $\therefore x^2 - 6x + 5 = 0$



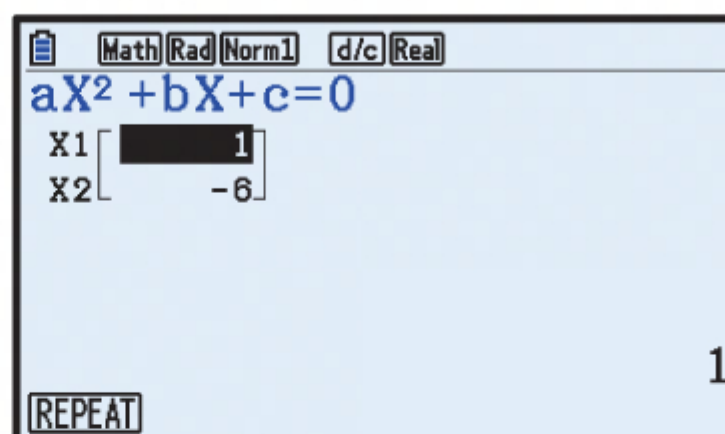
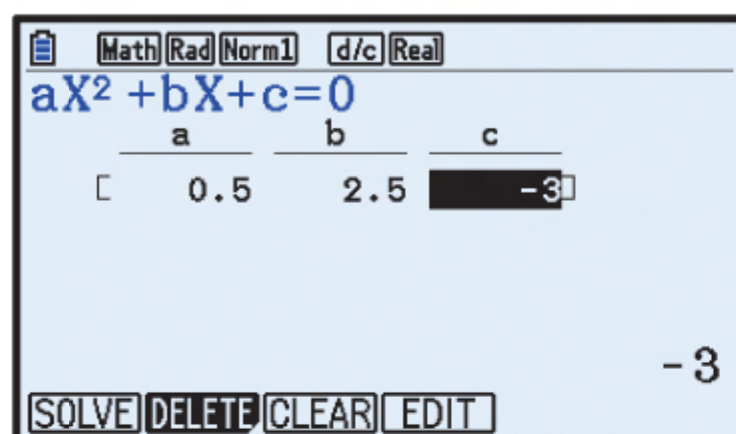
$$\therefore x = 1 \text{ or } 5$$

- d** If $y = 4$ then $2x^2 + 5x + 1 = 4$
 $\therefore 2x^2 + 5x - 3 = 0$



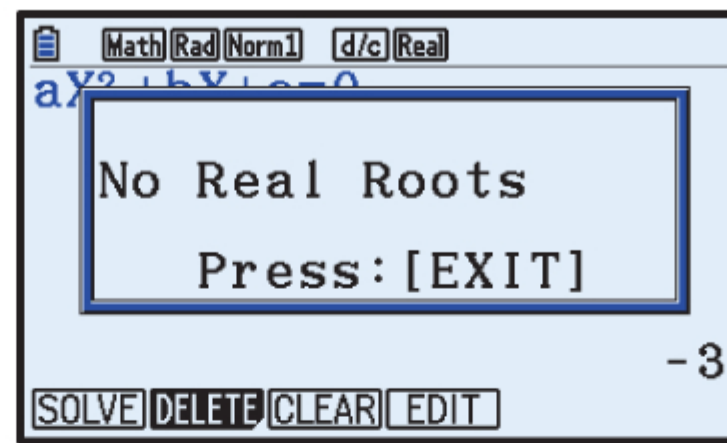
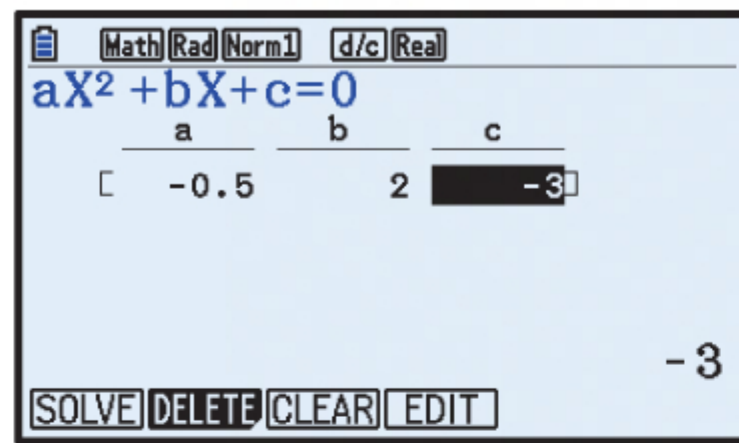
$$\therefore x = -3 \text{ or } \frac{1}{2}$$

- e** If $y = 1$ then $\frac{1}{2}x^2 + \frac{5}{2}x - 2 = 1$
 $\therefore \frac{1}{2}x^2 + \frac{5}{2}x - 3 = 0$



$$\therefore x = -6 \text{ or } 1$$

f If $y = 2$ then $-\frac{1}{2}x^2 + 2x - 1 = 2$
 $\therefore -\frac{1}{2}x^2 + 2x - 3 = 0$



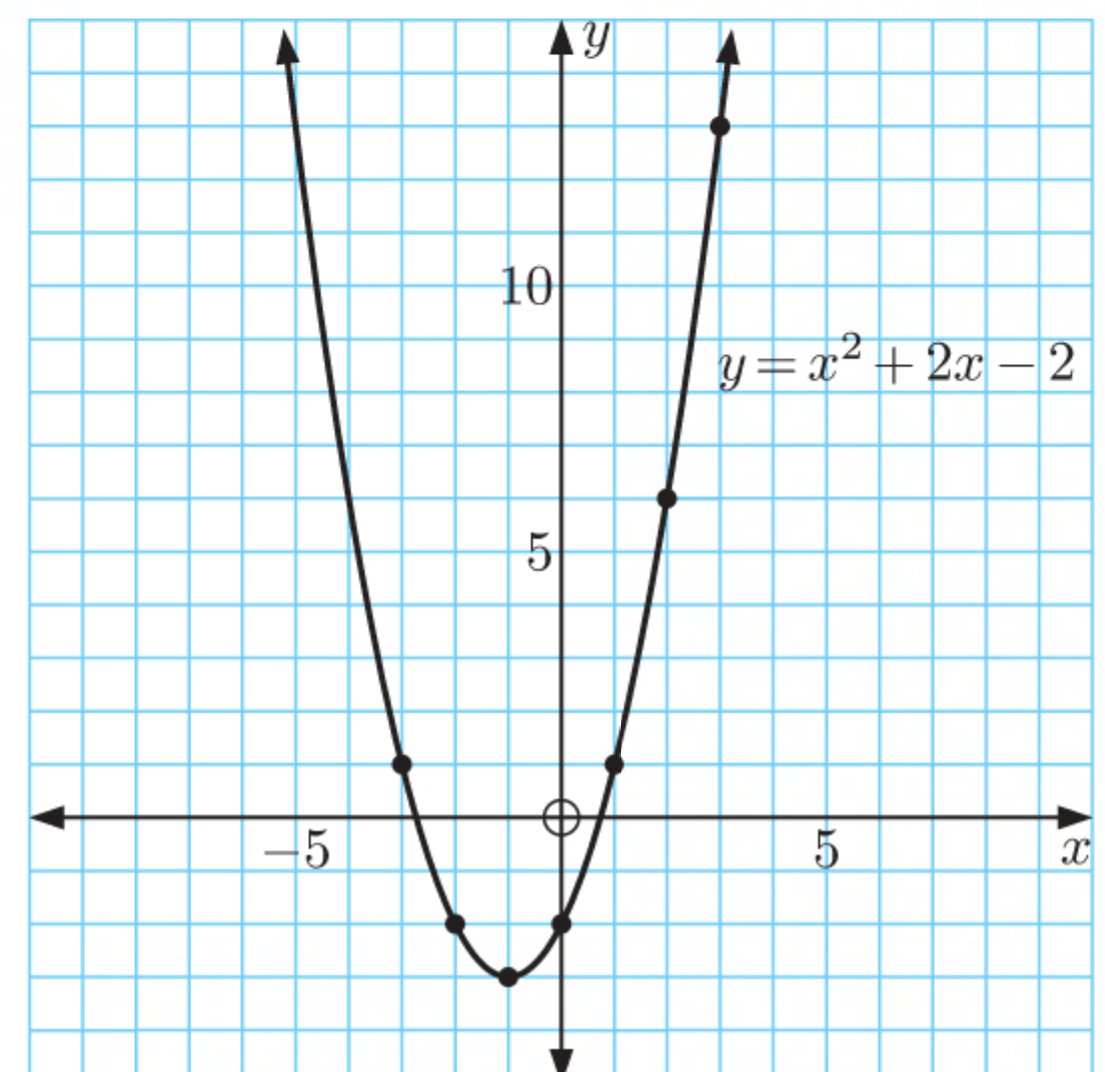
\therefore there are no real solutions.

EXERCISE 6B

1 a When $x = -3$, $y = (-3)^2 + 2(-3) - 2$
 $= 9 - 6 - 2$
 $= 1$

We can perform similar calculations for other values of x , to produce a table of values:

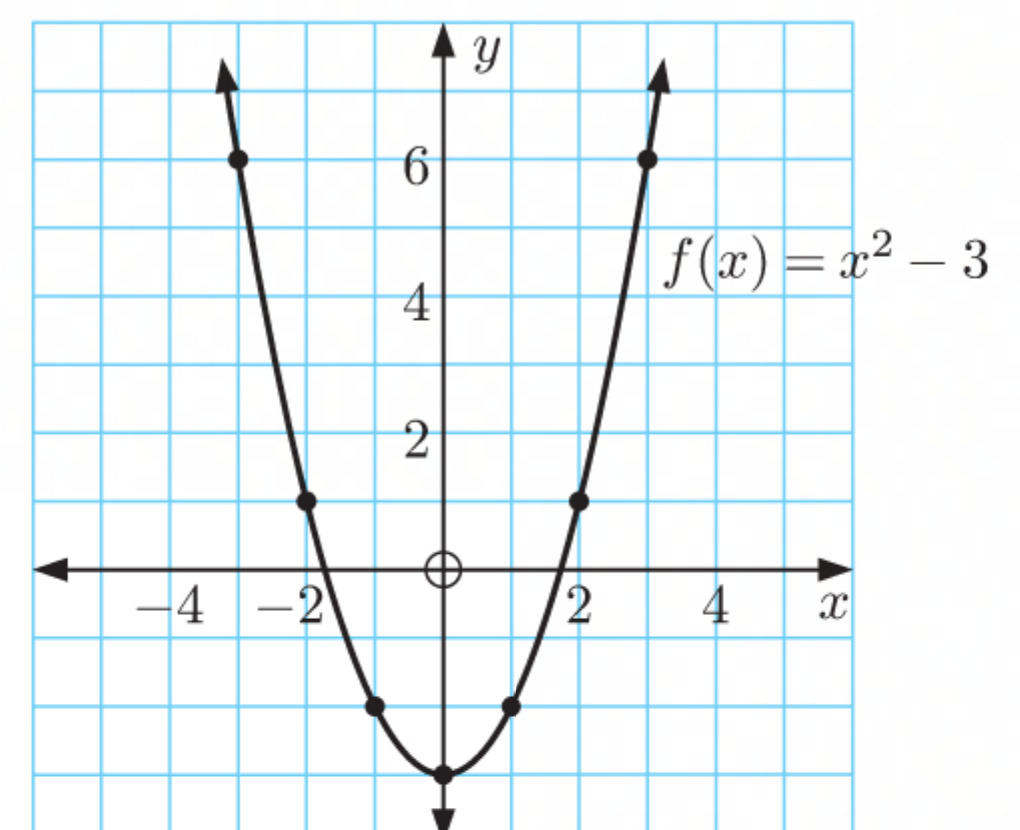
x	-3	-2	-1	0	1	2	3
y	1	-2	-3	-2	1	6	13



b $f(-3) = (-3)^2 - 3$
 $= 9 - 3$
 $= 6$

We can perform similar calculations for other values of x , to produce a table of values:

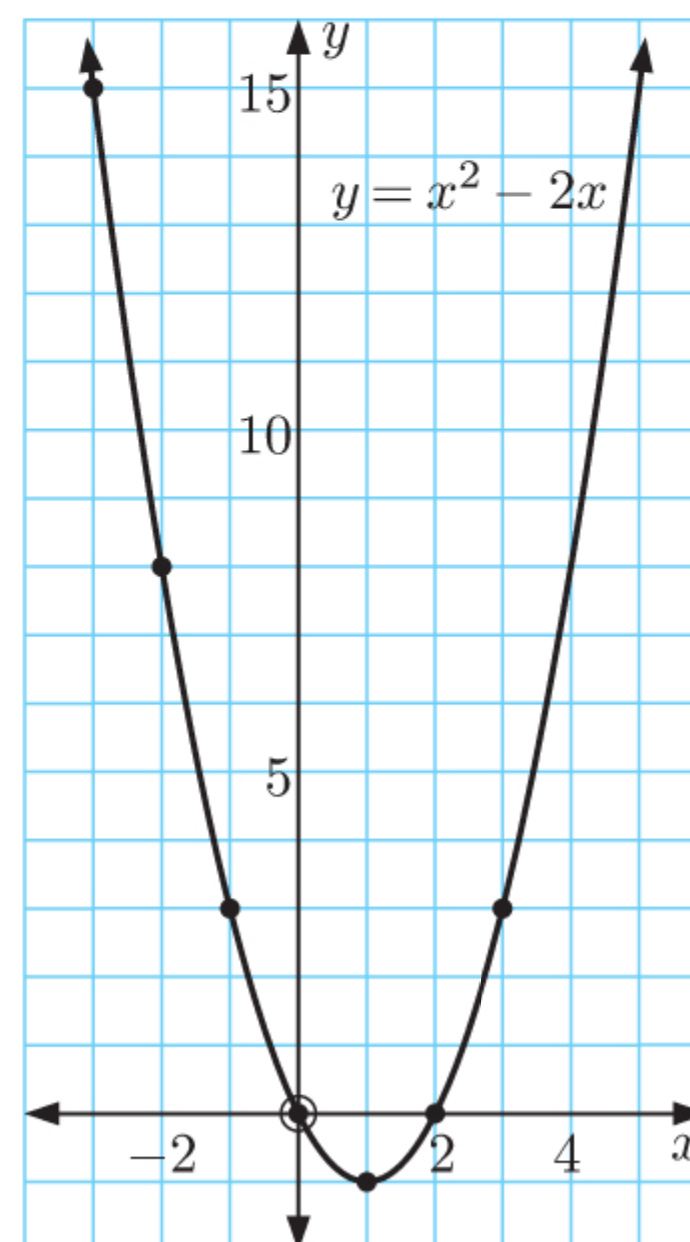
x	-3	-2	-1	0	1	2	3
$f(x)$	6	1	-2	-3	-2	1	6



c When $x = -3$, $y = (-3)^2 - 2(-3)$
 $= 9 + 6$
 $= 15$

We can perform similar calculations for other values of x , to produce a table of values:

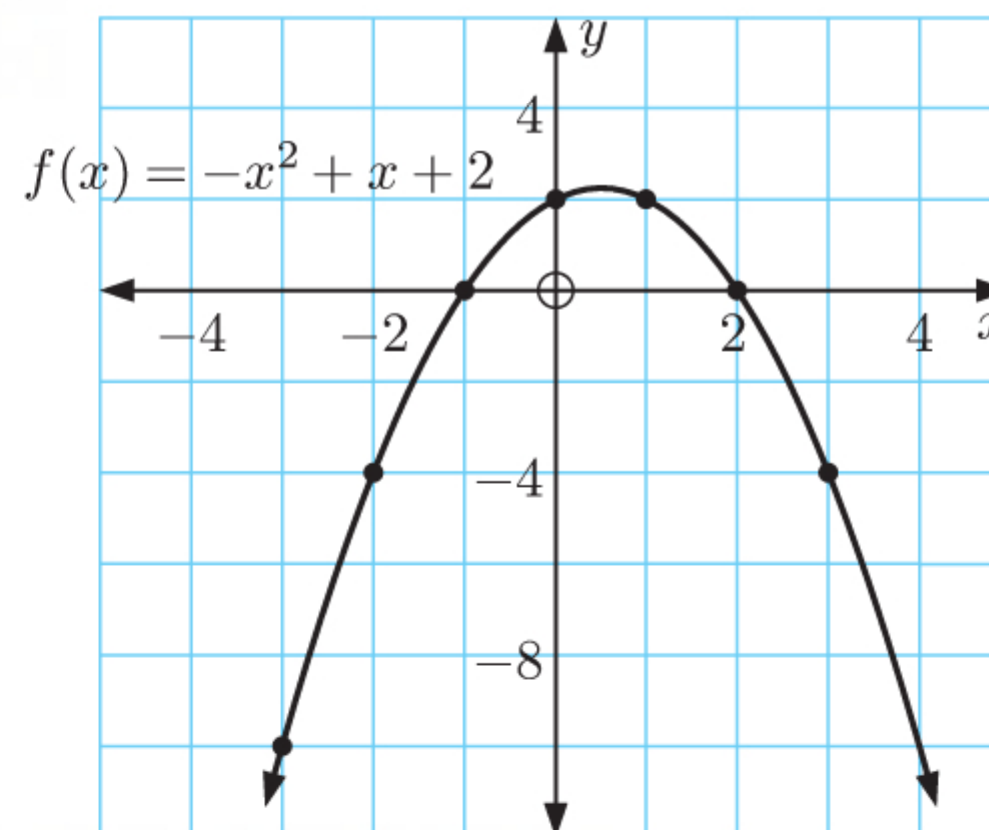
x	-3	-2	-1	0	1	2	3
y	15	8	3	0	-1	0	3



d $f(-3) = -(-3)^2 + (-3) + 2$
 $= -9 - 3 + 2$
 $= -10$

We can perform similar calculations for other values of x , to produce a table of values:

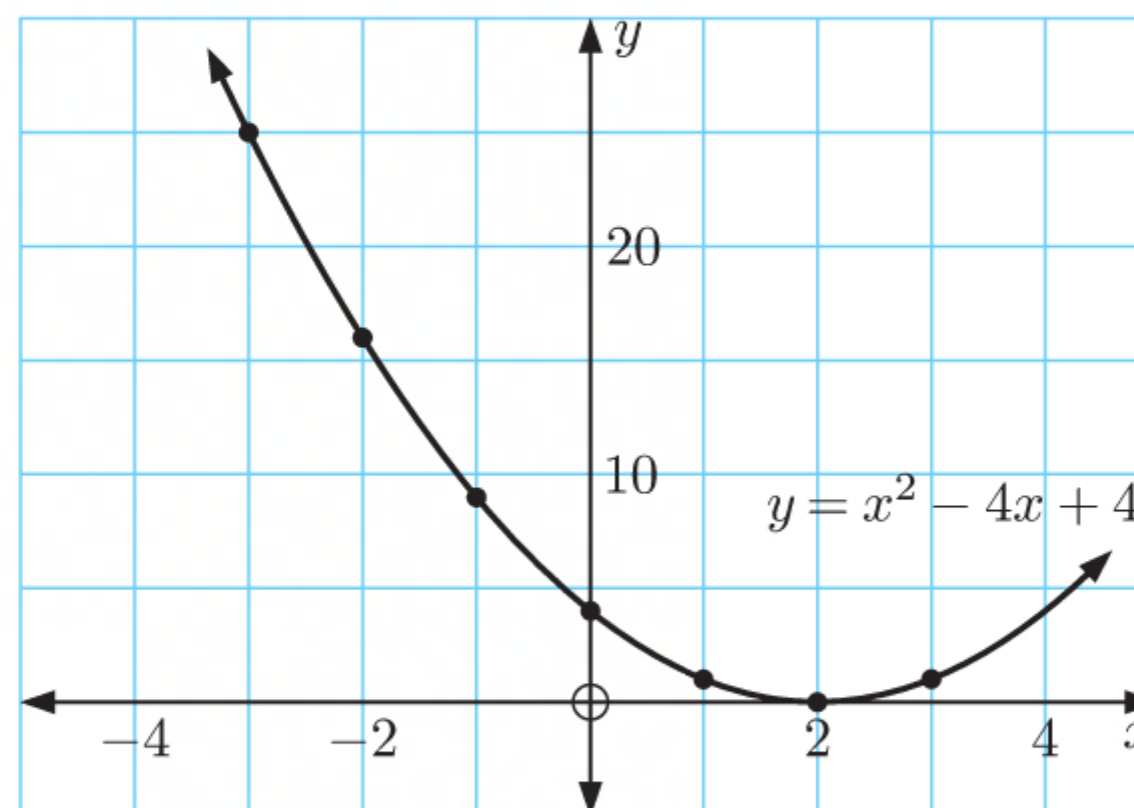
x	-3	-2	-1	0	1	2	3
$f(x)$	-10	-4	0	2	2	0	-4



e When $x = -3$, $y = (-3)^2 - 4(-3) + 4$
 $= 9 + 12 + 4$
 $= 25$

We can perform similar calculations for other values of x , to produce a table of values:

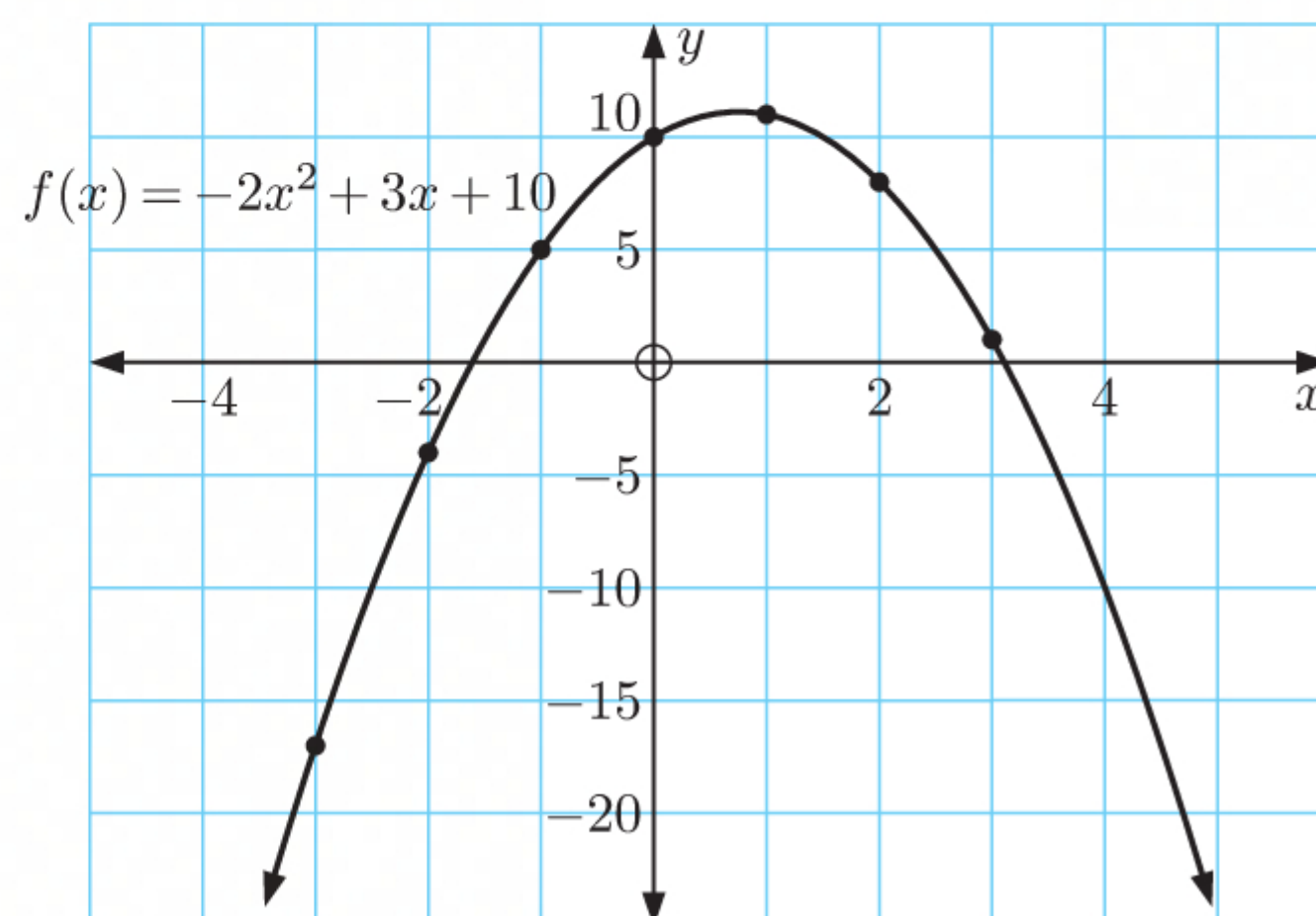
x	-3	-2	-1	0	1	2	3
y	25	16	9	4	1	0	1



f $f(-3) = -2(-3)^2 + 3(-3) + 10$
 $= -18 - 9 + 10$
 $= -17$

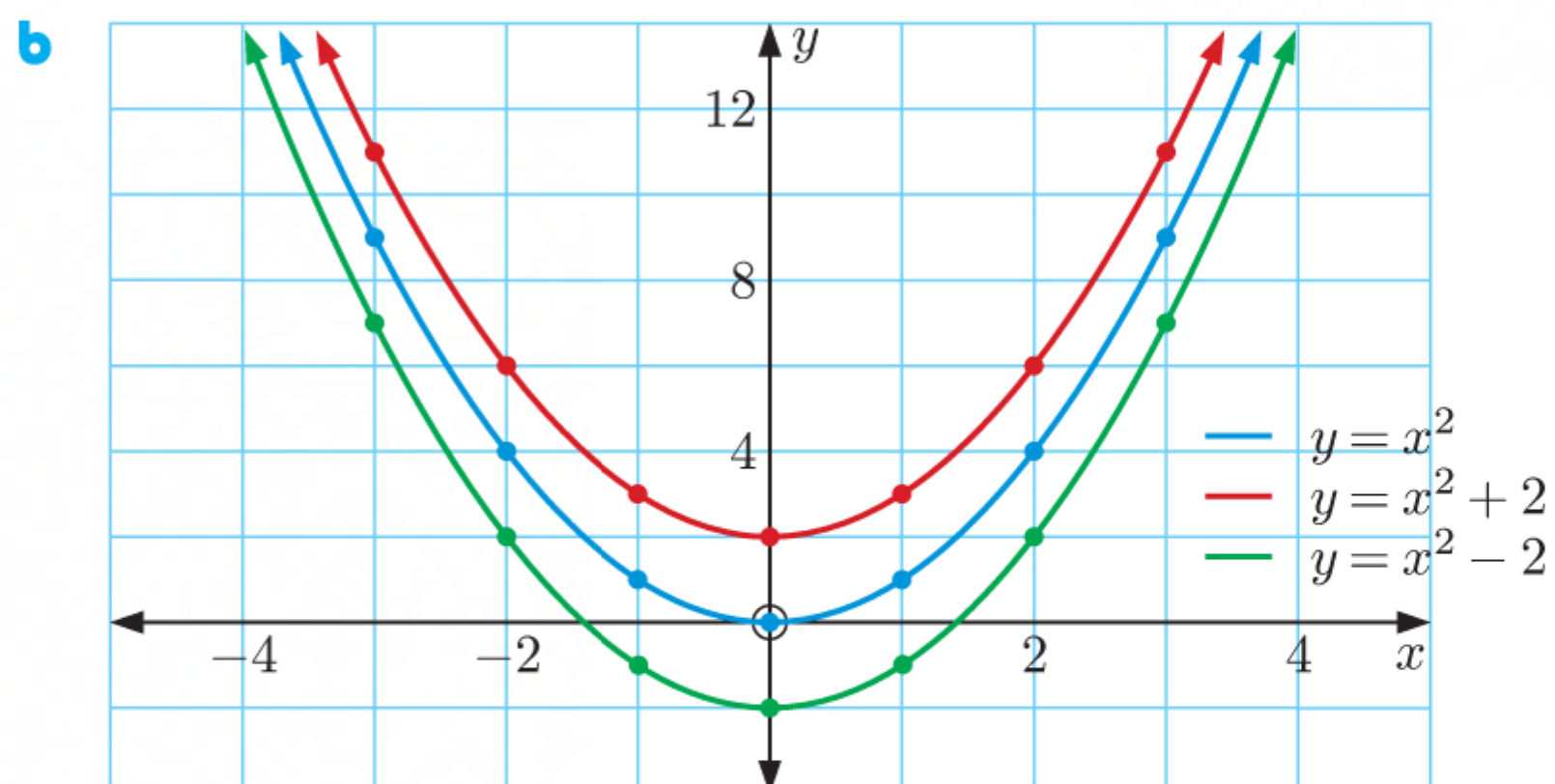
We can perform similar calculations for other values of x , to produce a table of values:

x	-3	-2	-1	0	1	2	3
$f(x)$	-17	-4	5	10	11	8	1



2 a

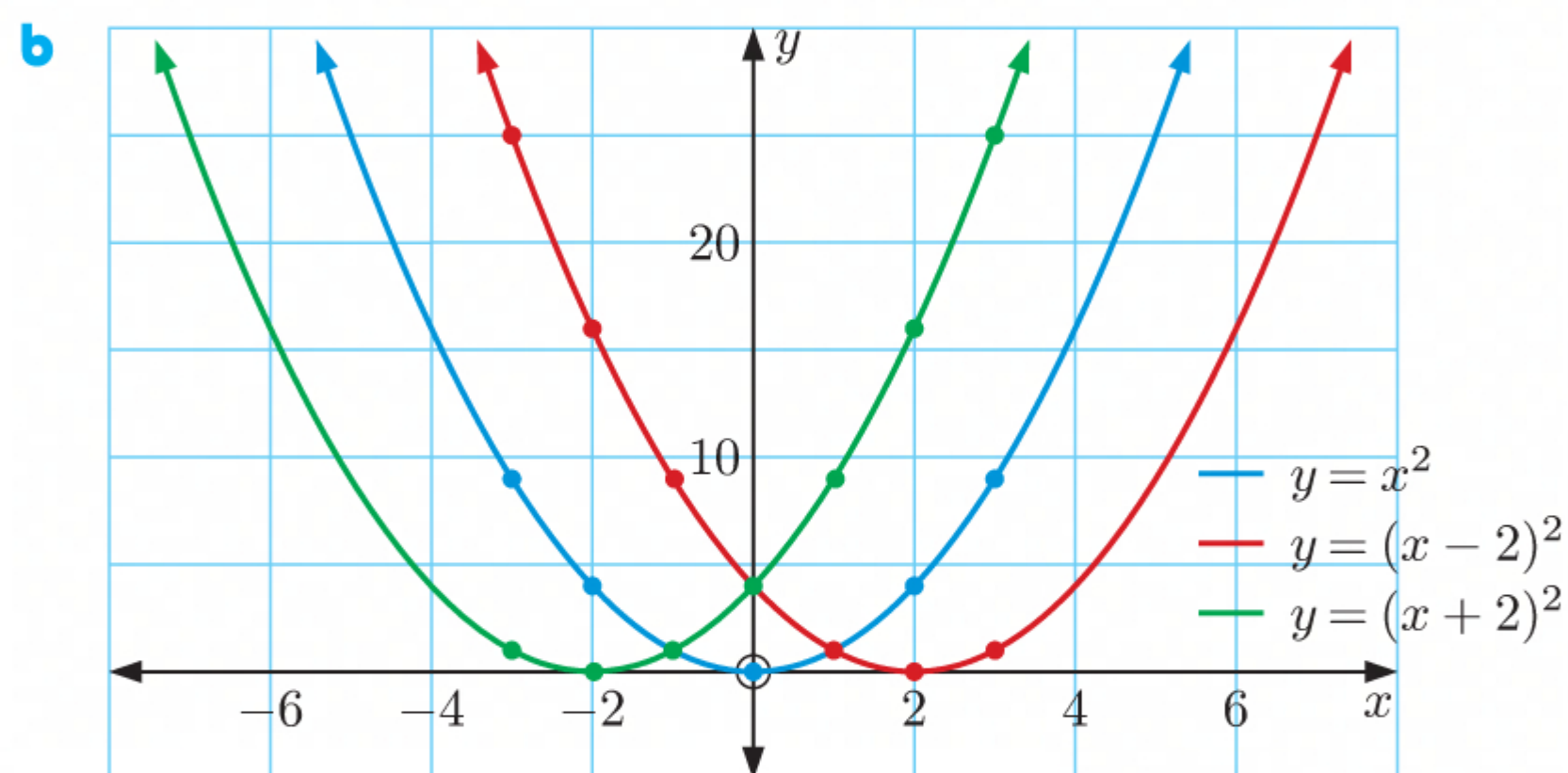
x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$x^2 + 2$	11	6	3	2	3	6	11
$x^2 - 2$	7	2	-1	-2	-1	2	7



- c** The graphs have the same shape. $y = x^2 + 2$ is $y = x^2$ translated 2 units up, and $y = x^2 - 2$ is $y = x^2$ translated 2 units down.

3 a

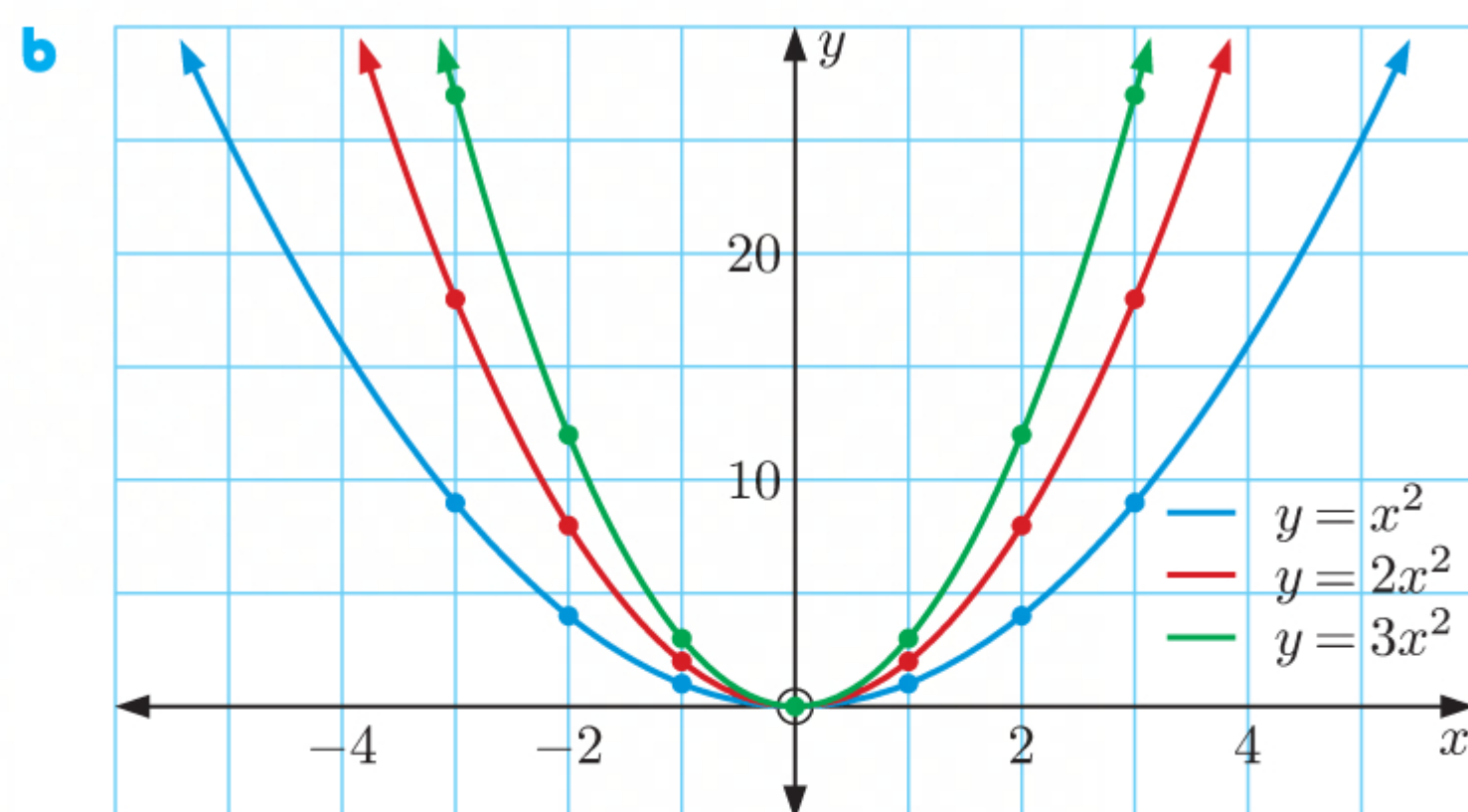
x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$(x+2)^2$	1	0	1	4	9	16	25
$(x-2)^2$	25	16	9	4	1	0	1



- c** The graphs have the same shape. $y = (x + 2)^2$ is $y = x^2$ translated 2 units left, and $y = (x - 2)^2$ is $y = x^2$ translated 2 units right.

4 a

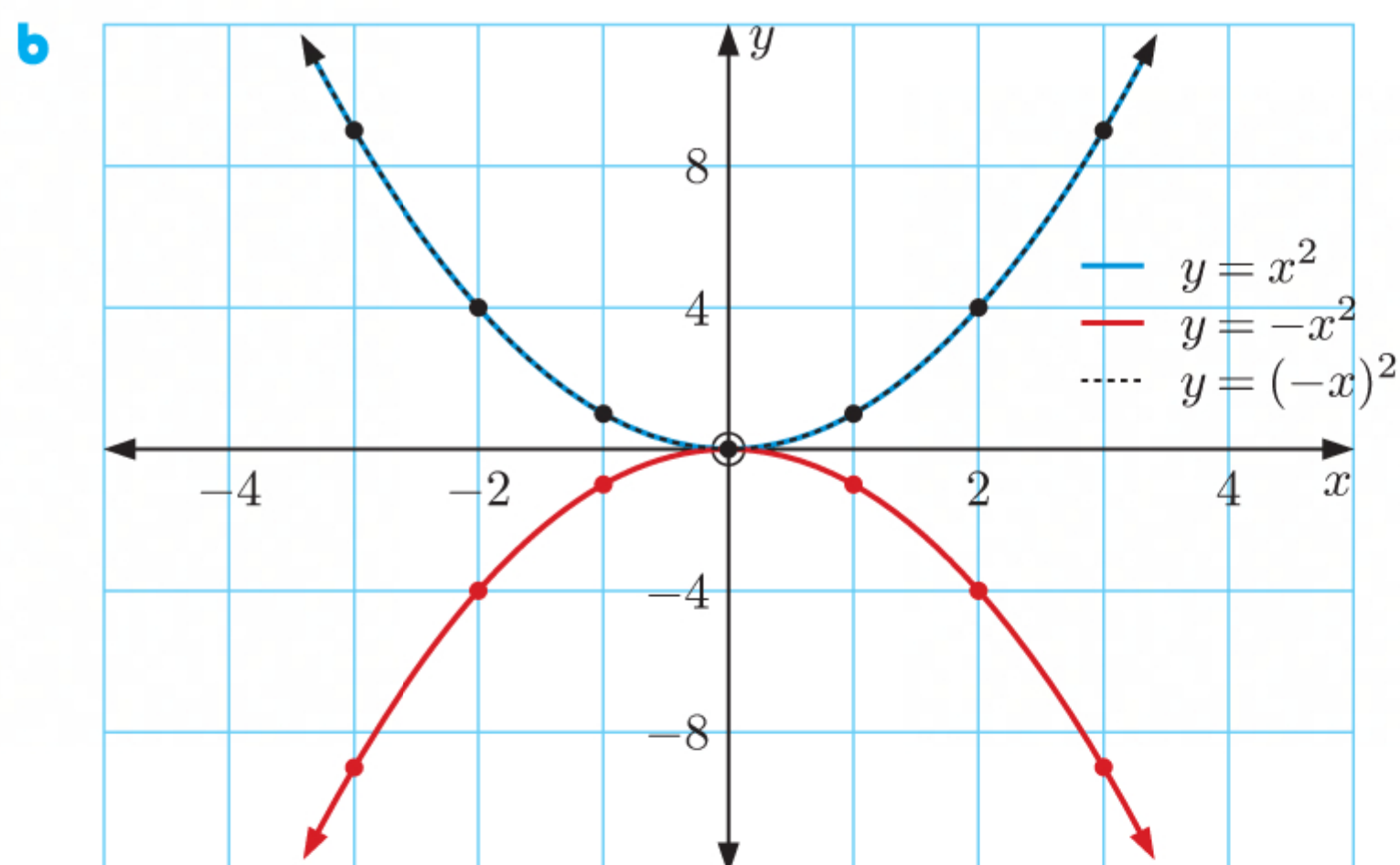
x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$2x^2$	18	8	2	0	2	8	18
$3x^2$	27	12	3	0	3	12	27



- c** As the coefficient of x^2 increases, the graphs become narrower.

5 a

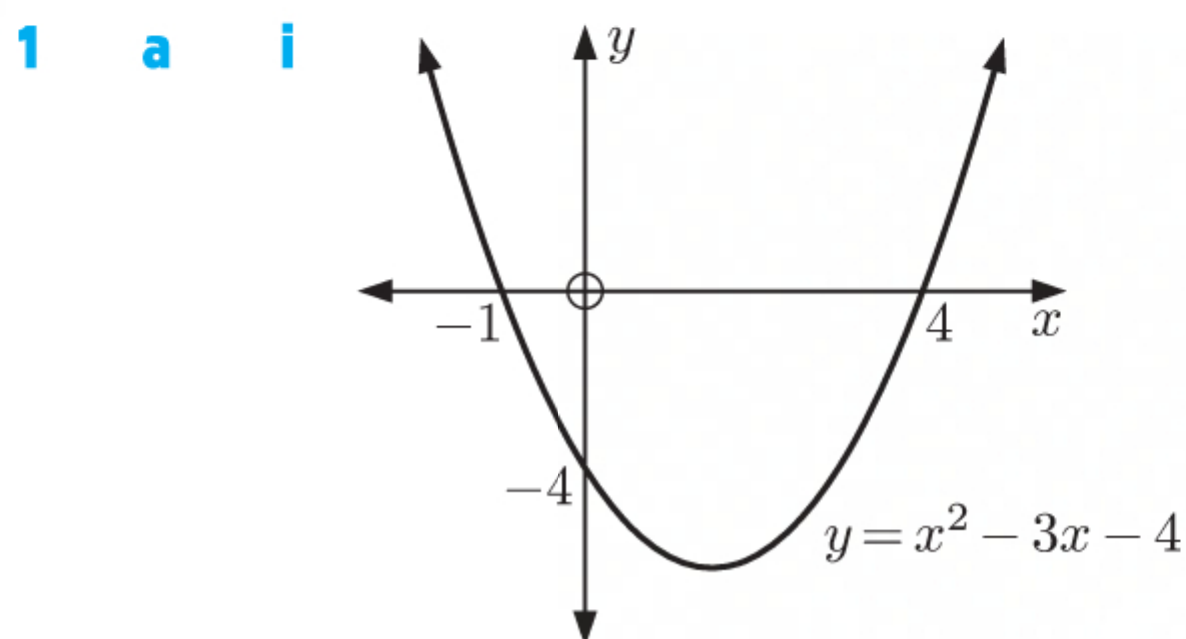
x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$-x^2$	-9	-4	-1	0	-1	-4	-9
$(-x)^2$	9	4	1	0	1	4	9



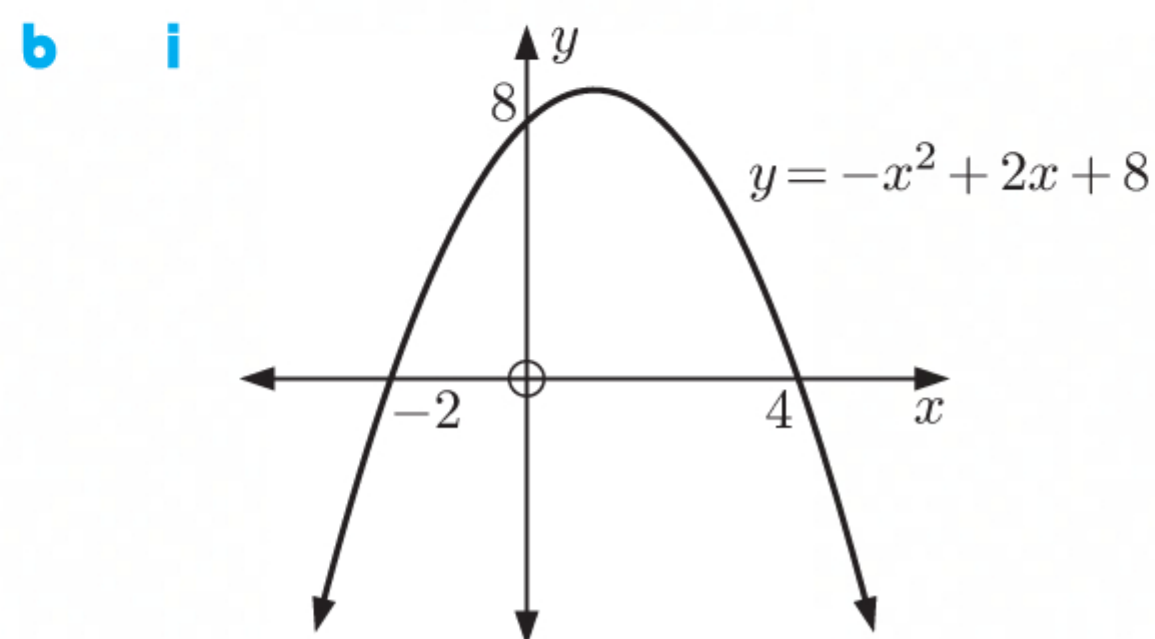
- c** $y = -x^2$ is a reflection of $y = x^2$ (or $y = (-x)^2$) in the x -axis.

INVESTIGATION 1

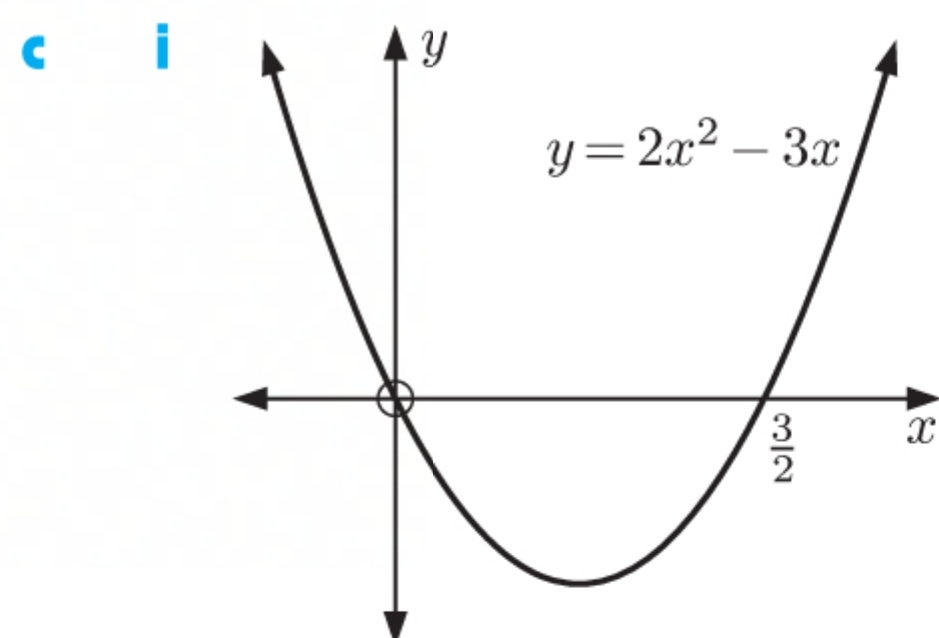
AXES INTERCEPTS



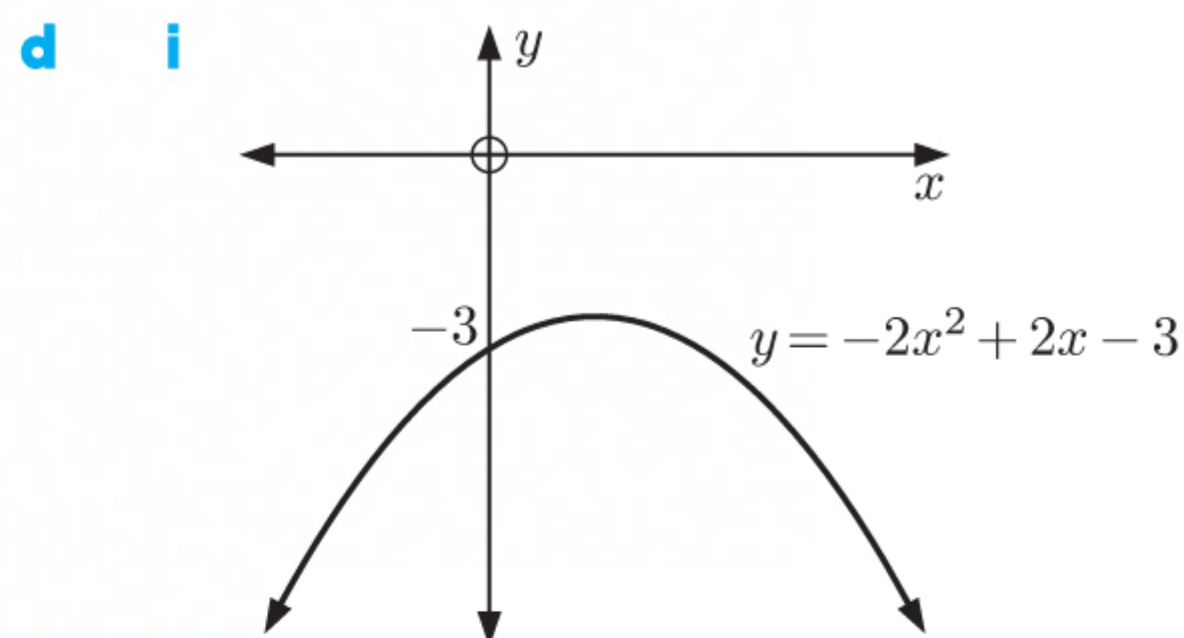
- ii** The y -intercept is -4 .
- iii** The x -intercepts are -1 and 4 .



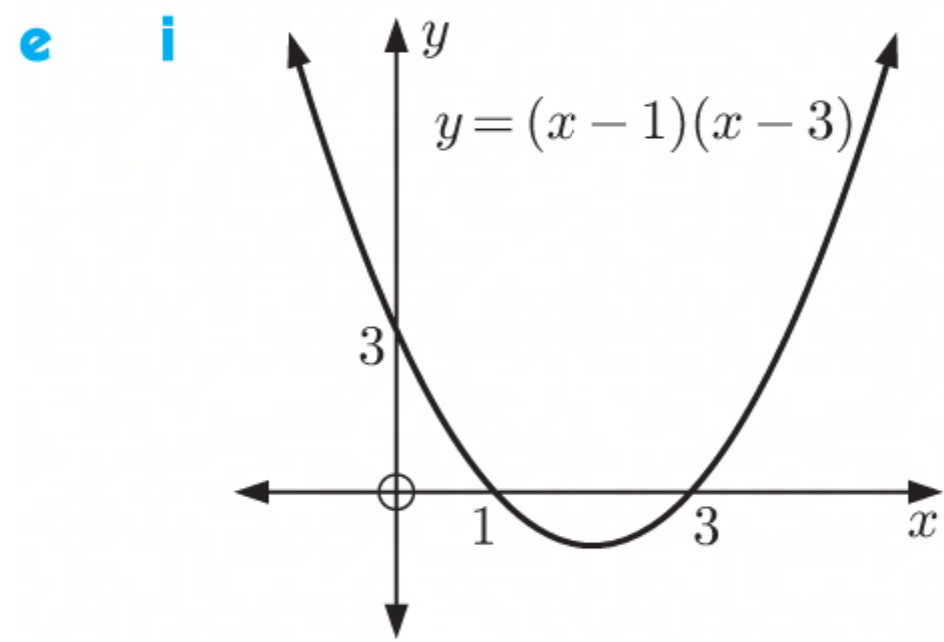
- ii** The y -intercept is 8 .
- iii** The x -intercepts are -2 and 4 .



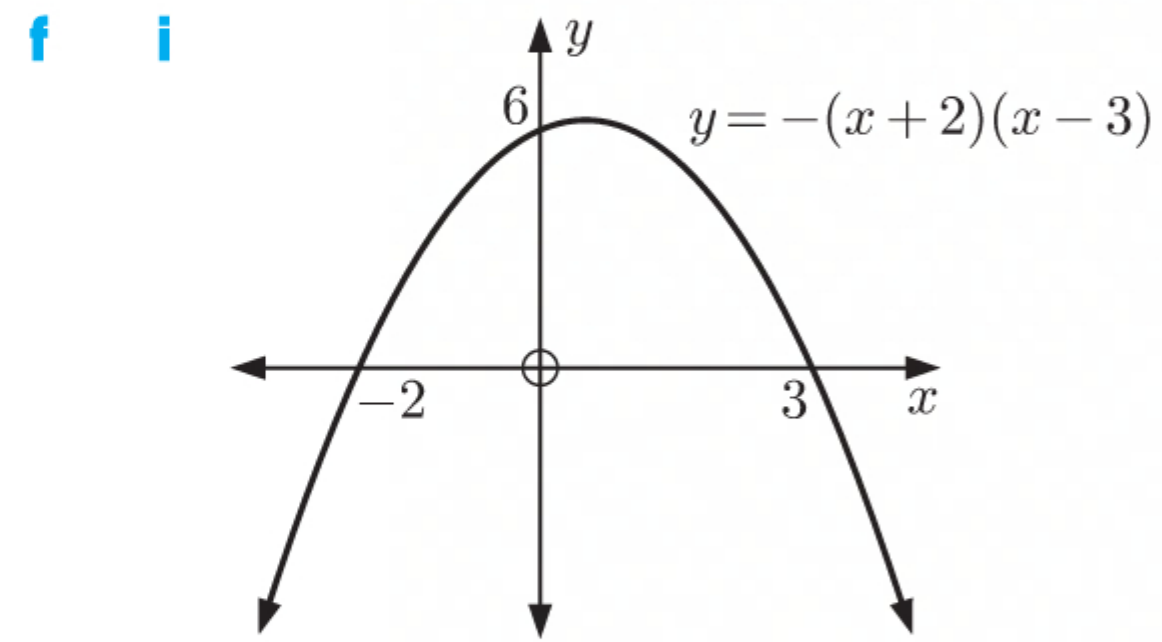
- ii** The y -intercept is 0 .
- iii** The x -intercepts are 0 and $\frac{3}{2}$.



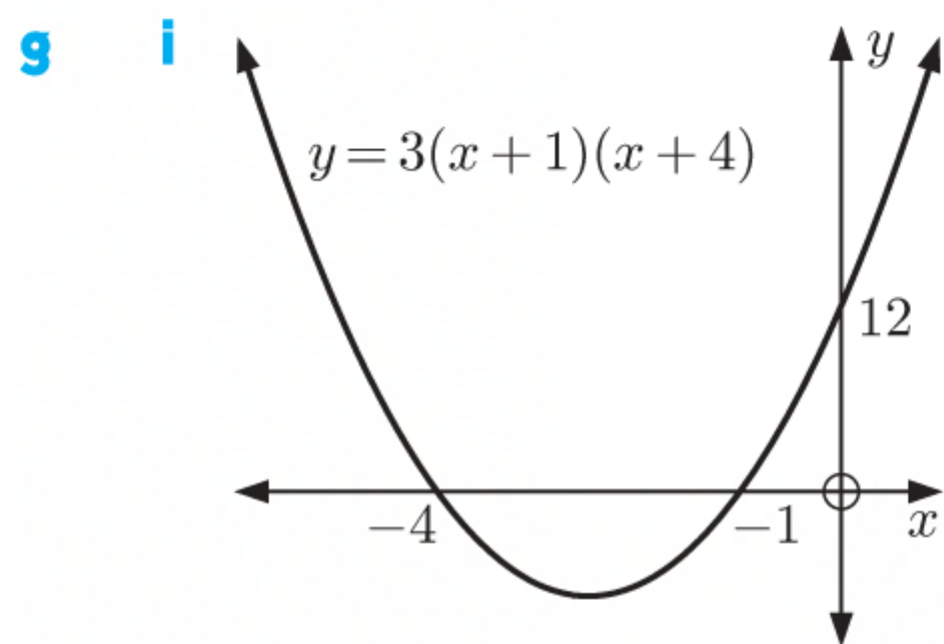
- ii** The y -intercept is -3 .
- iii** There are no x -intercepts.



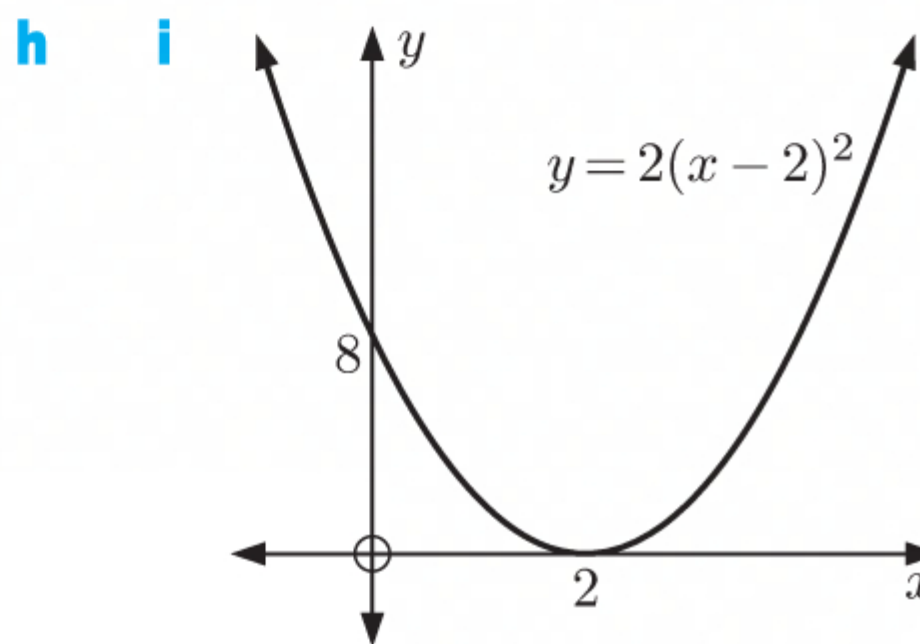
- ii** The y -intercept is 3.
- iii** The x -intercepts are 1 and 3.



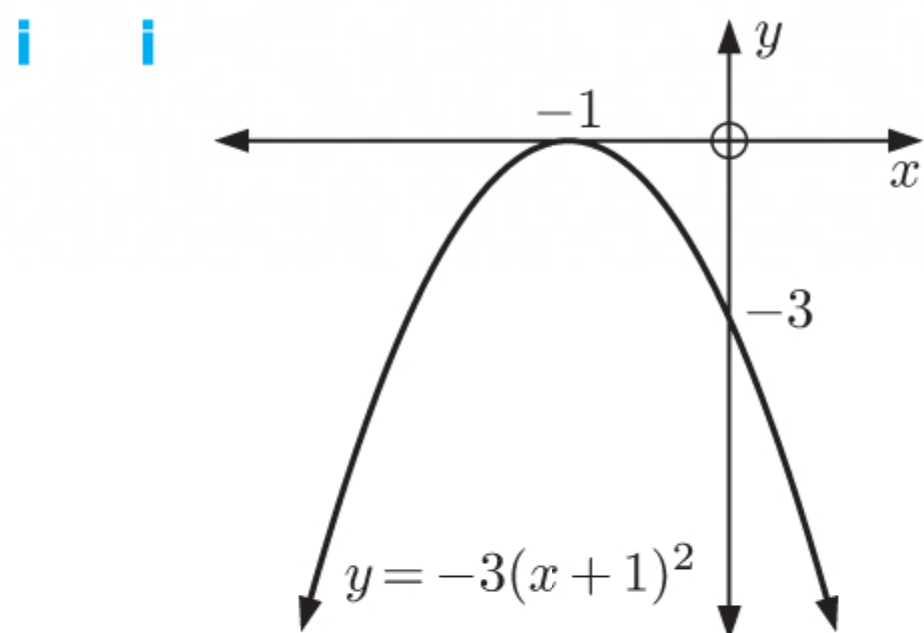
- ii** The y -intercept is 6.
- iii** The x -intercepts are -2 and 3.



- ii** The y -intercept is 12.
- iii** The x -intercepts are -4 and -1 .



- ii** The y -intercept is 8.
- iii** The x -intercept is 2.



- ii** The y -intercept is -3 .
- iii** The x -intercept is -1 .

- 2 a** For a quadratic function in the form $y = ax^2 + bx + c$, the y -intercept is c .
- b** For a quadratic function in the form $y = a(x - \alpha)(x - \beta)$, the x -intercepts are α and β .
- c** For quadratic functions in the form $y = a(x - \alpha)^2$, the x -intercept is α . The graph *touches* the x -axis at this point.

EXERCISE 6C

- 1 a** When $x = 0$, $y = 3$
 \therefore the y -intercept is 3.
- c** When $x = 0$, $y = -4$
 \therefore the y -intercept is -4 .
- b** When $x = 0$, $y = -1$
 \therefore the y -intercept is -1 .
- d** $f(0) = 1$
 \therefore the y -intercept is 1.

e When $x = 0$, $y = 5$
 \therefore the y -intercept is 5.

g When $x = 0$, $y = 8$
 \therefore the y -intercept is 8.

i When $x = 0$, $y = 2$
 \therefore the y -intercept is 2.

f When $x = 0$, $y = 0$
 \therefore the y -intercept is 0.

h $f(0) = -5$
 \therefore the y -intercept is -5 .

2 a When $x = 0$, $y = (1)(3)$
 $= 3$
 \therefore the y -intercept is 3.

c When $x = 0$, $y = (-7)^2$
 $= 49$
 \therefore the y -intercept is 49.

e When $x = 0$, $y = (0)(-4)$
 $= 0$
 \therefore the y -intercept is 0.

b $f(0) = (-2)(3)$
 $= -6$
 \therefore the y -intercept is -6 .

d When $x = 0$, $y = (5)(3)$
 $= 15$
 \therefore the y -intercept is 15.

f $f(0) = -(4)(-5)$
 $= 20$
 \therefore the y -intercept is 20.

3 a When $y = 0$, $(x - 2)(x - 5) = 0$
 $\therefore x - 2 = 0$ or $x - 5 = 0$ {null factor law}
 $\therefore x = 2$ or 5
 \therefore the x -intercepts are 2 and 5.

b When $y = 0$, $(x - 3)(x + 4) = 0$
 $\therefore x - 3 = 0$ or $x + 4 = 0$ {null factor law}
 $\therefore x = 3$ or -4
 \therefore the x -intercepts are 3 and -4 .

c When $y = 0$, $2(x + 6)(x + 3) = 0$
 $\therefore x + 6 = 0$ or $x + 3 = 0$ {null factor law}
 $\therefore x = -6$ or -3
 \therefore the x -intercepts are -6 and -3 .

d When $f(x) = 0$, $-(x - 7)(x + 1) = 0$
 $\therefore x - 7 = 0$ or $x + 1 = 0$ {null factor law}
 $\therefore x = 7$ or -1
 \therefore the x -intercepts are 7 and -1 .

e When $y = 0$, $x(x - 8) = 0$
 $\therefore x = 0$ or $x - 8 = 0$ {null factor law}
 $\therefore x = 0$ or 8
 \therefore the x -intercepts are 0 and 8.

f When $y = 0$, $-3(x + 5)(x - 5) = 0$
 $\therefore x + 5 = 0$ or $x - 5 = 0$ {null factor law}
 $\therefore x = -5$ or 5

\therefore the x -intercepts are -5 and 5 .

g When $y = 0$, $(2x - 3)(x + 1) = 0$
 $\therefore 2x - 3 = 0$ or $x + 1 = 0$ {null factor law}
 $\therefore 2x = 3$ or $x + 1 = 0$
 $\therefore x = \frac{3}{2}$ or $x = -1$

\therefore the x -intercepts are $\frac{3}{2}$ and -1 .

h When $y = 0$, $(3x + 1)(2x - 5) = 0$
 $\therefore 3x + 1 = 0$ or $2x - 5 = 0$ {null factor law}
 $\therefore 3x = -1$ or $2x = 5$
 $\therefore x = -\frac{1}{3}$ or $x = \frac{5}{2}$

\therefore the x -intercepts are $-\frac{1}{3}$ and $\frac{5}{2}$.

i When $y = 0$, $(x + 4)^2 = 0$
 $\therefore x + 4 = 0$
 $\therefore x = -4$

\therefore the x -intercept is -4 .

j When $f(x) = 0$, $7(x - 2)^2 = 0$
 $\therefore x - 2 = 0$
 $\therefore x = 2$

\therefore the x -intercept is 2 .

k When $y = 0$, $-4(x + 1)^2 = 0$
 $\therefore x + 1 = 0$
 $\therefore x = -1$

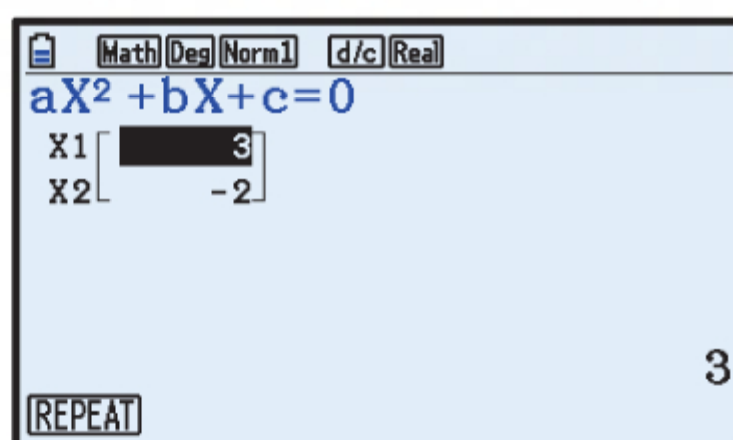
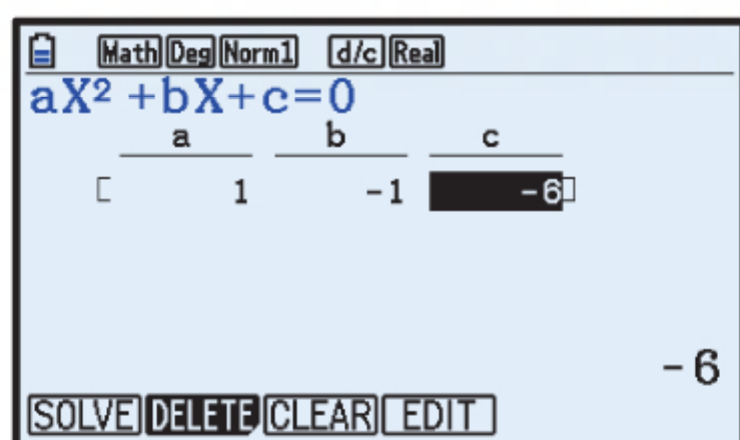
\therefore the x -intercept is -1 .

l When $y = 0$, $(4x + 3)^2 = 0$
 $\therefore 4x + 3 = 0$
 $\therefore 4x = -3$
 $\therefore x = -\frac{3}{4}$

\therefore the x -intercept is $-\frac{3}{4}$.

- 4 a** If a quadratic function cuts the x -axis twice, it has 2 zeros.
b If a quadratic function touches the x -axis, it has 1 zero.
c If a quadratic function lies entirely below the x -axis, it has no zeros.

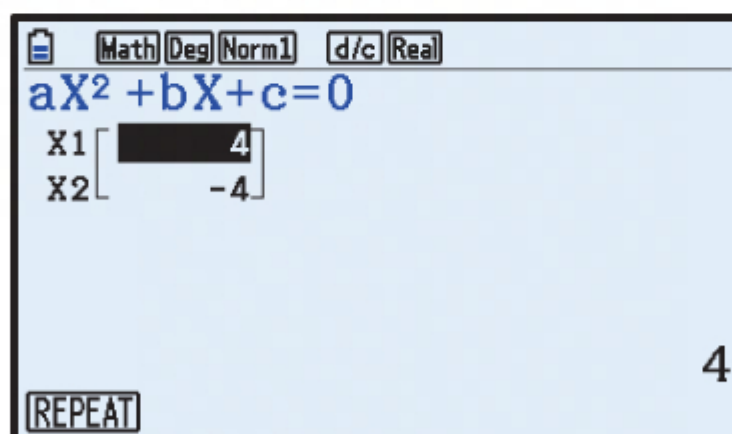
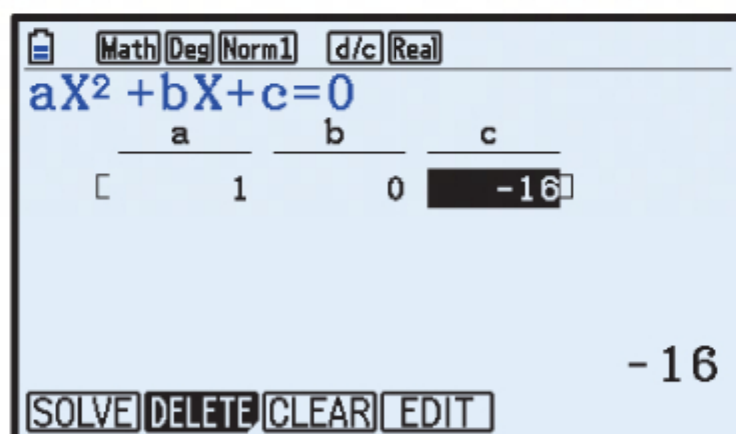
5 a When $y = 0$, $x^2 - x - 6 = 0$



$\therefore x = -2$ or 3

\therefore the zeros are -2 and 3 .

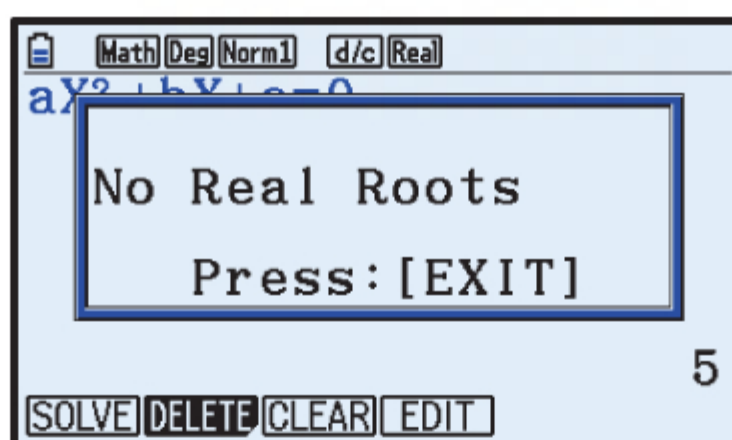
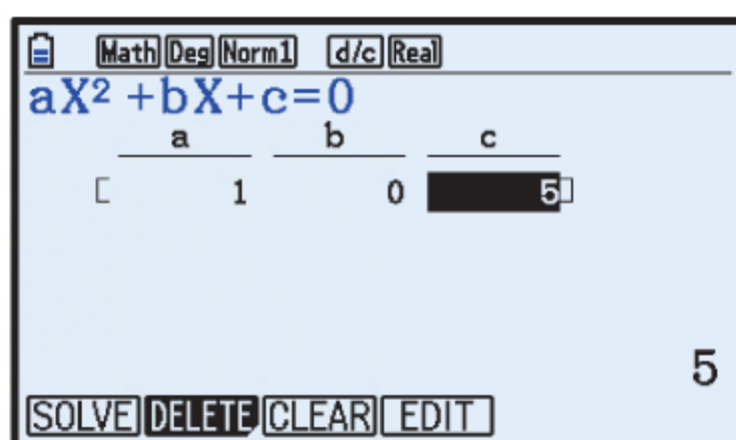
- b** When $y = 0$, $x^2 - 16 = 0$



$\therefore x = -4$ or 4

\therefore the zeros are -4 and 4 .

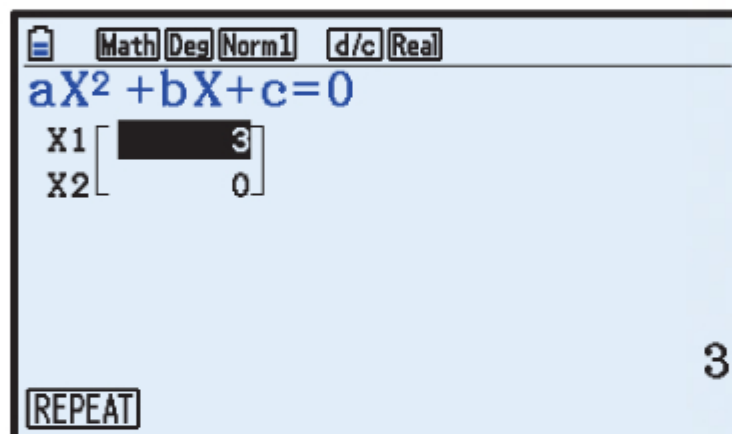
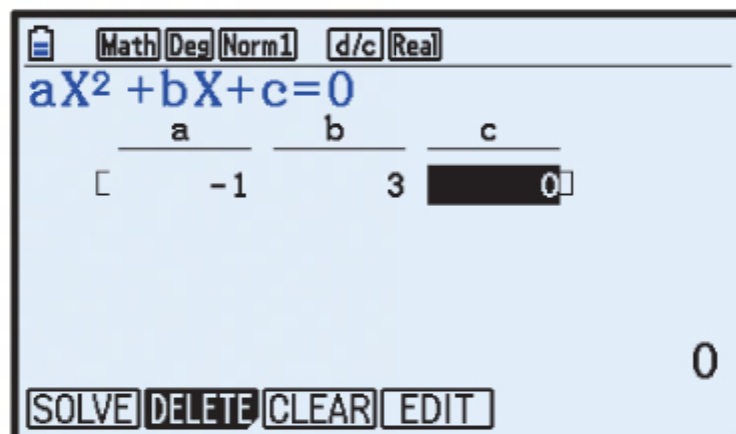
- c** When $y = 0$, $x^2 + 5 = 0$



There are no real solutions.

\therefore the function has no real zeros.

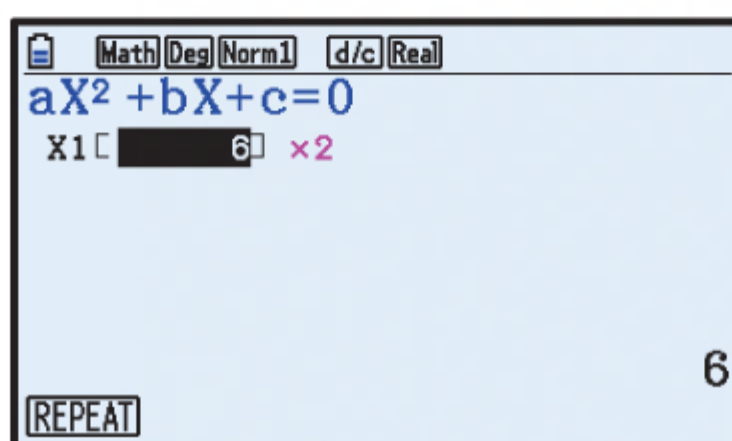
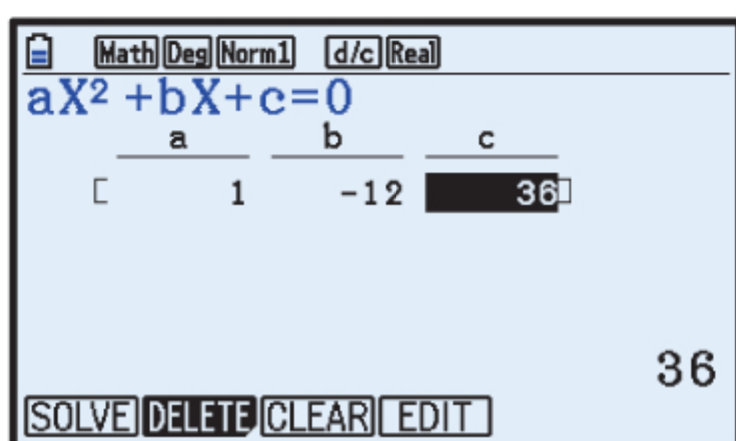
- d** When $f(x) = 0$, $3x - x^2 = 0$
 $\therefore -x^2 + 3x = 0$



$\therefore x = 0$ or 3

\therefore the zeros are 0 and 3 .

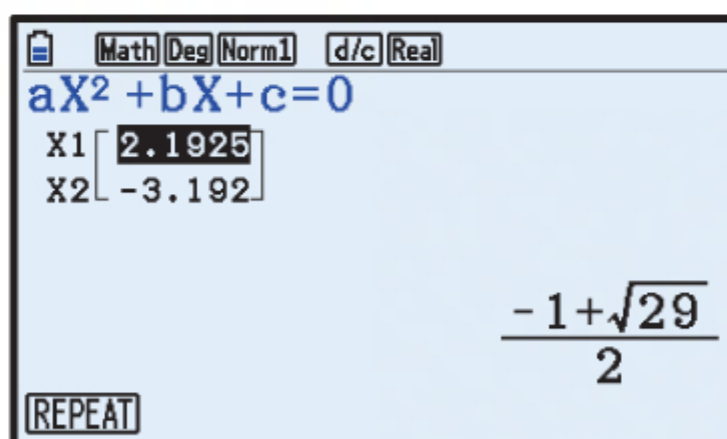
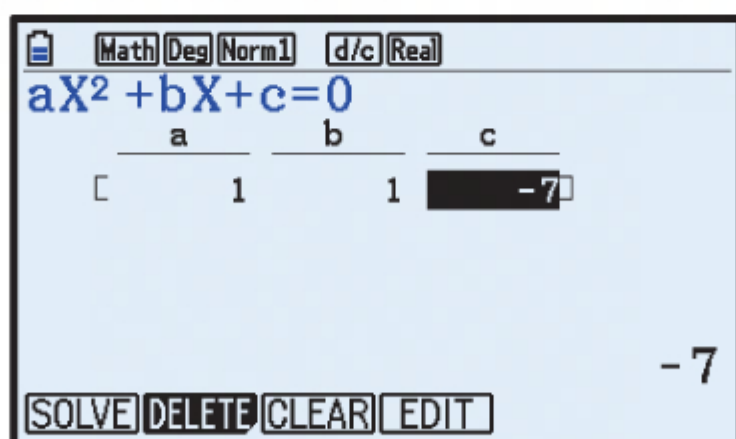
- e** When $y = 0$, $x^2 - 12x + 36 = 0$



$\therefore x = 6$

\therefore the zero is 6 .

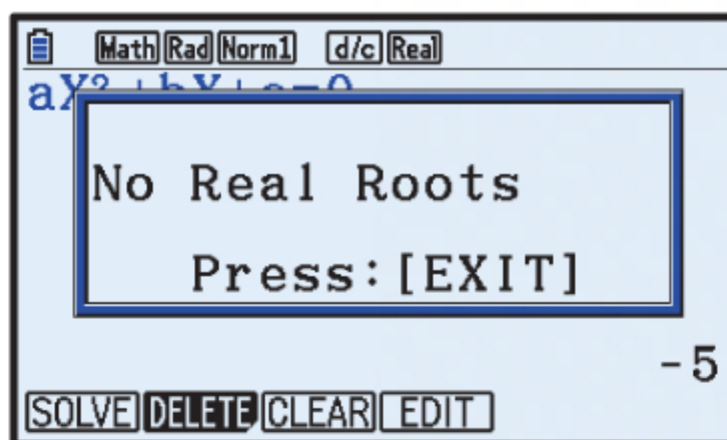
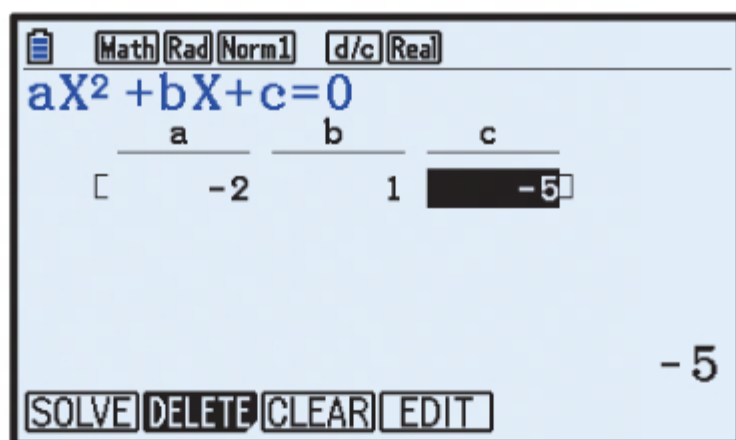
f When $y = 0$, $x^2 + x - 7 = 0$



$\therefore x \approx 2.19$ or -3.19

\therefore the zeros are ≈ 2.19 and ≈ -3.19 .

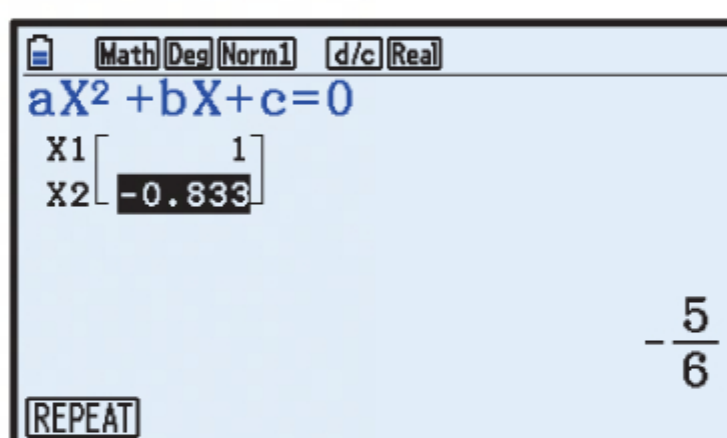
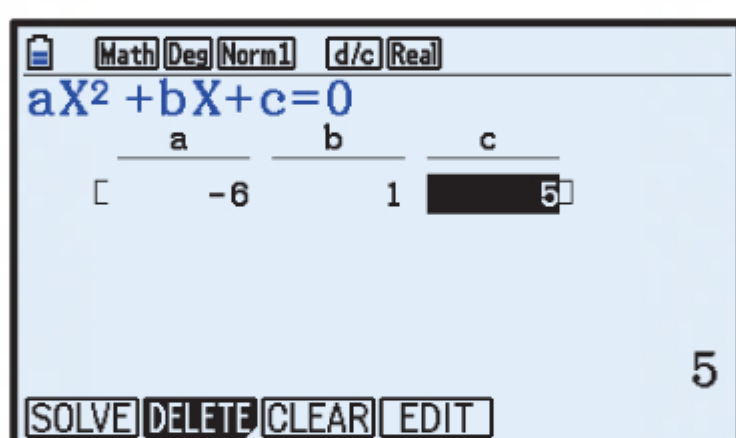
g When $y = 0$, $-2x^2 + x - 5 = 0$



There are no real solutions.

\therefore the function has no real zeros.

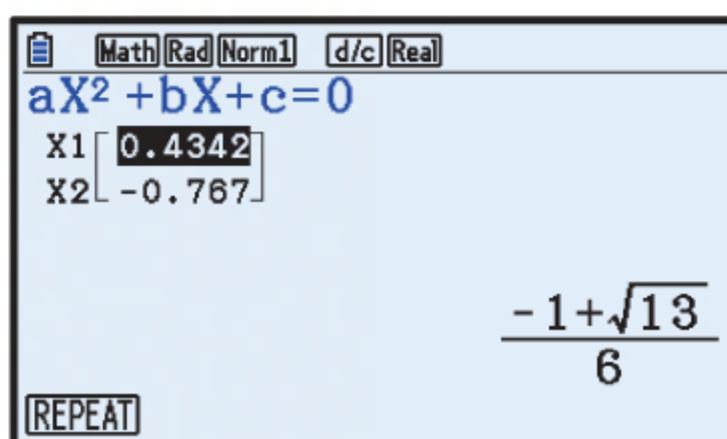
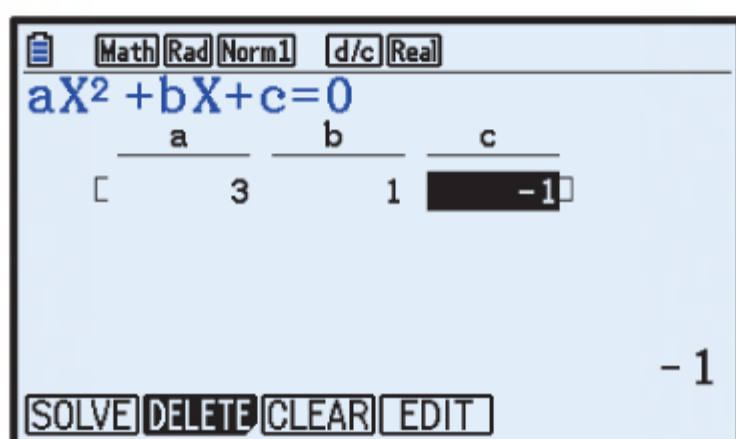
h When $y = 0$, $-6x^2 + x + 5 = 0$



$\therefore x = -\frac{5}{6}$ or 1

\therefore the zeros are $-\frac{5}{6}$ and 1 .

i When $f(x) = 0$, $3x^2 + x - 1 = 0$



$\therefore x \approx 0.434$ or -0.768

\therefore the zeros are ≈ 0.434 and ≈ -0.768 .

6 a When $x = 0$, $y = (2)(-1)$
 $= -2$

\therefore the y -intercept is -2 .

When $y = 0$, $(x + 2)(x - 1) = 0$

$\therefore x + 2 = 0$ or $x - 1 = 0$

$\therefore x = -2$ or 1

\therefore the x -intercepts are -2 and 1 .

c When $x = 0$, $y = (5)(-2)$
 $= -10$

\therefore the y -intercept is -10 .

When $y = 0$, $(x + 5)(x - 2) = 0$

$\therefore x + 5 = 0$ or $x - 2 = 0$

$\therefore x = -5$ or 2

\therefore the x -intercepts are -5 and 2 .

b $f(0) = (3)^2$
 $= 9$

\therefore the y -intercept is 9 .

When $f(x) = 0$, $(x + 3)^2 = 0$

$\therefore x + 3 = 0$

$\therefore x = -3$

\therefore the x -intercept is -3 .

d When $x = 0$, $y = (-2)(-5)$
 $= 10$

\therefore the y -intercept is 10 .

When $y = 0$, $(3x - 2)(x - 5) = 0$

$\therefore 3x - 2 = 0$ or $x - 5 = 0$

$\therefore 3x = 2$ or $x = 5$

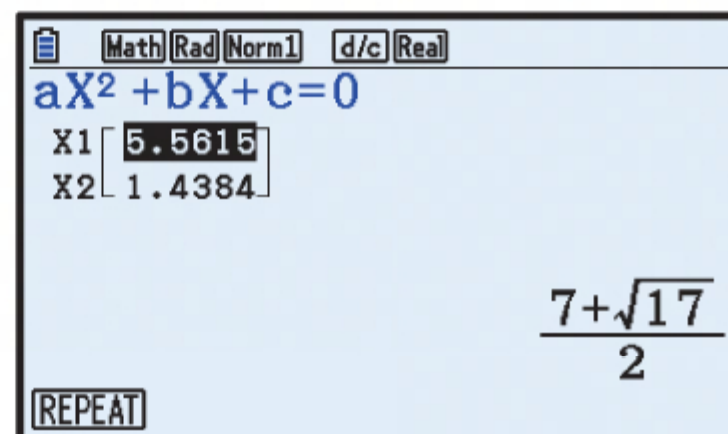
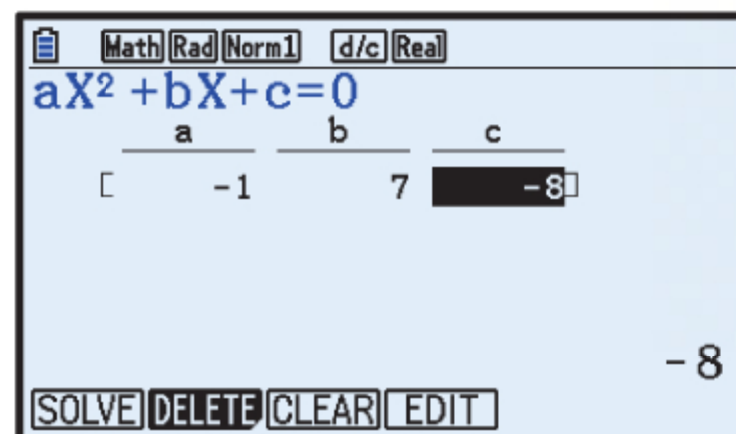
$\therefore x = \frac{2}{3}$ or $x = 5$

\therefore the x -intercepts are $\frac{2}{3}$ and 5 .

e When $x = 0$, $y = -8$

\therefore the y -intercept is -8 .

When $y = 0$, $-x^2 + 7x - 8 = 0$



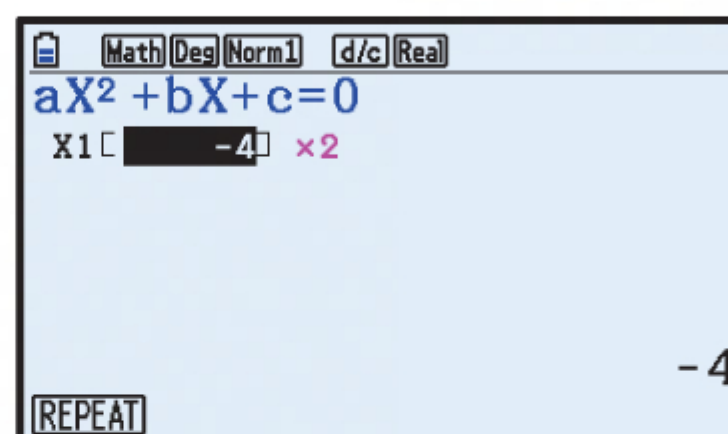
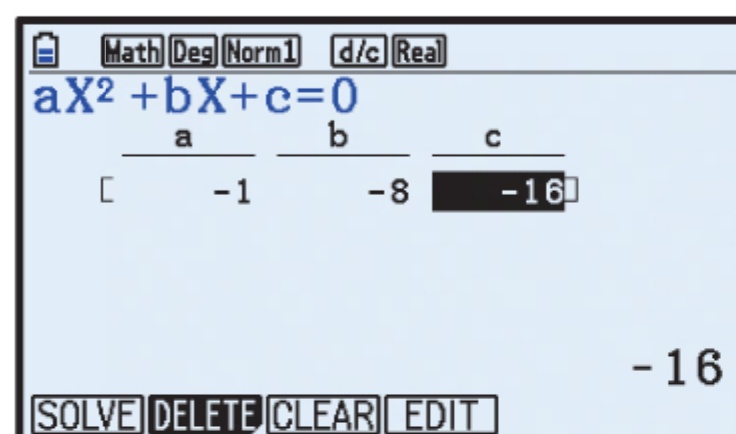
$\therefore x \approx 1.44$ or 5.56

\therefore the x -intercepts are ≈ 1.44 and ≈ 5.56 .

f When $x = 0$, $y = -16$

\therefore the y -intercept is -16 .

When $y = 0$, $-x^2 - 8x - 16 = 0$

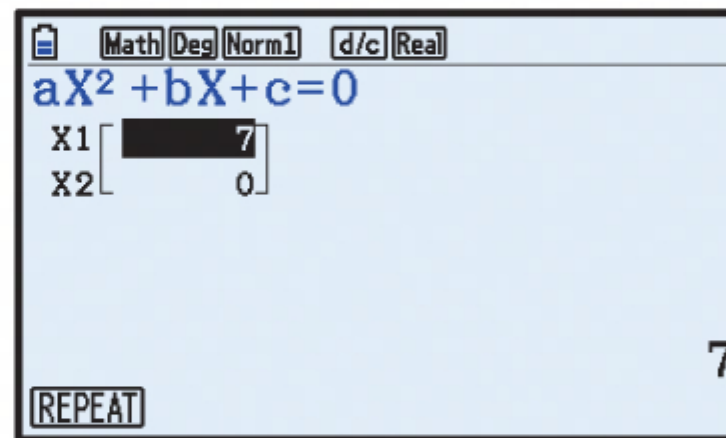
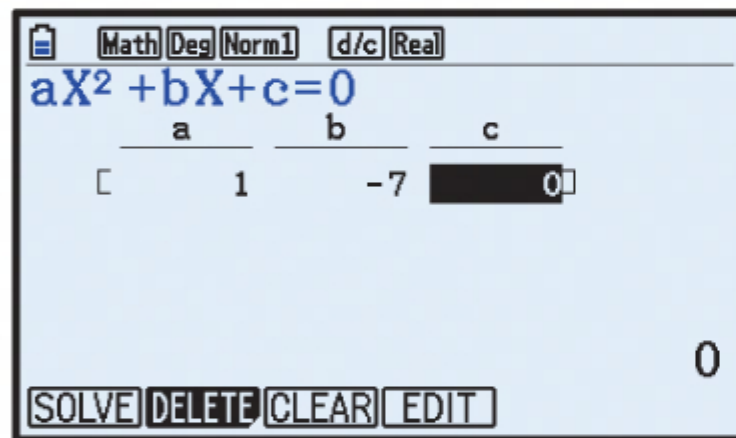


$\therefore x = -4$

\therefore the x -intercept is -4 .

- g** When $x = 0$, $y = 0$
 \therefore the y -intercept is 0.

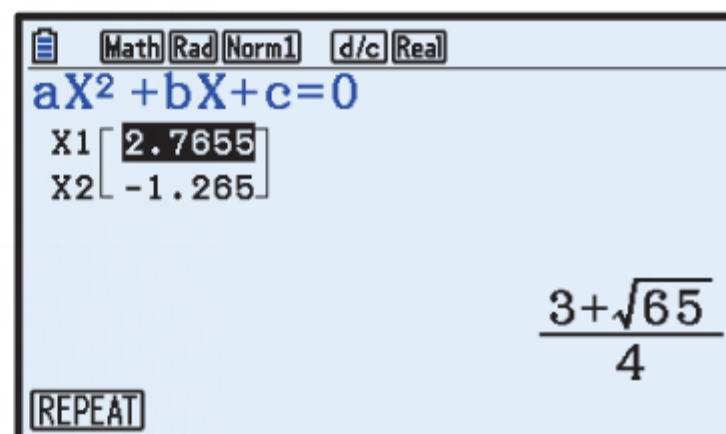
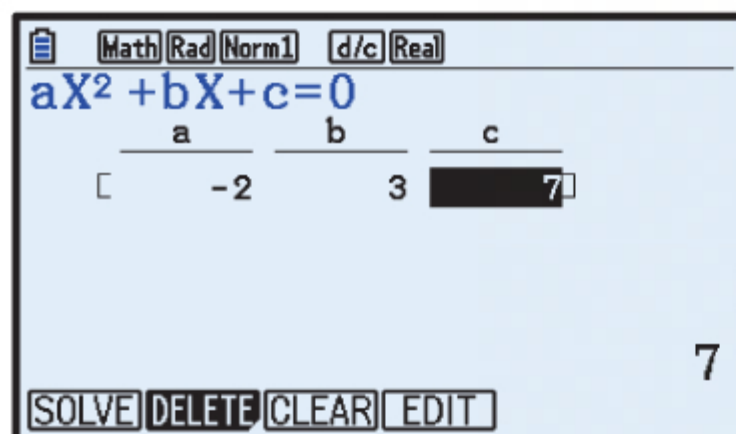
When $y = 0$, $x^2 - 7x = 0$



- $\therefore x = 0$ or 7
 \therefore the x -intercepts are 0 and 7.

- h** $f(0) = 7$
 \therefore the y -intercept is 7.

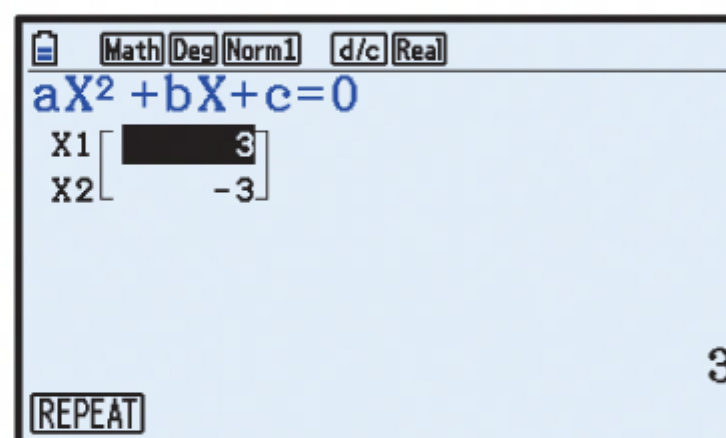
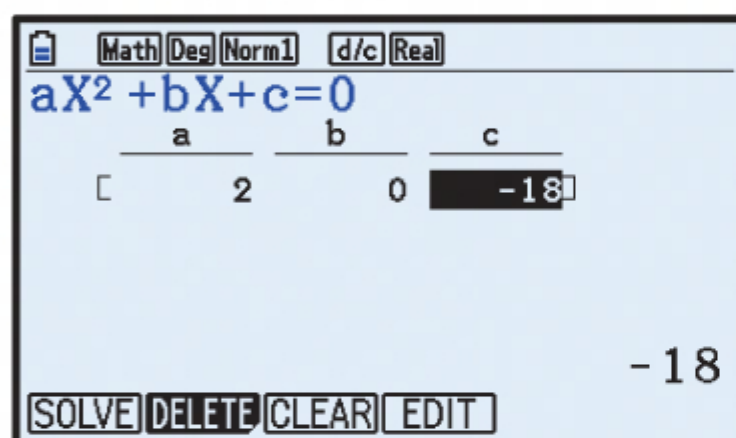
When $f(x) = 0$, $-2x^2 + 3x + 7 = 0$



- $\therefore x \approx -1.27$ or 2.77
 \therefore the x -intercepts are ≈ -1.27 and ≈ 2.77 .

- i** When $x = 0$, $y = -18$
 \therefore the y -intercept is -18 .

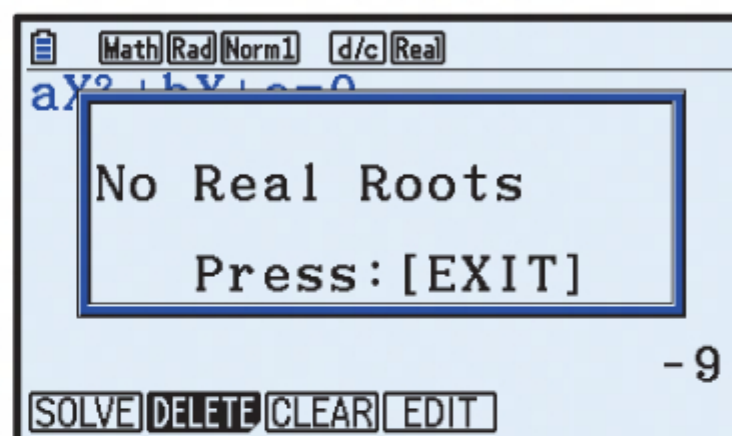
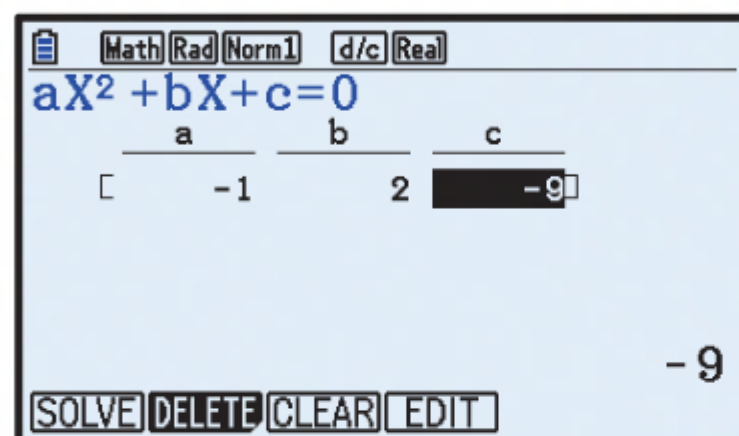
When $y = 0$, $2x^2 - 18 = 0$



- $\therefore x = -3$ or 3
 \therefore the x -intercepts are -3 and 3 .

- j When $x = 0$, $y = -9$
 \therefore the y -intercept is -9 .

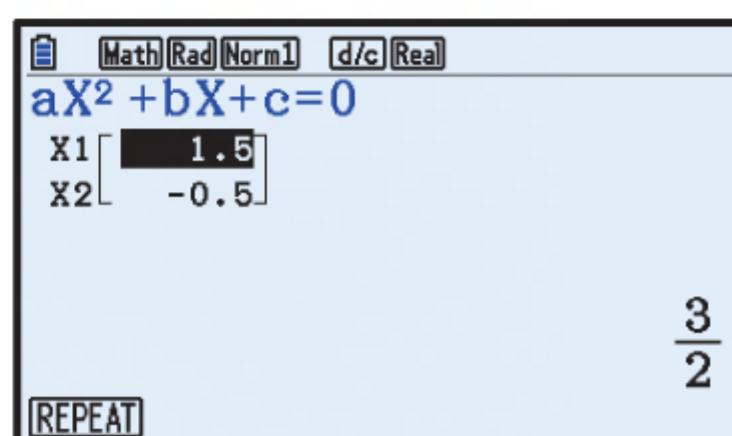
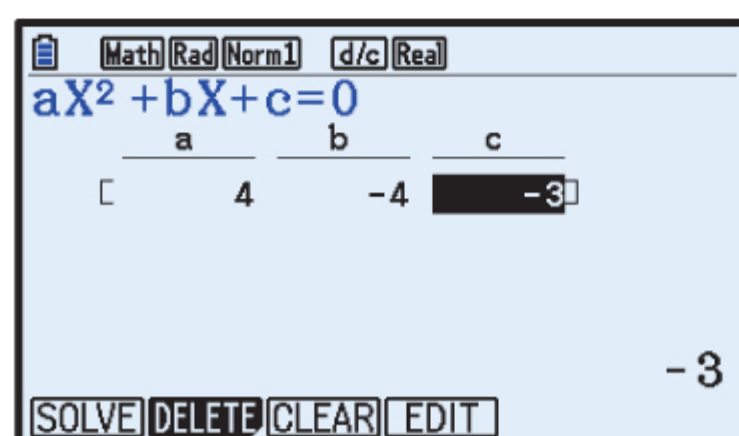
When $y = 0$, $-x^2 + 2x - 9 = 0$



There are no real solutions.
 \therefore there are no x -intercepts.

- k When $x = 0$, $y = -3$
 \therefore the y -intercept is -3 .

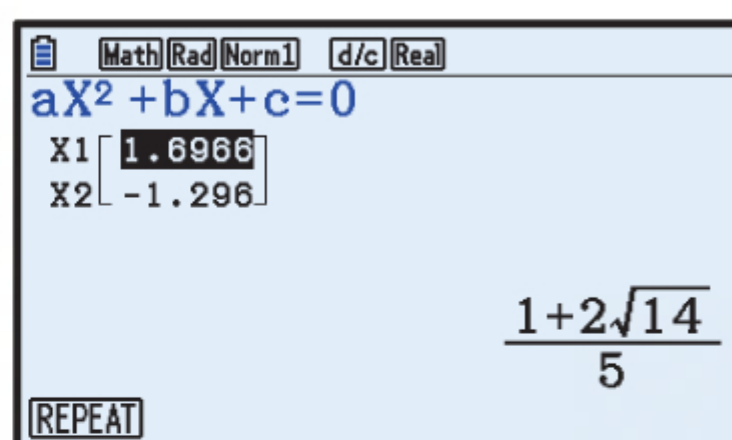
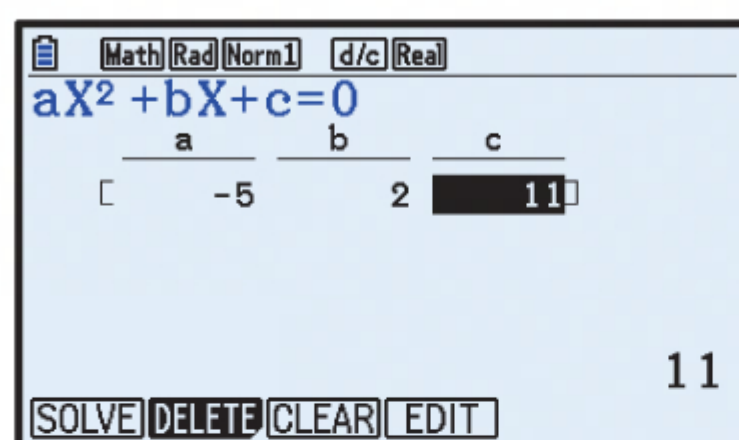
When $y = 0$, $4x^2 - 4x - 3 = 0$



$\therefore x = -\frac{1}{2}$ or $\frac{3}{2}$
 \therefore the x -intercepts are $-\frac{1}{2}$ and $\frac{3}{2}$.

- l When $x = 0$, $y = 11$
 \therefore the y -intercept is 11 .

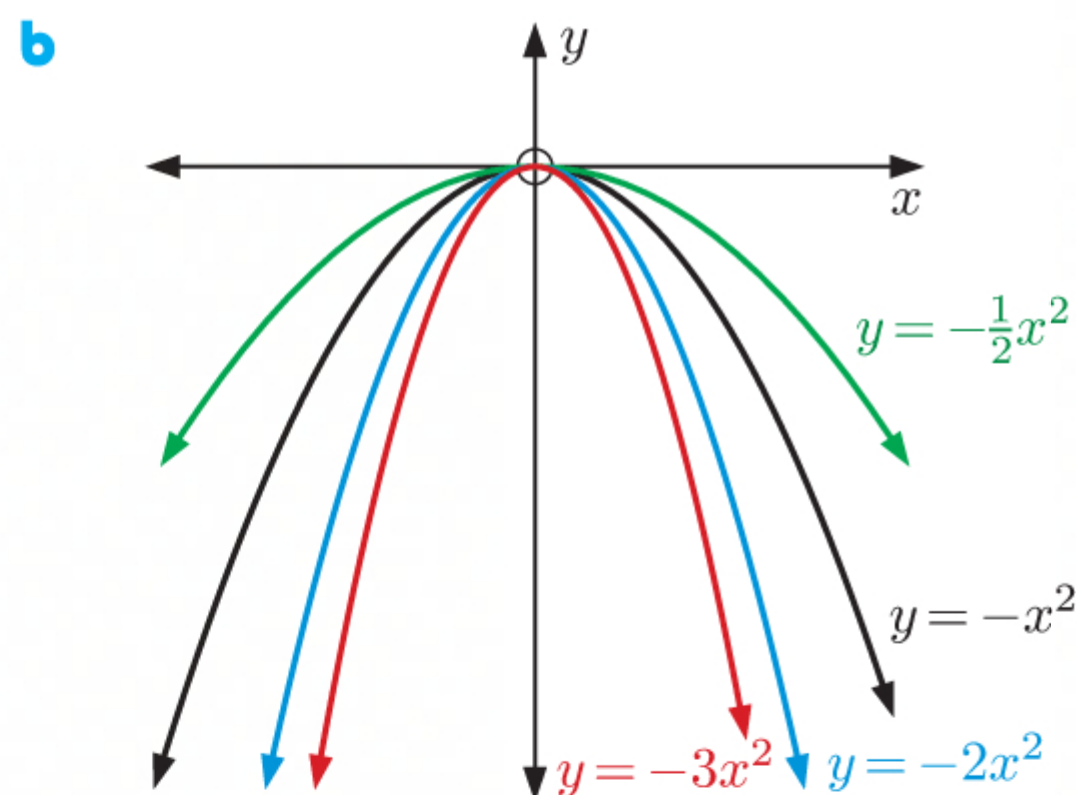
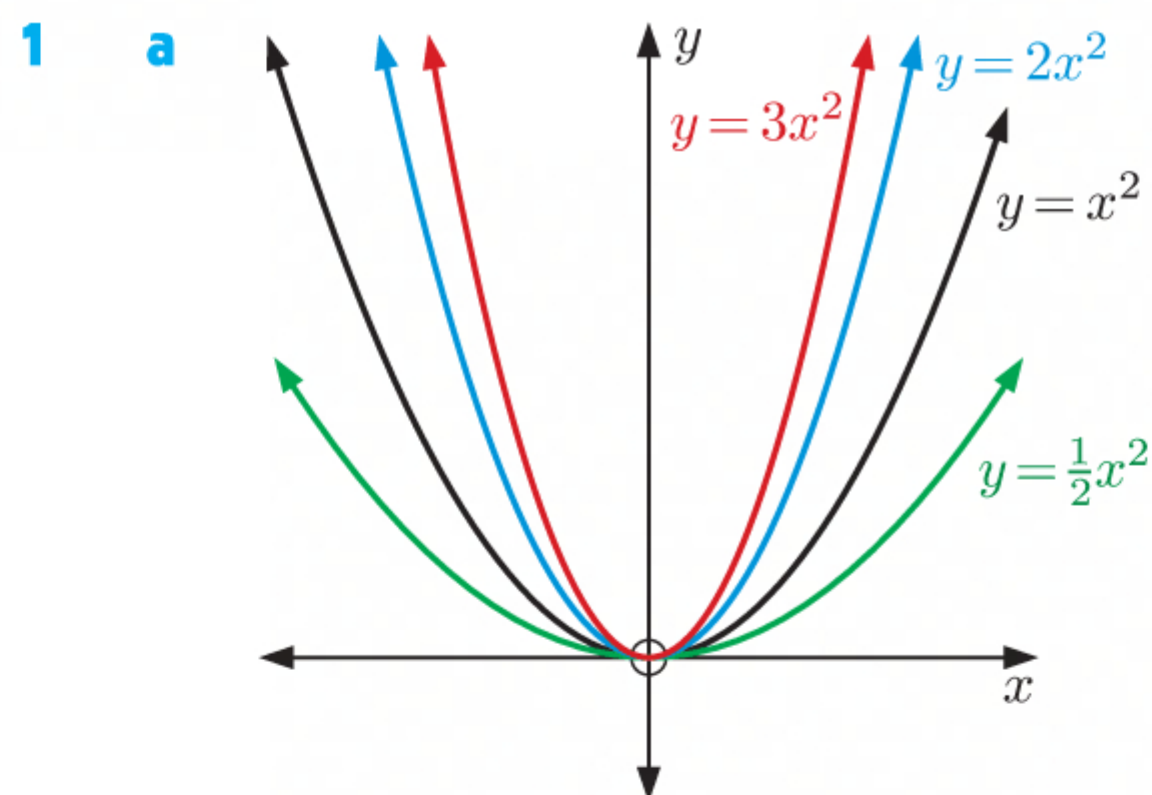
When $y = 0$, $-5x^2 + 2x + 11 = 0$




$\therefore x \approx -1.30$ or 1.70
 \therefore the x -intercepts are ≈ -1.30 and ≈ 1.70 .

INVESTIGATION 2

GRAPHS OF THE FORM $y = ax^2$



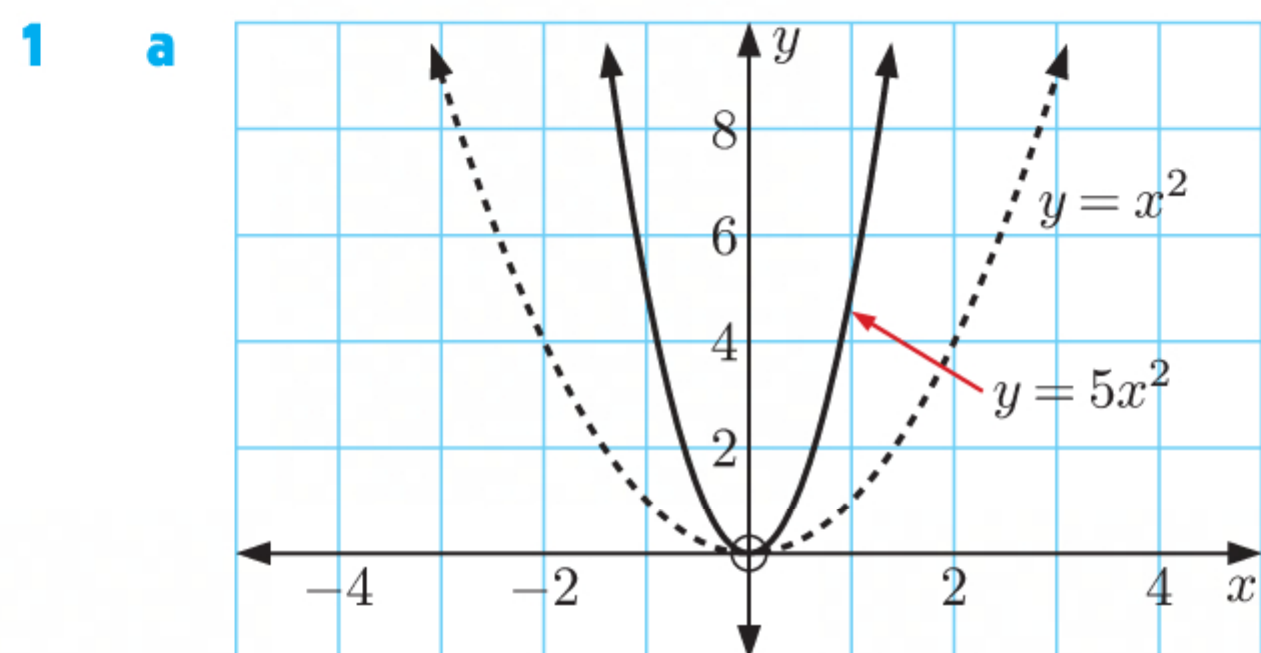
3 If $a > 0$, the graph opens upwards. It has the shape .

If $a < 0$, the graph opens downwards. It has the shape .

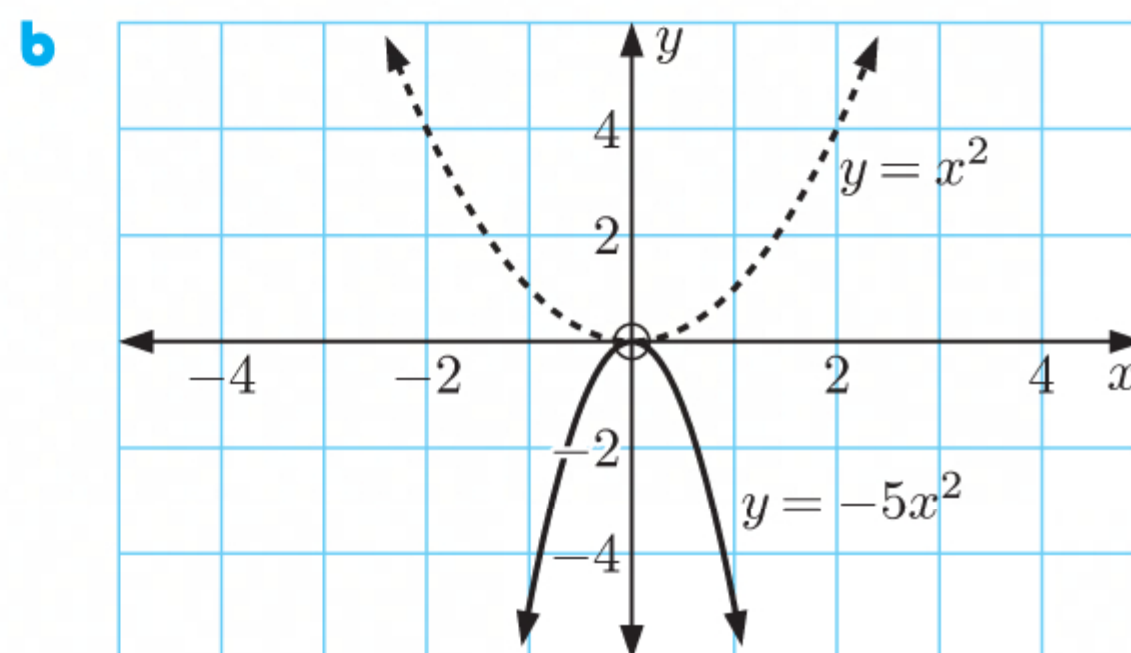
4 If $a < -1$ or $a > 1$, the graph is “thinner” than $y = x^2$.

If $-1 < a < 1$, $a \neq 0$, the graph is “wider” than $y = x^2$.

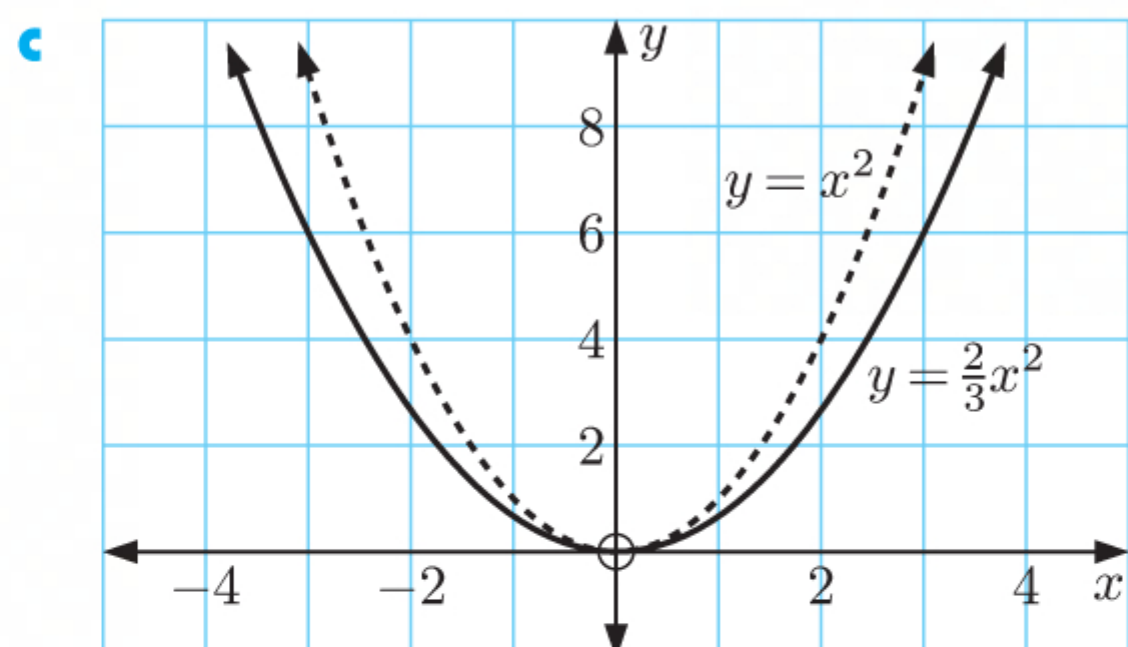
EXERCISE 6D



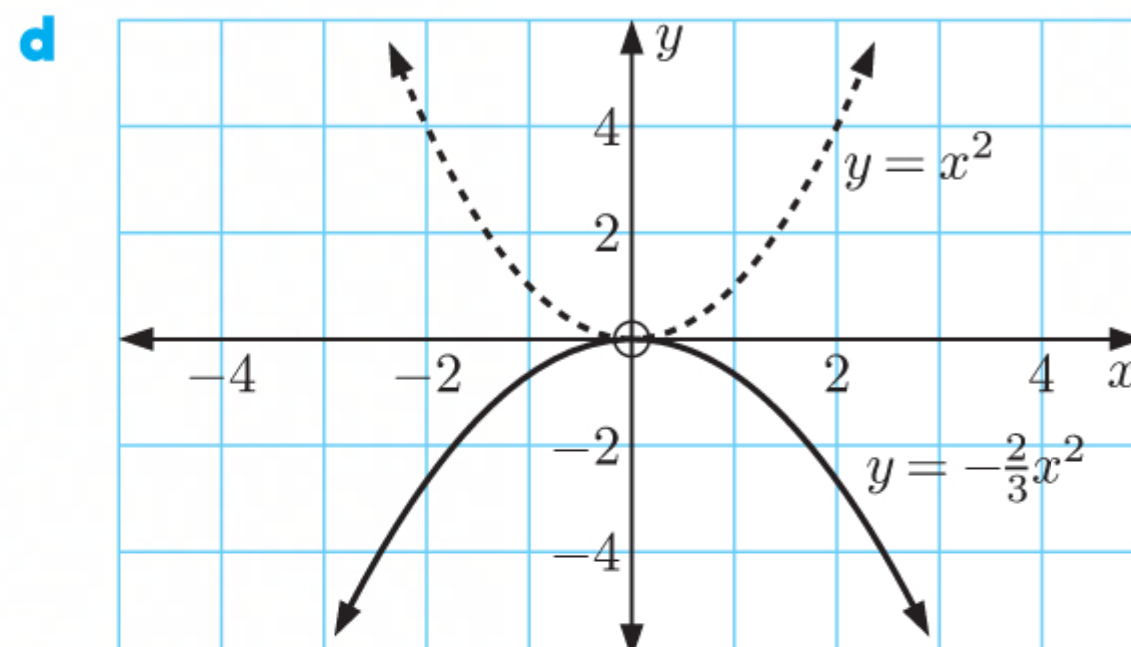
$y = 5x^2$ opens upwards and is “thinner” than $y = x^2$.



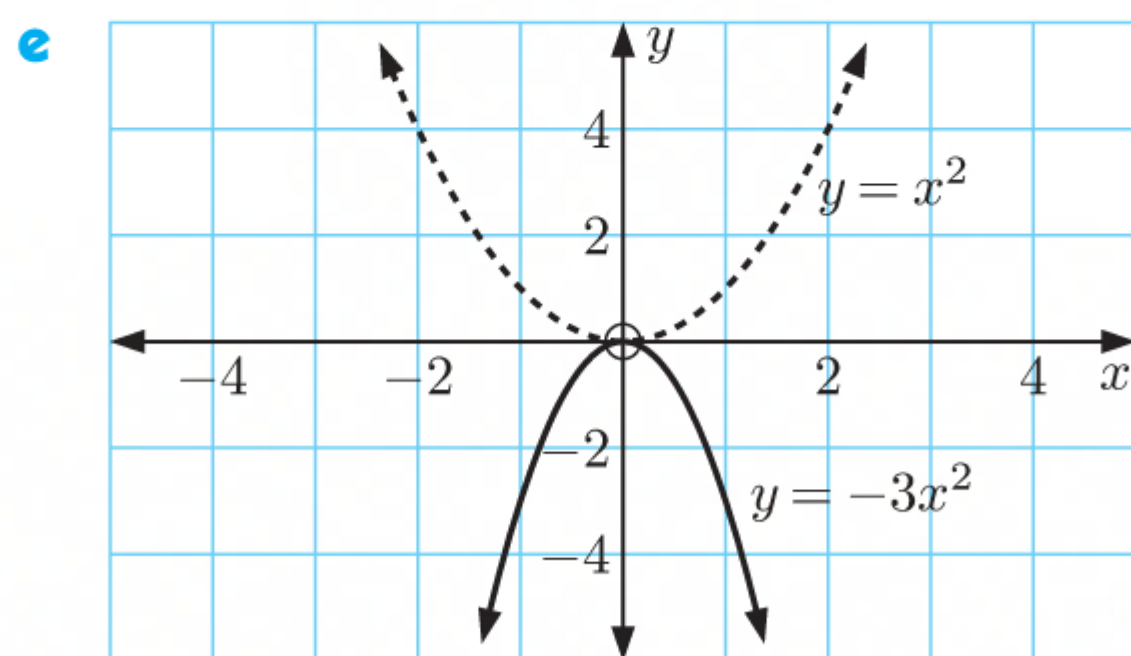
$y = -5x^2$ opens downwards and is “thinner” than $y = x^2$.



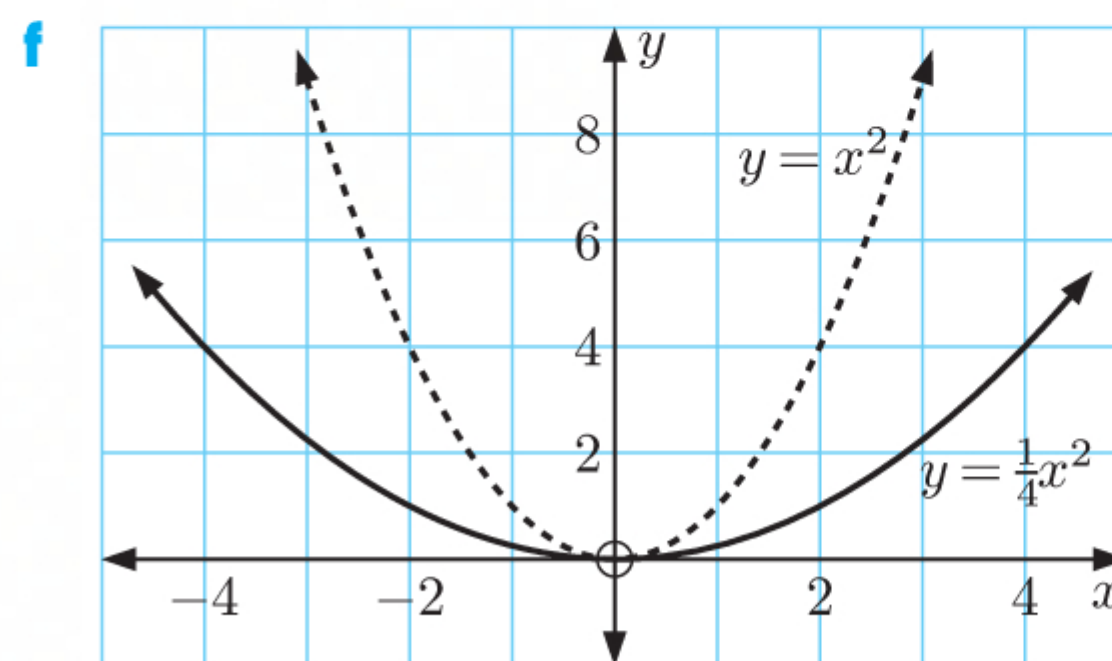
$y = \frac{2}{3}x^2$ opens upwards and is “wider” than $y = x^2$.



$y = -\frac{2}{3}x^2$ opens downwards and is “wider” than $y = x^2$.



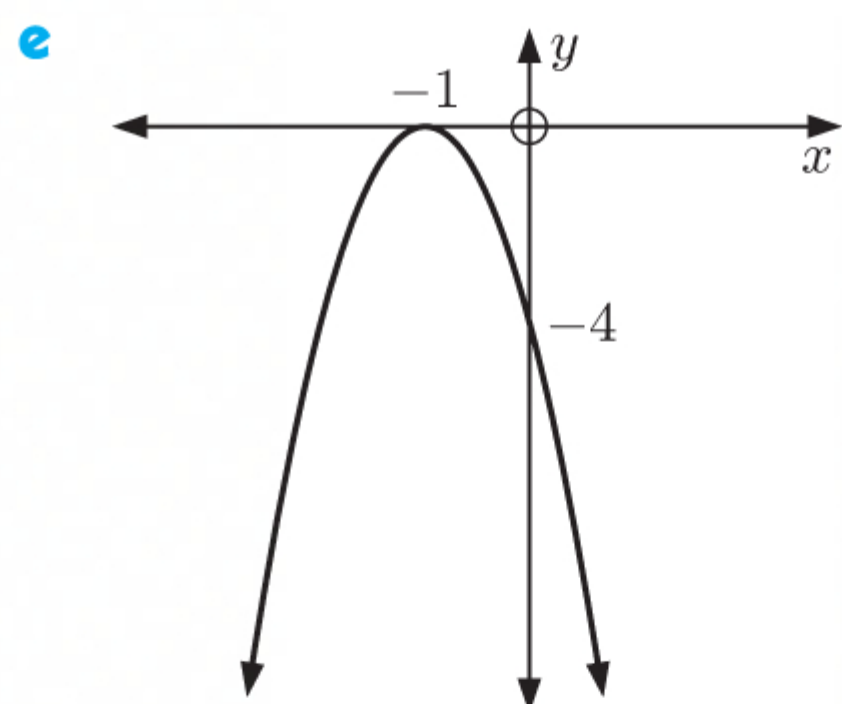
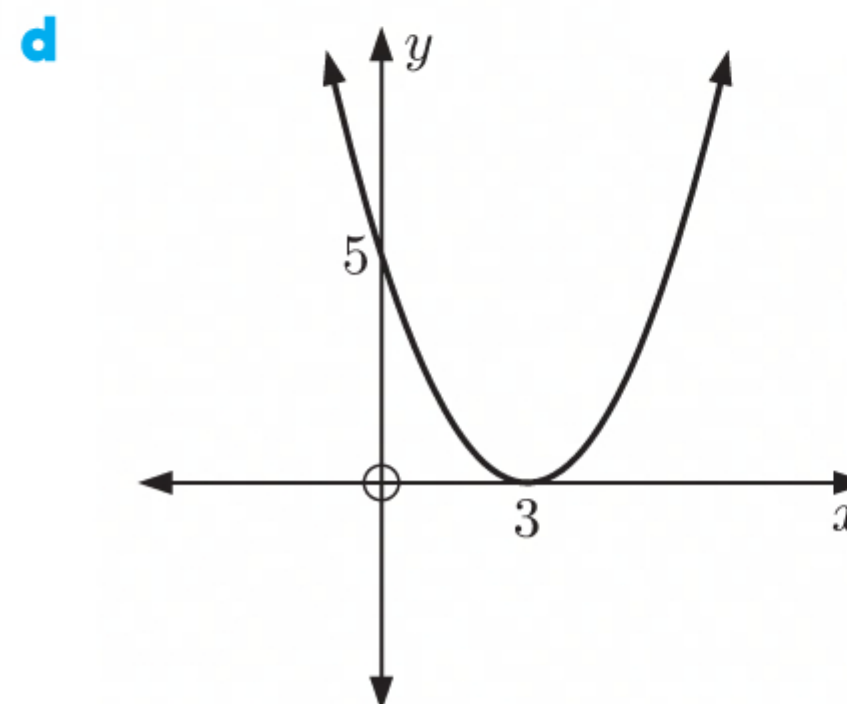
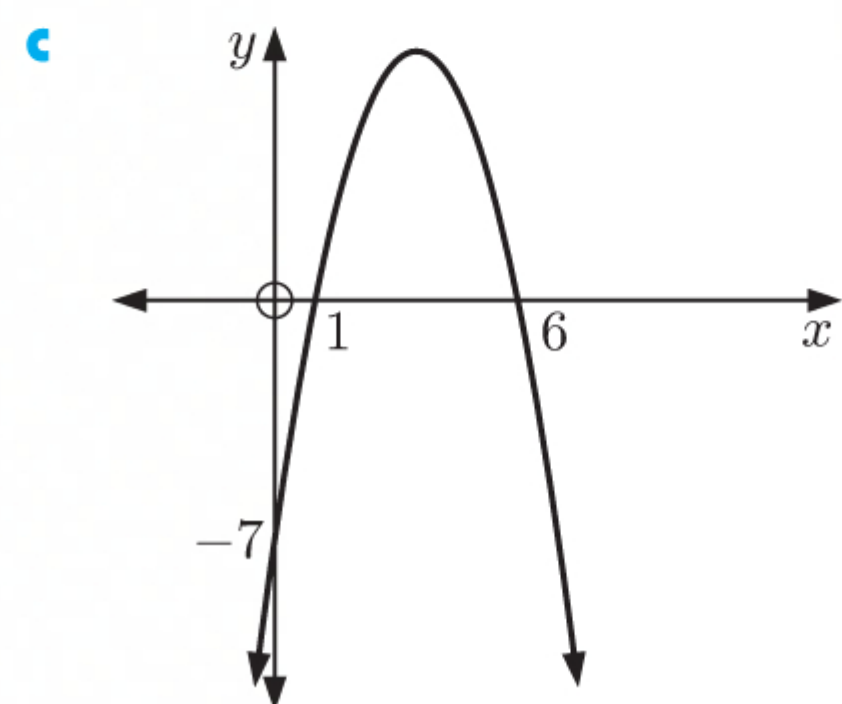
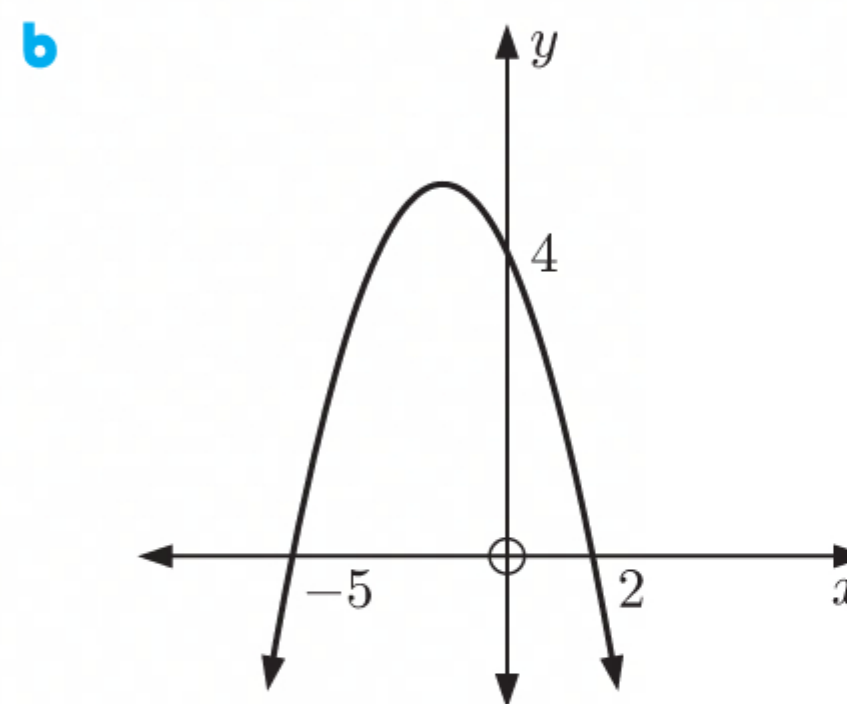
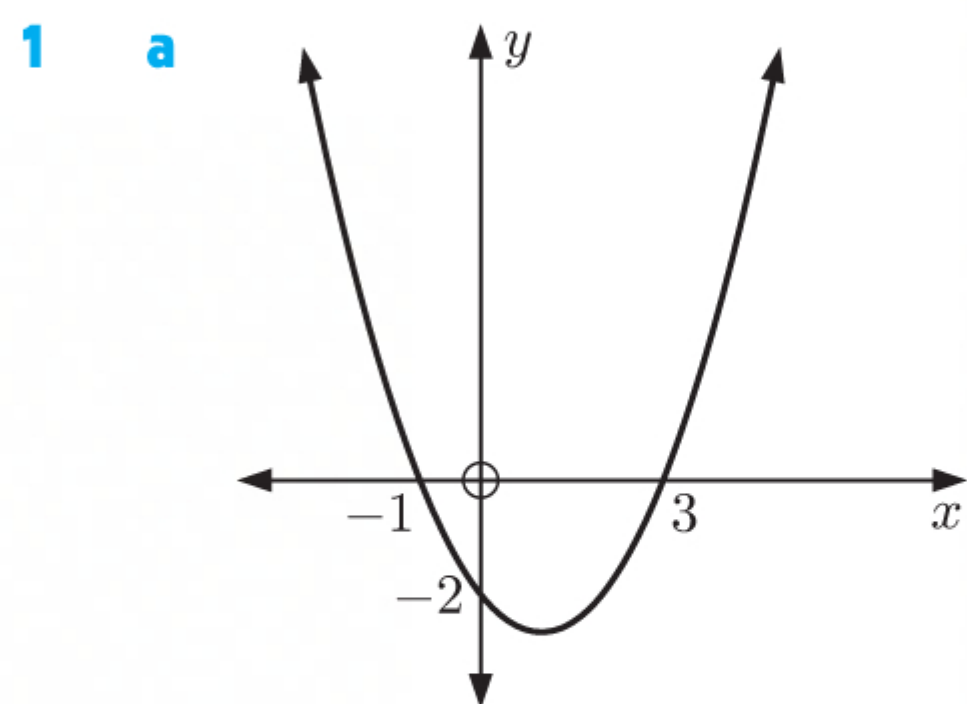
$y = -3x^2$ opens downwards and is “thinner” than $y = x^2$.




$y = \frac{1}{4}x^2$ opens upwards and is “wider” than $y = x^2$.

- 2**
- a** The vertex of $y = 3x^2$ is $(0, 0)$. Since $3 > 0$, $(0, 0)$ is a minimum turning point.
 - b** The vertex of $y = -6x^2$ is $(0, 0)$. Since $-6 < 0$, $(0, 0)$ is a maximum turning point.
 - c** The vertex of $y = -\frac{1}{3}x^2$ is $(0, 0)$. Since $-\frac{1}{3} < 0$, $(0, 0)$ is a maximum turning point.

EXERCISE 6E



- 2 a** Since $a = 1$ which is > 0 , the parabola has shape .

When $x = 0$, $y = -12$

\therefore the y -intercept is -12 .

When $y = 0$,

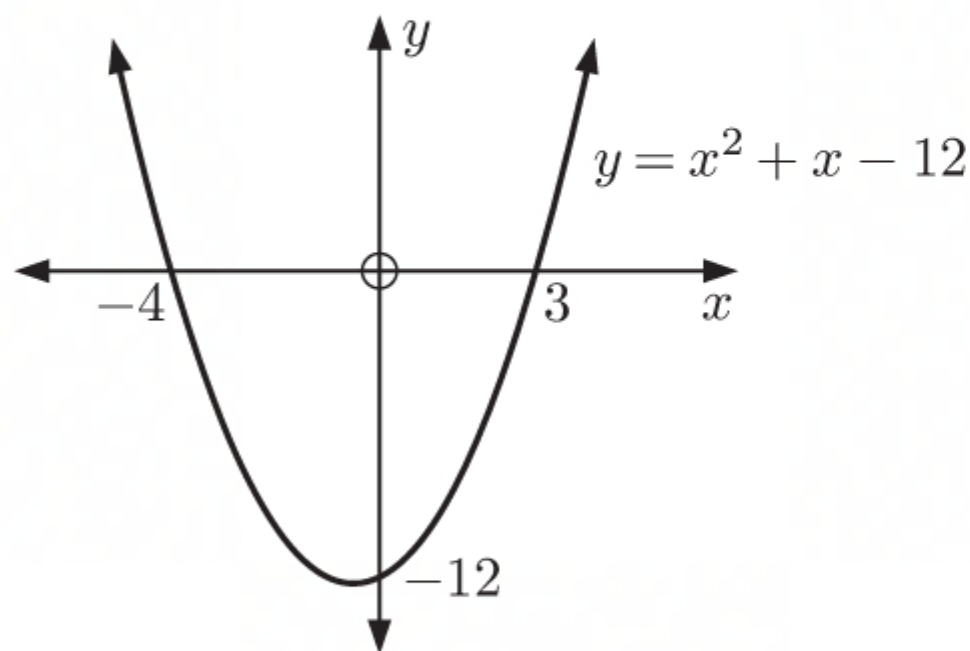
$$x^2 + x - 12 = 0$$


$$\therefore x = -4 \text{ or } 3$$

\therefore the x -intercepts are -4 and 3 .

Math Deg Norm1 d/c Real		
$aX^2 + bX + c = 0$		
a	b	c
1	1	-12
-12		
SOLVE DELETE CLEAR EDIT		

Math Deg Norm1 d/c Real		
$aX^2 + bX + c = 0$		
X1	3	
X2	-4	
3		
REPEAT		



- b** Since $a = 1$ which is > 0 , the parabola has shape .

When $x = 0$, $y = -5$

\therefore the y -intercept is -5 .

When $y = 0$,

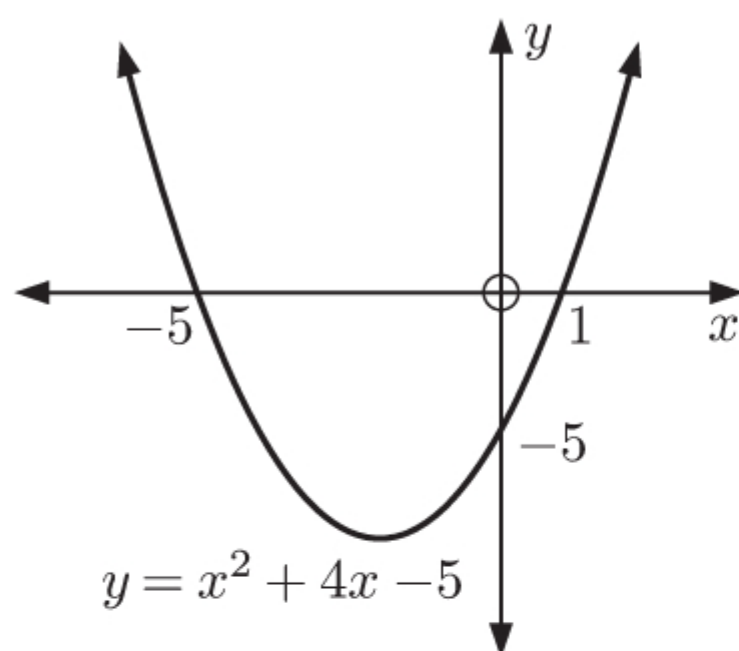
$$x^2 + 4x - 5 = 0$$


$$\therefore x = -5 \text{ or } 1$$

\therefore the x -intercepts are -5 and 1 .

Math Deg Norm1 d/c Real		
$aX^2 + bX + c = 0$		
a	b	c
1	4	-5
-5		
SOLVE DELETE CLEAR EDIT		

Math Deg Norm1 d/c Real		
$aX^2 + bX + c = 0$		
X1	1	
X2	-5	
1		
REPEAT		



- c Since $a = -1$ which is < 0 , the parabola has shape .

When $x = 0$, $y = -9$

\therefore the y -intercept is -9 .

When $y = 0$,

$$-x^2 + 6x - 9 = 0$$

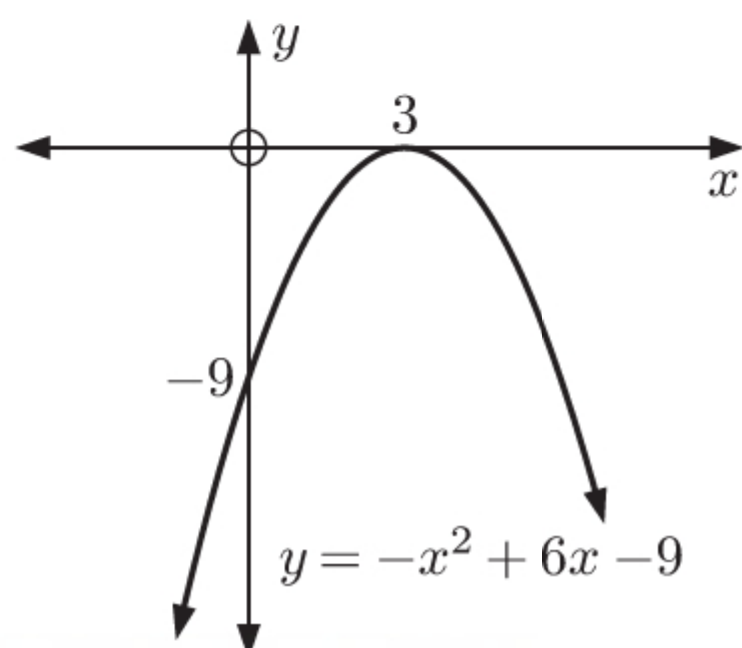
$$\therefore x = 3$$


\therefore the x -intercept is 3 (touching).

a	b	c
-1	6	-9

SOLVE DELETE CLEAR EDIT

REPEAT



- d Since $a = 1$ which is > 0 , the parabola has shape .

When $x = 0$, $y = 16$

\therefore the y -intercept is 16 .

When $y = 0$,

$$x^2 + 8x + 16 = 0$$

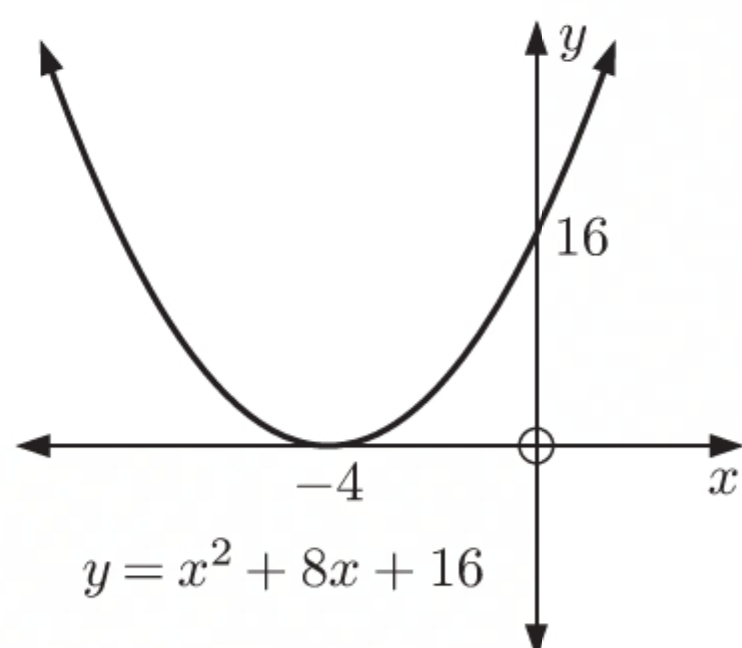
$$\therefore x = -4$$


\therefore the x -intercept is -4 (touching).

a	b	c
1	8	16

SOLVE DELETE CLEAR EDIT

REPEAT



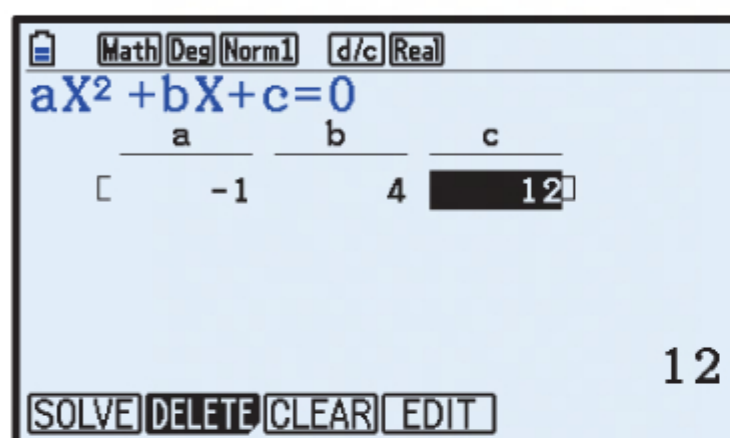
- e Since $a = -1$ which is < 0 , the parabola has shape .

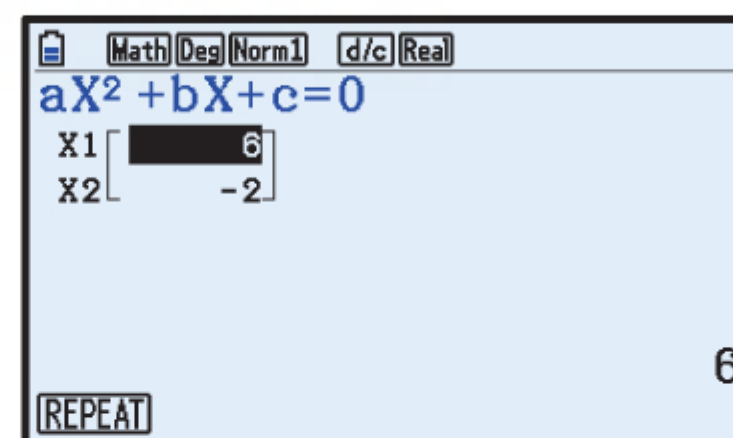
When $x = 0$, $y = 12$

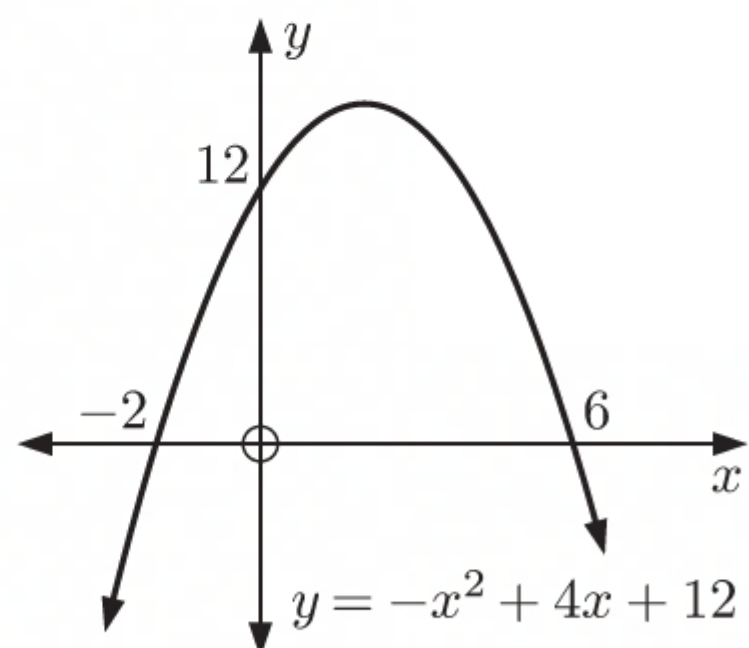
\therefore the y -intercept is 12.


When $y = 0$,
 $-x^2 + 4x + 12 = 0$
 $\therefore x = -2$ or 6

\therefore the x -intercepts are
 -2 and 6 .







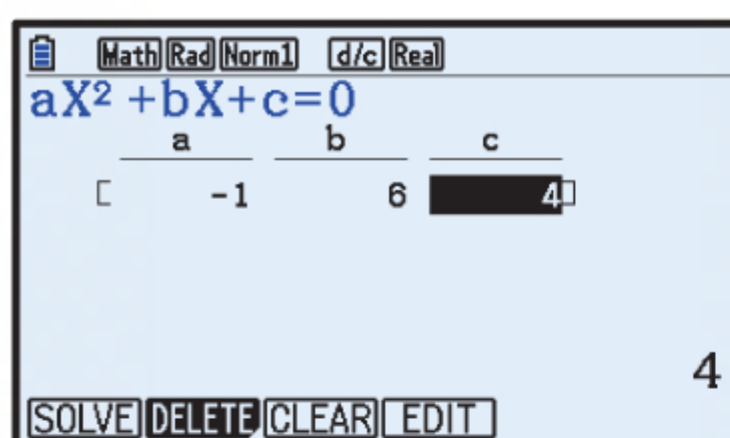
- f Since $a = -1$ which is < 0 , the parabola has shape .

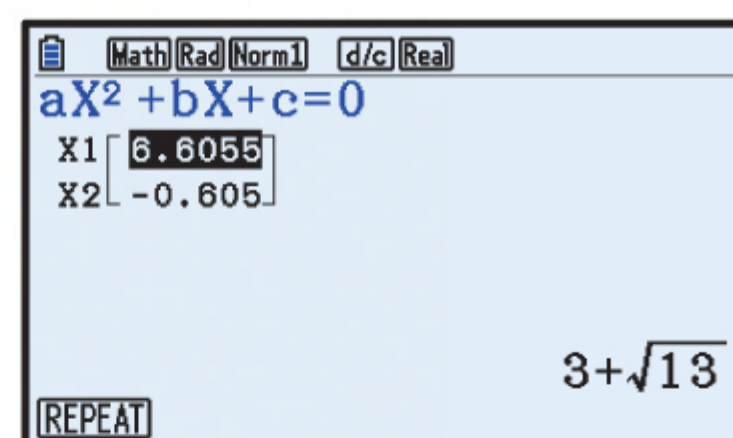
When $x = 0$, $y = 4$

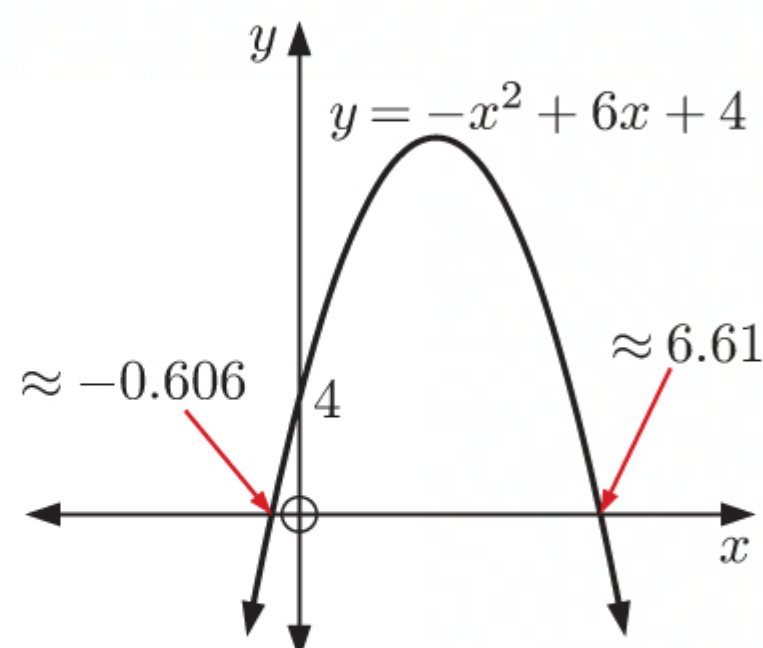
\therefore the y -intercept is 4.


When $y = 0$,
 $-x^2 + 6x + 4 = 0$
 $\therefore x \approx -0.606$
 or 6.61

\therefore the x -intercepts are
 ≈ -0.606 and ≈ 6.61 .







- g** Since $a = 2$ which is > 0 , the parabola has shape .

When $x = 0$, $y = -24$

\therefore the y -intercept is -24 .

When $y = 0$,

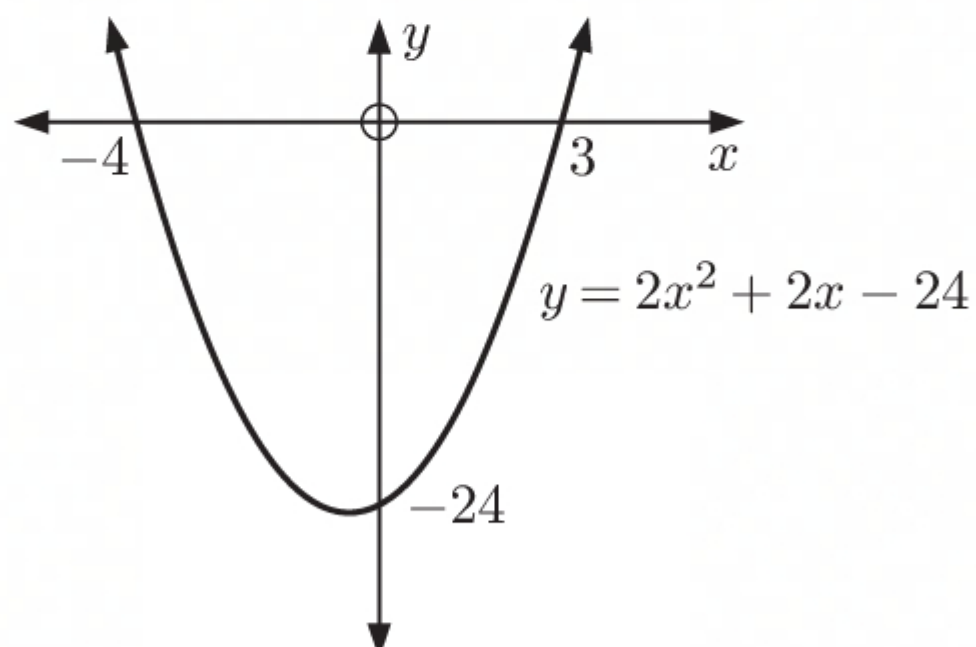
$$2x^2 + 2x - 24 = 0$$


$$\therefore x = -4 \text{ or } 3$$

\therefore the x -intercepts are -4 and 3 .

Math Deg Norm1 d/c Real		
$aX^2 + bX + c = 0$		
a	b	c
2	2	-24
-24		
SOLVE DELETE CLEAR EDIT		

Math Deg Norm1 d/c Real		
$aX^2 + bX + c = 0$		
X1	3	
X2	-4	
3		
REPEAT		



- h** Since $a = -2$ which is < 0 , the parabola has shape .

When $x = 0$, $y = 9$

\therefore the y -intercept is 9 .

When $y = 0$,

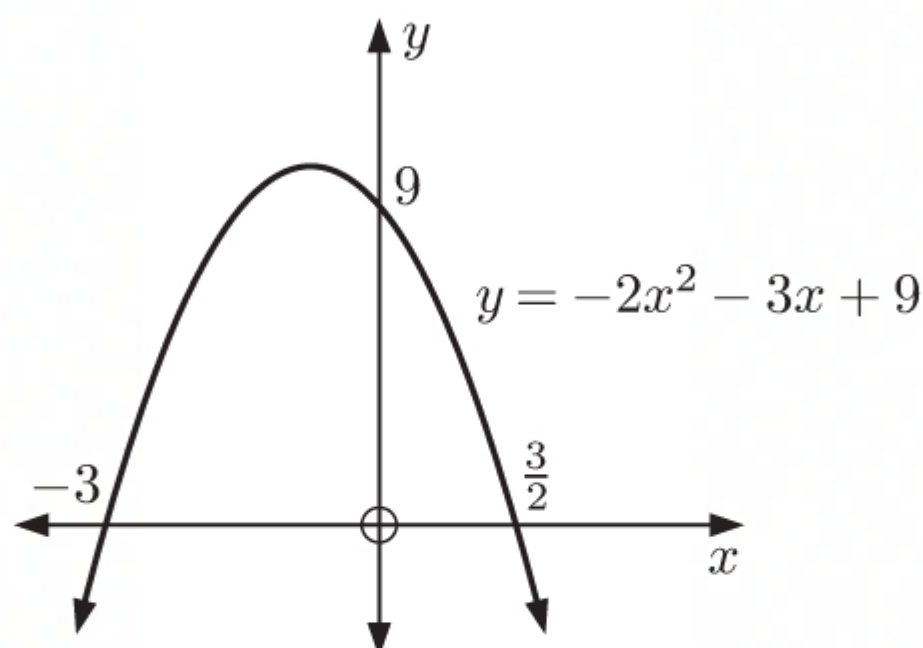
$$-2x^2 - 3x + 9 = 0$$


$$\therefore x = -3 \text{ or } \frac{3}{2}$$

\therefore the x -intercepts are -3 and $\frac{3}{2}$.

Math Deg Norm1 d/c Real		
$aX^2 + bX + c = 0$		
a	b	c
-2	-3	9
9		
SOLVE DELETE CLEAR EDIT		

Math Deg Norm1 d/c Real		
$aX^2 + bX + c = 0$		
X1	1.5	
X2	-3	
$\frac{3}{2}$		
REPEAT		



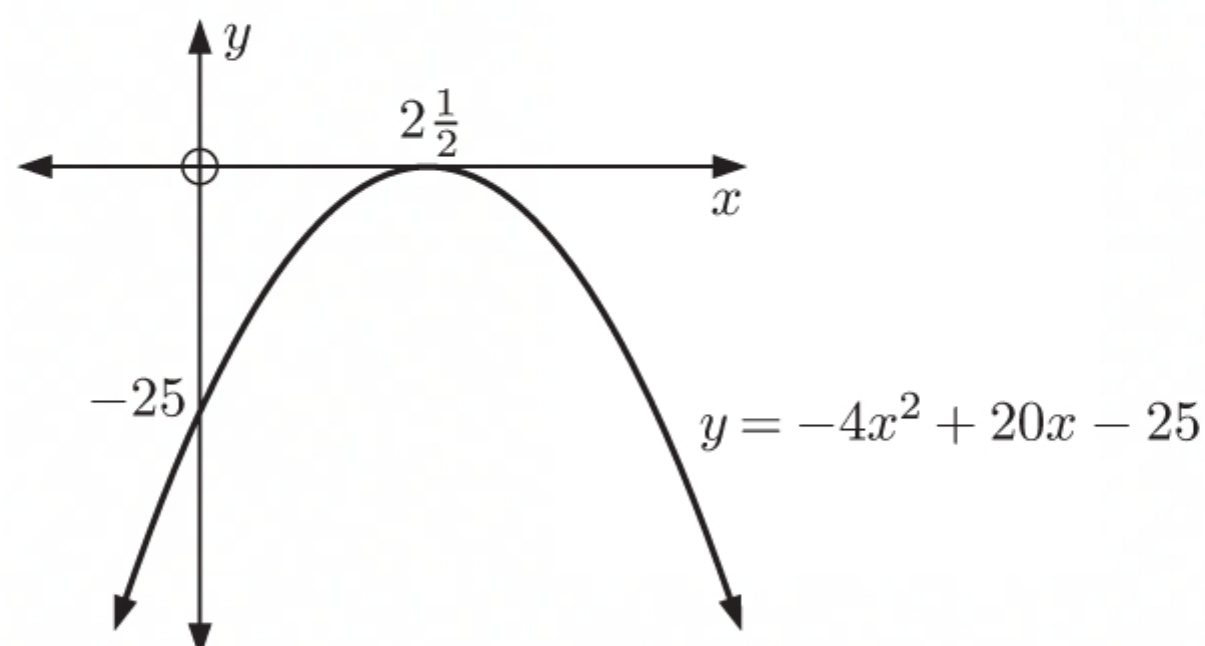
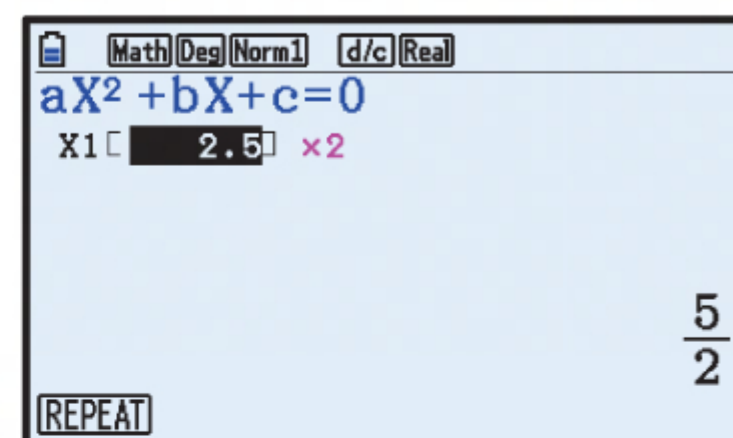
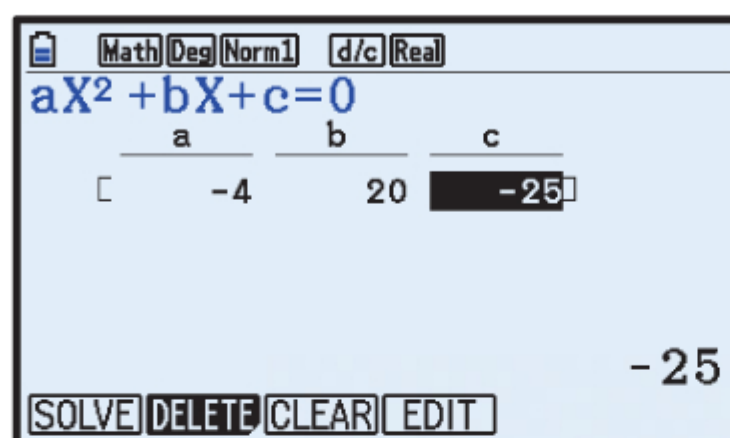
i Since $a = -4$ which is < 0 , the parabola has shape .


When $x = 0$, $y = -25$

\therefore the y -intercept is -25 .

When $y = 0$,
 $-4x^2 + 20x - 25 = 0$
 $\therefore x = 2\frac{1}{2}$

\therefore the x -intercept is $2\frac{1}{2}$
 (touching).



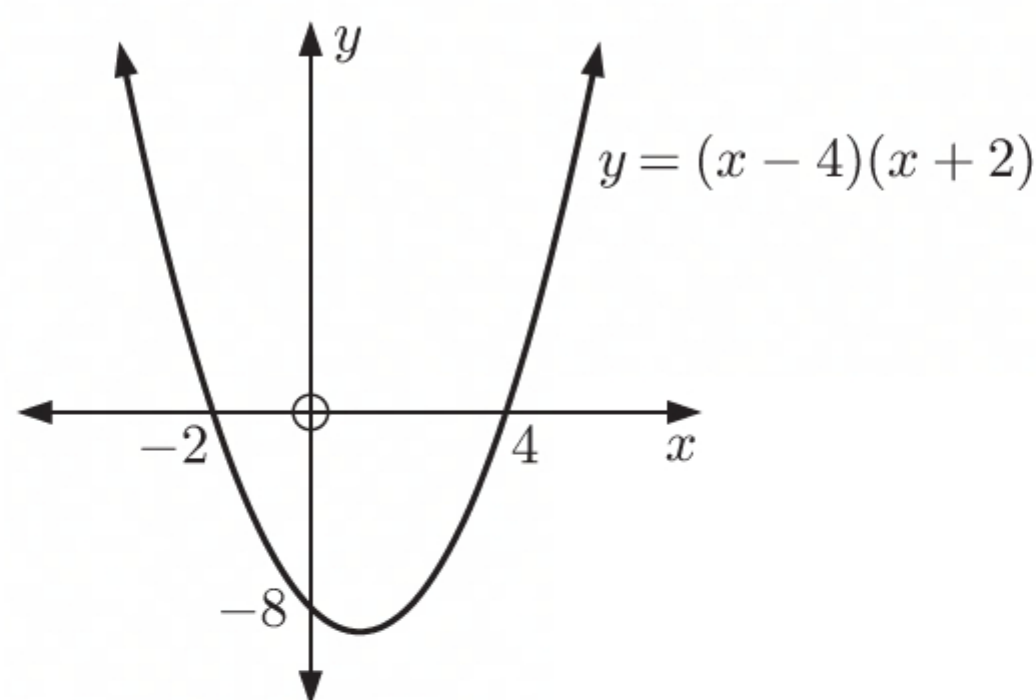
3 a Since $a = 1$ which is > 0 , the parabola has shape .


When $x = 0$, $y = (-4)(2)$
 $= -8$

\therefore the y -intercept is -8 .

When $y = 0$, $(x - 4)(x + 2) = 0$
 $\therefore x = 4$ or $x = -2$

\therefore the x -intercepts are 4 and -2 .



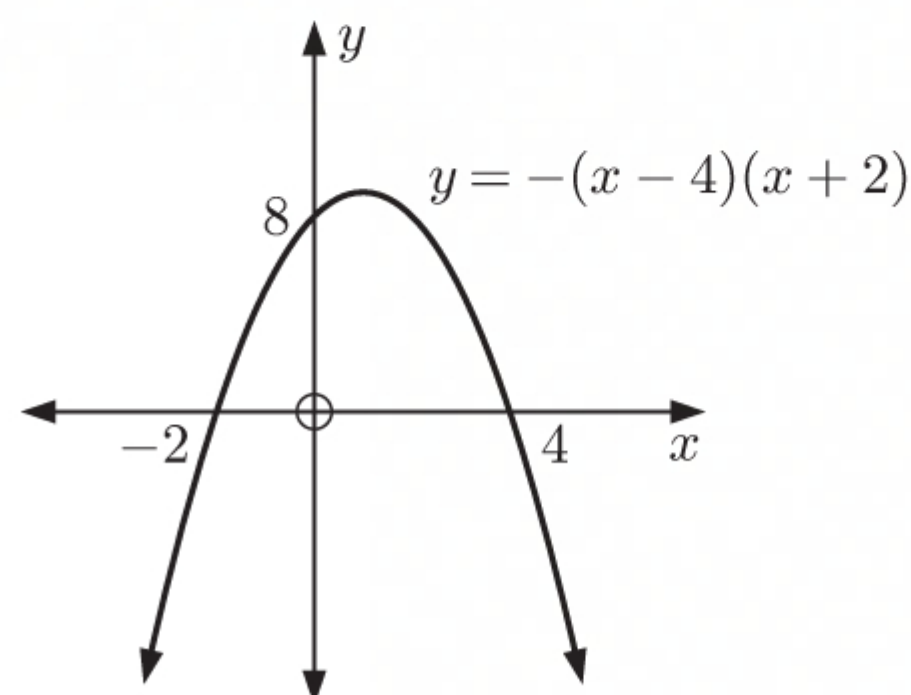
- b** Since $a = -1$ which is < 0 , the parabola has shape .


$$\begin{aligned}\text{When } x = 0, \quad y &= -(-4)(2) \\ &= 8\end{aligned}$$

\therefore the y -intercept is 8.

$$\begin{aligned}\text{When } y = 0, \quad -(x-4)(x+2) &= 0 \\ \therefore x &= 4 \text{ or } x = -2\end{aligned}$$

\therefore the x -intercepts are 4 and -2 .



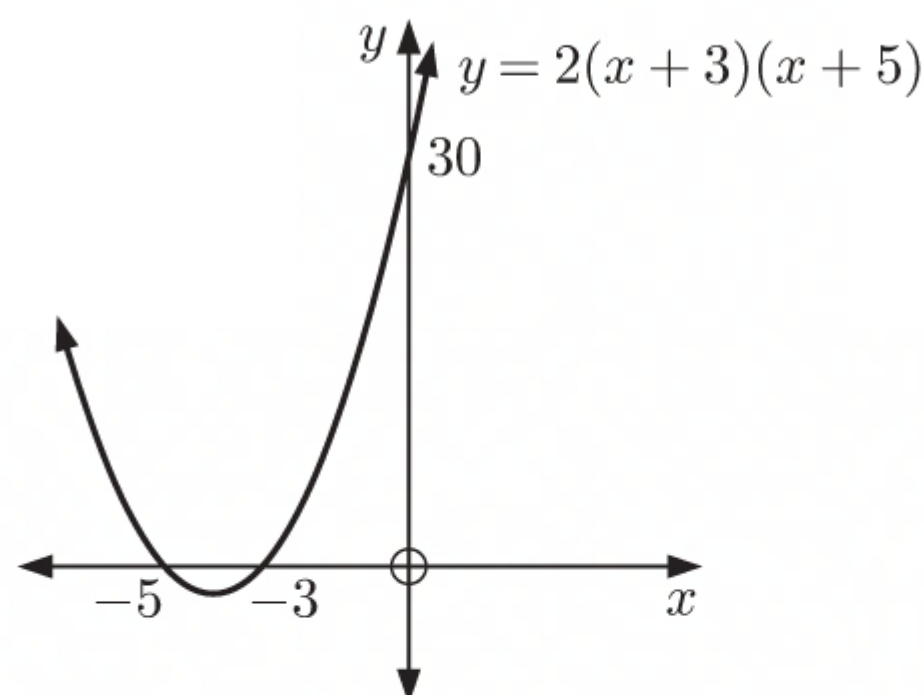
- c** Since $a = 2$ which is > 0 , the parabola has shape .


$$\begin{aligned}f(0) &= 2(3)(5) \\ &= 30\end{aligned}$$

\therefore the y -intercept is 30.

$$\begin{aligned}\text{When } f(x) = 0, \quad 2(x+3)(x+5) &= 0 \\ \therefore x &= -3 \text{ or } x = -5\end{aligned}$$

\therefore the x -intercepts are -3 and -5 .



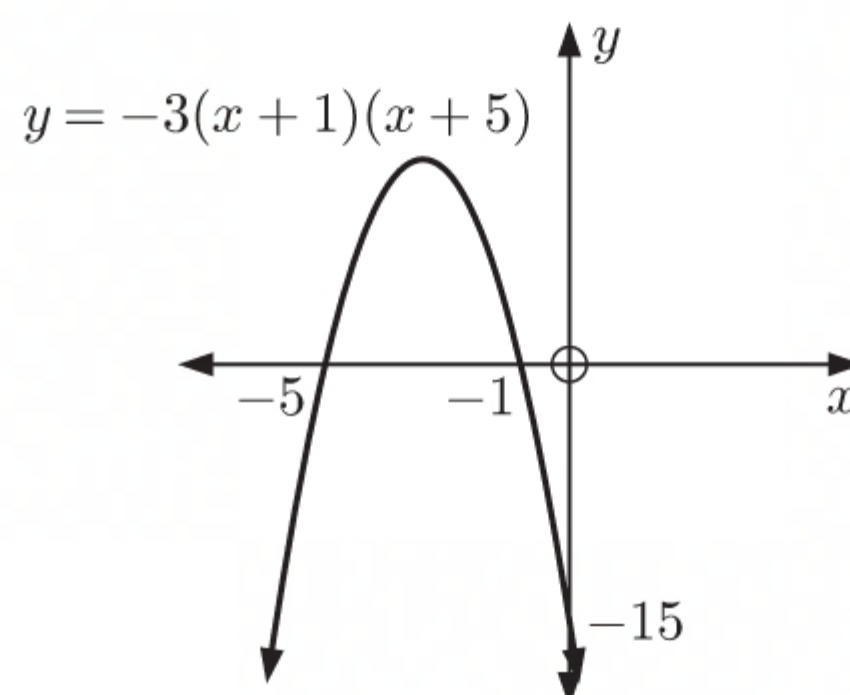
- d** Since $a = -3$ which is < 0 , the parabola has shape .

$$\begin{aligned}\text{When } x = 0, \quad y &= -3(1)(5) \\ &= -15\end{aligned}$$

\therefore the y -intercept is -15 .

$$\begin{aligned}\text{When } y = 0, \quad -3(x+1)(x+5) &= 0 \\ \therefore x &= -1 \text{ or } x = -5\end{aligned}$$

\therefore the x -intercepts are -1 and -5 .



e $f(x) = (3x - 2)(x + 4) = 3(x - \frac{2}{3})(x + 4)$

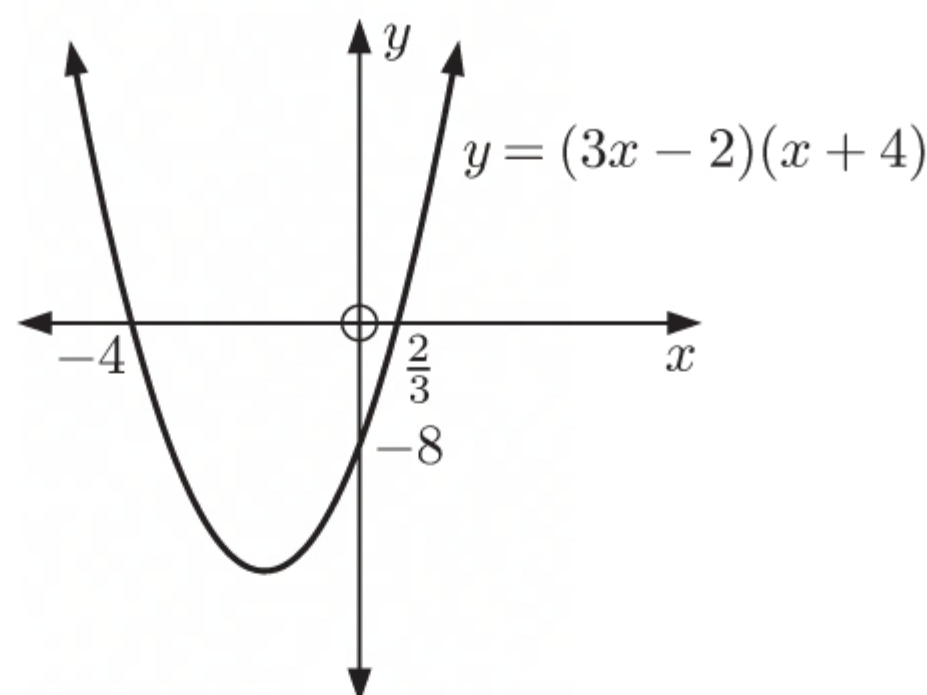
Since $a = 3$ which is > 0 , the parabola has shape .

$$\begin{aligned} f(0) &= (-2)(4) \\ &= -8 \end{aligned}$$


\therefore the y -intercept is -8 .

$$\begin{aligned} \text{When } f(x) = 0, \quad 3(x - \frac{2}{3})(x + 4) &= 0 \\ \therefore x &= \frac{2}{3} \text{ or } x = -4 \end{aligned}$$

\therefore the x -intercepts are $\frac{2}{3}$ and -4 .



f $f(x) = -(2x - 1)(x + 2) = -2(x - \frac{1}{2})(x + 2)$

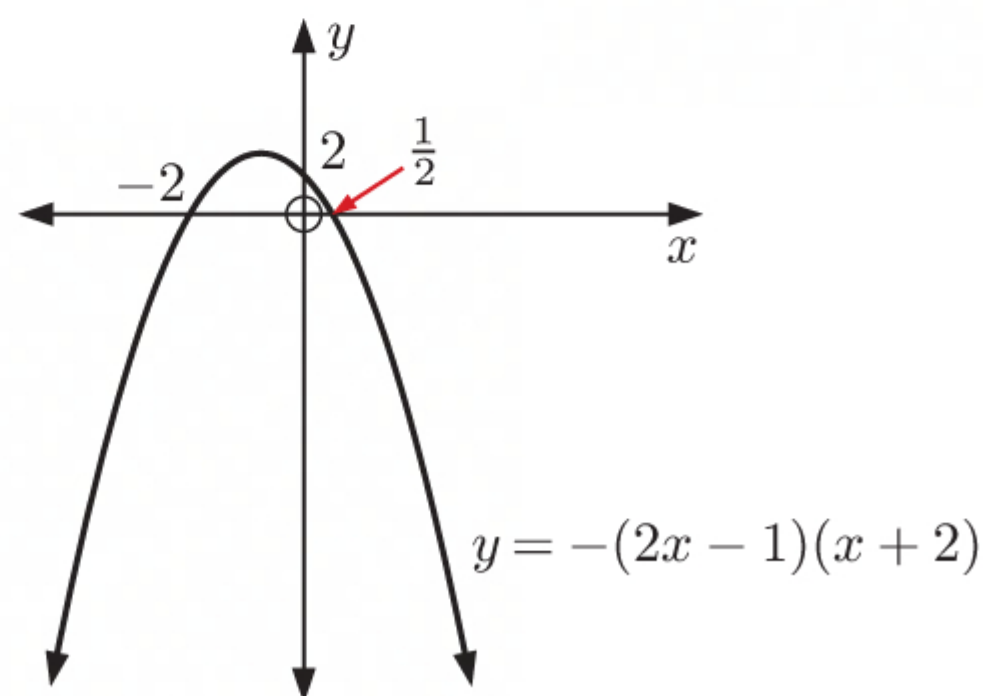
Since $a = -2$ which is < 0 , the parabola has shape .


$$\begin{aligned} f(0) &= -(-1)(2) \\ &= 2 \end{aligned}$$

\therefore the y -intercept is 2 .

$$\begin{aligned} \text{When } f(x) = 0, \quad -2(x - \frac{1}{2})(x + 2) &= 0 \\ \therefore x &= \frac{1}{2} \text{ or } x = -2 \end{aligned}$$

\therefore the x -intercepts are $\frac{1}{2}$ and -2 .



4 a Since $a = 3$ which is > 0 , the parabola has shape .

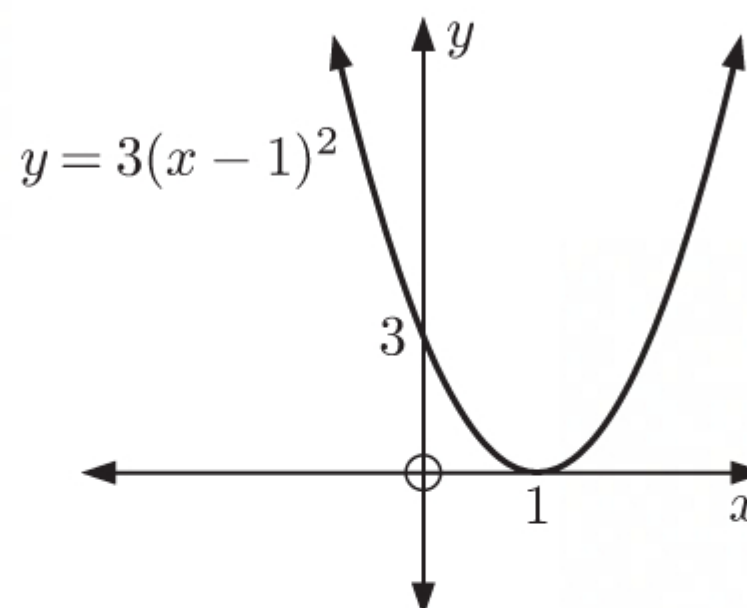
$$\begin{aligned} \text{When } x = 0, \quad y &= 3(-1)^2 \\ &= 3 \end{aligned}$$


\therefore the y -intercept is 3 .

$$\begin{aligned} \text{When } y = 0, \quad 3(x - 1)^2 &= 0 \\ \therefore x &= 1 \end{aligned}$$

\therefore the x -intercept is 1 .

There is only one x -intercept, so the graph *touches* the x -axis.



- b** Since $a = 2$ which is > 0 , the parabola has shape .

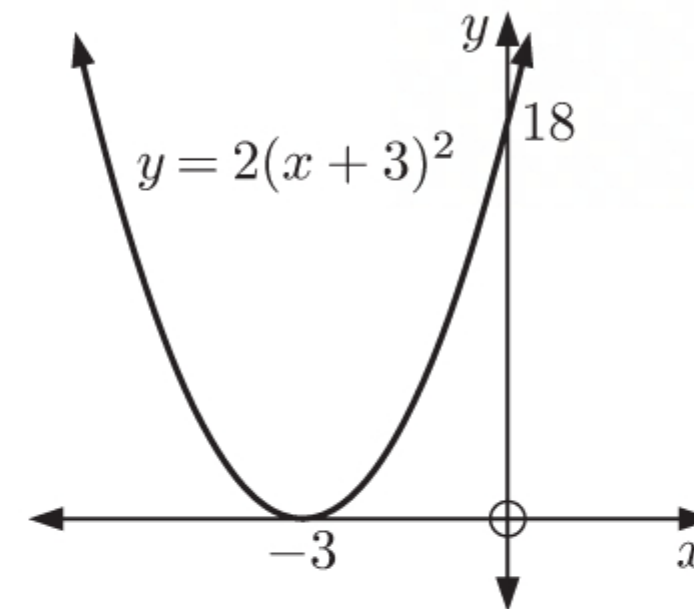
$$\begin{aligned}\text{When } x = 0, \quad y &= 2(3)^2 \\ &= 18\end{aligned}$$


\therefore the y -intercept is 18.

$$\begin{aligned}\text{When } y = 0, \quad 2(x + 3)^2 &= 0 \\ \therefore x &= -3\end{aligned}$$

\therefore the x -intercept is -3 .

There is only one x -intercept, so the graph *touches* the x -axis.



- c** Since $a = -\frac{1}{4}$ which is < 0 , the parabola has shape .

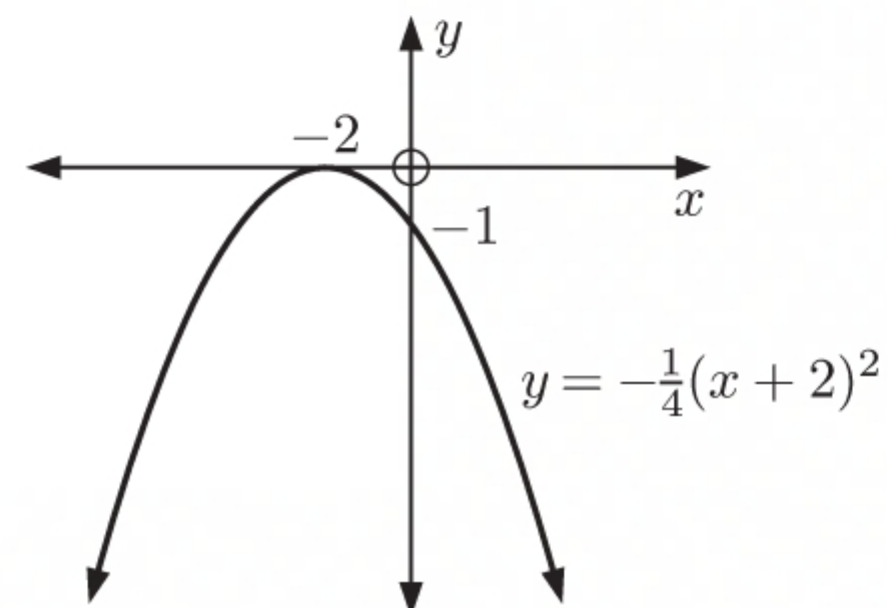
$$\begin{aligned}\text{When } x = 0, \quad y &= -\frac{1}{4}(2)^2 \\ &= -1\end{aligned}$$

\therefore the y -intercept is -1 .

$$\begin{aligned}\text{When } y = 0, \quad -\frac{1}{4}(x + 2)^2 &= 0 \\ \therefore x &= -2\end{aligned}$$

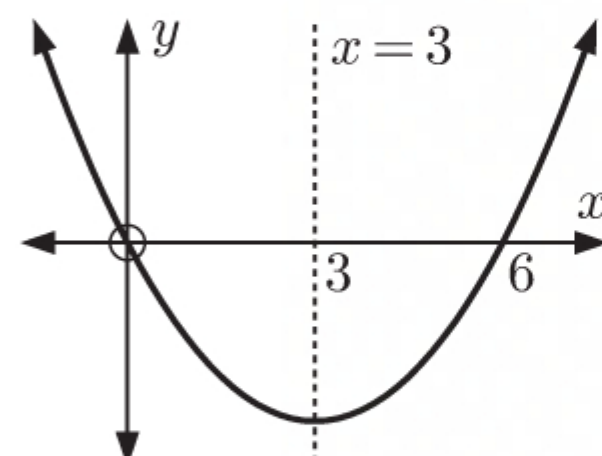
\therefore the x -intercept is -2 .

There is only one x -intercept, so the graph *touches* the x -axis.

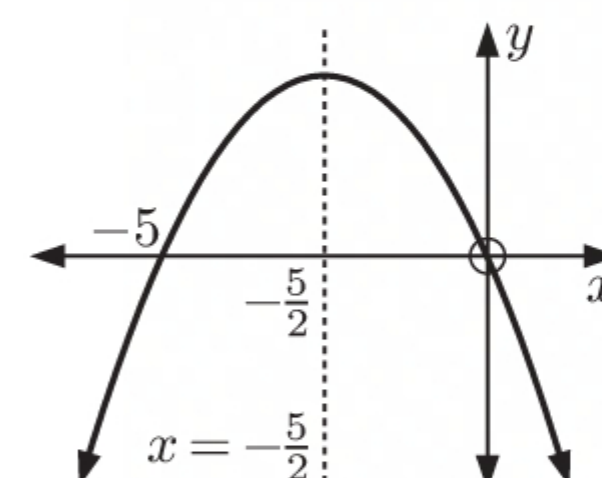


EXERCISE 6F

- 1 a** The x -intercepts are 0 and 6.
3 is halfway between 0 and 6, so the axis of symmetry is $x = 3$.

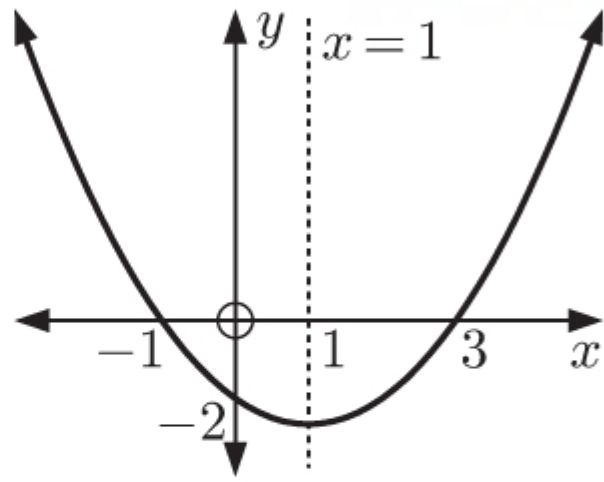


- b** The x -intercepts are -5 and 0 .
 $-\frac{5}{2}$ is halfway between -5 and 0 , so the axis of symmetry is $x = -\frac{5}{2}$.



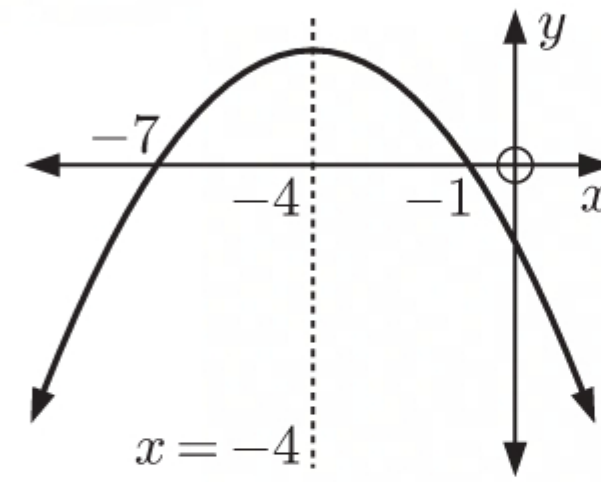
- c** The x -intercepts are -1 and 3 .

1 is halfway between -1 and 3 , so the axis of symmetry is $x = 1$.

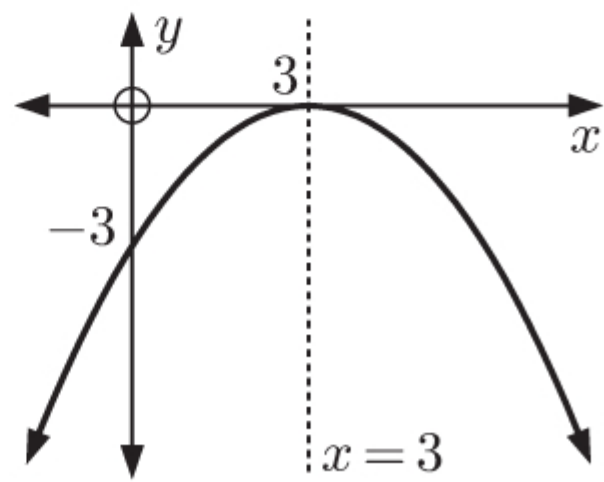


- d** The x -intercepts are -7 and -1 .

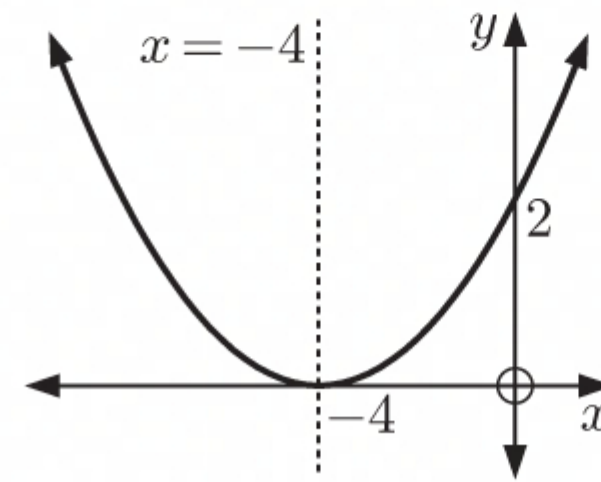
-4 is halfway between -7 and -1 , so the axis of symmetry is $x = -4$.



- e** The only x -intercept is 3 , so the axis of symmetry is $x = 3$.



- f** The only x -intercept is -4 , so the axis of symmetry is $x = -4$.



- 2 a** $y = 0$ when $(x - 2)(x - 6) = 0$
 $\therefore x = 2$ or $x = 6$

\therefore the x -intercepts are 2 and 6 .

4 is halfway between 2 and 6 , so the axis of symmetry is $x = 4$.

- b** $y = 0$ when $x(x + 4) = 0$
 $\therefore x = 0$ or $x = -4$

\therefore the x -intercepts are 0 and -4 .

-2 is halfway between 0 and -4 , so the axis of symmetry is $x = -2$.

- c** $y = 0$ when $-(x + 3)(x - 5) = 0$
 $\therefore x = -3$ or $x = 5$

\therefore the x -intercepts are -3 and 5 .

1 is halfway between -3 and 5 , so the axis of symmetry is $x = 1$.

- d** $y = 0$ when $(x - 3)(x - 8) = 0$
 $\therefore x = 3$ or $x = 8$

\therefore the x -intercepts are 3 and 8 .

$\frac{3+8}{2} = \frac{11}{2}$ is halfway between 3 and 8 , so the axis of symmetry is $x = \frac{11}{2}$.

- e** $y = 0$ when $2(x - 5)^2 = 0$
 $\therefore x = 5$

\therefore the only x -intercept is 5 , and so the axis of symmetry is $x = 5$.

$$\begin{aligned} \text{f } y = 0 \text{ when } -3(x+2)^2 &= 0 \\ \therefore x &= -2 \end{aligned}$$

\therefore the only x -intercept is -2 , and so the axis of symmetry is $x = -2$.

3 Let the other x -intercept be a .

Since the axis of symmetry $x = -3$ lies halfway between the x -intercepts a and 4 ,

$$\text{we have } \frac{a+4}{2} = -3$$

$$\therefore a+4 = -6$$

$$\therefore a = -10$$

So, the other x -intercept is -10 .

4 a $y = x^2 + 6x + 2$ has $a = 1$, $b = 6$, and $c = 2$.

$$\begin{aligned} \text{Now } -\frac{b}{2a} &= -\frac{6}{2(1)} \\ &= -3 \end{aligned}$$

\therefore the axis of symmetry is $x = -3$.

b $y = x^2 - 8x - 1$ has $a = 1$, $b = -8$, and $c = -1$.

$$\begin{aligned} \text{Now } -\frac{b}{2a} &= -\frac{-8}{2(1)} \\ &= \frac{8}{2} \\ &= 4 \end{aligned}$$

\therefore the axis of symmetry is $x = 4$.

c $f(x) = 2x^2 + 5x - 3$ has $a = 2$, $b = 5$, and $c = -3$.

$$\begin{aligned} \text{Now } -\frac{b}{2a} &= -\frac{5}{2(2)} \\ &= -\frac{5}{4} \end{aligned}$$

\therefore the axis of symmetry is $x = -\frac{5}{4}$.

d $y = -x^2 + 3x - 7$ has $a = -1$, $b = 3$, and $c = -7$.

$$\begin{aligned} \text{Now } -\frac{b}{2a} &= -\frac{3}{2(-1)} \\ &= \frac{3}{2} \end{aligned}$$

\therefore the axis of symmetry is $x = \frac{3}{2}$.

e $y = 2x^2 - 5$ has $a = 2$, $b = 0$, and $c = -5$.

$$\begin{aligned} \text{Now } -\frac{b}{2a} &= -\frac{0}{2(2)} \\ &= 0 \end{aligned}$$

\therefore the axis of symmetry is $x = 0$.

f $y = -5x^2 + 7x$ has $a = -5$, $b = 7$, and $c = 0$.

$$\begin{aligned}\text{Now } -\frac{b}{2a} &= -\frac{7}{2(-5)} \\ &= \frac{7}{10}\end{aligned}$$

\therefore the axis of symmetry is $x = \frac{7}{10}$.

g $y = -3x^2 - x + 4$ has $a = -3$, $b = -1$, and $c = 4$.

$$\begin{aligned}\text{Now } -\frac{b}{2a} &= -\frac{-1}{2(-3)} \\ &= -\frac{1}{6}\end{aligned}$$

\therefore the axis of symmetry is $x = -\frac{1}{6}$.

h $y = 10x - 3x^2$ has $a = -3$, $b = 10$, and $c = 0$.

$$\begin{aligned}\text{Now } -\frac{b}{2a} &= -\frac{10}{2(-3)} \\ &= \frac{10}{6} \\ &= \frac{5}{3}\end{aligned}$$

\therefore the axis of symmetry is $x = \frac{5}{3}$.

i $f(x) = \frac{1}{8}x^2 + x - 1$ has $a = \frac{1}{8}$, $b = 1$, and $c = -1$.

$$\begin{aligned}\text{Now } -\frac{b}{2a} &= -\frac{1}{2(\frac{1}{8})} \\ &= -\frac{1}{\frac{1}{4}} \\ &= -4\end{aligned}$$

\therefore the axis of symmetry is $x = -4$.

5 a $y = 0$ when $(x + 2)(x - 5) = 0$
 $\therefore x = -2$ or $x = 5$

\therefore the x -intercepts are -2 and 5 .

$\frac{-2 + 5}{2} = \frac{3}{2}$ lies halfway between -2 and 5 , so the axis of symmetry is $x = \frac{3}{2}$.

b $y = (x + 2)(x - 5)$
 $= x^2 - 5x + 2x - 10$
 $= x^2 - 3x - 10$ has $a = 1$, $b = -3$, and $c = -10$

$$\begin{aligned}\text{Now } -\frac{b}{2a} &= -\frac{-3}{2(1)} \\ &= \frac{3}{2}\end{aligned}$$

\therefore the axis of symmetry is $x = \frac{3}{2}$.

6 $f(x) = ax^2 + 6x - 4$ has $a = a$, $b = 6$, and $c = -4$.

The axis of symmetry is $x = -2$, so $-\frac{b}{2a} = -2$

$$\therefore -\frac{6}{2a} = -2$$

$$\therefore 4a = 6$$

$$\therefore a = \frac{3}{2}$$

7 a i $y = 2x^2 + 5x + 1$ has $a = 2$, $b = 5$, and $c = 1$.

$$\begin{aligned}\text{Now } -\frac{b}{2a} &= -\frac{5}{2(2)} \\ &= -\frac{5}{4}\end{aligned}$$

\therefore the axis of symmetry is $x = -\frac{5}{4}$.

ii $y = 2x^2 + 5x + 7$ has $a = 2$, $b = 5$, and $c = 7$.

$$\begin{aligned}\text{Now } -\frac{b}{2a} &= -\frac{5}{2(2)} \\ &= -\frac{5}{4}\end{aligned}$$

\therefore the axis of symmetry is $x = -\frac{5}{4}$.

iii $y = 2x^2 + 5x - 4$ has $a = 2$, $b = 5$, and $c = -4$.

$$\begin{aligned}\text{Now } -\frac{b}{2a} &= -\frac{5}{2(2)} \\ &= -\frac{5}{4}\end{aligned}$$

\therefore the axis of symmetry is $x = -\frac{5}{4}$.

The axes of symmetry are the same.

- b** The value of c only affects the vertical translation of the graph. It does not affect the axis of symmetry (since it is a vertical line). Hence $x = -\frac{b}{2a}$ depends only on a and b .

EXERCISE 6G

1 a $y = x^2 - 4x + 2$


has $a = 1$, $b = -4$, and $c = 2$.

$$\text{Now } -\frac{b}{2a} = -\frac{-4}{2(1)} = 2$$

\therefore the axis of symmetry is $x = 2$.

$$\begin{aligned}\text{When } x = 2, \quad y &= 2^2 - 4(2) + 2 \\ &= -2\end{aligned}$$

\therefore the vertex is $(2, -2)$.

$a > 0$, so the shape is 

\therefore the vertex $(2, -2)$ is a minimum turning point.

b $y = (x + 3)(x - 1)$


has x -intercepts -3 and 1 .

\therefore the axis of symmetry is

$$x = \frac{-3 + 1}{2} = -1.$$

$$\begin{aligned}\text{When } x = -1, \quad y &= (-1 + 3)(-1 - 1) \\ &= (2)(-2) \\ &= -4\end{aligned}$$

\therefore the vertex is $(-1, -4)$.

$a > 0$, so the shape is 

\therefore the vertex $(-1, -4)$ is a minimum turning point.

c $y = 2x^2 + 4$


has $a = 2$, $b = 0$, and $c = 4$.

Now $-\frac{b}{2a} = -\frac{0}{2(2)} = 0$

\therefore the axis of symmetry is $x = 0$.

When $x = 0$, $y = 4$

\therefore the vertex is $(0, 4)$.

$a > 0$, so the shape is 

\therefore the vertex $(0, 4)$ is a minimum turning point.

e $y = (x - 4)(x + 2)$


has x -intercepts 4 and -2 .

\therefore the axis of symmetry is

$$x = \frac{4 + (-2)}{2} = 1.$$

$$\begin{aligned} \text{When } x = 1, \quad y &= (1 - 4)(1 + 2) \\ &= (-3)(3) \\ &= -9 \end{aligned}$$

\therefore the vertex is $(1, -9)$.

$a > 0$, so the shape is 

\therefore the vertex $(1, -9)$ is a minimum turning point.

g $y = 2x^2 + 6x - 1$


has $a = 2$, $b = 6$, and $c = -1$.

Now $-\frac{b}{2a} = -\frac{6}{2(2)} = -\frac{3}{2}$

\therefore the axis of symmetry is $x = -\frac{3}{2}$.

$$\begin{aligned} \text{When } x = -\frac{3}{2}, \quad y &= 2\left(-\frac{3}{2}\right)^2 + 6\left(-\frac{3}{2}\right) - 1 \\ &= \frac{9}{2} - 9 - 1 \\ &= -\frac{11}{2} \end{aligned}$$

\therefore the vertex is $\left(-\frac{3}{2}, -\frac{11}{2}\right)$.

$a > 0$, so the shape is 

\therefore the vertex $\left(-\frac{3}{2}, -\frac{11}{2}\right)$ is a minimum turning point.

d $y = -3x^2 + 1$


has $a = -3$, $b = 0$, and $c = 1$.

Now $-\frac{b}{2a} = -\frac{0}{2(-3)} = 0$

\therefore the axis of symmetry is $x = 0$.

When $x = 0$, $y = 1$

\therefore the vertex is $(0, 1)$.

$a < 0$, so the shape is 

\therefore the vertex $(0, 1)$ is a maximum turning point.

f $y = -x^2 - 4x - 9$


has $a = -1$, $b = -4$, and $c = -9$.

Now $-\frac{b}{2a} = -\frac{-4}{2(-1)} = -2$

\therefore the axis of symmetry is $x = -2$.

$$\begin{aligned} \text{When } x = -2, \quad y &= -(-2)^2 - 4(-2) - 9 \\ &= -4 + 8 - 9 \\ &= -5 \end{aligned}$$

\therefore the vertex is $(-2, -5)$.

$a < 0$, so the shape is 

\therefore the vertex $(-2, -5)$ is a maximum turning point.

h $y = -2(x + 3)(x - 4)$


has x -intercepts -3 and 4 .

\therefore the axis of symmetry is

$$x = \frac{-3 + 4}{2} = \frac{1}{2}.$$

$$\begin{aligned} \text{When } x = \frac{1}{2}, \quad y &= -2\left(\frac{1}{2} + 3\right)\left(\frac{1}{2} - 4\right) \\ &= -2\left(\frac{7}{2}\right)\left(-\frac{7}{2}\right) \\ &= \frac{49}{2} \end{aligned}$$

\therefore the vertex is $\left(\frac{1}{2}, \frac{49}{2}\right)$.

$a < 0$, so the shape is 

\therefore the vertex $\left(\frac{1}{2}, \frac{49}{2}\right)$ is a maximum turning point.

i $y = -\frac{1}{2}x^2 + x - 5$


has $a = -\frac{1}{2}$, $b = 1$, and $c = -5$.

Now $-\frac{b}{2a} = -\frac{1}{2(-\frac{1}{2})} = 1$

\therefore the axis of symmetry is $x = 1$.

When $x = 1$, $y = -\frac{1}{2}(1)^2 + 1 - 5$
 $= -\frac{9}{2}$

\therefore the vertex is $(1, -\frac{9}{2})$.

$a < 0$, so the shape is 

\therefore the vertex $(1, -\frac{9}{2})$ is a maximum turning point.

- 2 a i** When $x = 0$, $y = (-1)(-7) = 7$
 \therefore the y -intercept is 7.

When $y = 0$,
 $(x - 1)(x - 7) = 0$
 $\therefore x = 1$ or 7

\therefore the x -intercepts are 1 and 7.

- ii** The axis of symmetry is halfway between the x -intercepts 1 and 7.

So, the axis of symmetry is $x = 4$.

- iii** When $x = 4$, $y = (4 - 1)(4 - 7)$
 $= (3)(-3)$
 $= -9$

So, the vertex is $(4, -9)$.

- b** $y = -x^2 - 6x - 8$ has $a = -1$, $b = -6$, and $c = -8$.

- i** When $x = 0$, $y = -8$
 \therefore the y -intercept is -8 .

When $y = 0$,
 $-x^2 - 6x - 8 = 0$
 Using technology, $x = -2$ or -4
 \therefore the x -intercepts are -2 and -4 .

- ii** $-\frac{b}{2a} = -\frac{-6}{2(-1)} = -3$

So, the axis of symmetry is $x = -3$.

- iii** When $x = -3$,
 $y = -(-3)^2 - 6(-3) - 8$
 $= -9 + 18 - 8$
 $= 1$

So, the vertex is $(-3, 1)$.

j $y = \frac{1}{4}x^2 - 7x + 6$


has $a = \frac{1}{4}$, $b = -7$, and $c = 6$.

Now $-\frac{b}{2a} = -\frac{-7}{2(\frac{1}{4})} = 14$

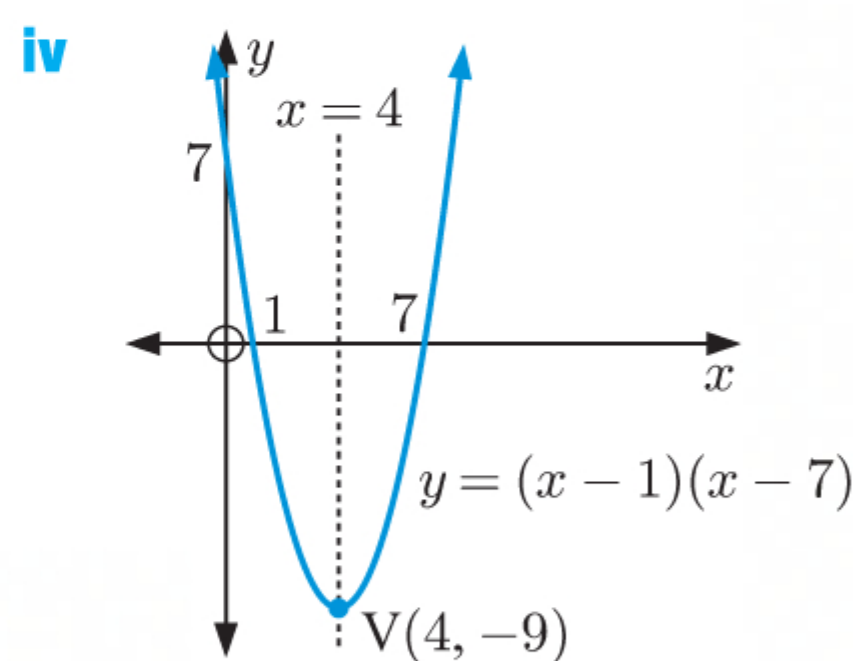
\therefore the axis of symmetry is $x = 14$.

When $x = 14$, $y = \frac{1}{4}(14)^2 - 7(14) + 6$
 $= -43$

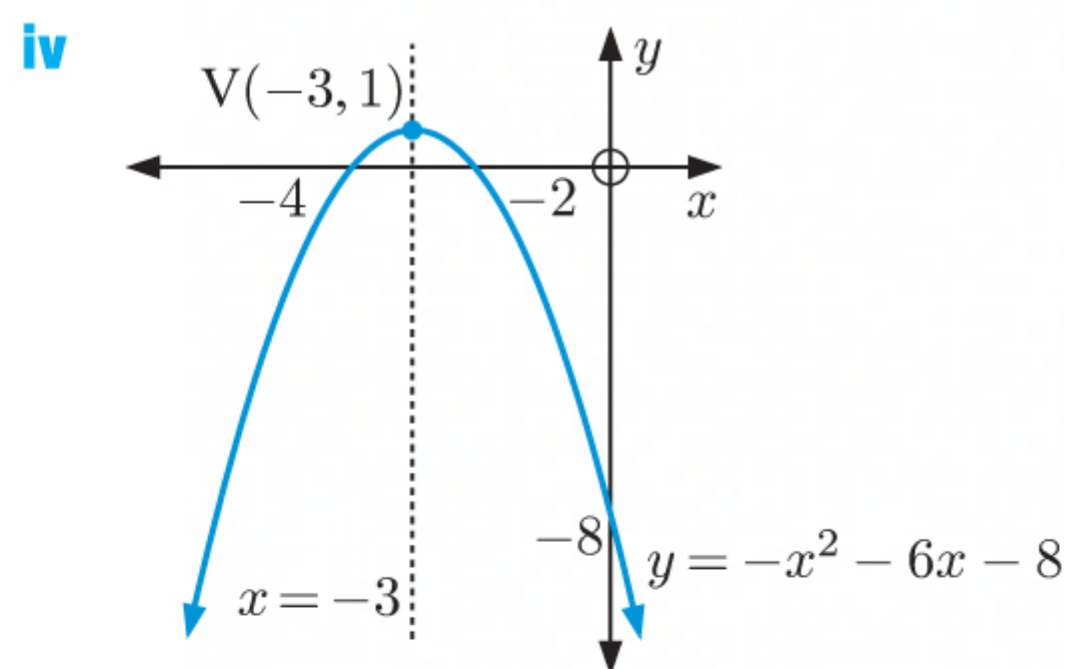
\therefore the vertex is $(14, -43)$.

$a > 0$, so the shape is 

\therefore the vertex $(14, -43)$ is a minimum turning point.



- v** The domain is $\{x \mid x \in \mathbb{R}\}$.
 The range is $\{y \mid y \geq -9\}$.



- v** The domain is $\{x \mid x \in \mathbb{R}\}$.
 The range is $\{y \mid y \leq 1\}$.

c $y = 6x - x^2$ has $a = -1$, $b = 6$, and $c = 0$.

i When $x = 0$, $y = 0$
 \therefore the y -intercept is 0.

$$\begin{aligned}\text{When } y = 0, \\ 6x - x^2 &= 0\end{aligned}$$

Using technology, $x = 0$ or 6

\therefore the x -intercepts are 0 and 6.

ii $-\frac{b}{2a} = -\frac{6}{2(-1)} = 3$

So, the axis of symmetry is $x = 3$.

iii When $x = 3$,

$$\begin{aligned}y &= 6(3) - 3^2 \\ &= 18 - 9 \\ &= 9\end{aligned}$$

So, the vertex is $(3, 9)$.

d i When $x = 0$, $y = -(-1)(-2) = -2$
 \therefore the y -intercept is -2 .

$$\begin{aligned}\text{When } y = 0, \\ -(x-1)(x-2) &= 0 \\ \therefore x &= 1 \text{ or } 2\end{aligned}$$

\therefore the x -intercepts are 1 and 2.

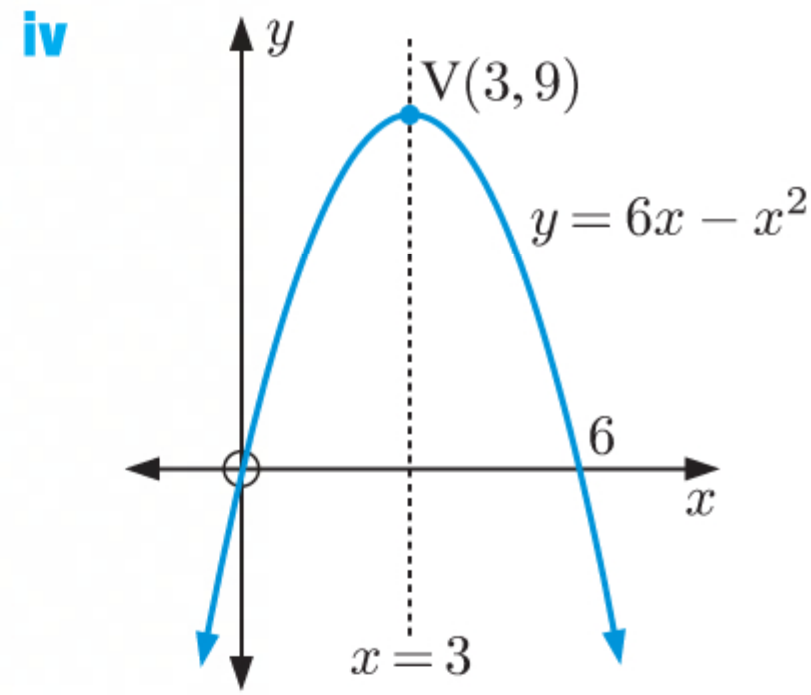
ii The axis of symmetry is halfway between the x -intercepts 1 and 2.

So, the axis of symmetry is $x = \frac{3}{2}$.

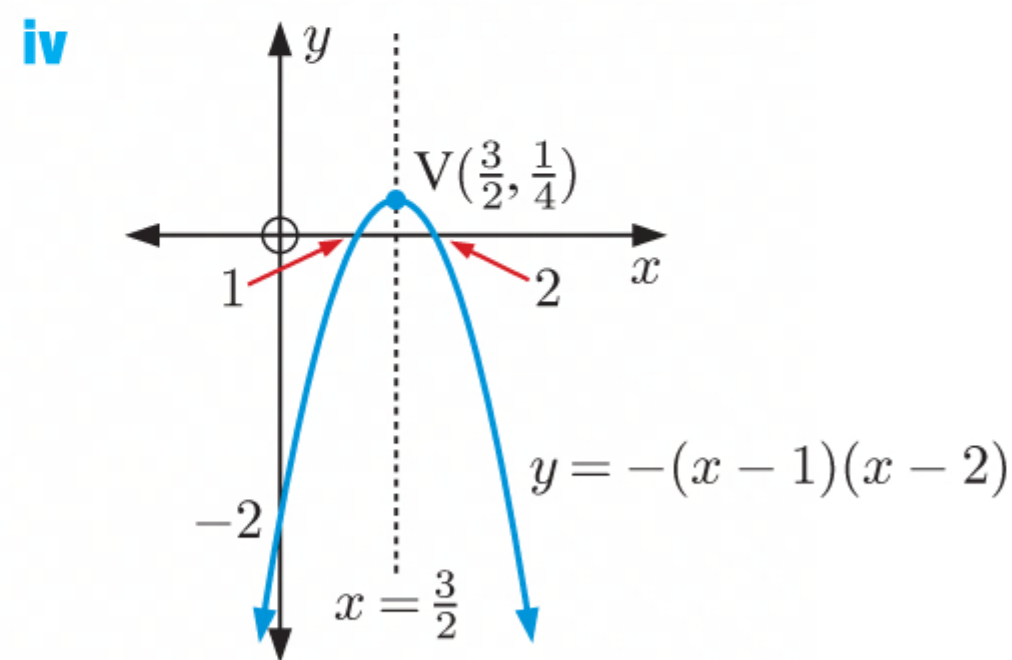
iii When $x = \frac{3}{2}$,

$$\begin{aligned}y &= -\left(\frac{3}{2} - 1\right)\left(\frac{3}{2} - 2\right) \\ &= -\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) \\ &= \frac{1}{4}\end{aligned}$$

So, the vertex is $\left(\frac{3}{2}, \frac{1}{4}\right)$.



v The domain is $\{x \mid x \in \mathbb{R}\}$.
 The range is $\{y \mid y \leq 9\}$.



v The domain is $\{x \mid x \in \mathbb{R}\}$.
 The range is $\{y \mid y \leq \frac{1}{4}\}$.

e $y = 2x^2 + 4x - 24$ has $a = 2$, $b = 4$, and $c = -24$.

i When $x = 0$, $y = -24$
 \therefore the y -intercept is -24 .

$$\begin{aligned} \text{When } y = 0, \\ 2x^2 + 4x - 24 = 0 \end{aligned}$$

Using technology, $x \approx -4.61$ or ≈ 2.61

\therefore the x -intercepts are ≈ -4.61 or ≈ 2.61 .

ii $-\frac{b}{2a} = -\frac{4}{2(2)} = -1$

So, the axis of symmetry is $x = -1$.

iii When $x = -1$,

$$\begin{aligned} y &= 2(-1)^2 + 4(-1) - 24 \\ &= 2 - 4 - 24 \\ &= -26 \end{aligned}$$

So, the vertex is $(-1, -26)$.

f $y = -3x^2 + 4x - 1$ has $a = -3$, $b = 4$, and $c = -1$.

i When $x = 0$, $y = -1$
 \therefore the y -intercept is -1 .

$$\begin{aligned} \text{When } y = 0, \\ -3x^2 + 4x - 1 = 0 \end{aligned}$$

Using technology, $x = \frac{1}{3}$ or 1
 \therefore the x -intercepts are $\frac{1}{3}$ and 1 .

ii $-\frac{b}{2a} = -\frac{4}{2(-3)} = \frac{2}{3}$

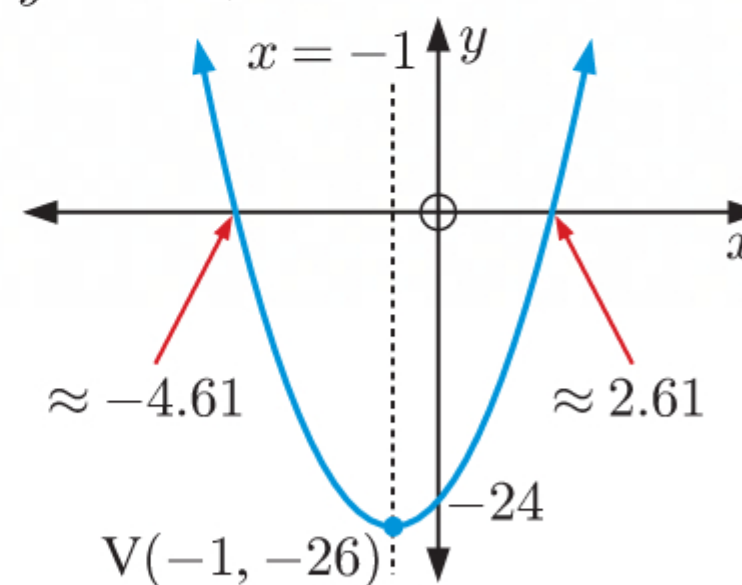
So, the axis of symmetry is $x = \frac{2}{3}$.

iii When $x = \frac{2}{3}$,

$$\begin{aligned} y &= -3\left(\frac{2}{3}\right)^2 + 4\left(\frac{2}{3}\right) - 1 \\ &= -\frac{4}{3} + \frac{8}{3} - 1 \\ &= \frac{1}{3} \end{aligned}$$

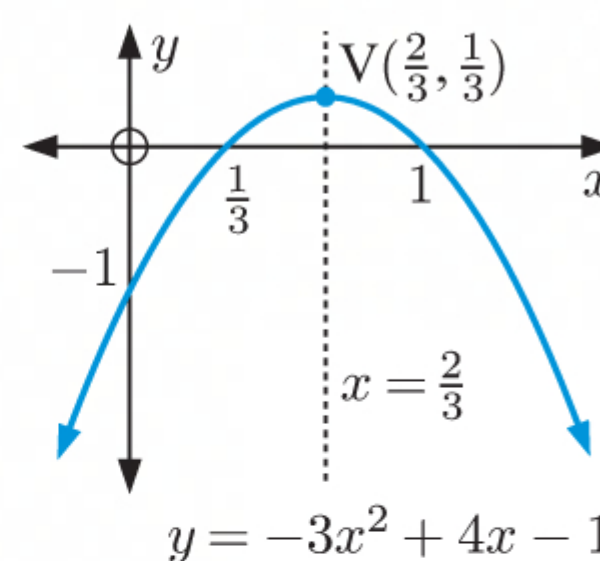
So, the vertex is $(\frac{2}{3}, \frac{1}{3})$.

iv $y = 2x^2 + 4x - 24$



v The domain is $\{x \mid x \in \mathbb{R}\}$.
 The range is $\{y \mid y \geq -26\}$.

iv



v The domain is $\{x \mid x \in \mathbb{R}\}$.
 The range is $\{y \mid y \leq \frac{1}{3}\}$.

g $y = 2x^2 - 5x + 2$ has $a = 2$, $b = -5$, and $c = 2$.

i When $x = 0$, $y = 2$
 \therefore the y -intercept is 2.

$$\begin{aligned}\text{When } y = 0, \\ 2x^2 - 5x + 2 &= 0\end{aligned}$$

Using technology, $x = \frac{1}{2}$ or 2

\therefore the x -intercepts are $\frac{1}{2}$ and 2.

ii $-\frac{b}{2a} = -\frac{-5}{2(2)} = \frac{5}{4}$

So, the axis of symmetry is $x = \frac{5}{4}$.

iii When $x = \frac{5}{4}$,

$$\begin{aligned}y &= 2\left(\frac{5}{4}\right)^2 - 5\left(\frac{5}{4}\right) + 2 \\ &= \frac{25}{8} - \frac{25}{4} + 2 \\ &= -\frac{9}{8}\end{aligned}$$

So, the vertex is $\left(\frac{5}{4}, -\frac{9}{8}\right)$.

h i When $x = 0$, $y = (-5)(1) = -5$
 \therefore the y -intercept is -5 .

$$\begin{aligned}\text{When } y = 0, \\ (2x - 5)(2x + 1) &= 0 \\ \therefore x &= \frac{5}{2} \text{ or } -\frac{1}{2}\end{aligned}$$

\therefore the x -intercepts are $\frac{5}{2}$ and $-\frac{1}{2}$.

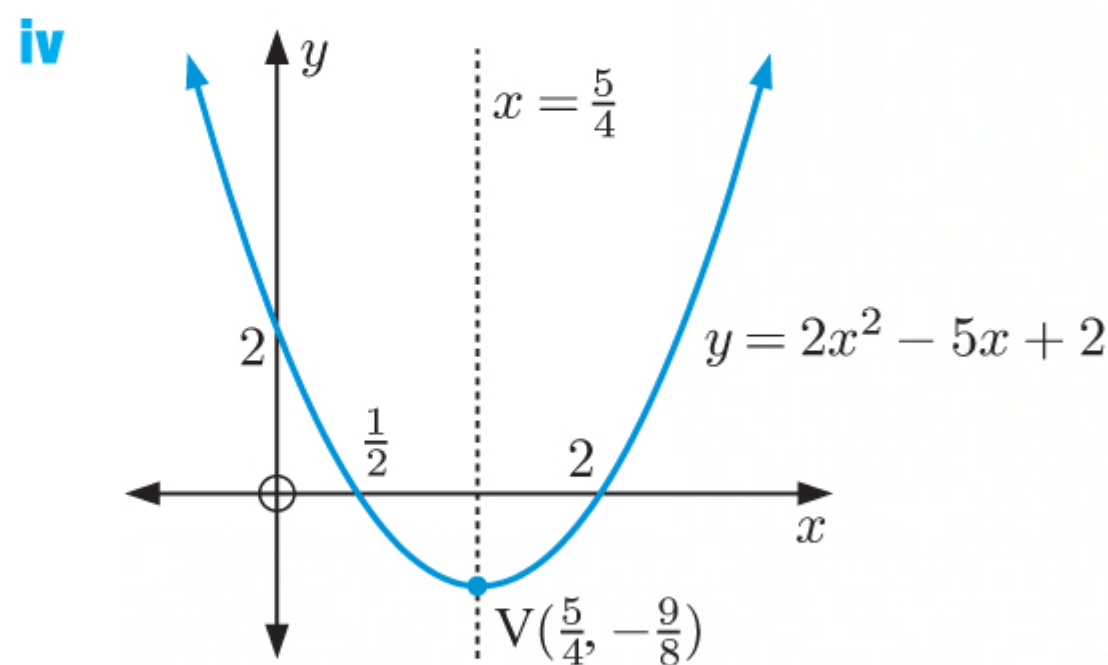
ii The axis of symmetry is halfway between the x -intercepts $\frac{5}{2}$ and $-\frac{1}{2}$.

So, the axis of symmetry is $x = 1$.

iii When $x = 1$,

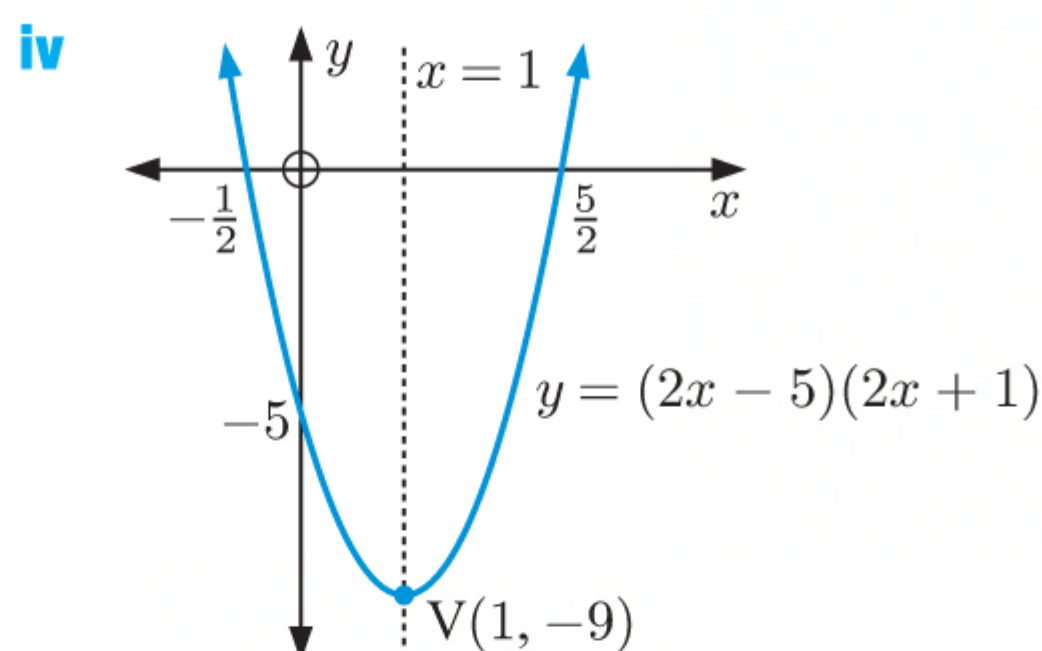
$$\begin{aligned}y &= (2(1) - 5)(2(1) + 1) \\ &= (-3)(3) \\ &= -9\end{aligned}$$

So, the vertex is $(1, -9)$.



v The domain is $\{x \mid x \in \mathbb{R}\}$.

The range is $\{y \mid y \geq -\frac{9}{8}\}$.



v The domain is $\{x \mid x \in \mathbb{R}\}$.

The range is $\{y \mid y \geq -9\}$.

i $y = -\frac{1}{4}x^2 + 2x - 3$ has $a = -\frac{1}{4}$, $b = 2$, and $c = -3$.

i When $x = 0$, $y = -3$
 \therefore the y -intercept is -3 .

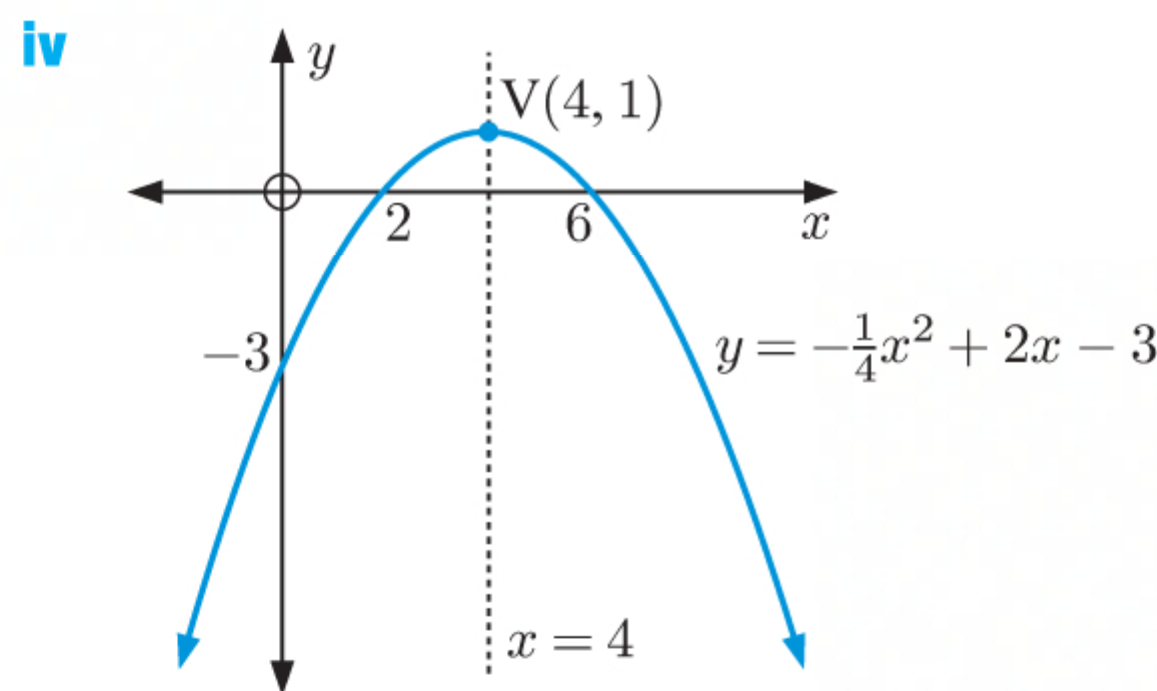
When $y = 0$,
 $-\frac{1}{4}x^2 + 2x - 3 = 0$
 Using technology, $x = 2$ or 6
 \therefore the x -intercepts are 2 and 6 .

ii $-\frac{b}{2a} = -\frac{2}{2(-\frac{1}{4})} = 4$

So, the axis of symmetry is $x = 4$.

iii When $x = 4$,
 $y = -\frac{1}{4}(4)^2 + 2(4) - 3$
 $= -4 + 8 - 3$
 $= 1$

So, the vertex is $(4, 1)$.



v The domain is $\{x \mid x \in \mathbb{R}\}$.
 The range is $\{y \mid y \leq 1\}$.

3 a $y = x^2 - 8x + 5$ has $a = 1$, $b = -8$, and $c = 5$.

Now $-\frac{b}{2a} = -\frac{-8}{2(1)} = 4$

\therefore the axis of symmetry is $x = 4$.

When $x = 4$, $y = 4^2 - 8(4) + 5$
 $= 16 - 32 + 5$
 $= -11$

So, the vertex is $(4, -11)$.

$a > 0$, so the vertex is a minimum turning point.

\therefore the minimum value of $y = x^2 - 8x + 5$ is $y = -11$ when $x = 4$.

b $y = -x^2 - 10x + 4$ has $a = -1$, $b = -10$, and $c = 4$.

Now $-\frac{b}{2a} = -\frac{-10}{2(-1)} = -5$

\therefore the axis of symmetry is $x = -5$.

When $x = -5$, $y = -(-5)^2 - 10(-5) + 4$
 $= -25 + 50 + 4$
 $= 29$

So, the vertex is $(-5, 29)$.

$a < 0$, so the vertex is a maximum turning point.

\therefore the maximum value of $y = -x^2 - 10x + 4$ is $y = 29$ when $x = -5$.

- c** $y = 3x^2 + 3x - 2$ has $a = 3$, $b = 3$, and $c = -2$.

$$\text{Now } -\frac{b}{2a} = -\frac{3}{2(3)} = -\frac{1}{2}$$

\therefore the axis of symmetry is $x = -\frac{1}{2}$.

$$\begin{aligned}\text{When } x = -\frac{1}{2}, \quad y &= 3\left(-\frac{1}{2}\right)^2 + 3\left(-\frac{1}{2}\right) - 2 \\ &= \frac{3}{4} - \frac{3}{2} - 2 \\ &= -\frac{11}{4}\end{aligned}$$

So, the vertex is $\left(-\frac{1}{2}, -\frac{11}{4}\right)$.

$a > 0$, so the vertex is a minimum turning point.

\therefore the minimum value of $y = 3x^2 + 3x - 2$ is $y = -\frac{11}{4}$ when $x = -\frac{1}{2}$.

- d** $y = -\frac{1}{2}x^2 + 7x - 4$ has $a = -\frac{1}{2}$, $b = 7$, and $c = -4$.

$$\text{Now } -\frac{b}{2a} = -\frac{7}{2\left(-\frac{1}{2}\right)} = 7$$

\therefore the axis of symmetry is $x = 7$.

$$\begin{aligned}\text{When } x = 7, \quad y &= -\frac{1}{2}(7)^2 + 7(7) - 4 \\ &= -\frac{49}{2} + 49 - 4 \\ &= \frac{41}{2}\end{aligned}$$

So, the vertex is $\left(7, \frac{41}{2}\right)$.

$a < 0$, so the vertex is a maximum turning point.

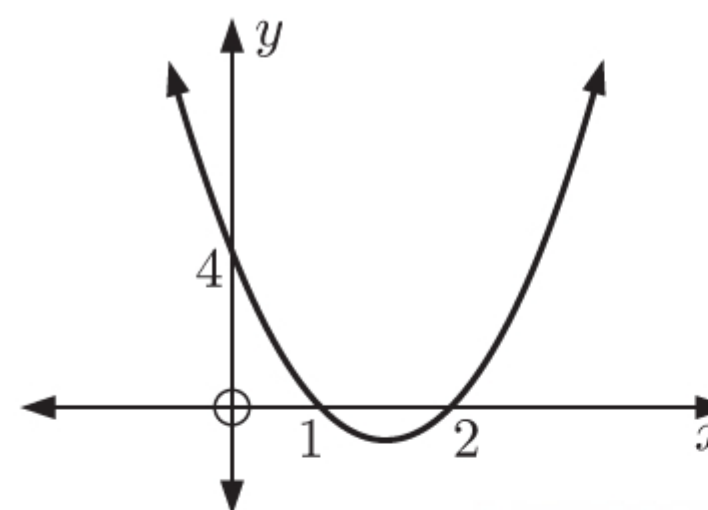
\therefore the maximum value of $y = -\frac{1}{2}x^2 + 7x - 4$ is $y = \frac{41}{2}$ when $x = 7$.

EXERCISE 6H

- 1 a** Since the x -intercepts are 1 and 2, $y = a(x-1)(x-2)$.

$$\begin{aligned}\text{When } x = 0, \quad y &= 4 \\ \therefore 4 &= a(-1)(-2) \\ \therefore a &= 2\end{aligned}$$

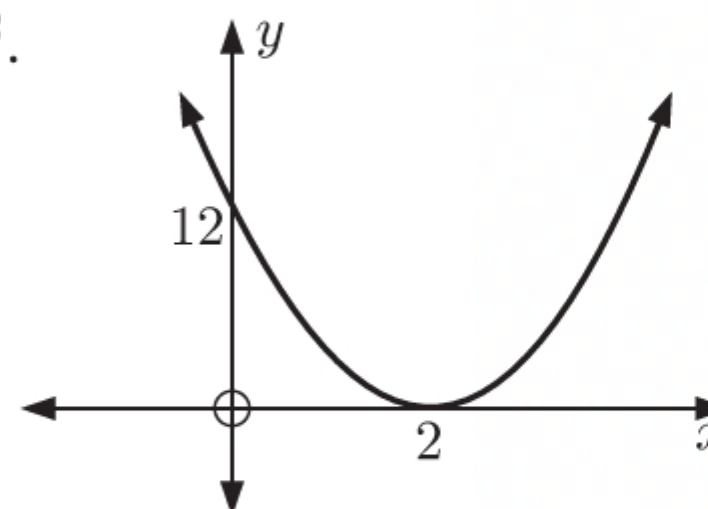
The quadratic is $y = 2(x-1)(x-2)$.



- b** The graph touches the x -axis at $x = 2$, so $y = a(x-2)^2$.

$$\begin{aligned}\text{When } x = 0, \quad y &= 12 \\ \therefore 12 &= a(-2)^2 \\ \therefore a &= 3\end{aligned}$$

The quadratic is $y = 3(x-2)^2$.



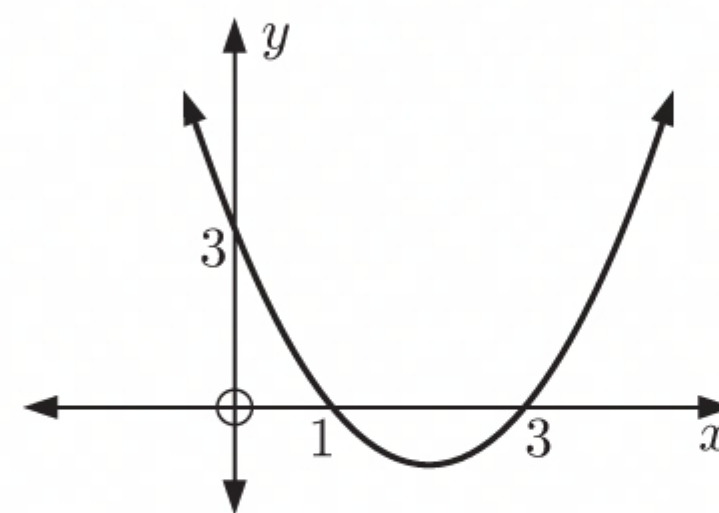
- c** Since the x -intercepts are 1 and 3, $y = a(x - 1)(x - 3)$.

When $x = 0$, $y = 3$

$$\therefore 3 = a(-1)(-3)$$

$$\therefore a = 1$$

The quadratic is $y = (x - 1)(x - 3)$.



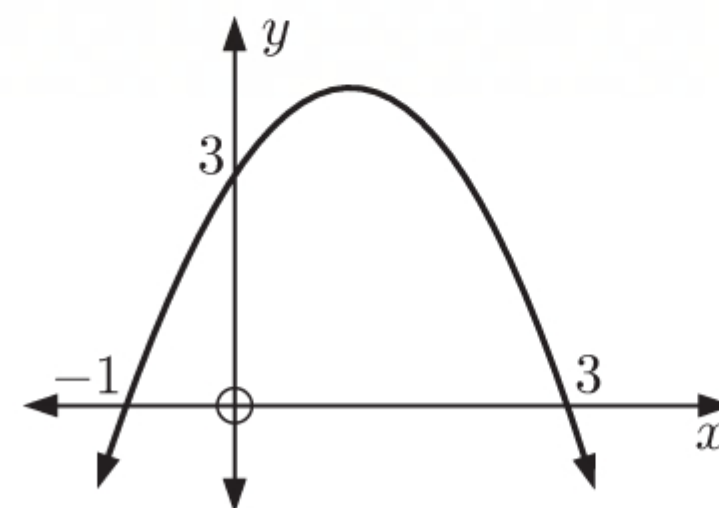
- d** Since the x -intercepts are -1 and 3 , $y = a(x + 1)(x - 3)$.

When $x = 0$, $y = 3$

$$\therefore 3 = a(1)(-3)$$

$$\therefore a = -1$$

The quadratic is $y = -(x + 1)(x - 3)$.



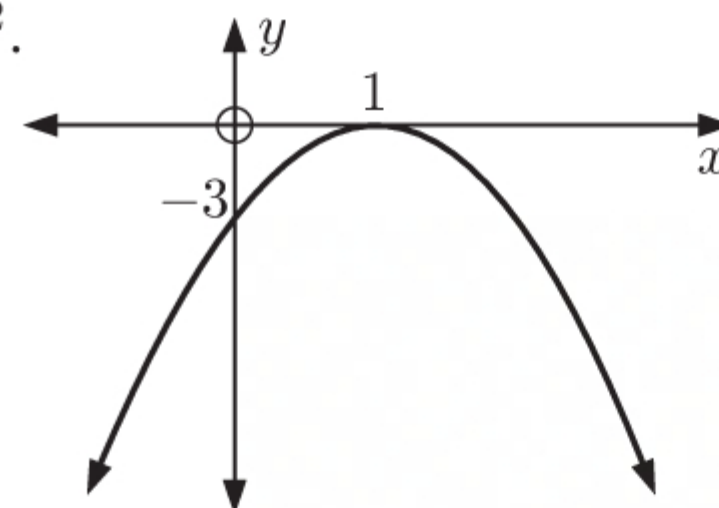
- e** The graph touches the x -axis at $x = 1$, so $y = a(x - 1)^2$.

When $x = 0$, $y = -3$

$$\therefore -3 = a(-1)^2$$

$$\therefore a = -3$$

The quadratic is $y = -3(x - 1)^2$.



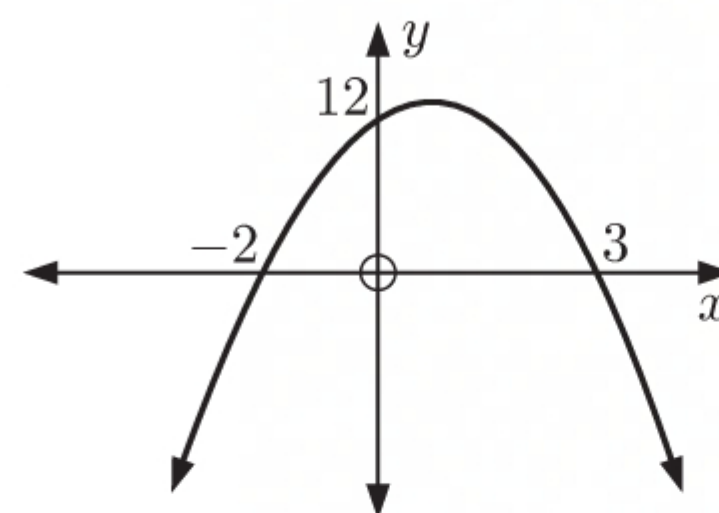
- f** Since the x -intercepts are -2 and 3 , $y = a(x + 2)(x - 3)$.

When $x = 0$, $y = 12$

$$\therefore 12 = a(2)(-3)$$

$$\therefore a = -2$$

The quadratic is $y = -2(x + 2)(x - 3)$.



- 2 a** The axis of symmetry $x = 3$ lies midway between the x -intercepts.

\therefore the other x -intercept is 4.

\therefore the quadratic has the form

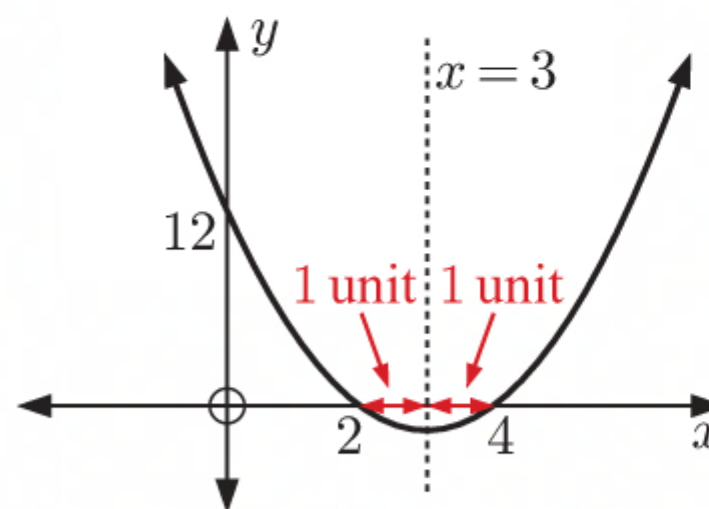
$$y = a(x - 2)(x - 4) \quad \text{where } a > 0$$

But when $x = 0$, $y = 12$

$$\therefore 12 = a(-2)(-4)$$

$$\therefore a = \frac{3}{2}$$

The quadratic is $y = \frac{3}{2}(x - 2)(x - 4)$.



- b** The axis of symmetry $x = -1$ lies midway between the x -intercepts.
 \therefore the other x -intercept is 2.
 \therefore the quadratic has the form

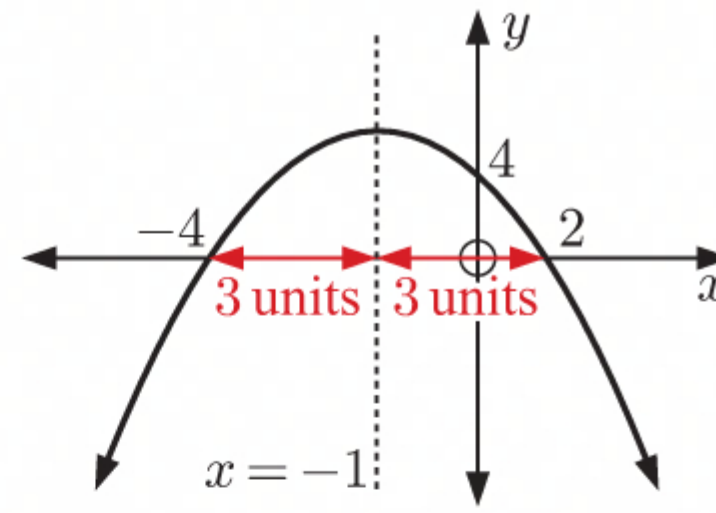
$$y = a(x + 4)(x - 2) \quad \text{where } a < 0$$

But when $x = 0$, $y = 4$

$$\therefore 4 = a(4)(-2)$$

$$\therefore a = -\frac{1}{2}$$

The quadratic is $y = -\frac{1}{2}(x + 4)(x - 2)$.



- c** The graph touches the x -axis at $x = -3$.
 \therefore the quadratic has the form

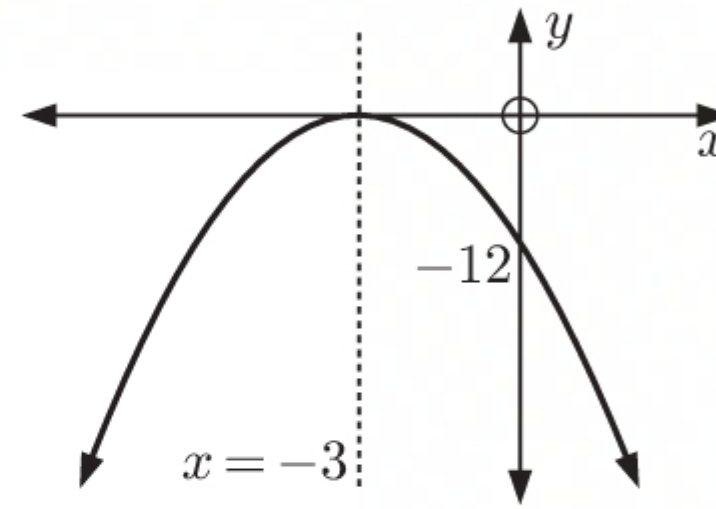
$$y = a(x + 3)^2 \quad \text{where } a < 0$$

But when $x = 0$, $y = -12$

$$\therefore -12 = a(3)^2$$

$$\therefore a = -\frac{4}{3}$$

The quadratic is $y = -\frac{4}{3}(x + 3)^2$.



- 3 a** Since the x -intercepts are 5 and 1, the quadratic has the form $y = a(x - 5)(x - 1)$, $a \neq 0$.

When $x = 2$, $y = -9$

$$\therefore -9 = a(2 - 5)(2 - 1)$$

$$\therefore -9 = a(-3)(1)$$

$$\therefore a = 3$$

$$\begin{aligned} \text{The quadratic is } y &= 3(x - 5)(x - 1) \\ &= 3(x^2 - 6x + 5) \\ &= 3x^2 - 18x + 15 \end{aligned}$$

- b** Since the x -intercepts are 2 and $-\frac{1}{2}$, the quadratic has the form $y = a(x - 2)(2x + 1)$, $a \neq 0$.

When $x = 3$, $y = -14$

$$\therefore -14 = a(3 - 2)(2(3) + 1)$$

$$\therefore -14 = a(1)(7)$$

$$\therefore a = -2$$

$$\begin{aligned} \text{The quadratic is } y &= -2(x - 2)(2x + 1) \\ &= -2(2x^2 - 3x - 2) \\ &= -4x^2 + 6x + 4 \end{aligned}$$

- c** Since the graph touches the x -axis at 3, the quadratic has the form $y = a(x - 3)^2$, $a \neq 0$.

When $x = -2$, $y = -25$

$$\therefore -25 = a(-2 - 3)^2$$

$$\therefore -25 = a(-5)^2$$

$$\therefore a = -1$$

$$\begin{aligned} \text{The quadratic is } y &= -(x - 3)^2 \\ &= -(x^2 - 6x + 9) \\ &= -x^2 + 6x - 9 \end{aligned}$$

- d** Since the graph touches the x -axis at -2 , the quadratic has the form $y = a(x + 2)^2$, $a \neq 0$.

When $x = -1$, $y = 4$

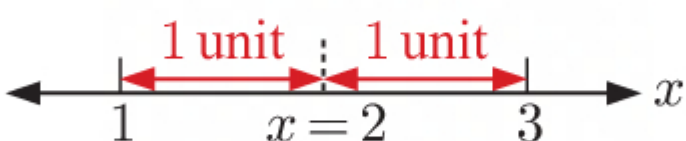
$$\therefore 4 = a(-1 + 2)^2$$

$$\therefore 4 = a(1)^2$$

$$\therefore a = 4$$

$$\begin{aligned}\text{The quadratic is } y &= 4(x + 2)^2 \\ &= 4(x^2 + 4x + 4) \\ &= 4x^2 + 16x + 16\end{aligned}$$

- e** The axis of symmetry $x = 2$ lies midway between the x -intercepts.

\therefore the other x -intercept is 1. 

Since the x -intercepts are 3 and 1, the quadratic has the form $y = a(x - 3)(x - 1)$, $a \neq 0$.

When $x = 5$, $y = 12$

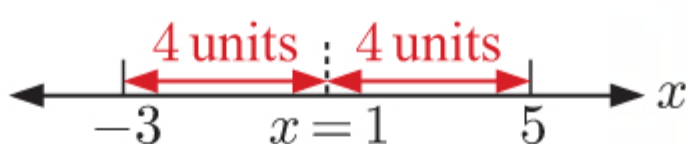
$$\therefore 12 = a(5 - 3)(5 - 1)$$

$$\therefore 12 = a(2)(4)$$

$$\therefore a = \frac{3}{2}$$

$$\begin{aligned}\text{The quadratic is } y &= \frac{3}{2}(x - 3)(x - 1) \\ &= \frac{3}{2}(x^2 - 4x + 3) \\ &= \frac{3}{2}x^2 - 6x + \frac{9}{2}\end{aligned}$$

- f** The axis of symmetry $x = 1$ lies midway between the x -intercepts.

\therefore the other x -intercept is -3 . 

Since the x -intercepts are 5 and -3 , the quadratic has the form $y = a(x - 5)(x + 3)$, $a \neq 0$.

When $x = 2$, $y = 5$

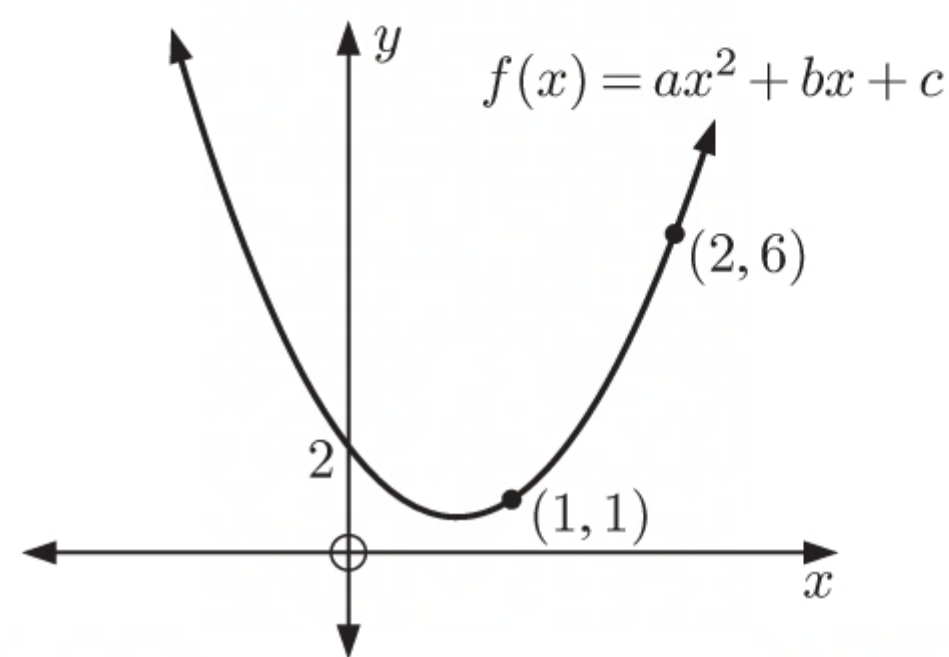
$$\therefore 5 = a(2 - 5)(2 + 3)$$

$$\therefore 5 = a(-3)(5)$$

$$\therefore a = -\frac{1}{3}$$

$$\begin{aligned}\text{The quadratic is } y &= -\frac{1}{3}(x - 5)(x + 3) \\ &= -\frac{1}{3}(x^2 - 2x - 15) \\ &= -\frac{1}{3}x^2 + \frac{2}{3}x + 5\end{aligned}$$

- 4 a The y -intercept of $y = f(x)$ is 2 $\therefore c = 2$.



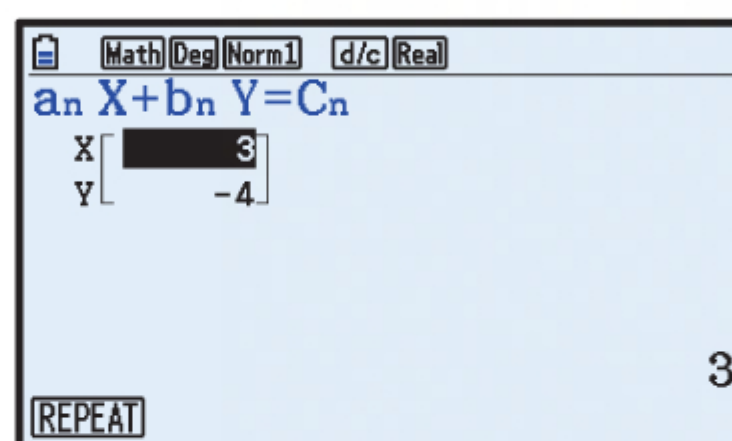
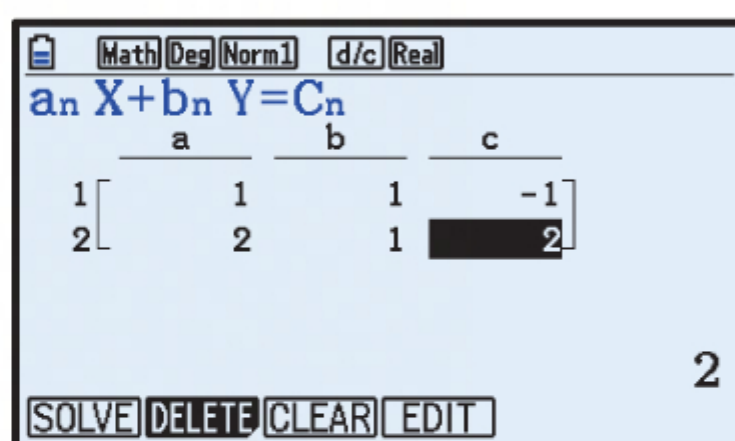
b $f(x) = ax^2 + bx + 2$

$$f(1) = 1 \quad \therefore 1 = a(1)^2 + b(1) + 2 \quad \text{or} \quad a + b = -1$$

$$f(2) = 6 \quad \therefore 6 = a(2)^2 + b(2) + 2 \quad \text{or} \quad 4a + 2b = 4$$

$$\therefore 2a + b = 2$$

- c We solve the system of equations $\begin{cases} a + b = -1 \\ 2a + b = 2 \end{cases}$ simultaneously using technology.



We find that $a = 3$ and $b = -4$.

So, the function is $f(x) = 3x^2 - 4x + 2$.

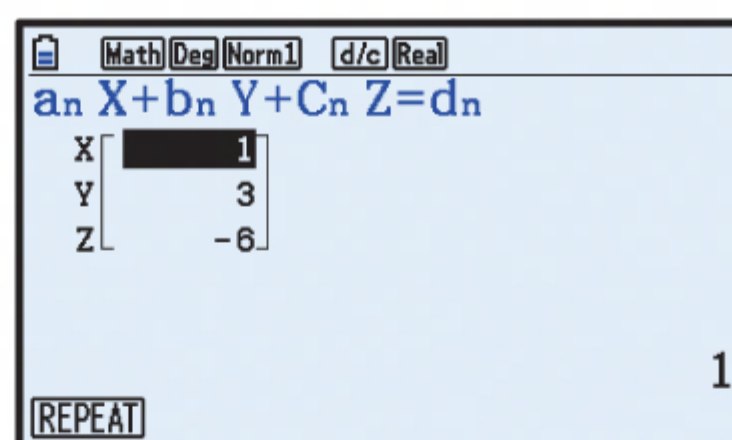
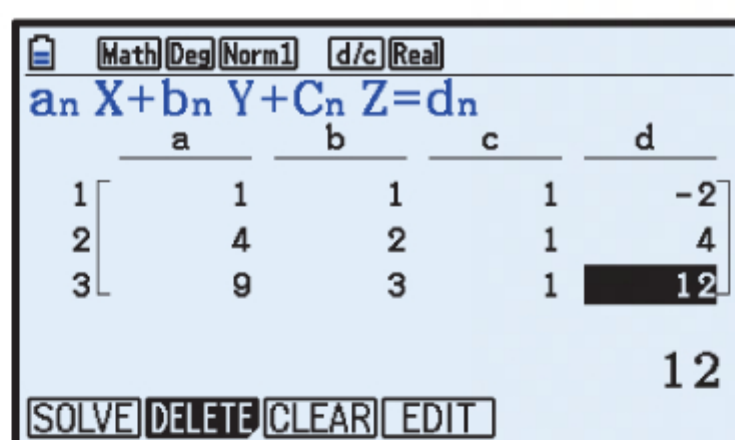
- 5 a Let the quadratic function be $y = ax^2 + bx + c$.

$$\text{When } x = 1, y = -2 \quad \therefore -2 = a(1)^2 + b(1) + c \quad \text{or} \quad a + b + c = -2$$

$$\text{When } x = 2, y = 4 \quad \therefore 4 = a(2)^2 + b(2) + c \quad \text{or} \quad 4a + 2b + c = 4$$

$$\text{When } x = 3, y = 12 \quad \therefore 12 = a(3)^2 + b(3) + c \quad \text{or} \quad 9a + 3b + c = 12$$

- We solve the system of equations $\begin{cases} a + b + c = -2 \\ 4a + 2b + c = 4 \\ 9a + 3b + c = 12 \end{cases}$ simultaneously using technology.



We find that $a = 1$, $b = 3$, and $c = -6$.

So, the function is $y = x^2 + 3x - 6$.

- b** Let the quadratic function be $y = ax^2 + bx + c$.

When $x = -1$, $y = 3$ $\therefore 3 = a(-1)^2 + b(-1) + c$ or $a - b + c = 3$

When $x = 2$, $y = 9$ $\therefore 9 = a(2)^2 + b(2) + c$ or $4a + 2b + c = 9$

When $x = 4$, $y = -7$ $\therefore -7 = a(4)^2 + b(4) + c$ or $16a + 4b + c = -7$

We solve the system of equations
$$\begin{cases} a - b + c = 3 \\ 4a + 2b + c = 9 \\ 16a + 4b + c = -7 \end{cases}$$
 simultaneously using technology.

	a	b	c	d
1	1	-1	1	3
2	4	2	1	9
3	16	4	1	-7

-7

SOLVE DELETE CLEAR EDIT

	X	Y	Z
1	-2	4	9

-2

REPEAT

We find that $a = -2$, $b = 4$, and $c = 9$.

So, the function is $y = -2x^2 + 4x + 9$.

- c** Let the quadratic function be $y = ax^2 + bx + c$.

When $x = -8$, $y = 4$ $\therefore 4 = a(-8)^2 + b(-8) + c$ or $64a - 8b + c = 4$

When $x = -4$, $y = -8$ $\therefore -8 = a(-4)^2 + b(-4) + c$ or $16a - 4b + c = -8$

When $x = 6$, $y = 32$ $\therefore 32 = a(6)^2 + b(6) + c$ or $36a + 6b + c = 32$

We solve the system of equations
$$\begin{cases} 64a - 8b + c = 4 \\ 16a - 4b + c = -8 \\ 36a + 6b + c = 32 \end{cases}$$
 simultaneously using technology.

	a	b	c	d
1	64	-8	1	4
2	16	-4	1	-8
3	36	6	1	32

32

SOLVE DELETE CLEAR EDIT

	X	Y	Z
1	0.5	3	-4

$\frac{1}{2}$

REPEAT

We find that $a = \frac{1}{2}$, $b = 3$, and $c = -4$.

So, the function is $y = \frac{1}{2}x^2 + 3x - 4$.

- 6 Let the quadratic function be $y = ax^2 + bx + c$.

When $x = -1$, $y = 7$ $\therefore 7 = a(-1)^2 + b(-1) + c$ or $a - b + c = 7$

When $x = 2$, $y = 1$ $\therefore 1 = a(2)^2 + b(2) + c$ or $4a + 2b + c = 1$

When $x = 3$, $y = -1$ $\therefore -1 = a(3)^2 + b(3) + c$ or $9a + 3b + c = -1$

We solve the system of equations
$$\begin{cases} a - b + c = 7 \\ 4a + 2b + c = 1 \\ 9a + 3b + c = -1 \end{cases}$$
 simultaneously using technology.

	a	b	c	d
1	1	-1	1	7
2	4	2	1	1
3	9	3	1	-1

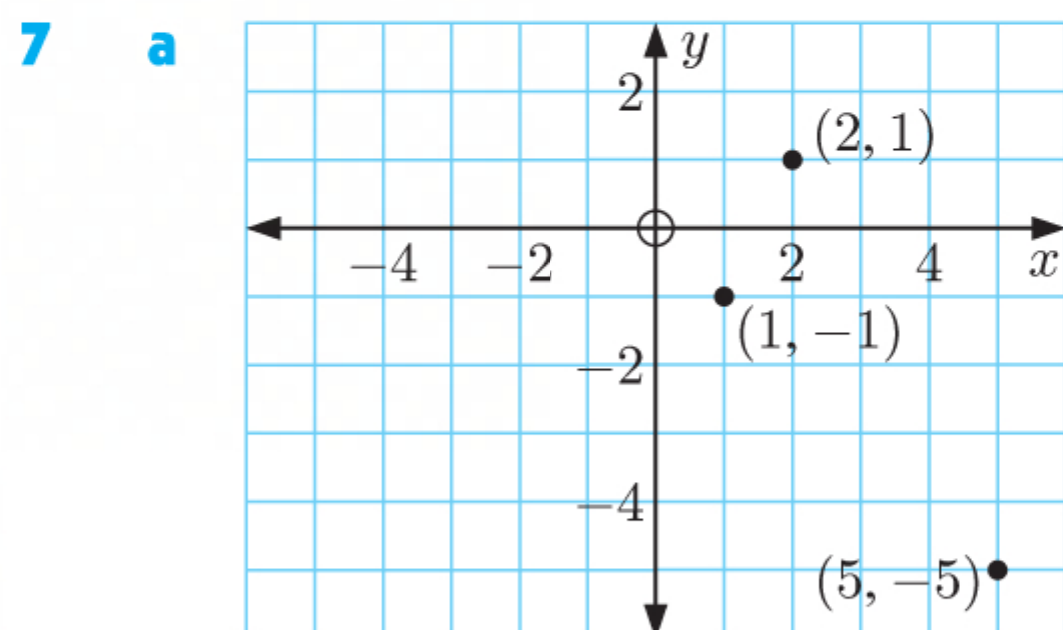
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	X	Y	Z
1	0	-2	5

REPEAT

We find that $a = 0$, $b = -2$, and $c = 5$.

So, the three points lie on the line $y = -2x + 5$, but not on a quadratic.



- b i The graph of the quadratic through these points must open downwards, so $a < 0$.
 ii The y -intercept is below the x -axis, so $c < 0$.
 iii The vertex has x -coordinate $-\frac{b}{2a}$ and is > 0 . Since $a < 0$, $b > 0$.

c When $x = 1$, $y = -1$ $\therefore -1 = a(1)^2 + b(1) + c$ or $a + b + c = -1$

When $x = 2$, $y = 1$ $\therefore 1 = a(2)^2 + b(2) + c$ or $4a + 2b + c = 1$

When $x = 5$, $y = -5$ $\therefore -5 = a(5)^2 + b(5) + c$ or $25a + 5b + c = -5$

We solve the system of equations
$$\begin{cases} a + b + c = -1 \\ 4a + 2b + c = 1 \\ 25a + 5b + c = -5 \end{cases}$$
 simultaneously using technology.

	a	b	c	d
1	1	1	1	-1
2	4	2	1	1
3	25	5	1	-5

SOLVE DELETE CLEAR EDIT

	X	Y	Z
1	-1	5	-5

REPEAT

We find that $a = -1$, $b = 5$, and $c = -5$.

- d** $y = -x^2 + 5x - 5$ has $a = -1$, $b = 5$, and $c = -5$.

$$\text{Now } -\frac{b}{2a} = -\frac{5}{2(-1)} = \frac{5}{2}$$

\therefore the axis of symmetry is $x = \frac{5}{2}$.

$$\begin{aligned} \text{When } x = \frac{5}{2}, \quad y &= -\left(\frac{5}{2}\right)^2 + 5\left(\frac{5}{2}\right) - 5 \\ &= -\frac{25}{4} + \frac{25}{2} - 5 \\ &= \frac{5}{4} \end{aligned}$$

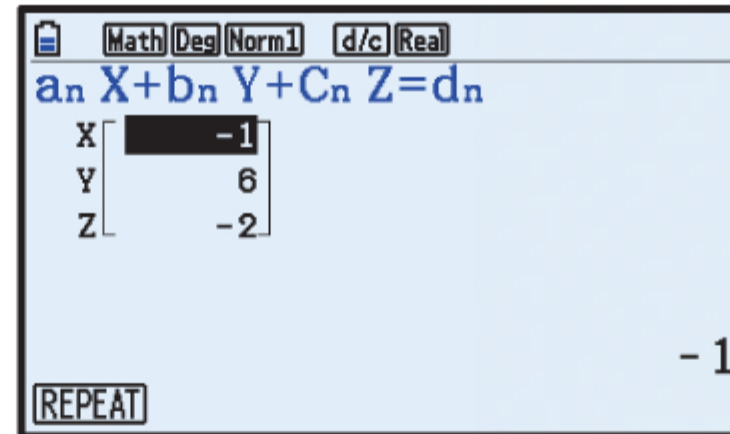
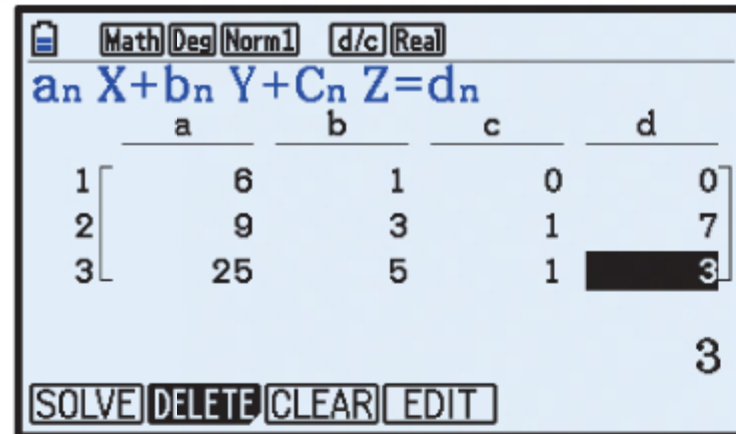
So, the vertex has coordinates $\left(\frac{5}{2}, \frac{5}{4}\right)$.

- 8 a** The vertex has x -coordinate 3 $\therefore -\frac{b}{2a} = 3$
 $\therefore -b = 6a$
 $\therefore 6a + b = 0$

$$\text{When } x = 3, \quad y = 7 \quad \therefore 7 = a(3)^2 + b(3) + c \quad \text{or} \quad 9a + 3b + c = 7$$

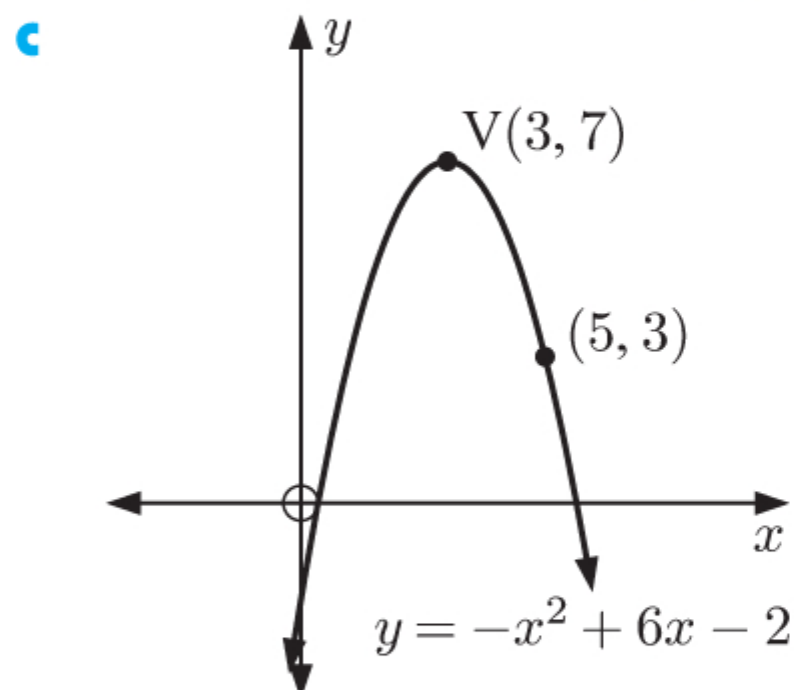
$$\text{When } x = 5, \quad y = 3 \quad \therefore 3 = a(5)^2 + b(5) + c \quad \text{or} \quad 25a + 5b + c = 3$$

- b** We solve the system of equations
$$\begin{cases} 6a + b = 0 \\ 9a + 3b + c = 7 \\ 25a + 5b + c = 3 \end{cases}$$
 simultaneously using technology.



We find that $a = -1$, $b = 6$, and $c = -2$.

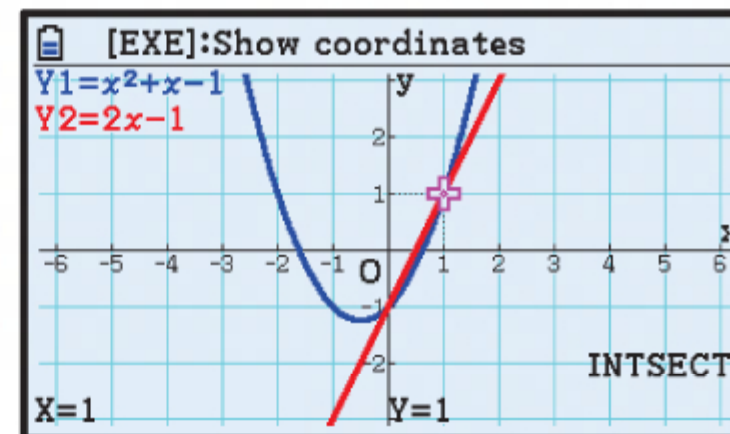
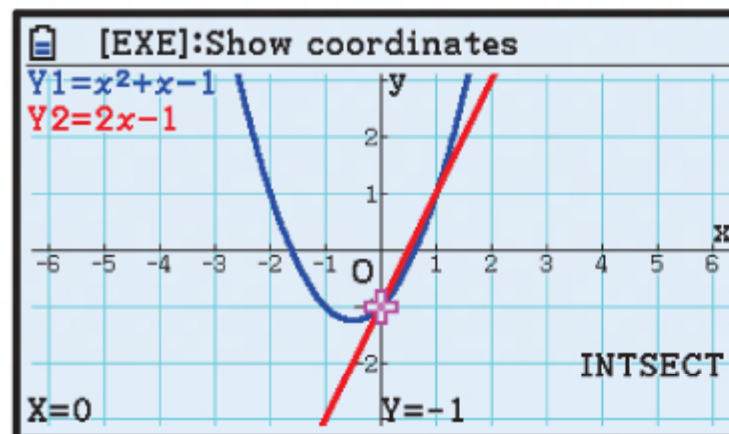
So, the function is $f(x) = -x^2 + 6x - 2$.



EXERCISE 6I

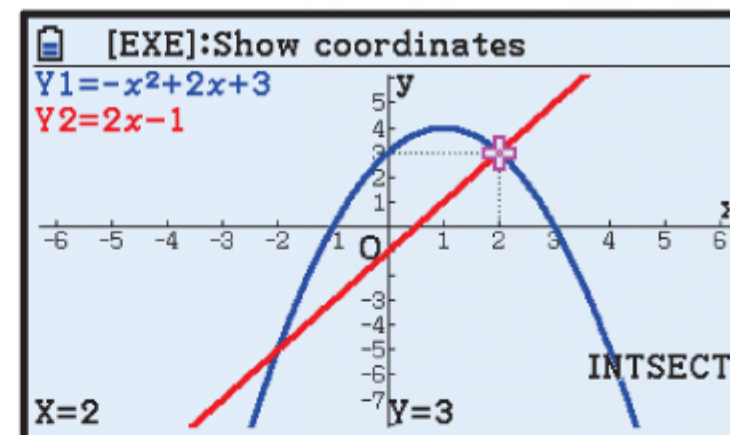
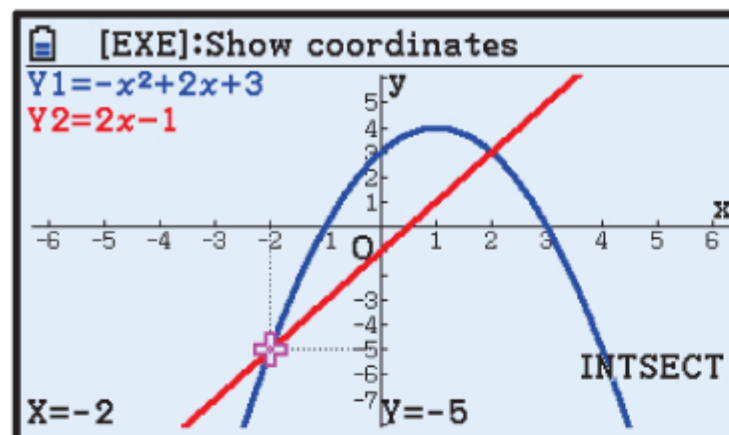
- 1 a We graph $Y_1 = X^2 + X - 1$ and $Y_2 = 2X - 1$ on the same set of axes.

The graphs intersect at $(0, -1)$ and $(1, 1)$.



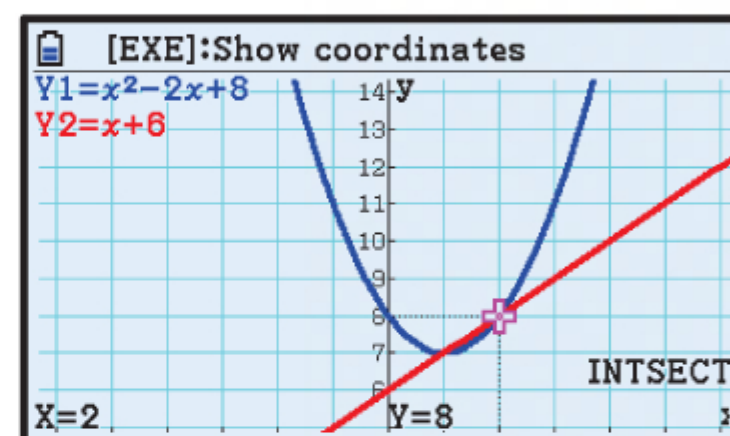
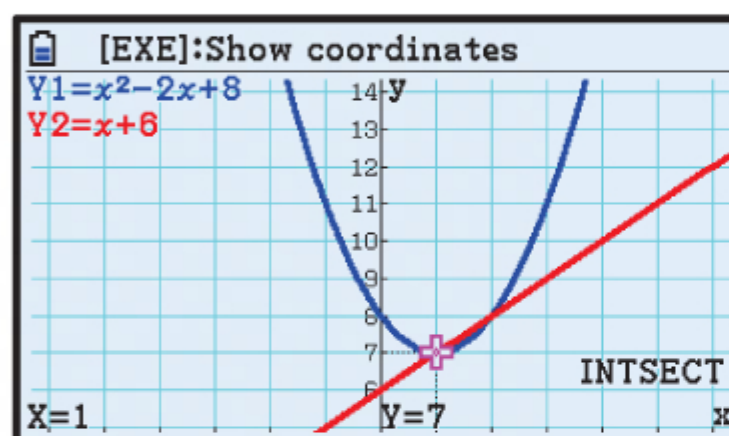
- b We graph $Y_1 = -X^2 + 2X + 3$ and $Y_2 = 2X - 1$ on the same set of axes.

The graphs intersect at $(-2, -5)$ and $(2, 3)$.



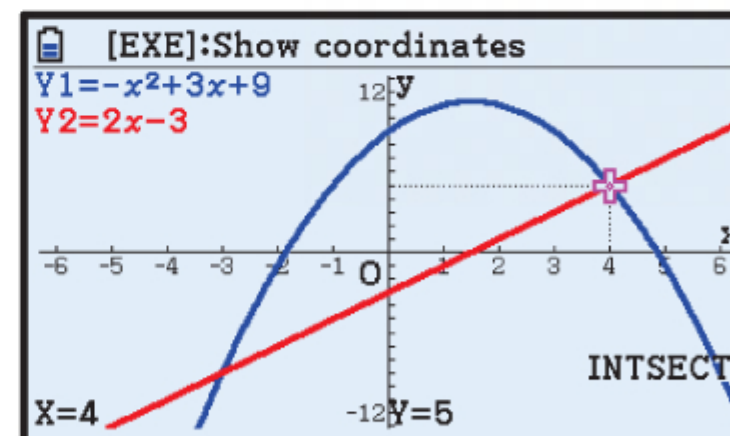
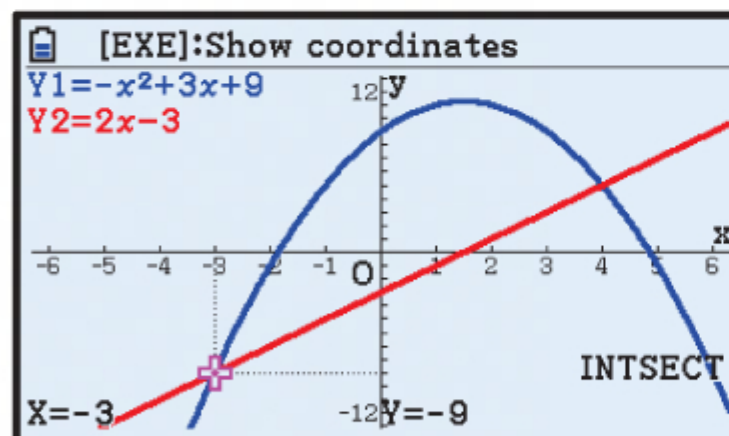
- c We graph $Y_1 = X^2 - 2X + 8$ and $Y_2 = X + 6$ on the same set of axes.

The graphs intersect at $(1, 7)$ and $(2, 8)$.



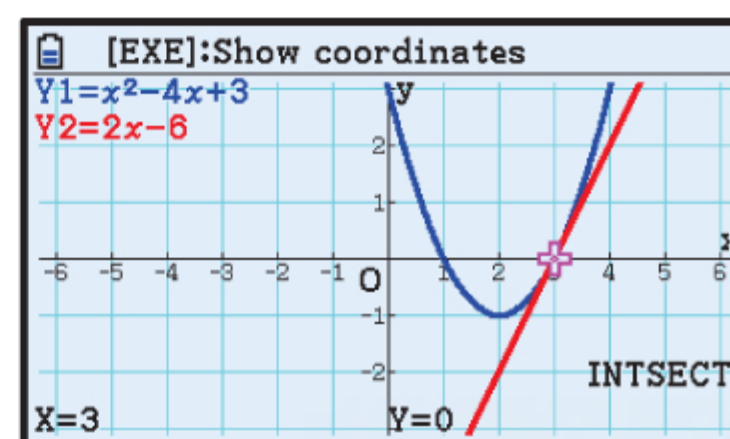
- d We graph $Y_1 = -X^2 + 3X + 9$ and $Y_2 = 2X - 3$ on the same set of axes.

The graphs intersect at $(-3, -9)$ and $(4, 5)$.



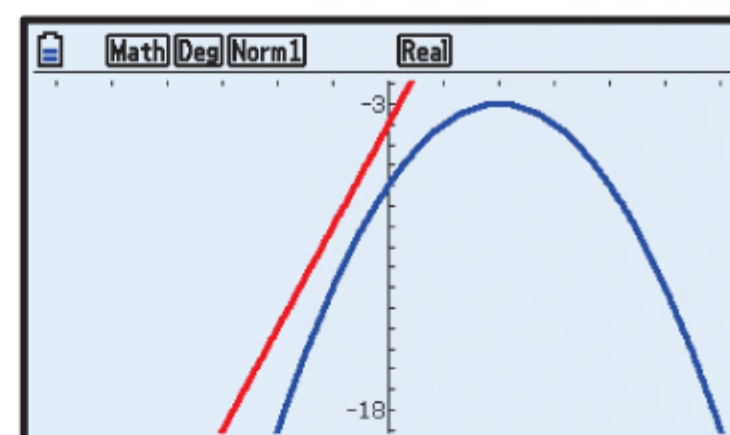
- e We graph $Y_1 = X^2 - 4X + 3$ and $Y_2 = 2X - 6$ on the same set of axes.

The graphs intersect at $(3, 0)$.



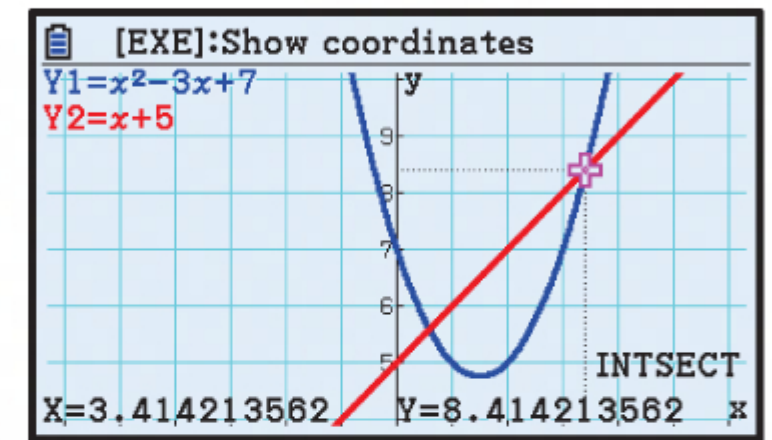
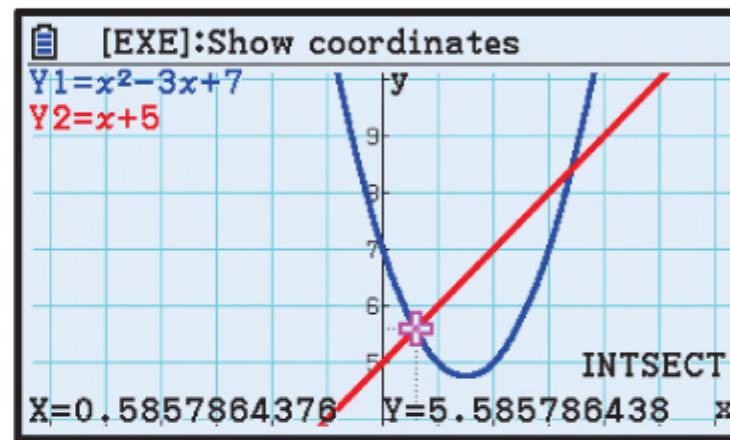
- f We graph $Y_1 = -X^2 + 4X - 7$ and $Y_2 = 5X - 4$ on the same set of axes.

The graphs do not intersect.



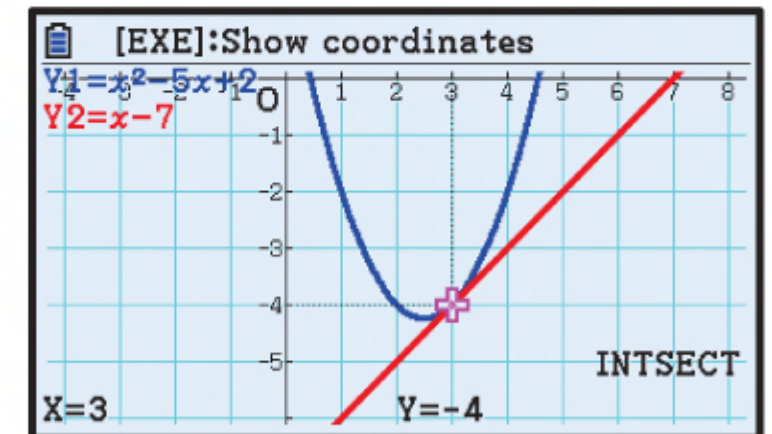
- 2 a** We graph
 $Y_1 = X^2 - 3X + 7$ and
 $Y_2 = X + 5$ on the same set of axes.

The graphs intersect at
 $(0.586, 5.59)$ and
 $(3.41, 8.41)$.



- b** We graph $Y_1 = X^2 - 5X + 2$ and $Y_2 = X - 7$ on the same set of axes.

The graphs intersect at $(3.00, -4.00)$.



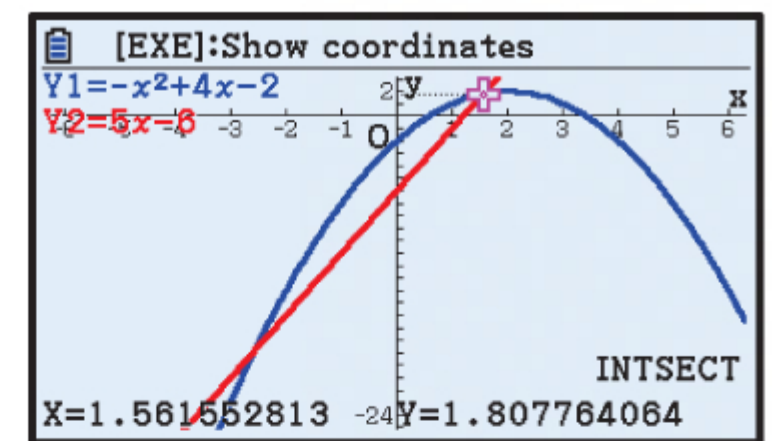
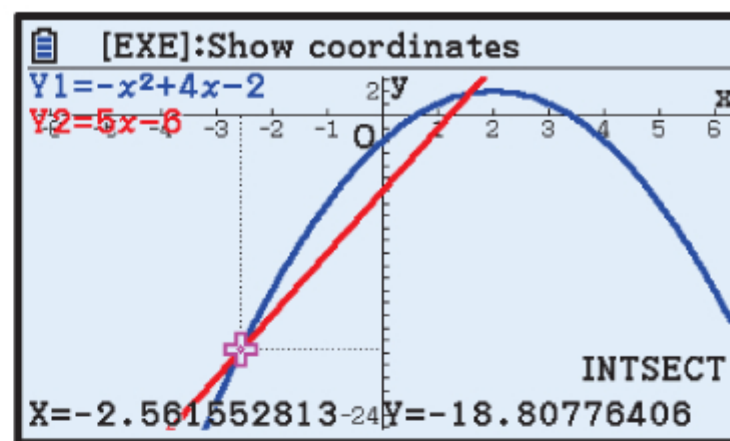
- c** We graph $Y_1 = -X^2 - 2X + 4$ and $Y_2 = X + 8$ on the same set of axes.

The graphs do not intersect.



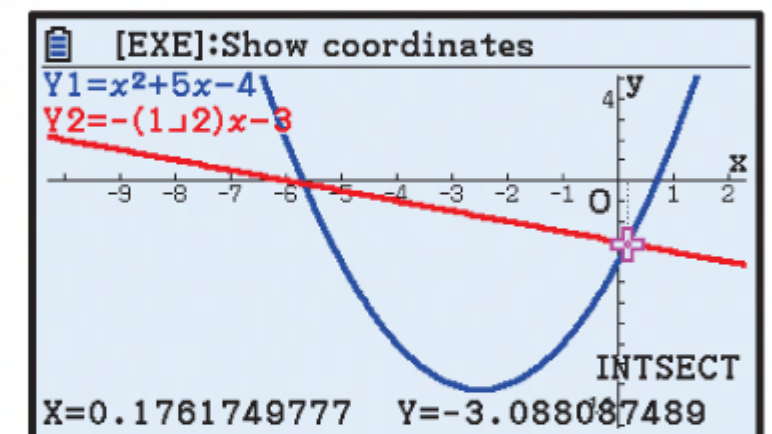
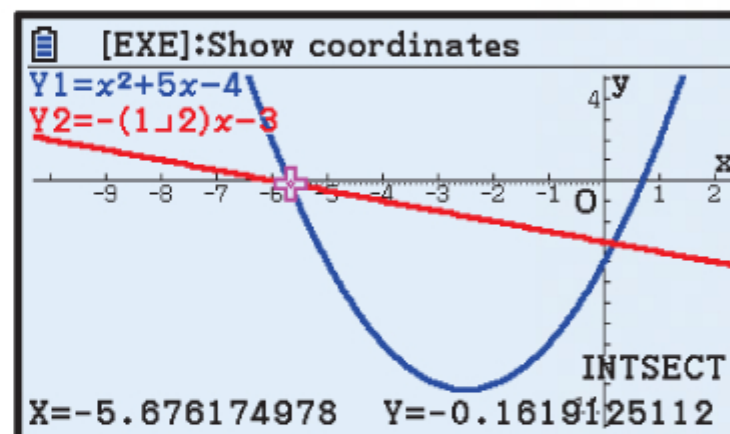
- d** We graph
 $Y_1 = -X^2 + 4X - 2$ and
 $Y_2 = 5X - 6$ on the same set of axes.

The graphs intersect at
 $(-2.56, -18.8)$ and
 $(1.56, 1.81)$.



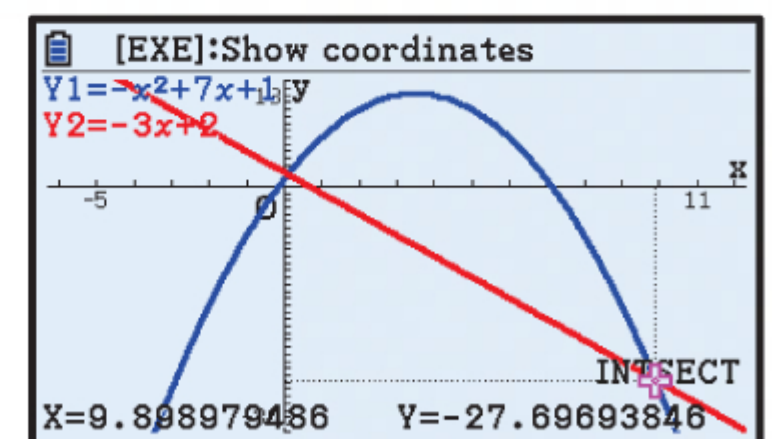
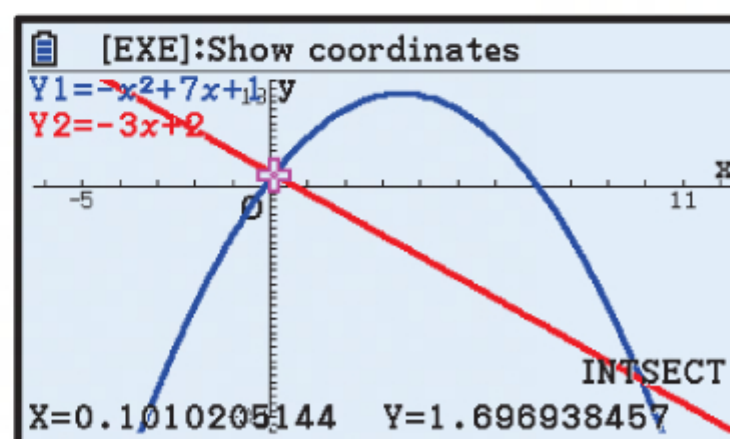
- e** We graph
 $Y_1 = X^2 + 5X - 4$ and
 $Y_2 = -\frac{1}{2}X - 3$ on the same set of axes.

The graphs intersect at
 $(-5.68, -0.162)$ and
 $(0.176, -3.09)$.



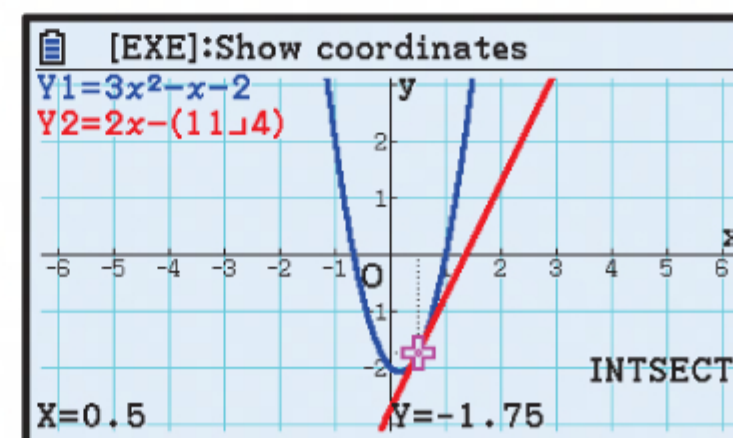
- f** We graph
 $Y_1 = -X^2 + 7X + 1$ and
 $Y_2 = -3X + 2$ on the same set of axes.

The graphs intersect at
 $(0.101, 1.70)$ and
 $(9.90, -27.7)$.



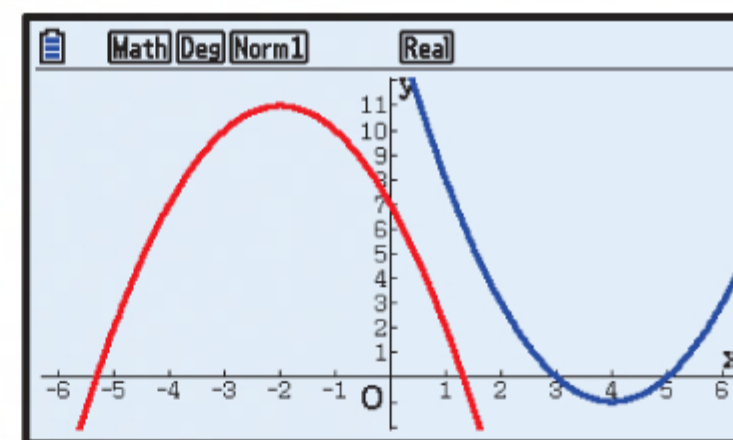
- g** We graph $Y_1 = 3X^2 - X - 2$ and $Y_2 = 2X - \frac{11}{4}$ on the same set of axes.

The graphs intersect at $(0.500, -1.75)$.



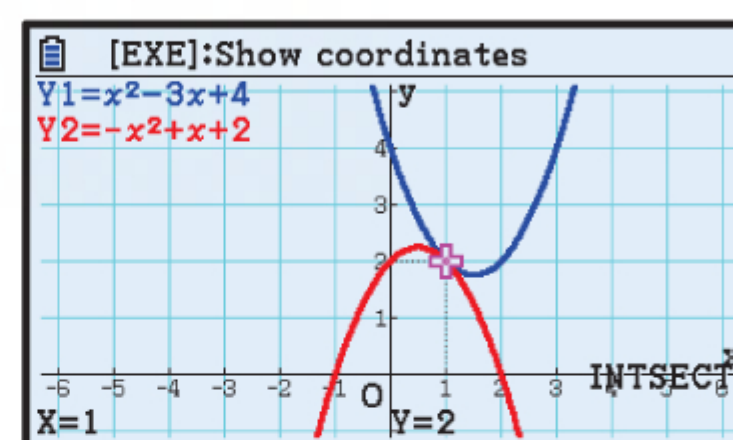
- 3 a** We graph $Y_1 = X^2 - 8X + 15$ and $Y_2 = -X^2 - 4X + 7$ on the same set of axes.

The graphs do not intersect.



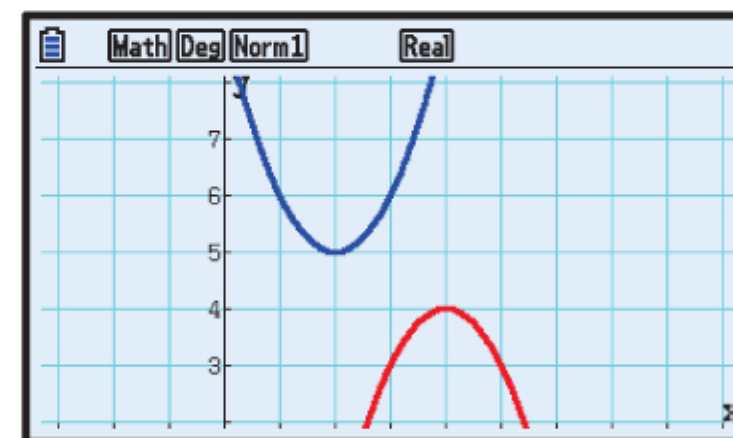
- b** We graph $Y_1 = X^2 - 3X + 4$ and $Y_2 = -X^2 + X + 2$ on the same set of axes.

The graphs intersect at $(1.00, 2.00)$.



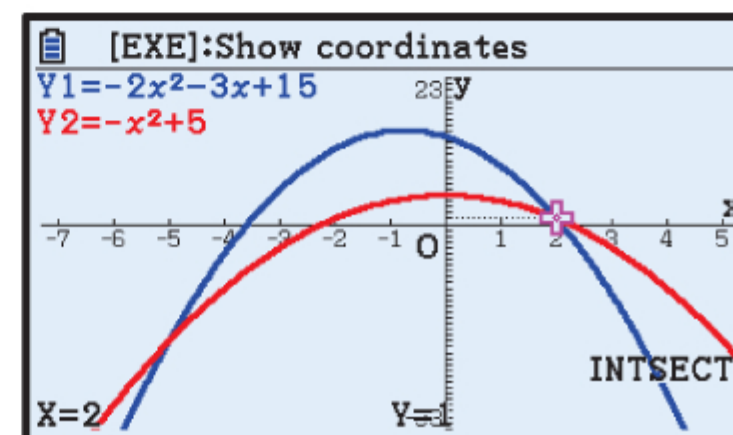
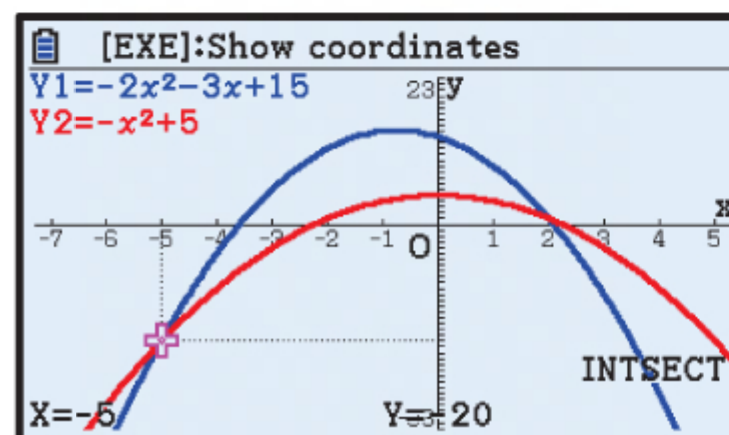
- c** We graph $Y_1 = X^2 - 4X + 9$ and $Y_2 = -X^2 + 8X - 12$ on the same set of axes.

The graphs do not intersect.



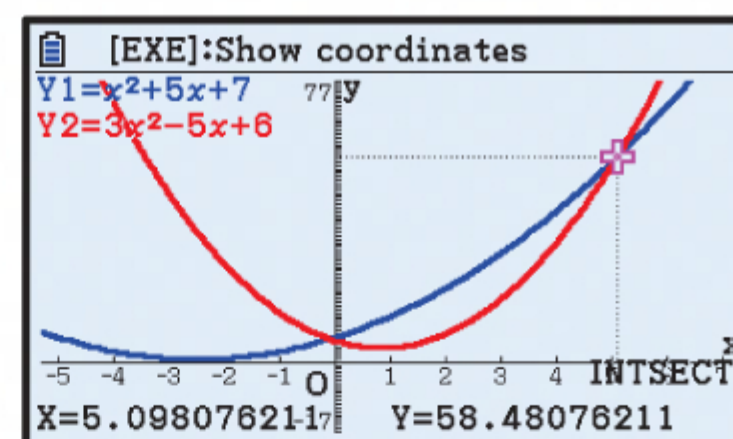
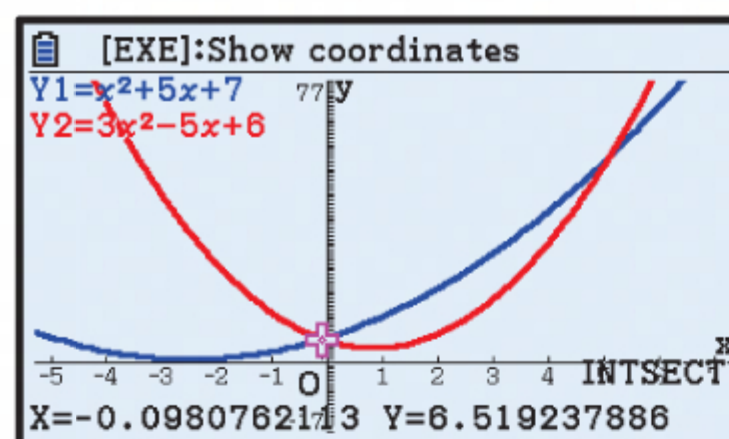
- d** We graph $Y_1 = -2X^2 - 3X + 15$ and $Y_2 = -X^2 + 5$ on the same set of axes.

The graphs intersect at $(-5.00, -20.0)$ and $(2.00, 1.00)$.



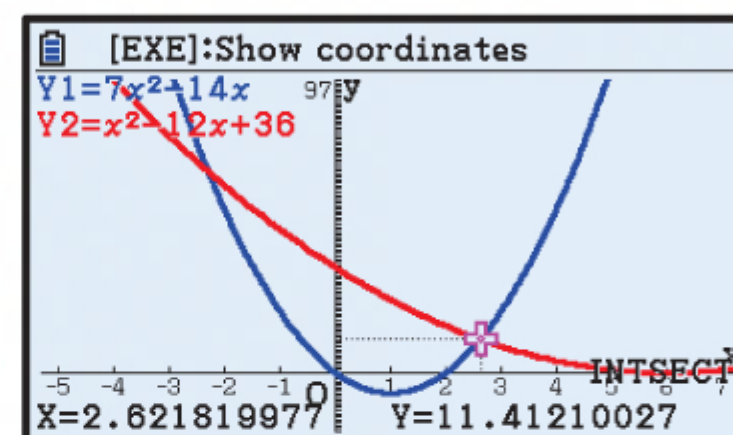
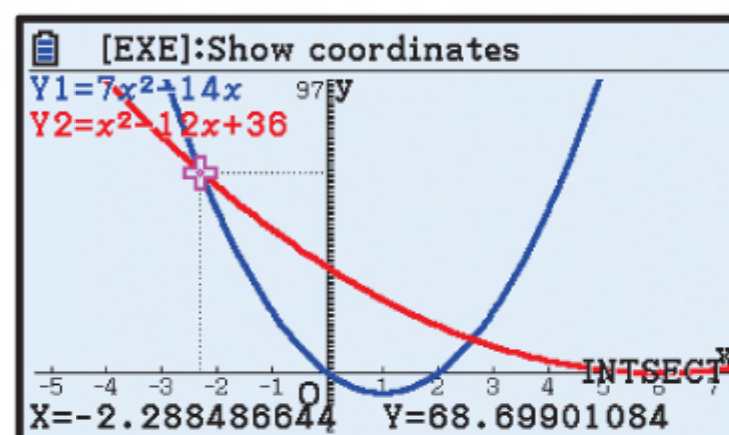
- e** We graph $Y_1 = X^2 + 5X + 7$ and $Y_2 = 3X^2 - 5X + 6$ on the same set of axes.

The graphs intersect at $(-0.0981, 6.52)$ and $(5.10, 58.5)$.



- f** We graph $Y_1 = 7X^2 - 14X$ and $Y_2 = X^2 - 12X + 36$ on the same set of axes.

The graphs intersect at $(-2.29, 68.7)$ and $(2.62, 11.4)$.




EXERCISE 6J

1 $H(t) = -4t^2 + 4t + 3$ metres, $t \geq 0$

a $H(0) = -4(0)^2 + 4(0) + 3$
 $= 3$

\therefore the pier is 3 m above the water.

b For the quadratic function $H(t)$, $a = -4$, $b = 4$, and $c = 3$.

Since $a < 0$, the shape is .

The maximum height occurs when $t = -\frac{b}{2a} = -\frac{4}{2(-4)} = \frac{1}{2}$

So, it takes Andrew 0.5 seconds to reach the maximum height of his dive.

c $H(\frac{1}{2}) = -4(\frac{1}{2})^2 + 4(\frac{1}{2}) + 3$
 $= -1 + 2 + 3$
 $= 4$

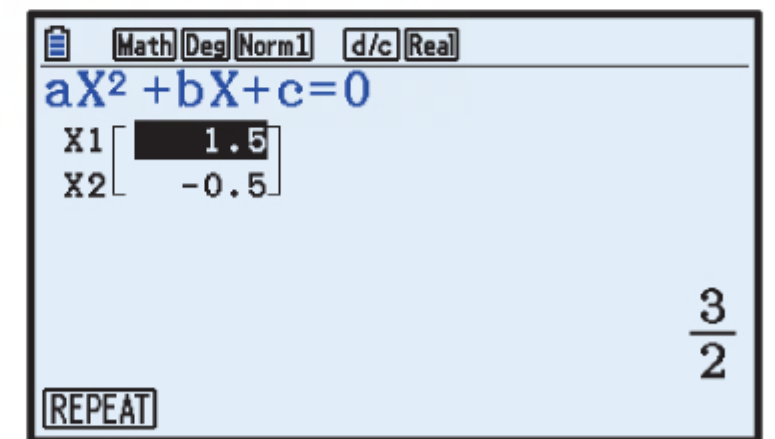
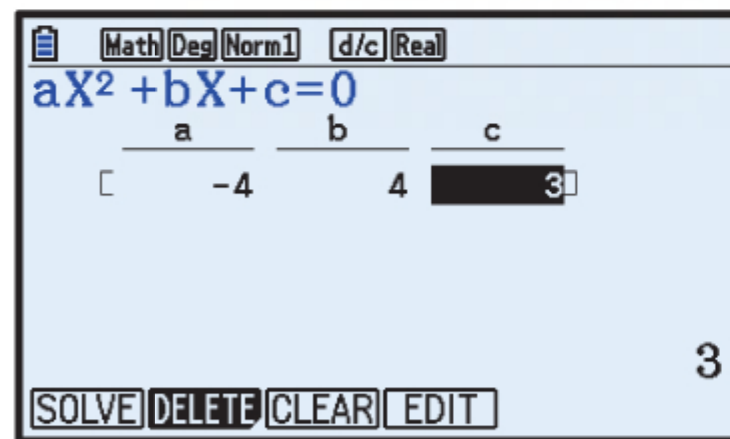
So, Andrew is 4 m above the water at his highest point.

d Andrew hits the water when

$$H(t) = 0$$

$$\therefore -4t^2 + 4t + 3 = 0$$

$$\therefore t = -\frac{1}{2} \text{ or } \frac{3}{2}$$




Since t must be positive, Andrew hits the water after 1.5 seconds.

2 $P(x) = -x^2 + 20x$ dollars, $0 \leq x \leq 25$

a $P(7) = -(7)^2 + 20(7)$
 $= -49 + 140$
 $= 91$

\therefore Jasmine makes a profit of \$91 if she makes 7 necklaces per day.

b For the quadratic function $P(x)$, $a = -1$, $b = 20$, and $c = 0$.

Since $a < 0$, the shape is .

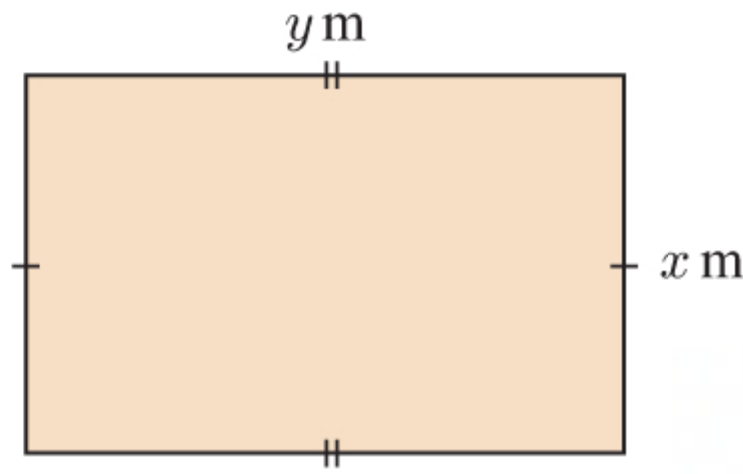
The maximum profit occurs when $x = -\frac{b}{2a} = -\frac{20}{2(-1)} = 10$

So, Jasmine should make 10 necklaces per day to maximise her profit.

c $P(10) = -(10)^2 + 20(10)$
 $= -100 + 200$
 $= 100$

So, the maximum daily profit that Jasmine can make is \$100.

- 3 a Let the other side be y m long.



The perimeter is 200 m.

$$\therefore 2x + 2y = 200$$

$$\therefore x + y = 100$$

$$\therefore y = 100 - x$$

The area $A = xy$

$$\therefore A = x(100 - x)$$

$$\therefore A = 100x - x^2$$

- 4 Let the dimensions of the paddock be x m \times y m.

If 1000 m of fence is available, then

$$2x + y = 1000 \quad \{\text{perimeter}\}$$

$$\therefore y = 1000 - 2x$$

The area of the enclosure $A = xy$

$$\begin{aligned} \text{Since } y = 1000 - 2x, \quad A &= x(1000 - 2x) \\ &= 1000x - 2x^2 \end{aligned}$$

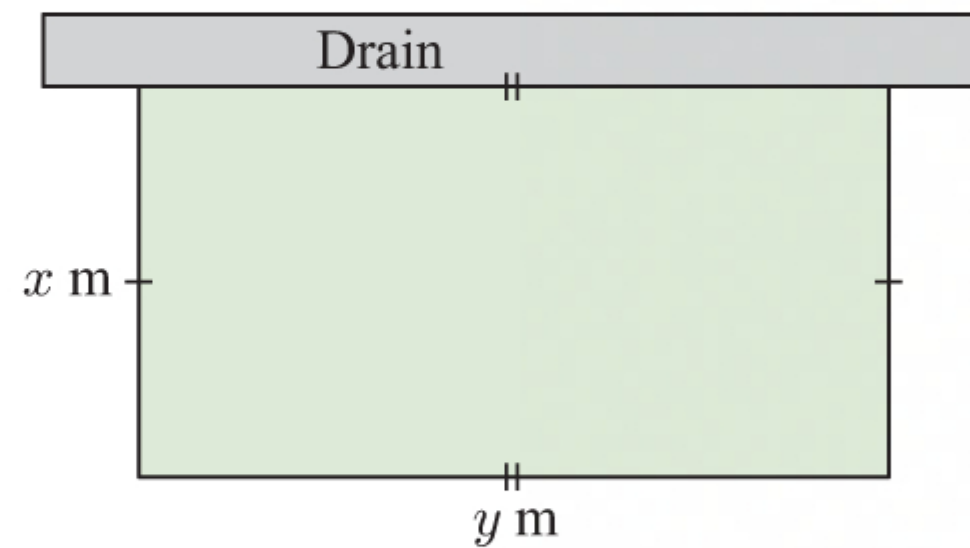
$$\therefore A = -2x^2 + 1000x$$

A is a quadratic and $a < 0$, so its shape is

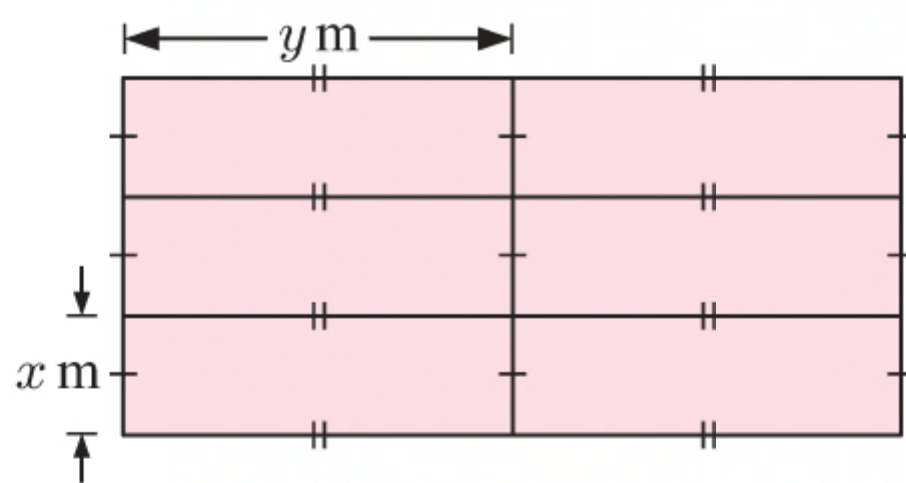
So, the area is maximised when $x = -\frac{b}{2a} = -\frac{1000}{2(-2)} = 250$

and when $x = 250$, $y = 1000 - 2(250) = 500$

So, the paddock has a maximum area when the dimensions are 250 m \times 500 m.



- 5 a



The length of fence required for this enclosure is $9x + 8y$. If 1800 m is available for this enclosure, then $9x + 8y = 1800$.

- b If $9x + 8y = 1800$, then $y = \frac{1800 - 9x}{8}$.


The area of each pen is $A = xy$.

Substituting $y = \frac{1800 - 9x}{8}$ into A we get

$$A = x \left(\frac{1800 - 9x}{8} \right)$$

$$\therefore A = \frac{1800x}{8} - \frac{9x^2}{8}$$

$$\therefore A = -\frac{9}{8}x^2 + 225x \text{ m}^2$$

- c The area A is a quadratic with $a < 0$, so its shape is 

So, the area is maximised when $x = -\frac{b}{2a} = -\frac{225}{2(-\frac{9}{8})} = 100$

$$\text{and when } x = 100, \quad y = \frac{1800 - 9(100)}{8} = 112.5$$

Hence, the area is maximised when the dimensions are $100 \text{ m} \times 112.5 \text{ m}$.

- 6 a Let $x \text{ m} \times y \text{ m}$ be the dimensions of a single pen as shown below.

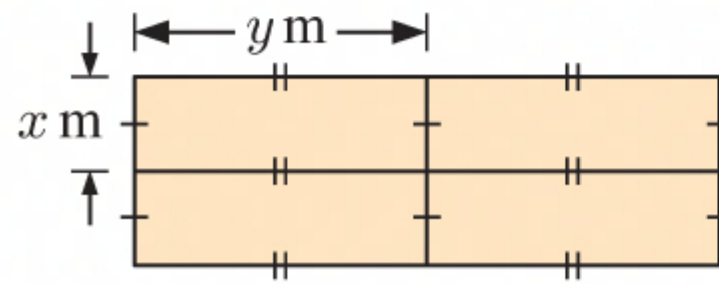
So, the total length of fencing required is $6x + 6y$.

If there is 500 m of fencing available, then

$$6x + 6y = 500$$

$$\therefore x + y = 83\frac{1}{3}$$


$$\therefore y = 83\frac{1}{3} - x \quad \dots (1)$$



The area of each pen will be $A = xy$ and substituting equation (1), we have

$$A = x(83\frac{1}{3} - x)$$

$$\therefore A = -x^2 + 83\frac{1}{3}x$$

which is a quadratic with $a < 0$, so its shape is 

So, A is maximised when $x = -\frac{b}{2a} = -\frac{83\frac{1}{3}}{2(-1)} = 41\frac{2}{3}$

$$\text{and when } x = 41\frac{2}{3}, \quad y = 83\frac{1}{3} - 41\frac{2}{3} = 41\frac{2}{3}$$

So, the dimensions that maximise the area are $41\frac{2}{3} \text{ m} \times 41\frac{2}{3} \text{ m}$.

- b Let $x \text{ m} \times y \text{ m}$ be the dimensions of a single pen as shown below.

So, the total length of fencing required is $5x + 8y$.

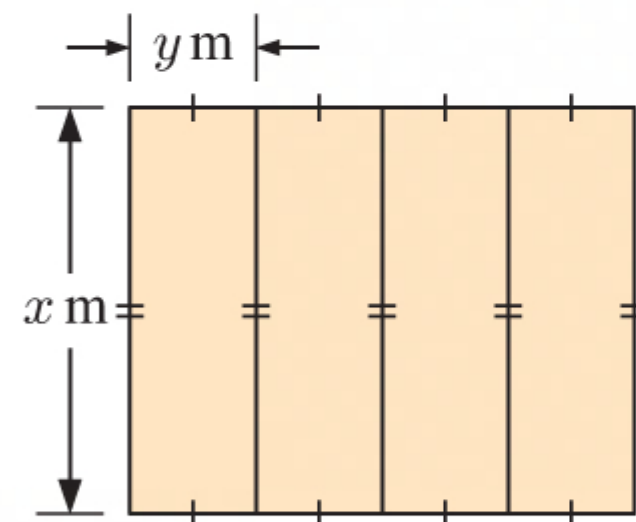
If there is 500 m of fencing available, then

$$5x + 8y = 500$$

$$\therefore 8y = 500 - 5x$$

$$\therefore y = \frac{500 - 5x}{8}$$

$$\therefore y = 62\frac{1}{2} - \frac{5}{8}x \quad \dots (1)$$



The area of each pen will be $A = xy$ and substituting equation (1), we have

$$A = x(62\frac{1}{2} - \frac{5}{8}x)$$

$\therefore A = -\frac{5}{8}x^2 + 62\frac{1}{2}x$ which is a quadratic with $a < 0$, so its shape is 

So, A is maximised when $x = -\frac{b}{2a} = -\frac{62\frac{1}{2}}{2(-\frac{5}{8})} = 50$

$$\text{and when } x = 50, \quad y = 62\frac{1}{2} - \frac{5}{8}(50) = 31\frac{1}{4}$$

So, the dimensions that maximise the area are $50 \text{ m} \times 31\frac{1}{4} \text{ m}$.

7 $H(x) = -0.015x^2 + x + 1.7$

- a** $H(x)$ is a quadratic function with $a < 0$.

\therefore the graph of $H(x)$ is a parabola that opens downwards.


b $H(0) = -0.015(0)^2 + 0 + 1.7$
 $= 1.7$

\therefore the javelin was released 1.7 m above the ground.

c $H(20) = -0.015(20)^2 + 20 + 1.7$
 $= -6 + 20 + 1.7$
 $= 15.7$

\therefore the javelin is 15.7 m high after travelling 20 m horizontally.

- d** For the quadratic function $H(x)$, $a = -0.015$, $b = 1$, and $c = 1.7$.

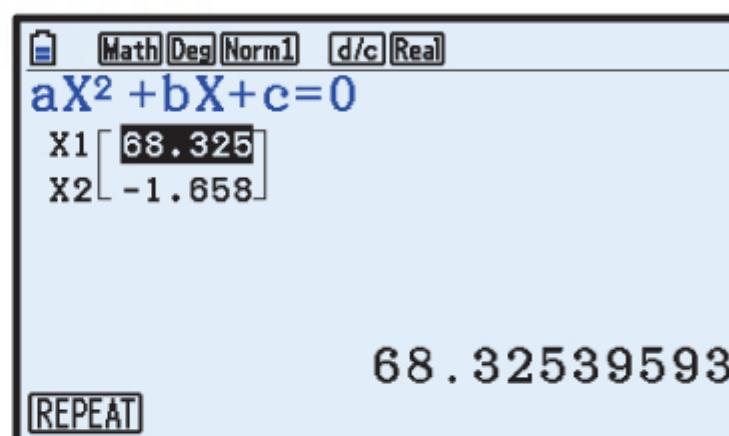
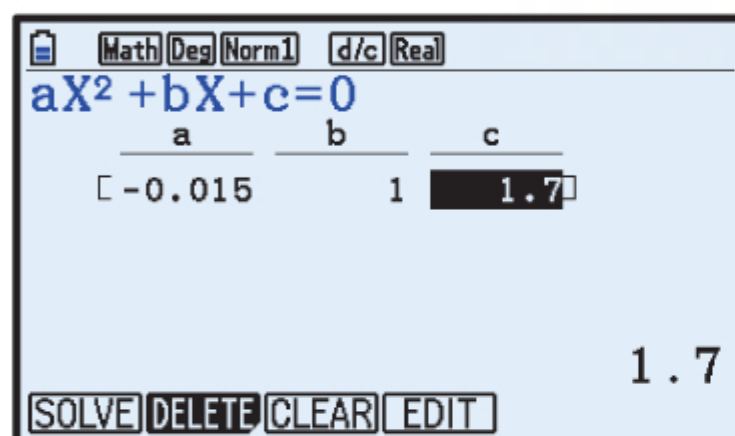
Since $a < 0$, the shape is .

The maximum height occurs when $x = -\frac{b}{2a} = -\frac{1}{2(-0.015)} = \frac{100}{3}$

Now $H(\frac{100}{3}) = -0.015(\frac{100}{3})^2 + \frac{100}{3} + 1.7$
 $= -\frac{50}{3} + \frac{100}{3} + 1.7$
 ≈ 18.4

\therefore the maximum height reached by the javelin is about 18.4 m.

- e** The javelin hits the ground when $H(x) = 0$
 $\therefore -0.015x^2 + x + 1.7 = 0$



$\therefore x \approx -1.66$ or 68.3

Since x must be positive, the javelin hits the ground after travelling about 68.3 m horizontally.

\therefore it is reasonable to use this model for $0 \leq x \leq 68.3$.

- f** From **e**, the javelin travels about 68.3 m before hitting the ground.

\therefore the distance recorded for the throw $\approx 68.3 - 3.4$
 ≈ 64.9 m

8 a i When $x = 0$, $y = 8$
 $\therefore 8 = a(0)^2 + b(0) + c$
 $\therefore c = 8$

ii The axis of symmetry is halfway between the x -intercepts -3 and 3 .

So, the axis of symmetry is $x = 0$.

$$\therefore -\frac{b}{2a} = 0$$

$$\therefore b = 0$$

iii When $x = -3$, $y = 0$
 $\therefore 0 = a(-3)^2 + 0(-3) + 8$
 $\therefore 0 = 9a + 8$
 $\therefore 9a = -8$
 $\therefore a = -\frac{8}{9}$

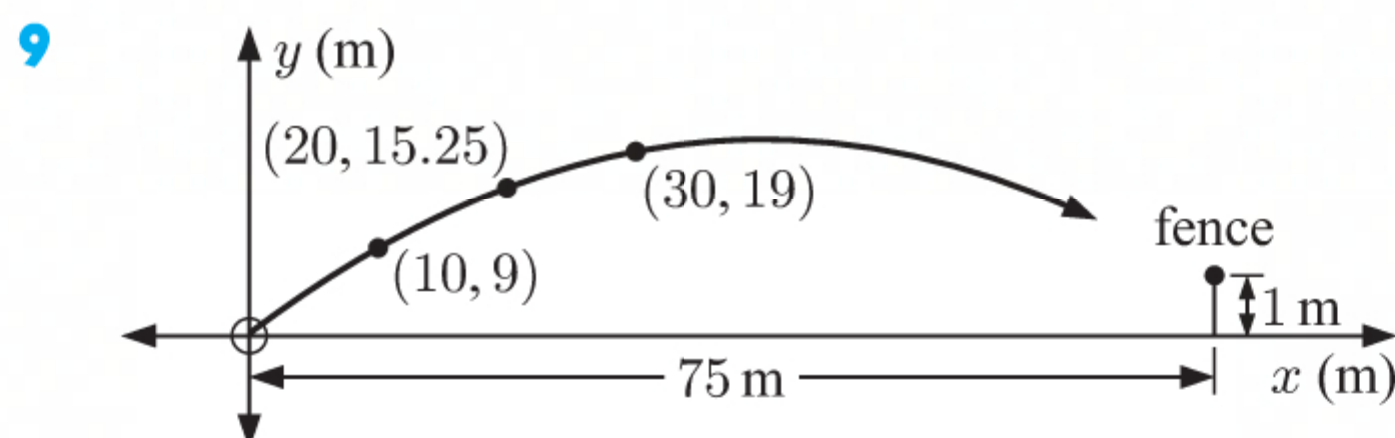
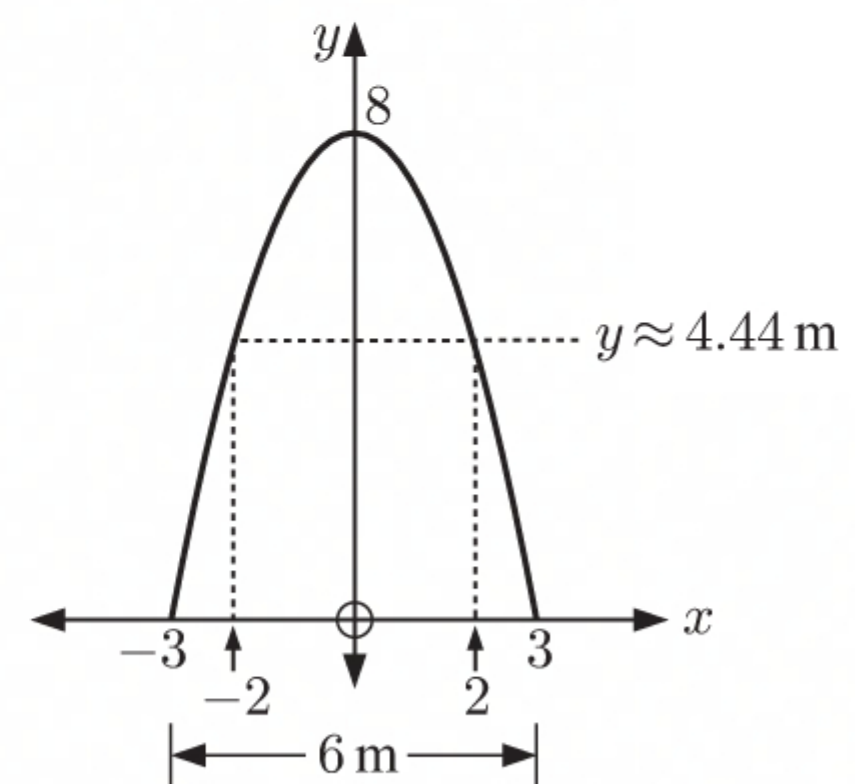
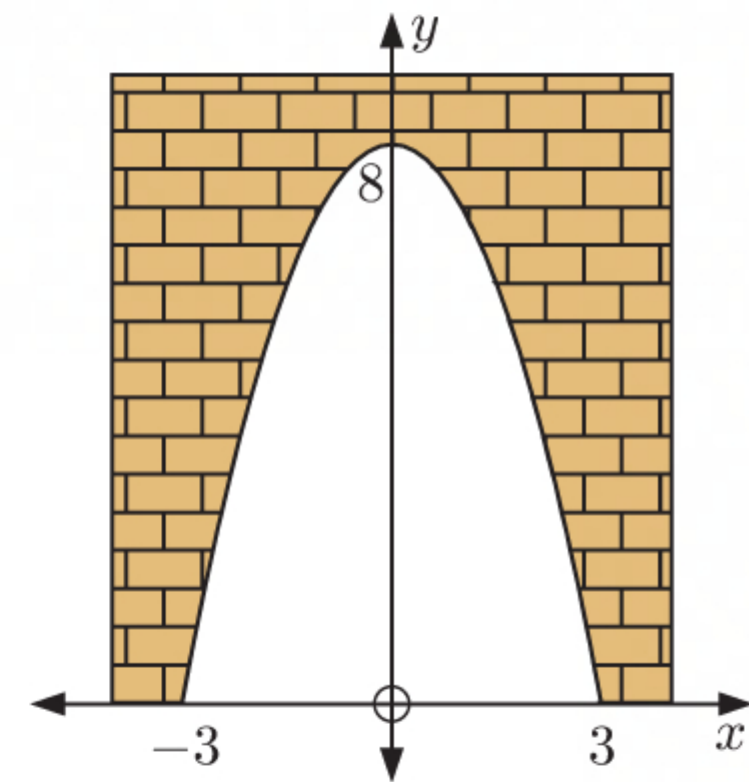
So, the quadratic model is $y = -\frac{8}{9}x^2 + 8$.

b The semi-trailer is 4 m wide, so we use the equation in **a** to find the height of the tunnel when it is 4 m wide.

$$\begin{aligned} \text{When } x = \pm 2, \quad y &= -\frac{8}{9}(2)^2 + 8 \\ &= -\frac{32}{9} + 8 \\ &= \frac{40}{9} \approx 4.44 \text{ m} \end{aligned}$$

For heights greater than 4.44 m, the tunnel is less than 4 m wide. But the semi-trailer is 5 m high.

\therefore the semi-trailer will not fit through the tunnel.



a Let the quadratic model be $y = ax^2 + bx + c$.

$$\begin{array}{lll} \text{When } x = 10, y = 9 & \therefore 9 = a(10)^2 + b(10) + c & \text{or } 100a + 10b + c = 9 \\ \text{When } x = 20, y = 15.25 & \therefore 15.25 = a(20)^2 + b(20) + c & \text{or } 400a + 20b + c = 15.25 \\ \text{When } x = 30, y = 19 & \therefore 19 = a(30)^2 + b(30) + c & \text{or } 900a + 30b + c = 19 \end{array}$$

We solve the system of equations
$$\begin{cases} 100a + 10b + c = 9 \\ 400a + 20b + c = 15.25 \\ 900a + 30b + c = 19 \end{cases}$$
 simultaneously using technology.


	a	b	c	d
1	100	10	1	9
2	400	20	1	15.25
3	900	30	1	19

	X	Y	Z
1	-0.012	1	0.25

We find that $a = -\frac{1}{80}$, $b = 1$, and $c = \frac{1}{4}$.

So, the quadratic model is $y = -\frac{1}{80}x^2 + x + \frac{1}{4}$.

- b** For the quadratic model y , $a = -\frac{1}{80}$, $b = 1$, and $c = \frac{1}{4}$.

Since $a < 0$, the shape is .

The maximum height occurs when $x = -\frac{b}{2a} = -\frac{1}{2(-\frac{1}{80})} = 40$

$$\begin{aligned} \text{Now, when } x = 40, \quad y &= -\frac{1}{80}(40)^2 + 40 + \frac{1}{4} \\ &= -20 + 40 + \frac{1}{4} \\ &= 20.25 \end{aligned}$$

\therefore the maximum height reached by the ball is 20.25 m.

- c** When $x = 75$, $y = -\frac{1}{80}(75)^2 + 75 + \frac{1}{4}$
- $$\begin{aligned} &= -\frac{1125}{16} + 75 + \frac{1}{4} \\ &= 4.9375 \end{aligned}$$

So, the ball is 4.9375 m above the ground after travelling 75 m horizontally.

\therefore the ball will clear the boundary fence.

- 10 a** The top of the cliff is 60 m above sea level, so $H(0) = 60$
- $$\begin{aligned} \therefore 60 &= a(0)^2 + b(0) + c \\ \therefore c &= 60 \end{aligned}$$

- b** The vertex has t -coordinate 2 $\therefore -\frac{b}{2a} = 2$
- $$\begin{aligned} \therefore -b &= 4a \\ \therefore 4a + b &= 0 \end{aligned}$$

$$\begin{aligned} H(2) = 80 \quad \therefore 80 &= a(2)^2 + b(2) + 60 \quad \text{or} \quad 4a + 2b = 20 \\ \therefore 2a + b &= 10 \end{aligned}$$

- c** We solve the system of equations $\begin{cases} 4a + b = 0 \\ 2a + b = 10 \end{cases}$ simultaneously using technology.

	a	b	c
1	4	1	0
2	2	1	10

	X	Y
1	-5	20

We find that $a = -5$ and $b = 20$.

d $H(t) = -5t^2 + 20t + 60$
 $\therefore H(3) = -5(3)^2 + 20(3) + 60$
 $= -45 + 60 + 60$
 $= 75$

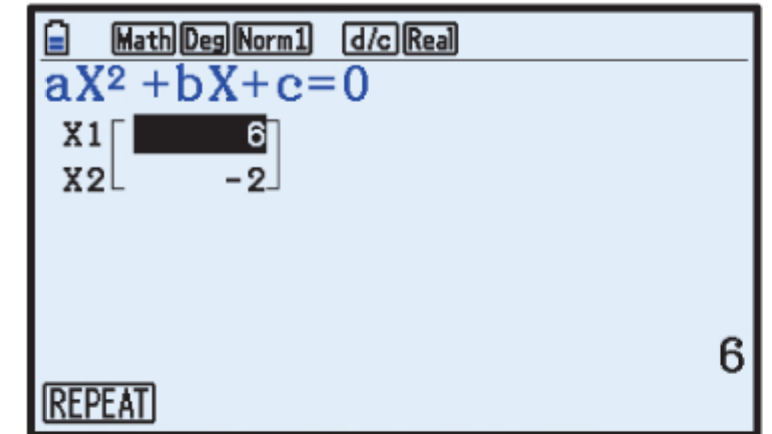
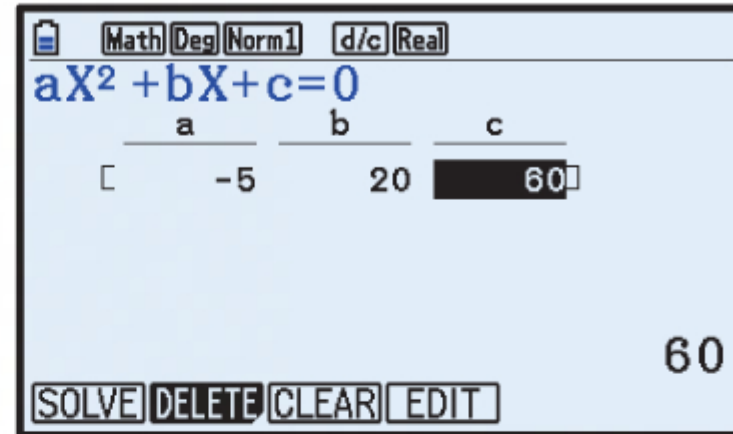
After 3 seconds, the stone is 75 m above sea level.

e The stone hits the water when

$$H(t) = 0$$

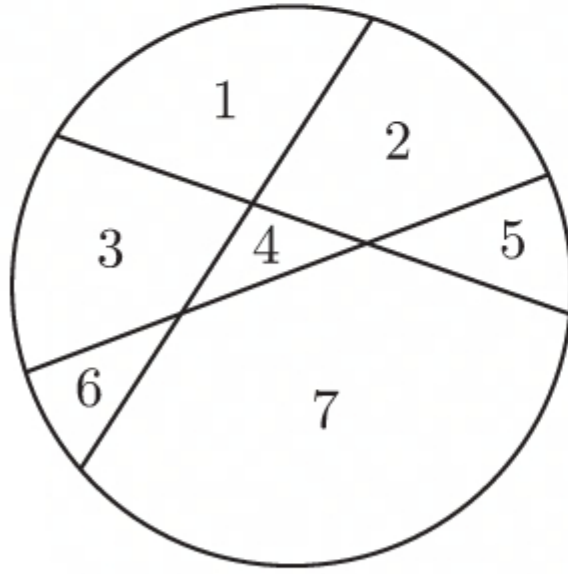
$$\therefore -5t^2 + 20t + 60 = 0$$

$$\therefore t = -2 \text{ or } 6$$



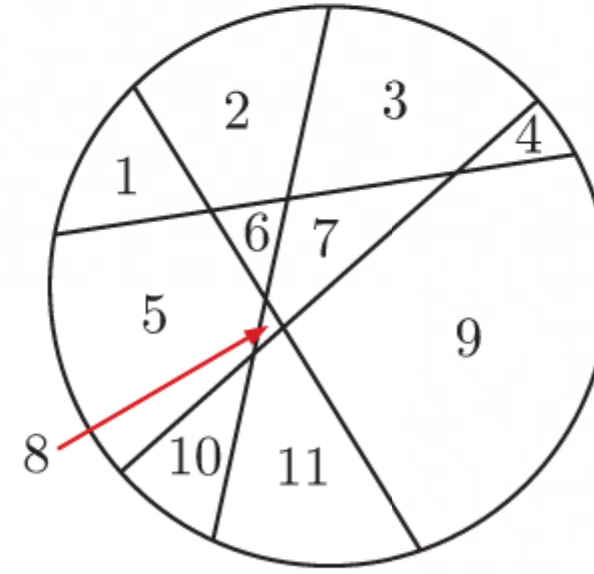
Since t must be positive, the stone hits the water after 6 seconds.

11 a With 3 cuts we can make a maximum of 7 pieces.



$$\therefore P(3) = 7$$

With 4 cuts we can make a maximum of 11 pieces.



$$\therefore P(4) = 11$$

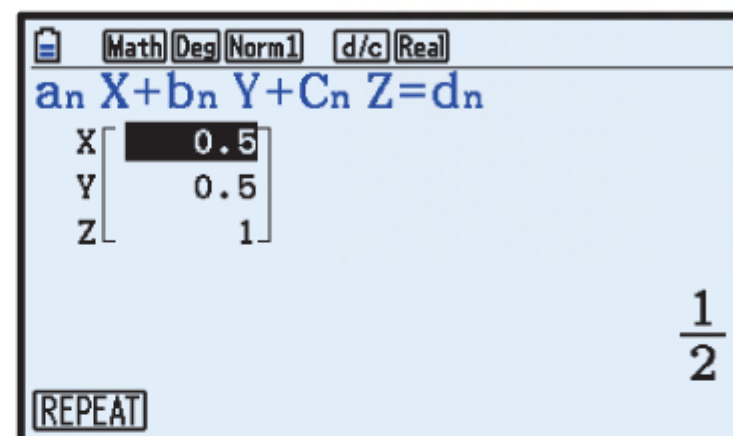
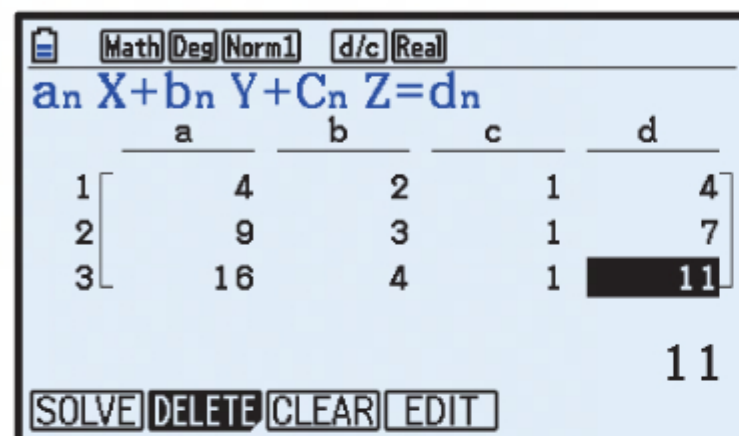
b $P(n) = an^2 + bn + c$

$$P(2) = 4 \quad \therefore 4 = a(2)^2 + b(2) + c \quad \text{or} \quad 4a + 2b + c = 4$$

$$P(3) = 7 \quad \therefore 7 = a(3)^2 + b(3) + c \quad \text{or} \quad 9a + 3b + c = 7$$

$$P(4) = 11 \quad \therefore 11 = a(4)^2 + b(4) + c \quad \text{or} \quad 16a + 4b + c = 11$$

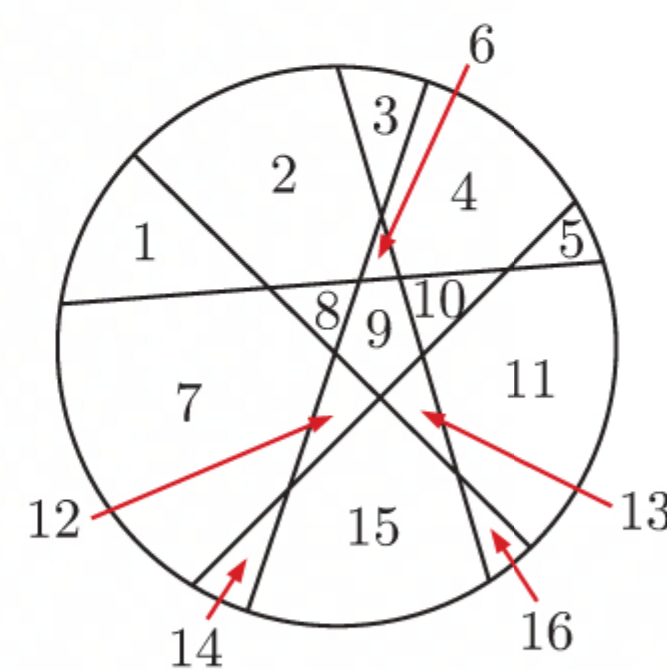
We solve the system of equations
$$\begin{cases} 4a + 2b + c = 4 \\ 9a + 3b + c = 7 \\ 16a + 4b + c = 11 \end{cases}$$
 simultaneously using technology.



We find that $a = \frac{1}{2}$, $b = \frac{1}{2}$, and $c = 1$.

$$\begin{aligned}
 \text{c} \quad P(n) &= \frac{1}{2}n^2 + \frac{1}{2}n + 1 \\
 \therefore P(5) &= \frac{1}{2}(5)^2 + \frac{1}{2}(5) + 1 \\
 &= \frac{25}{2} + \frac{5}{2} + 1 \\
 &= 16
 \end{aligned}$$

With 5 cuts we can make a maximum of 16 pieces.

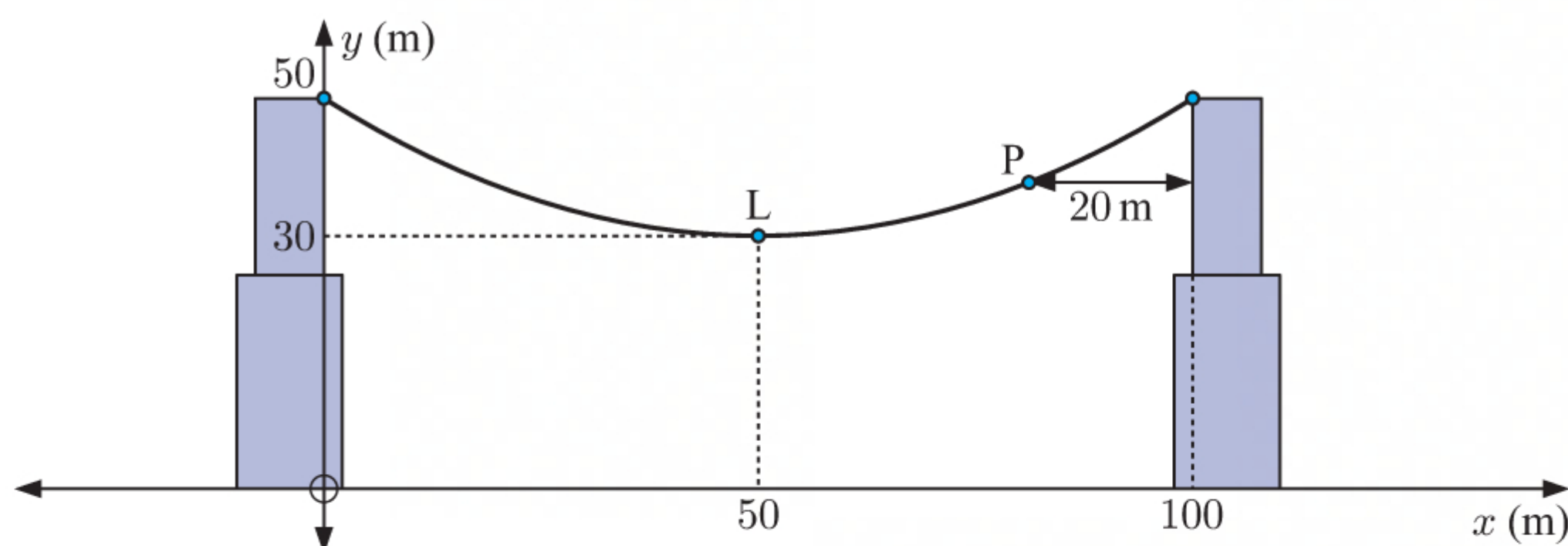


$$\begin{aligned}
 \text{d} \quad P(12) &= \frac{1}{2}(12)^2 + \frac{1}{2}(12) + 1 \\
 &= 72 + 6 + 1 \\
 &= 79
 \end{aligned}$$

\therefore with 12 cuts we can make a maximum of 79 pieces.

- e We can make 0 cuts, or any positive whole number of cuts.
 \therefore the model can be used for $n \geq 0$, $n \in \mathbb{Z}$.

12



- a Let the quadratic model be $y = ax^2 + bx + c$.

When $x = 0$, $y = 50$

$$\therefore 50 = a(0)^2 + b(0) + c$$

$$\therefore c = 50$$

The vertex L has x -coordinate 50 $\therefore -\frac{b}{2a} = 50$

$$\therefore -b = 100a$$

$$\therefore 100a + b = 0$$

When $x = 50$, $y = 30$ $\therefore 30 = a(50)^2 + b(50) + 50$ or $2500a + 50b = -20$

$$\therefore 250a + 5b = -2$$

We solve the system of equations $\begin{cases} 100a + b = 0 \\ 250a + 5b = -2 \end{cases}$ simultaneously using technology.

We find that $a = \frac{1}{125}$ and $b = -\frac{4}{5}$.

So, the quadratic model is $y = \frac{1}{125}x^2 - \frac{4}{5}x + 50$.

- b** Point P has x -coordinate $100 - 20 = 80$.

$$\begin{aligned}\text{When } x = 80, \quad y &= \frac{1}{125}(80)^2 - \frac{4}{5}(80) + 50 \\ &= \frac{256}{5} - 64 + 50 \\ &= 37.2\end{aligned}$$

\therefore the tightrope is 37.2 m above ground level at point P.

- c** The platforms are 100 apart.

\therefore the quadratic model is valid for $0 \leq x \leq 100$.

ACTIVITY 3

PROJECTILE MOTION

1 $x = 20t$, $y = -4.9t^2 + 14.7t + 1$, $t \geq 0$

a i When $t = 1$, $x = 20(1) = 20$ and $y = -4.9(1)^2 + 14.7(1) + 1$

$$\begin{aligned}&= -4.9 + 14.7 + 1 \\ &= 10.8\end{aligned}$$


After 1 second, the ball has horizontal component 20 m and vertical component 10.8 m.

ii When $t = 2$, $x = 20(2) = 40$ and $y = -4.9(2)^2 + 14.7(2) + 1$

$$\begin{aligned}&= -19.6 + 29.4 + 1 \\ &= 10.8\end{aligned}$$

After 2 seconds, the ball has horizontal component 40 m and vertical component 10.8 m.

- b i** For the quadratic function y , $a = -4.9$, $b = 14.7$, and $c = 1$.

Since $a < 0$, the shape is .

The maximum height occurs when $t = -\frac{b}{2a} = -\frac{14.7}{2(-4.9)} = 1.5$

So, the ball is at its highest point after 1.5 seconds.

ii When $t = 1.5$, $y = -4.9(1.5)^2 + 14.7(1.5) + 1$

$$\begin{aligned}&= -11.025 + 22.05 + 1 \\ &= 12.025\end{aligned}$$

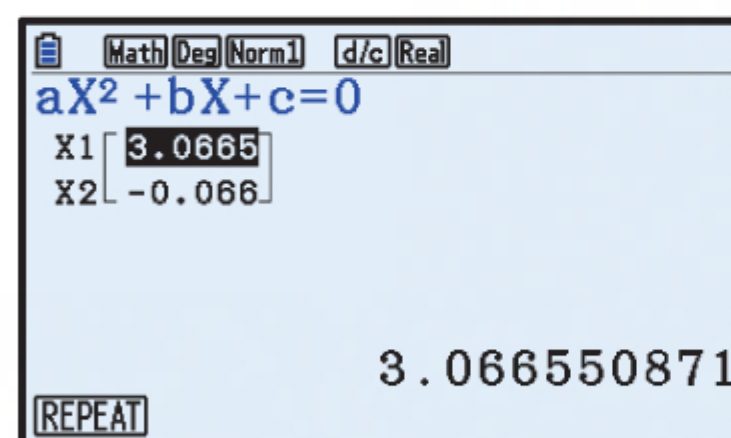
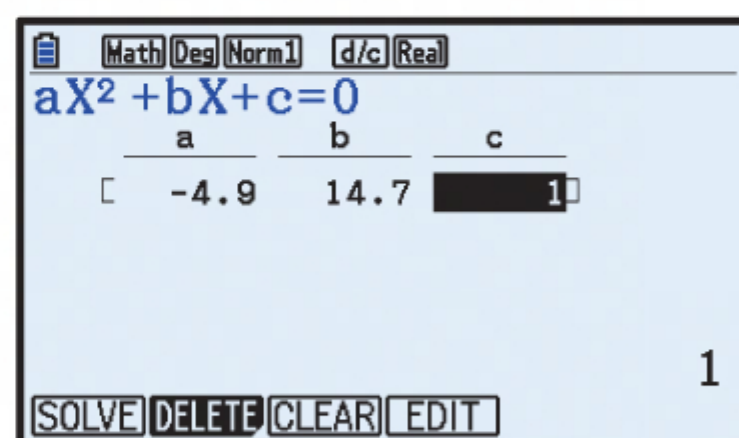
\therefore the maximum height reached by the ball is 12.025 m.

iii When $t = 1.5$, $x = 20(1.5) = 30$

\therefore the ball travelled 30 m horizontally before reaching its maximum height.

- c i** The ball hits the ground when $y = 0$

$$\therefore -4.9t^2 + 14.7t + 1 = 0$$



$$\therefore t \approx -0.0666 \text{ or } 3.07$$

Since t must be positive, the ball hits the ground after about 3.07 seconds.

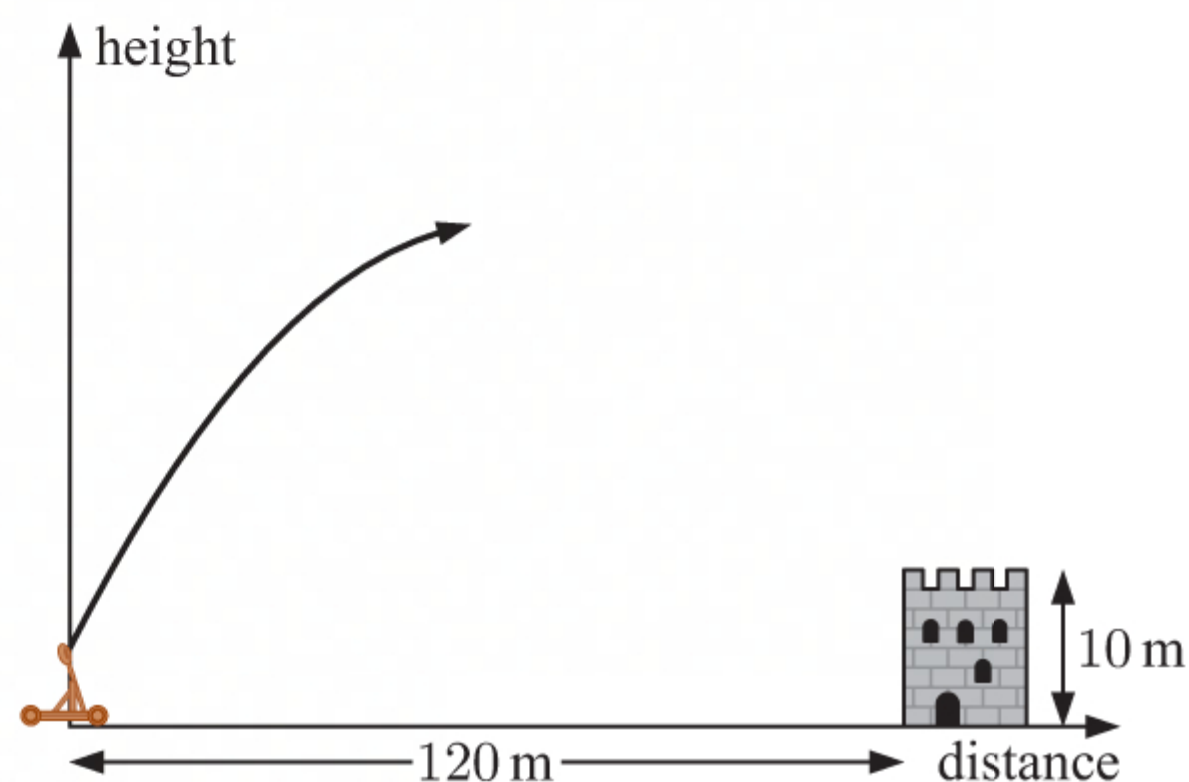
ii When $t \approx 3.07$, $x \approx 20(3.07) \approx 61.4$

\therefore the ball travelled about 61.4 m horizontally before hitting the ground.


2 $x = 30t$, $y = -4.9t^2 + 20t + 5$, $t \geq 0$

a When $t = 1$, $x = 30(1) = 30$
 and $y = -4.9(1)^2 + 20(1) + 5$
 $= -4.9 + 20 + 5$
 $= 20.1$

After 1 second, the stone has horizontal component 30 m and vertical component 20.1 m.



b For the quadratic function y , $a = -4.9$, $b = 20$, and $c = 5$.

Since $a < 0$, the shape is .

The maximum height occurs when $t = -\frac{b}{2a} = -\frac{20}{2(-4.9)} \approx 2.04$

When $t \approx 2.04$, $y \approx -4.9(2.04)^2 + 20(2.04) + 5$
 ≈ 25.4

\therefore the maximum height reached by the stone is about 25.4 m.

c The stone has travelled 120 m when $x = 120$
 $\therefore 30t = 120$
 $\therefore t = 4$

When $t = 4$, $y = -4.9(4)^2 + 20(4) + 5$
 $= -78.4 + 80 + 5$
 $= 6.6$

Since the castle is 10 m high, the stone will hit the castle.

3 A trajectory of 45° appears to give the greatest range for the cannon.

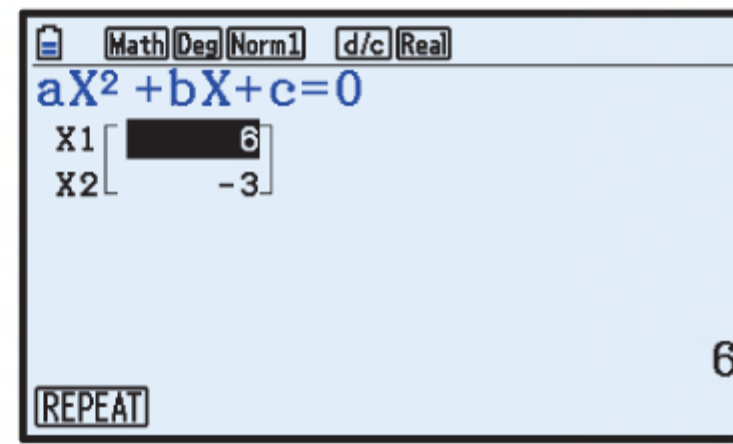
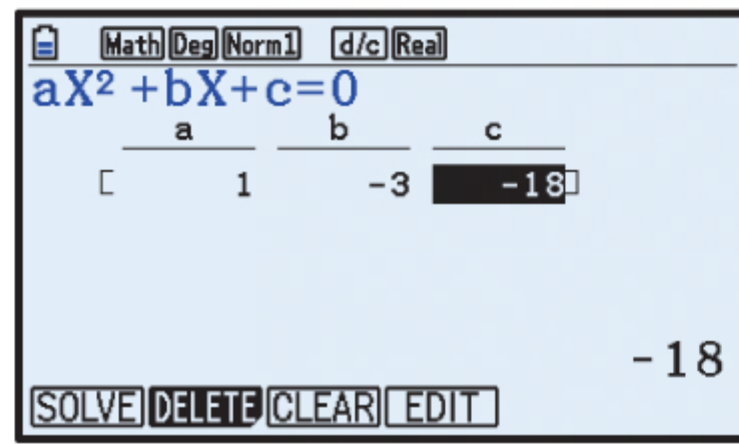
REVIEW SET 6A

1 $f(x) = x^2 - 3x - 15$

a $f(0) = 0^2 - 3(0) - 15$
 $= -15$

b $f(1) = 1^2 - 3(1) - 15$
 $= 1 - 3 - 15$
 $= -17$

c $f(x) = 3$ when $x^2 - 3x - 15 = 3$
 $\therefore x^2 - 3x - 18 = 0$

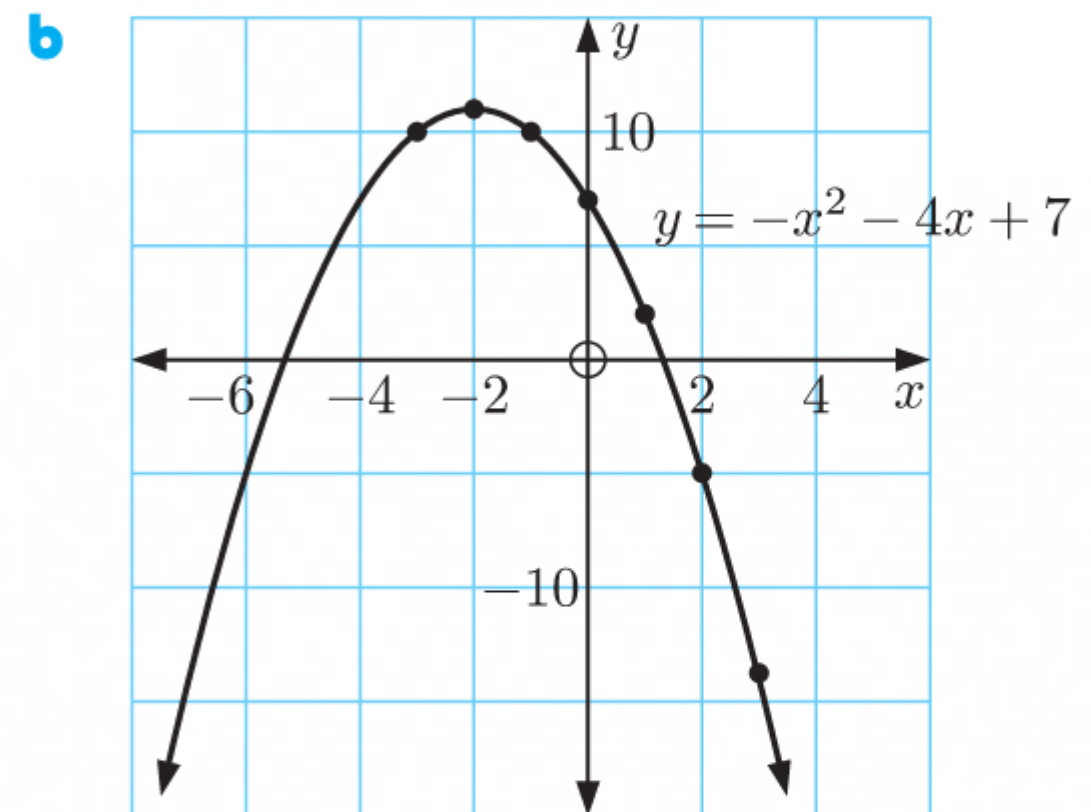


$\therefore x = -3$ or 6

2 $y = -x^2 - 4x + 7$

a

x	-3	-2	-1	0	1	2	3
y	10	11	10	7	2	-5	-14



3 a When $x = 0$, $y = 9$
 \therefore the y -intercept is 9.

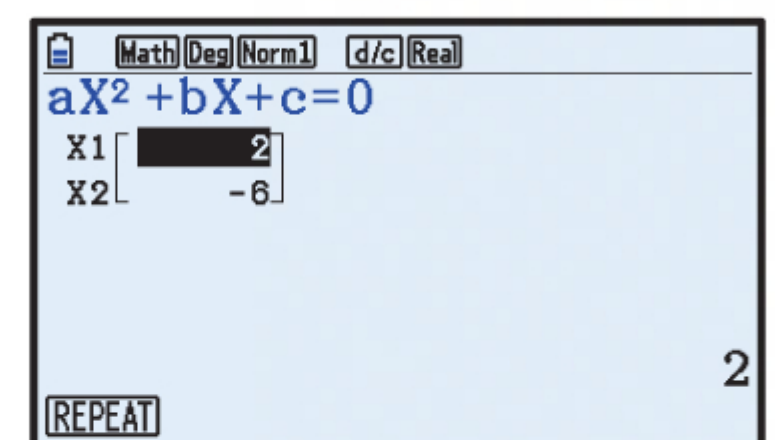
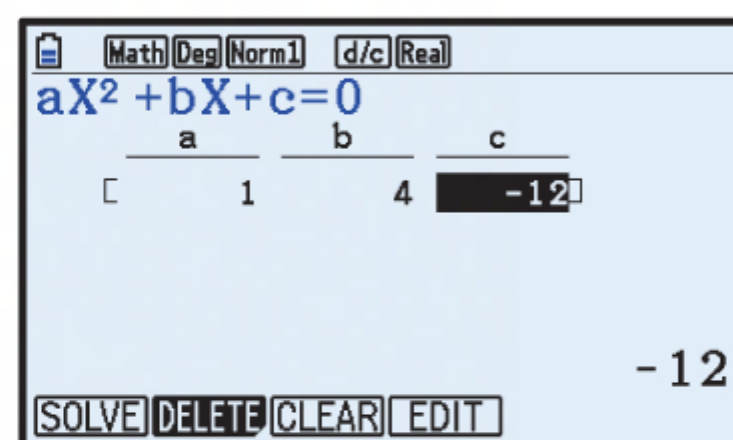
b $f(0) = (5)(-6)$
 $= -30$
 \therefore the y -intercept is -30 .

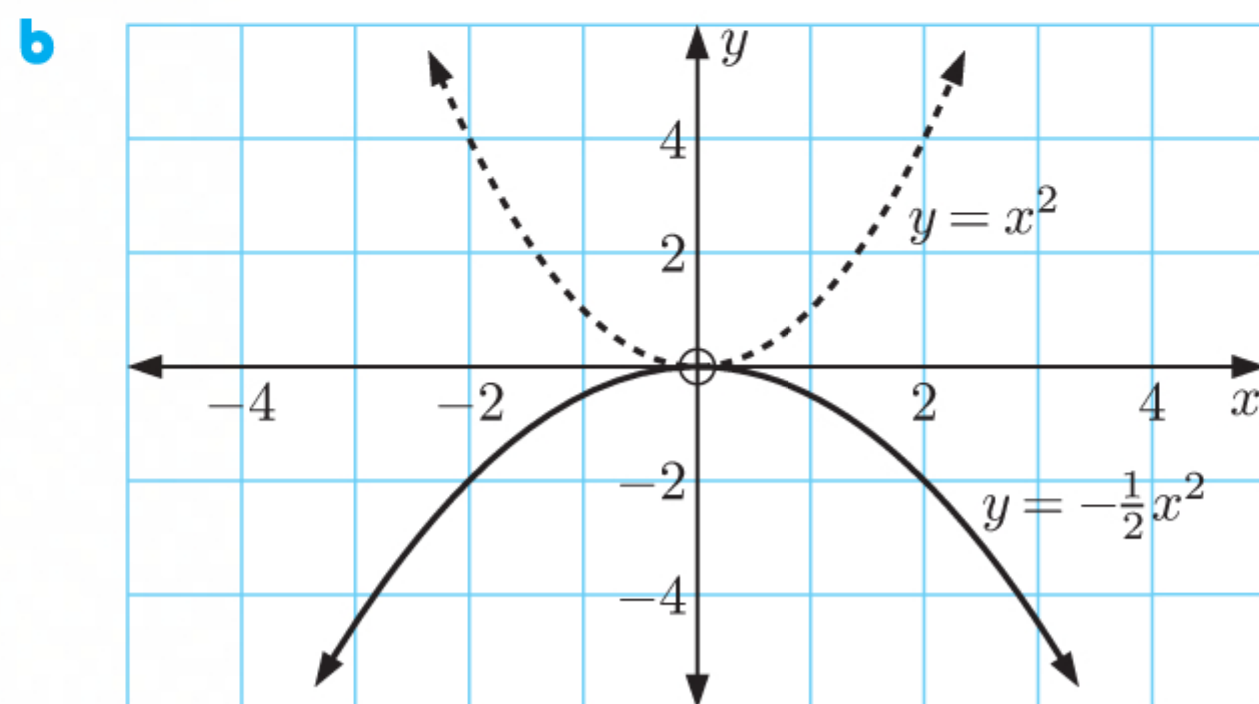
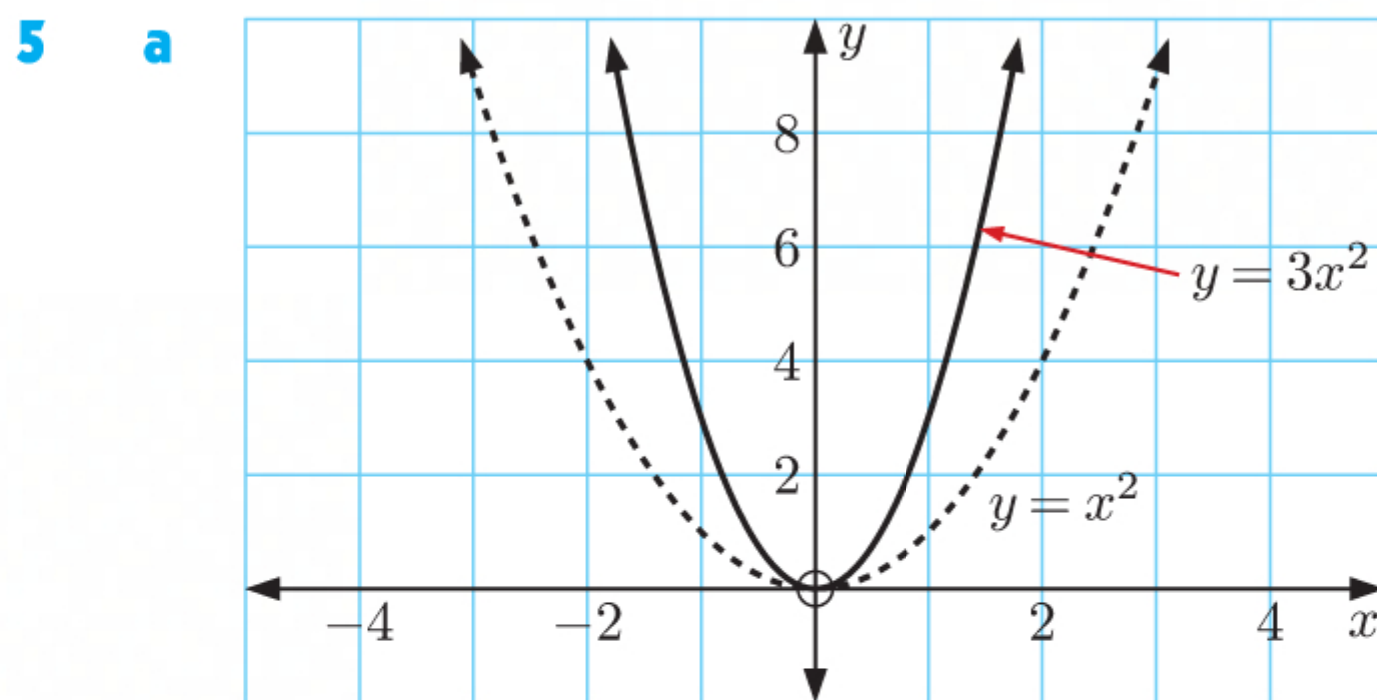
c When $x = 0$, $y = -(-4)^2$
 $= -16$
 \therefore the y -intercept is -16 .

4 a When $y = 0$, $(3x - 5)(x + 2) = 0$
 $\therefore x = \frac{5}{3}$ or -2
 \therefore the zeros are $\frac{5}{3}$ and -2 .

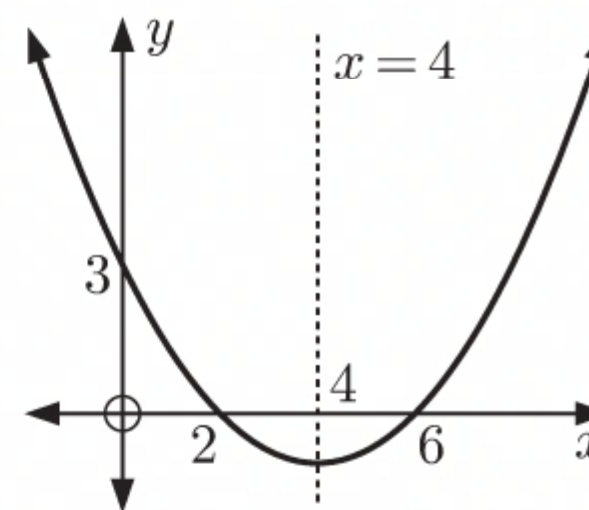
b When $y = 0$, $-\frac{1}{2}(x - 7)^2 = 0$
 $\therefore x = 7$
 \therefore the zero is 7.

c When $f(x) = 0$,
 $x^2 + 4x - 12 = 0$
 $\therefore x = -6$ or 2
 \therefore the zeros are -6 and 2 .

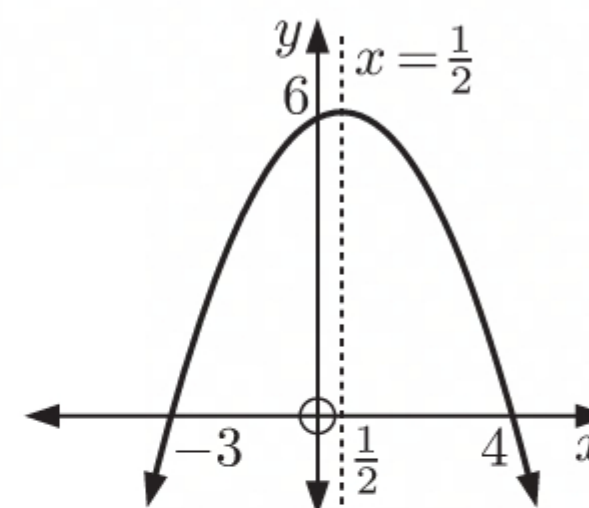




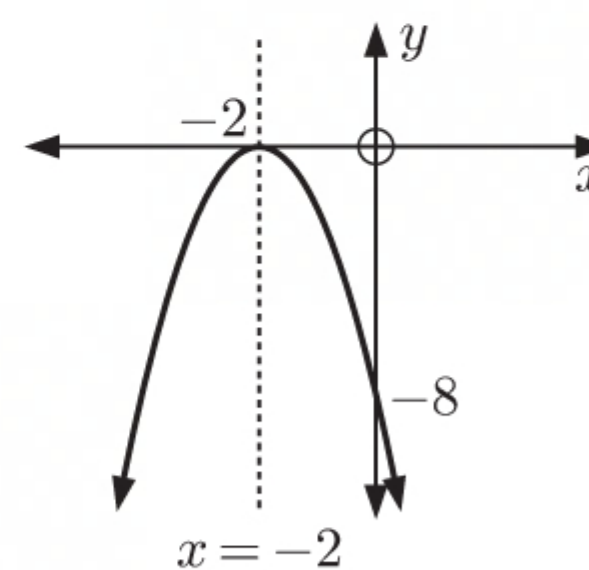
- 6 a** The x -intercepts are 2 and 6.
4 is halfway between 2 and 6, so the axis of symmetry is $x = 4$.




- b** The x -intercepts are -3 and 4 .
 $\frac{1}{2}$ is halfway between -3 and 4 , so the axis of symmetry is $x = \frac{1}{2}$.



- c** The only x -intercept is -2 , so the axis of symmetry is $x = -2$.



7 $y = -2(x - 1)(x + 3)$

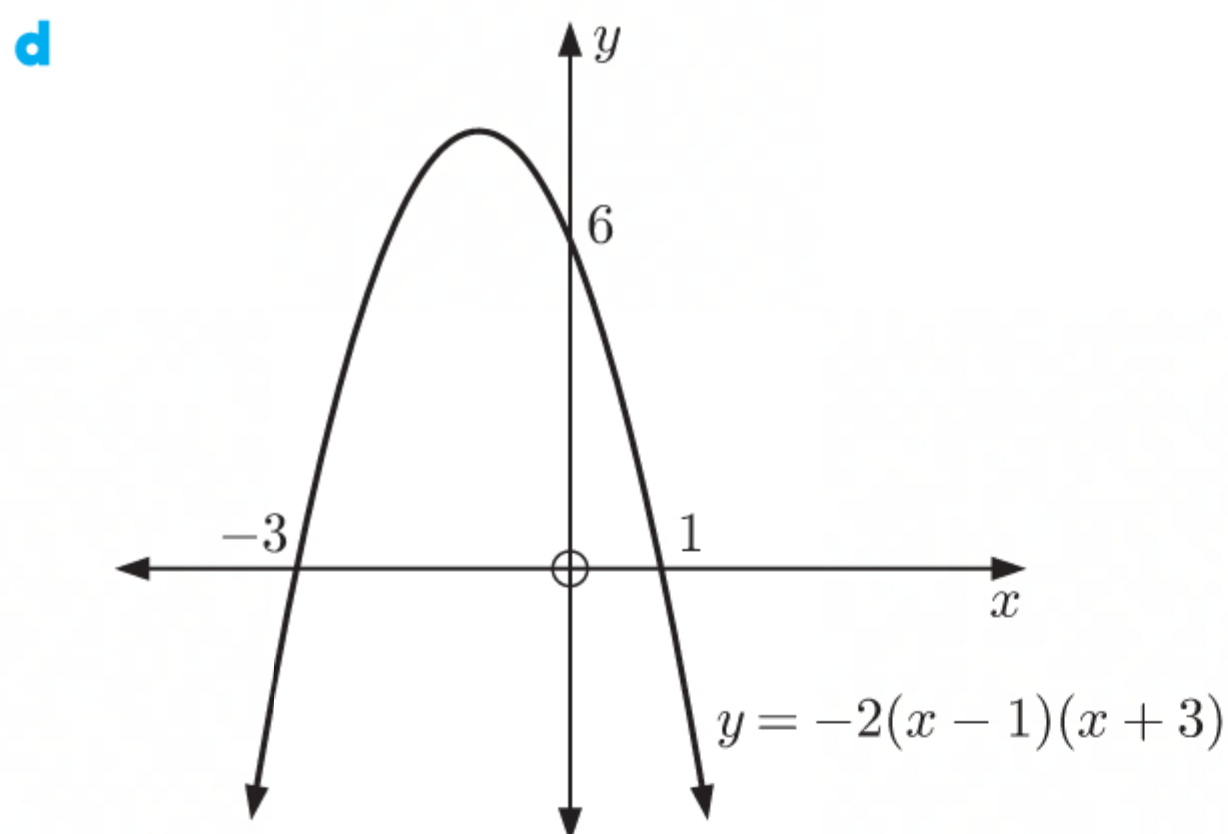
- a** Since $a = -2$ which is < 0 , the parabola has shape .
 \therefore the parabola opens downwards.

b When $x = 0$, $y = -2(-1)(3)$
 $= 6$

\therefore the y -intercept is 6.

c When $y = 0$, $-2(x - 1)(x + 3) = 0$
 $\therefore x = 1$ or -3

\therefore the x -intercepts are 1 and -3 .



8 a $y = 0$ when $(x + 1)(x - 4) = 0$
 $\therefore x = -1$ or 4

\therefore the x -intercepts are -1 and 4 .

$\frac{3}{2}$ is halfway between -1 and 4 , so the axis of symmetry is $x = \frac{3}{2}$.

b $y = x^2 - 7x - 3$ has $a = 1$, $b = -7$, and $c = -3$.

$$\text{Now } -\frac{b}{2a} = -\frac{-7}{2(1)}$$

$$= \frac{7}{2}$$

\therefore the axis of symmetry is $x = \frac{7}{2}$.

c $f(x) = -2x^2 + 3x - 5$ has $a = -2$, $b = 3$, and $c = -5$.

$$\text{Now } -\frac{b}{2a} = -\frac{3}{2(-2)}$$

$$= \frac{3}{4}$$

\therefore the axis of symmetry is $x = \frac{3}{4}$.

9 a $y = -x^2 + 8x + 5$ has $a = -1$, $b = 8$, and $c = 5$.

$$\text{Now } -\frac{b}{2a} = -\frac{8}{2(-1)}$$

$$= 4$$

\therefore the axis of symmetry is $x = 4$.

$$\text{When } x = 4, \quad y = -4^2 + 8(4) + 5$$

$$= -16 + 32 + 5$$

$$= 21$$

\therefore the vertex is $(4, 21)$.

b $f(x) = (x - 5)(x + 4)$ has x -intercepts -4 and 5 .

\therefore the axis of symmetry is $x = \frac{-4 + 5}{2} = \frac{1}{2}$.

$$\begin{aligned} f\left(\frac{1}{2}\right) &= \left(\frac{1}{2} - 5\right)\left(\frac{1}{2} + 4\right) \\ &= \left(-\frac{9}{2}\right)\left(\frac{9}{2}\right) \\ &= -\frac{81}{4} \end{aligned}$$

\therefore the vertex is $\left(\frac{1}{2}, -\frac{81}{4}\right)$.

c $f(x) = 3x^2 + 12x - 4$ has $a = 3$, $b = 12$, and $c = -4$.

$$\begin{aligned} \text{Now } -\frac{b}{2a} &= -\frac{12}{2(3)} \\ &= -2 \end{aligned}$$

\therefore the axis of symmetry is $x = -2$.

$$\begin{aligned} f(-2) &= 3(-2)^2 + 12(-2) - 4 \\ &= 12 - 24 - 4 \\ &= -16 \end{aligned}$$

\therefore the vertex is $(-2, -16)$.

10 $y = x^2 - 2x - 15$

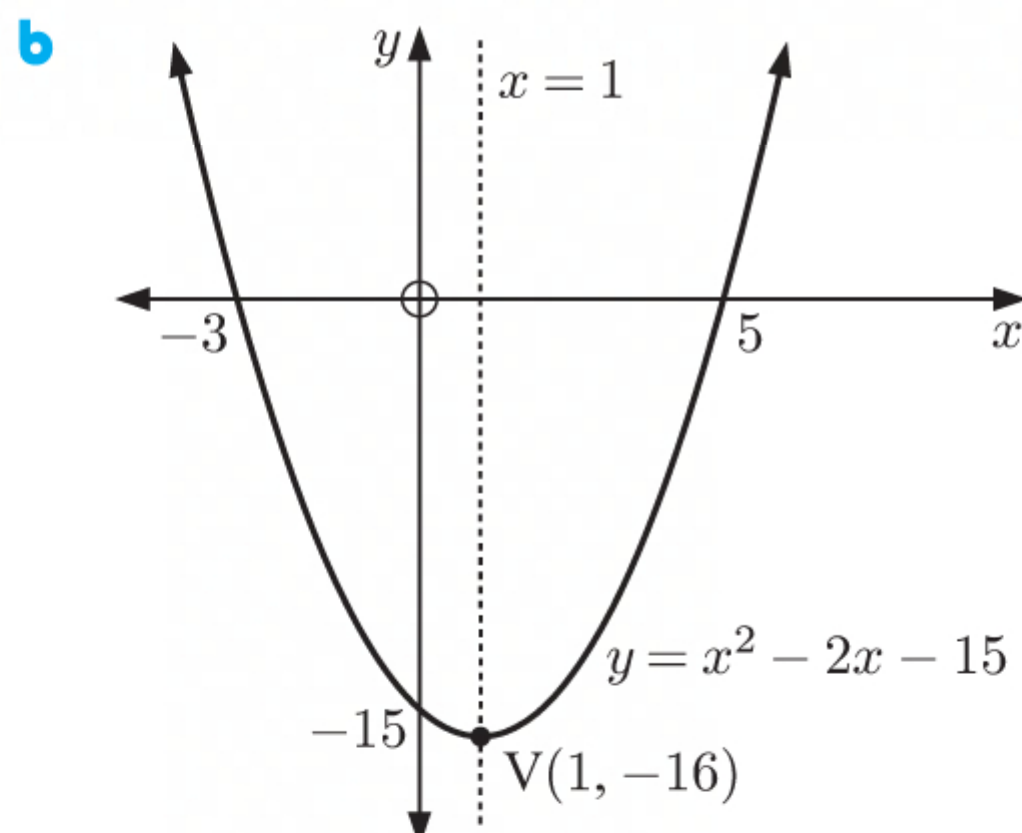
a i When $x = 0$, $y = -15$
 \therefore the y -intercept is -15 .

ii When $y = 0$,
 $x^2 - 2x - 15 = 0$
 $\therefore x = -3$ or 5
 \therefore the x -intercepts are
 -3 and 5 .

iii 1 is halfway between -3 and 5 , so the axis of symmetry is $x = 1$.

iv When $x = 1$, $y = 1^2 - 2(1) - 15$
 $= 1 - 2 - 15$
 $= -16$

\therefore the vertex is $(1, -16)$.



- 11 a** The vertex has x -coordinate 2.

\therefore the axis of symmetry is $x = 2$.

The axis of symmetry lies midway between the x -intercepts.

\therefore the other x -intercept is -1 .

\therefore the quadratic has the form

$$y = a(x + 1)(x - 5) \quad \text{where } a > 0$$

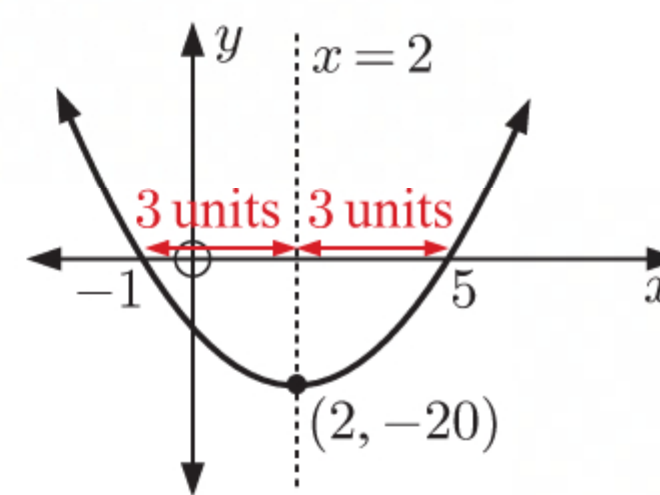
But when $x = 2$, $y = -20$

$$\therefore -20 = a(2 + 1)(2 - 5)$$

$$\therefore -20 = a(3)(-3)$$

$$\therefore a = \frac{20}{9}$$

The quadratic is $y = \frac{20}{9}(x + 1)(x - 5)$.



- b** The axis of symmetry $x = 4$ lies midway between the x -intercepts.

\therefore the other x -intercept is 1.

\therefore the quadratic has the form

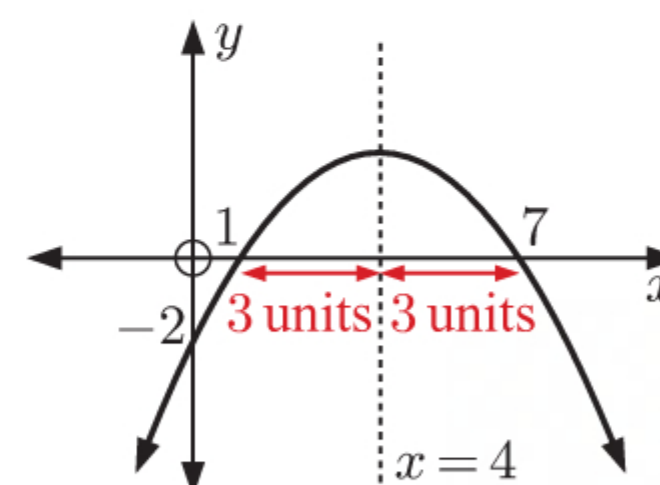
$$y = a(x - 1)(x - 7) \quad \text{where } a < 0$$

But when $x = 0$, $y = -2$

$$\therefore -2 = a(-1)(-7)$$

$$\therefore a = -\frac{2}{7}$$

The quadratic is $y = -\frac{2}{7}(x - 1)(x - 7)$.



- c** The graph touches the x -axis at $x = -3$.

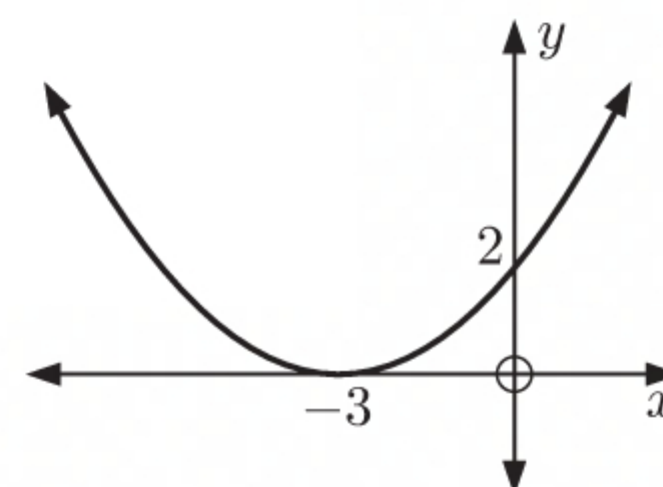
\therefore the quadratic has the form $y = a(x + 3)^2$ where $a > 0$.

But when $x = 0$, $y = 2$

$$\therefore 2 = a(3)^2$$

$$\therefore a = \frac{2}{9}$$

The quadratic is $y = \frac{2}{9}(x + 3)^2$.



- 12 a** Since the x -intercepts are -1 and 2 , the quadratic has the form $y = a(x + 1)(x - 2)$, $a \neq 0$.

When $x = 0$, $y = -6$

$$\therefore -6 = a(1)(-2)$$

$$\therefore a = 3$$

The quadratic is $y = 3(x + 1)(x - 2)$

$$= 3(x^2 - x - 2)$$

$$= 3x^2 - 3x - 6$$

- b** Since the graph touches the x -axis at 4, the quadratic has the form $y = a(x - 4)^2$, $a \neq 0$.

When $x = 2$, $y = 12$

$$\therefore 12 = a(2 - 4)^2$$

$$\therefore 12 = a(-2)^2$$

$$\therefore a = 3$$

$$\begin{aligned}\text{The quadratic is } y &= 3(x - 4)^2 \\ &= 3(x^2 - 8x + 16) \\ &= 3x^2 - 24x + 48\end{aligned}$$

13 a $f(0) = 8$

$$\therefore 8 = a(0)^2 + b(0) + c$$

$$\therefore c = 8$$

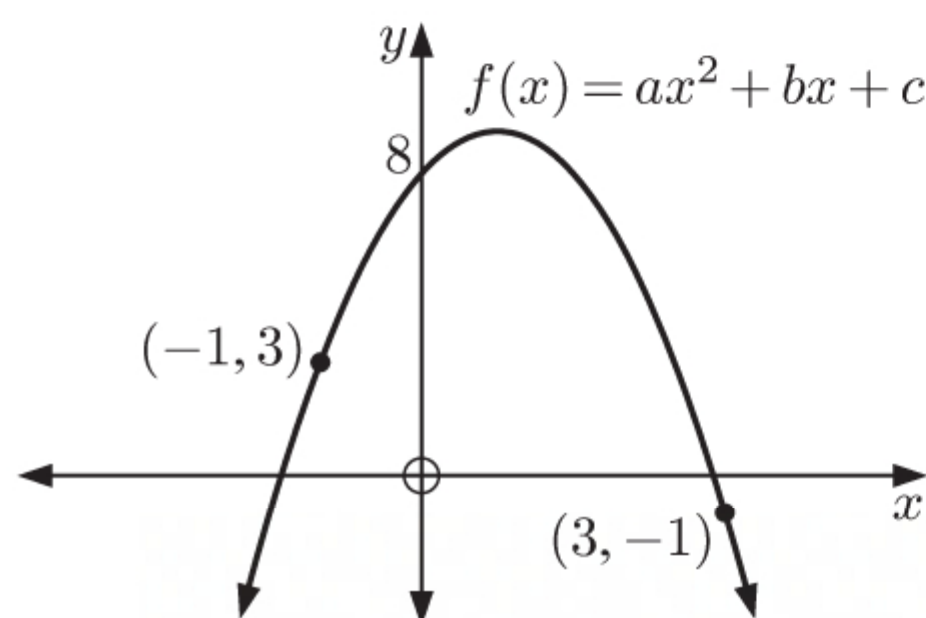
b $f(-1) = 3$

$$\therefore 3 = a(-1)^2 + b(-1) + 8 \quad \text{or} \quad a - b = -5$$

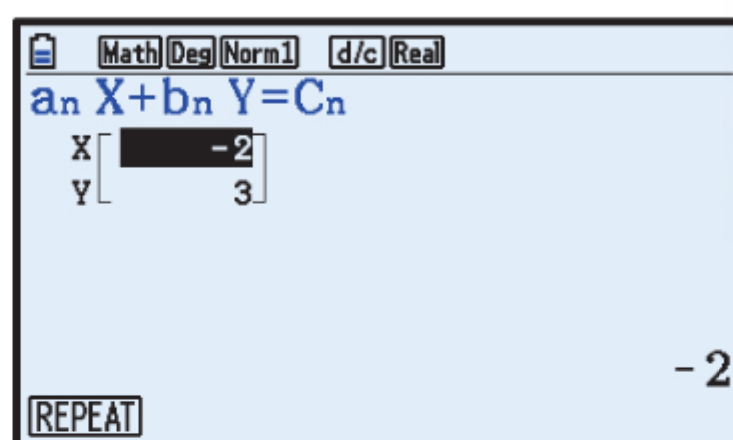
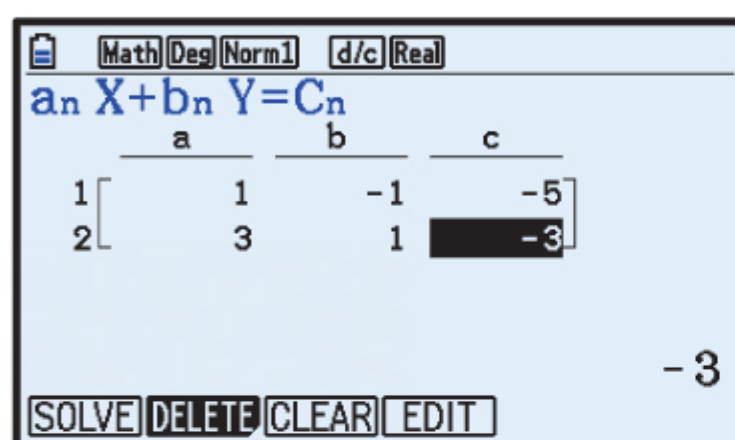
$$f(3) = -1$$

$$\therefore -1 = a(3)^2 + b(3) + 8 \quad \text{or} \quad 9a + 3b = -9$$

$$\therefore 3a + b = -3$$



- c** We solve the system of equations $\begin{cases} a - b = -5 \\ 3a + b = -3 \end{cases}$ simultaneously using technology.



We find that $a = -2$ and $b = 3$.

So, the function is $f(x) = -2x^2 + 3x + 8$.

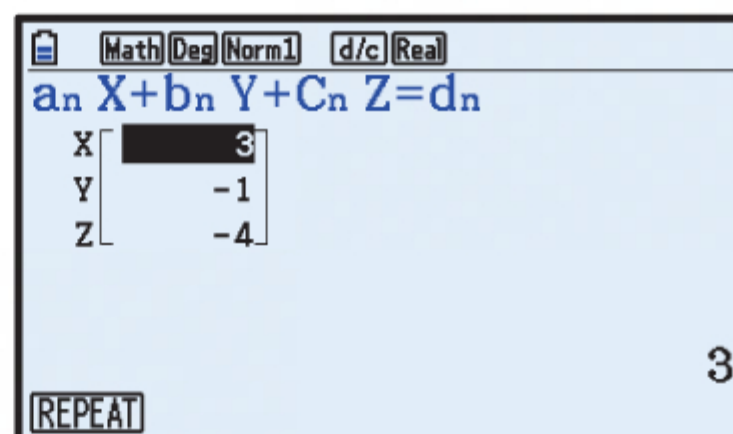
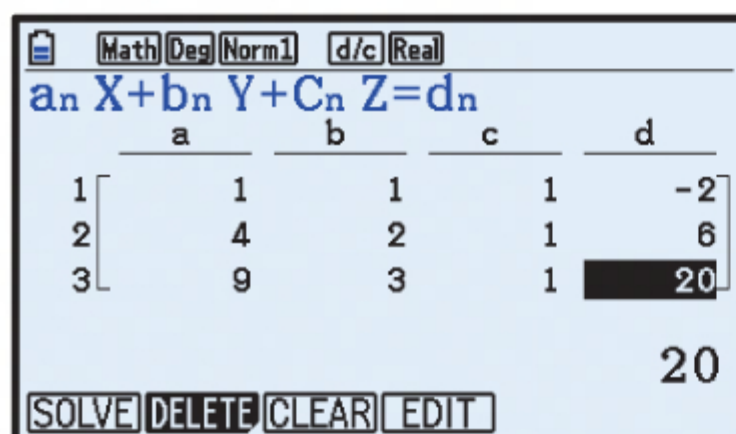
- 14** Let the equation of the quadratic be $y = ax^2 + bx + c$.

When $x = 1$, $y = -2$ $\therefore -2 = a(1)^2 + b(1) + c$ or $a + b + c = -2$

When $x = 2$, $y = 6$ $\therefore 6 = a(2)^2 + b(2) + c$ or $4a + 2b + c = 6$

When $x = 3$, $y = 20$ $\therefore 20 = a(3)^2 + b(3) + c$ or $9a + 3b + c = 20$

We solve the system of equations $\begin{cases} a + b + c = -2 \\ 4a + 2b + c = 6 \\ 9a + 3b + c = 20 \end{cases}$ simultaneously using technology.

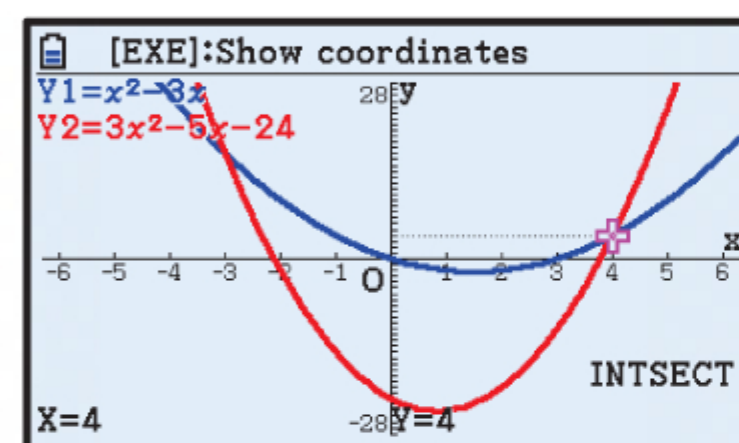
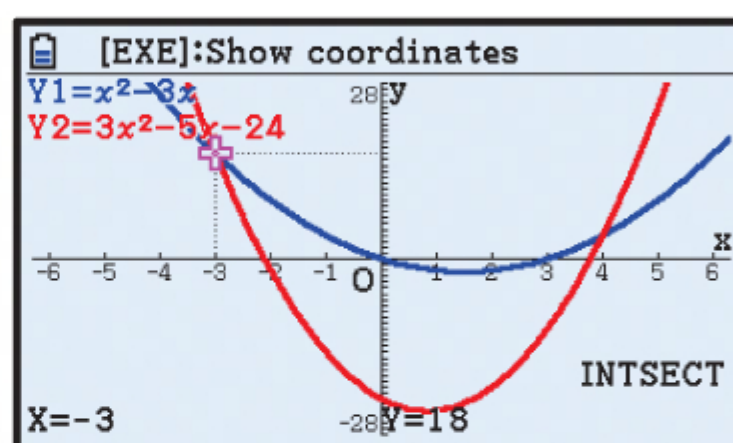


We find that $a = 3$, $b = -1$, and $c = -4$.

So, the quadratic has equation $y = 3x^2 - x - 4$.

- 15** We graph $Y_1 = X^2 - 3X$ and $Y_2 = 3X^2 - 5X - 24$ on the same set of axes.

The graphs intersect at $(-3, 18)$ and $(4, 4)$.




- 16** $h = -4.9t^2 + 19.6t + 1.4$ metres

a When $t = 1$, $h = -4.9(1)^2 + 19.6(1) + 1.4$
 $= -4.9 + 19.6 + 1.4$
 $= 16.1$

After 1 second, the ball is 16.1 m above the ground.

- b** For the quadratic function h , $a = -4.9$, $b = 19.6$, and $c = 1.4$.

Since $a < 0$, the shape is .

The maximum height occurs when $t = -\frac{b}{2a} = -\frac{19.6}{2(-4.9)} = 2$

When $t = 2$, $h = -4.9(2)^2 + 19.6(2) + 1.4$
 $= -19.6 + 39.2 + 1.4$
 $= 21$

\therefore the maximum height reached by the ball is 21 m.

- 17 a** Let the other side of a field be y m long as shown.

So, the total length of fencing required is $3x + 4y$.

If there is 2000 m of fencing available, then

$$3x + 4y = 2000$$

$$\therefore 4y = 2000 - 3x$$

$$\therefore y = 500 - \frac{3}{4}x \quad \dots (*)$$


The total area of the fields $A(x) = x(2y)$

$$= 2xy \text{ m}^2$$

Substituting equation (*), $A(x) = 2x(500 - \frac{3}{4}x)$

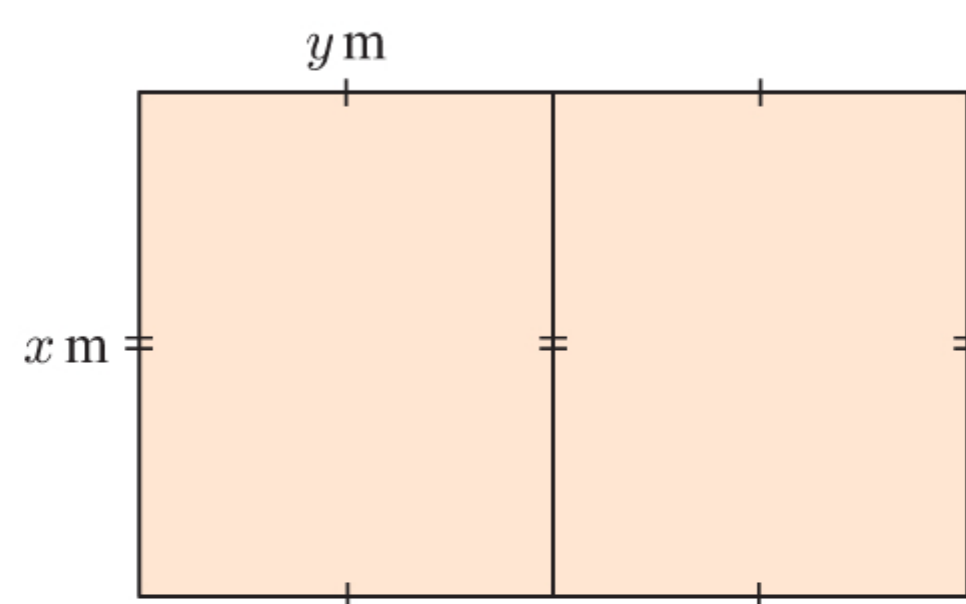
$$= (1000x - \frac{3}{2}x^2) \text{ m}^2$$

- b** The area $A(x)$ is a quadratic with $a = -\frac{3}{2}$, $b = 1000$, and $c = 0$.

Since $a < 0$, the shape is .

The maximum area occurs when $x = -\frac{b}{2a} = -\frac{1000}{2(-\frac{3}{2})} = \frac{1000}{3}$.

When $x = \frac{1000}{3}$, $y = 500 - \frac{3}{4}(\frac{1000}{3})$ {using (*)}
 $= 500 - 250$
 $= 250$



$$\begin{aligned}\text{Also } A\left(\frac{1000}{3}\right) &= 1000\left(\frac{1000}{3}\right) - \frac{3}{2}\left(\frac{1000}{3}\right)^2 \\ &= \frac{1\,000\,000}{3} - \frac{500\,000}{3} \\ &= \frac{500\,000}{3} = 166\,666\frac{2}{3}\end{aligned}$$

\therefore the maximum possible total area of the two fields is $166\,666\frac{2}{3} \text{ m}^2$ when the fields are 250 m by $333\frac{1}{3} \text{ m}$.

18 a $S(3) = 1 + 2 + 3 = 6$

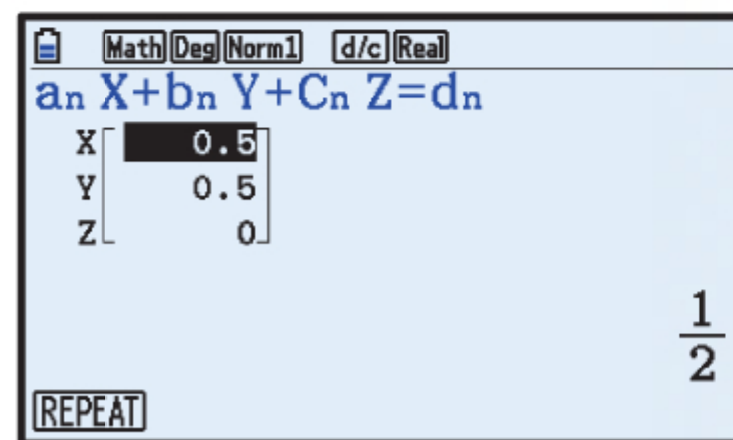
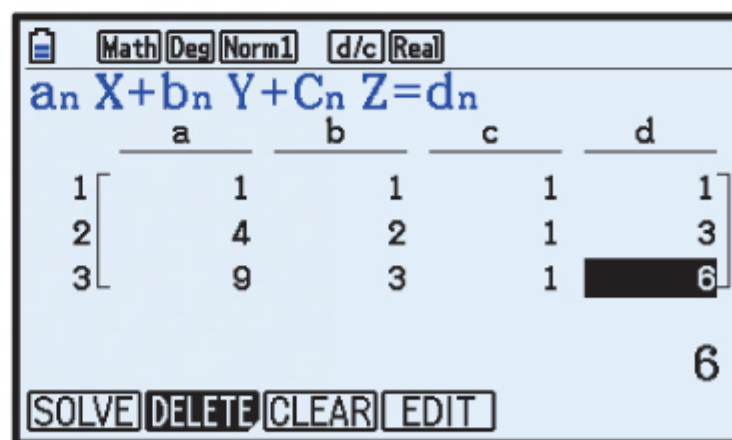
b $S(n) = an^2 + bn + c$

$$S(1) = 1 \quad \therefore 1 = a(1)^2 + b(1) + c \quad \text{or} \quad a + b + c = 1$$

$$S(2) = 3 \quad \therefore 3 = a(2)^2 + b(2) + c \quad \text{or} \quad 4a + 2b + c = 3$$

$$S(3) = 6 \quad \therefore 6 = a(3)^2 + b(3) + c \quad \text{or} \quad 9a + 3b + c = 6$$

We solve the system of equations
$$\begin{cases} a + b + c = 1 \\ 4a + 2b + c = 3 \\ 9a + 3b + c = 6 \end{cases}$$
 simultaneously using technology.



We find that $a = \frac{1}{2}$, $b = \frac{1}{2}$, and $c = 0$.

c $S(6) = 1 + 2 + 3 + 4 + 5 + 6 = 21$

$$\begin{aligned}\text{The model in b is } S(n) &= \frac{1}{2}n^2 + \frac{1}{2}n \\ \therefore S(6) &= \frac{1}{2}(6)^2 + \frac{1}{2}(6) \\ &= 18 + 3 \\ &= 21 \quad \checkmark\end{aligned}$$

d
$$\begin{aligned}S(60) &= \frac{1}{2}(60)^2 + \frac{1}{2}(60) \\ &= 1800 + 30 \\ &= 1830\end{aligned}$$

e
$$\begin{aligned}S(2\frac{1}{2}) &= \frac{1}{2}(2\frac{1}{2})^2 + \frac{1}{2}(2\frac{1}{2}) \\ &= \frac{25}{8} + \frac{5}{4} \\ &= 4.375\end{aligned}$$

This result is not meaningful as we cannot find the sum of the first “two and a half” positive integers.

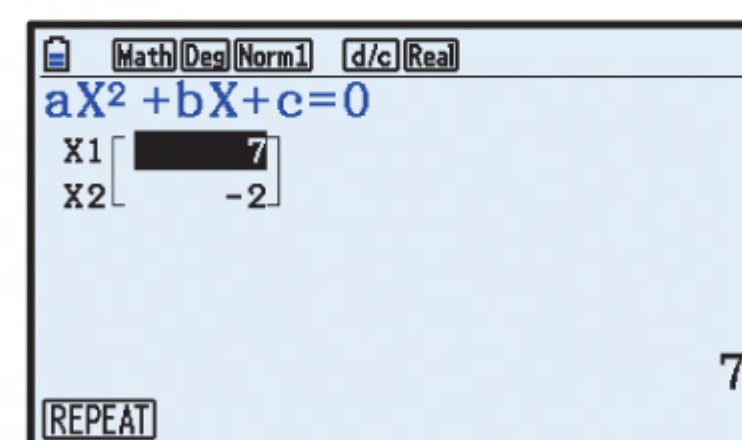
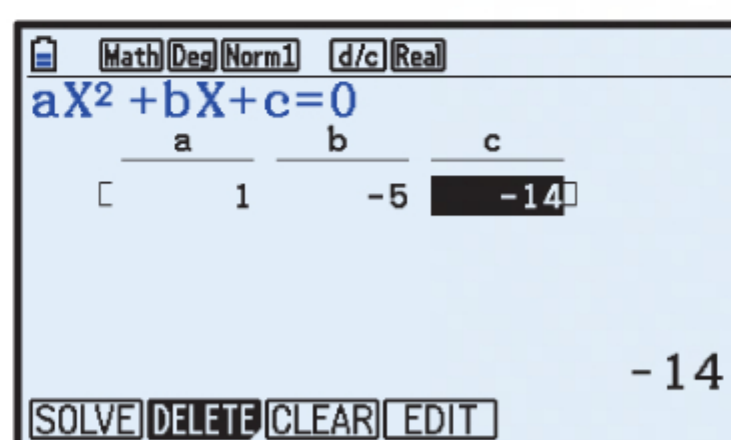
f It is appropriate to use this model when $n \in \mathbb{Z}^+$.

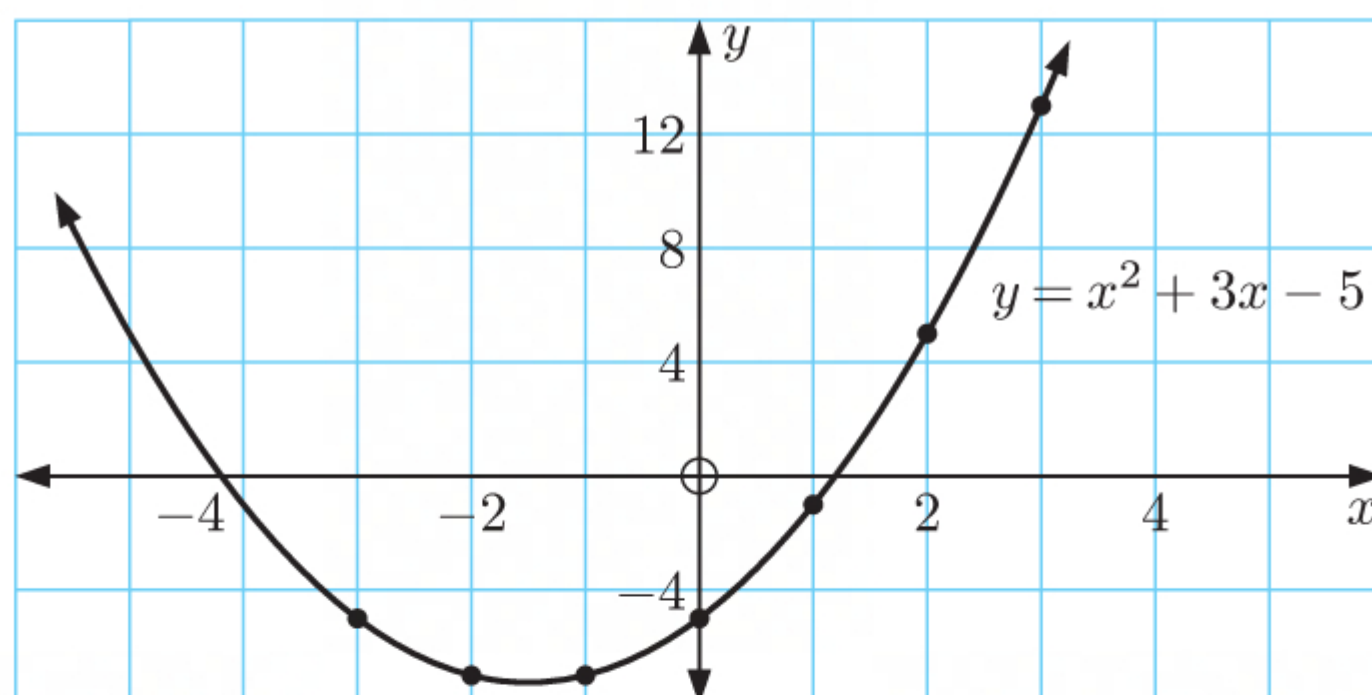
REVIEW SET 6B

$$\begin{aligned}
 1 \quad f(1) &= -2(1)^2 + 13(1) - 4 \\
 &= -2 + 13 - 4 \\
 &= 7
 \end{aligned}$$

$\therefore (1, 5)$ does not satisfy the function $f(x) = -2x^2 + 13x - 4$.

$$\begin{aligned}
 2 \quad \text{If } g(x) &= 5 \text{ then} \\
 x^2 - 5x - 9 &= 5 \\
 \therefore x^2 - 5x - 14 &= 0 \\
 \therefore x &= -2 \text{ or } x = 7
 \end{aligned}$$



$$\begin{array}{|c|c|c|c|c|c|c|c|}
 \hline
 3 & x & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
 \hline
 & y & -5 & -7 & -7 & -5 & -1 & 5 & 13 \\
 \hline
 \end{array}$$


$$\begin{aligned}
 4 \quad a \quad \text{When } x &= 0, \quad y = (5)(-1) \\
 &= -5
 \end{aligned}$$

\therefore the y -intercept is -5 .

$$\begin{aligned}
 \text{When } y &= 0, \quad (x + 5)(x - 1) = 0 \\
 \therefore x &= -5 \text{ or } 1
 \end{aligned}$$

\therefore the x -intercepts are -5 and 1 .

$$\begin{aligned}
 b \quad \text{When } x &= 0, \quad y = -2(-3)(4) \\
 &= 24
 \end{aligned}$$

\therefore the y -intercept is 24 .

$$\begin{aligned}
 \text{When } y &= 0, \quad -2(2x - 3)(x + 4) = 0 \\
 \therefore x &= \frac{3}{2} \text{ or } -4
 \end{aligned}$$

\therefore the x -intercepts are $\frac{3}{2}$ and -4 .

• $f(0) = -3$

∴ the y -intercept is -3 .

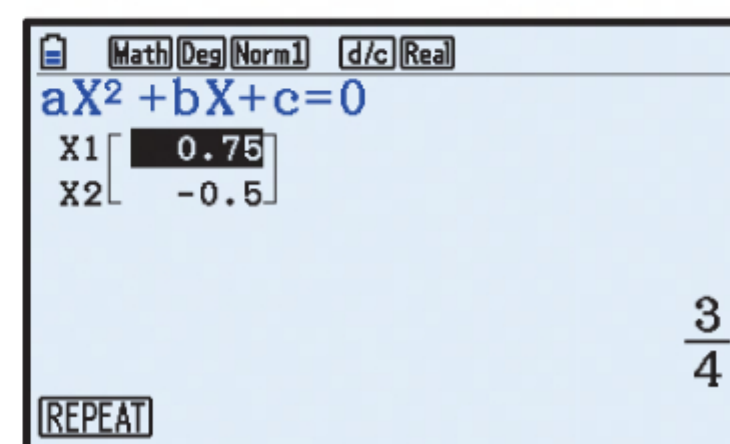
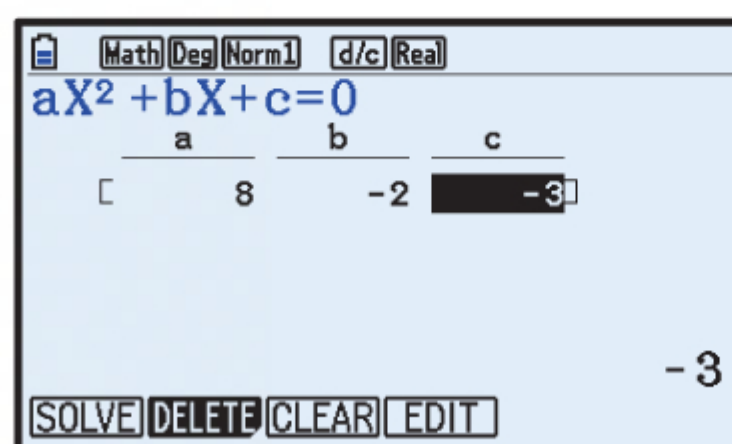
When $f(x) = 0$,

$$8x^2 - 2x - 3 = 0$$

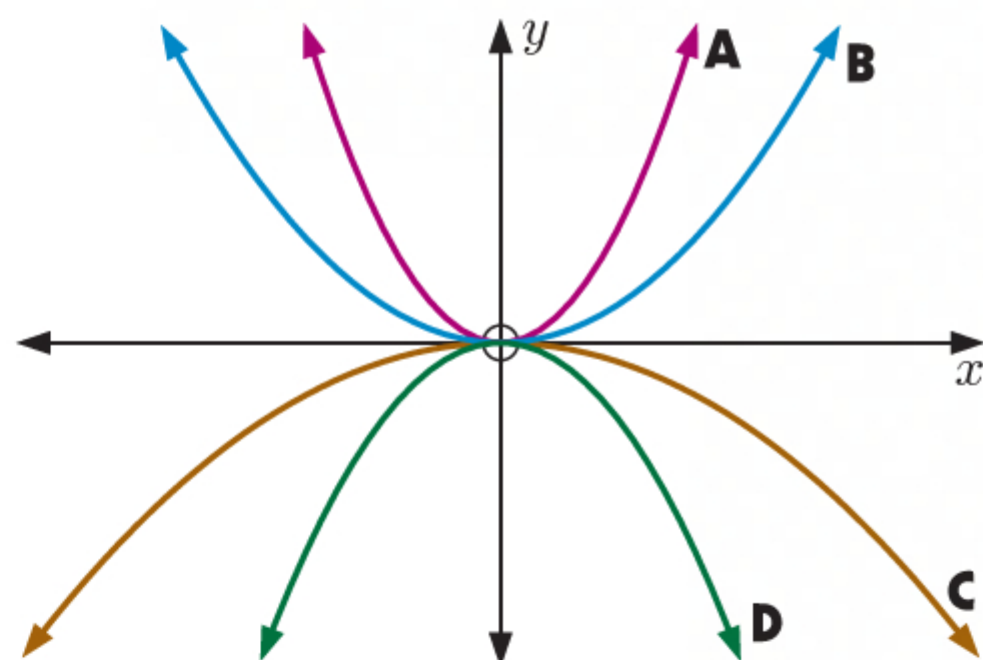
$$\therefore x = -\frac{1}{2} \text{ or } \frac{3}{4}$$

∴ the x -intercepts are

$$-\frac{1}{2} \text{ and } \frac{3}{4}.$$



5



$y = -\frac{1}{2}x^2$ opens downwards and is “wider” than $y = x^2$.

$y = 3x^2$ opens upwards and is “thinner” than $y = x^2$.

$y = -2x^2$ opens downwards and is “thinner” than $y = x^2$.


a The graph of $y = x^2$ is **B**.

c The graph of $y = 3x^2$ is **A**.

b The graph of $y = -\frac{1}{2}x^2$ is **C**.

d The graph of $y = -2x^2$ is **D**.

6 $y = 3(x - 2)^2$

a Since $a = 3$ which is > 0 , the parabola has shape .

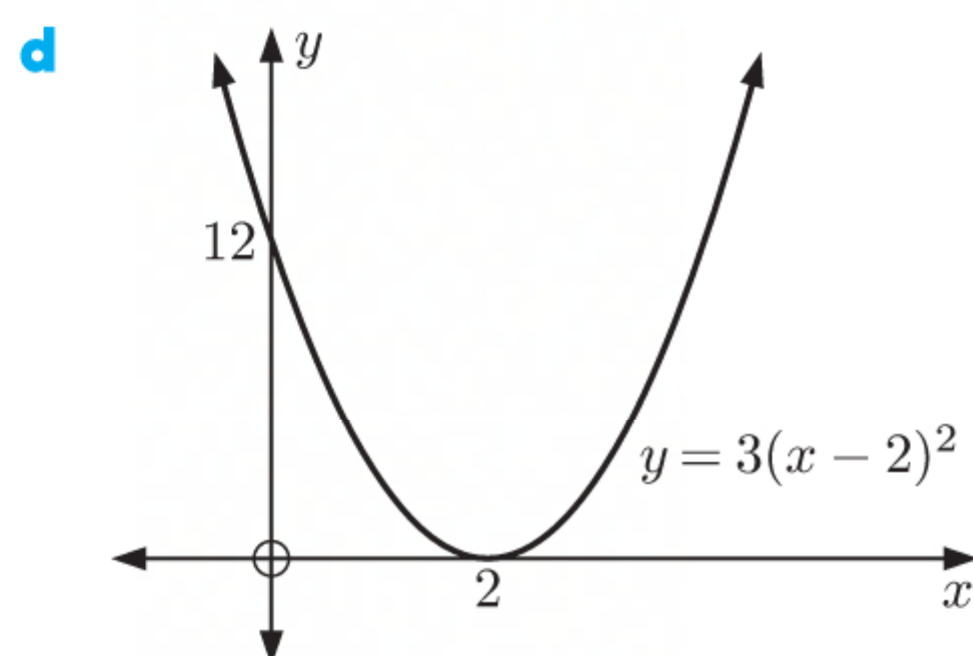
∴ the parabola opens upwards.

b When $x = 0$, $y = 3(-2)^2$
 $= 12$

∴ the y -intercept is 12.

c When $y = 0$, $3(x - 2)^2 = 0$
 $\therefore x = 2$

∴ the x -intercept is 2.



7 a $y = 0$ when $(x - 2)(x - 9) = 0$

$$\therefore x = 2 \text{ or } 9$$

∴ the x -intercepts are 2 and 9.

$\frac{11}{2}$ is halfway between 2 and 9, so the axis of symmetry is $x = \frac{11}{2}$.

b $y = -x^2 + 8x - 1$ has $a = -1$, $b = 8$, and $c = -1$.

$$\begin{aligned}\text{Now } -\frac{b}{2a} &= -\frac{8}{2(-1)} \\ &= 4\end{aligned}$$

\therefore the axis of symmetry is $x = 4$.

c $y = \frac{2}{3}x^2 - x + \frac{1}{3}$ has $a = \frac{2}{3}$, $b = -1$, and $c = \frac{1}{3}$.

$$\begin{aligned}\text{Now } -\frac{b}{2a} &= -\frac{-1}{2(\frac{2}{3})} \\ &= \frac{3}{4}\end{aligned}$$

\therefore the axis of symmetry is $x = \frac{3}{4}$.

8 Let the other x -intercept be a . Since the axis of symmetry $x = 6$ lies halfway between the x -intercepts a and -3 , we have $\frac{a + (-3)}{2} = 6$

$$\therefore a - 3 = 12$$

$$\therefore a = 15$$

So, the other x -intercept is 15.


9 $y = -3x^2 + 8x + 7$ has $a = -3$, $b = 8$, and $c = 7$.

$$\text{Now } -\frac{b}{2a} = -\frac{8}{2(-3)} = \frac{4}{3}$$

\therefore the axis of symmetry is $x = \frac{4}{3}$.

$$\begin{aligned}\text{When } x = \frac{4}{3}, \quad y &= -3\left(\frac{4}{3}\right)^2 + 8\left(\frac{4}{3}\right) + 7 \\ &= -\frac{16}{3} + \frac{32}{3} + 7 \\ &= \frac{37}{3}\end{aligned}$$

\therefore the vertex is $(\frac{4}{3}, \frac{37}{3})$.

$a < 0$, so the shape is .

\therefore the vertex $(\frac{4}{3}, \frac{37}{3})$ is a maximum turning point.

10 a $y = -x^2 + 10x - 9$ has $a = -1$, $b = 10$, and $c = -9$.

$$\text{Now } -\frac{b}{2a} = -\frac{10}{2(-1)} = 5$$

\therefore the axis of symmetry is $x = 5$.

$$\begin{aligned}\text{When } x = 5, \quad y &= -5^2 + 10(5) - 9 \\ &= -25 + 50 - 9 \\ &= 16\end{aligned}$$

So, the vertex is $(5, 16)$.

$a < 0$, so the vertex is a maximum turning point.

\therefore the maximum value of $y = -x^2 + 10x - 9$ is $y = 16$ when $x = 5$.

b $y = 2x^2 - 2x + 5$ has $a = 2$, $b = -2$, and $c = 5$.

Now $-\frac{b}{2a} = -\frac{-2}{2(2)} = \frac{1}{2}$

\therefore the axis of symmetry is $x = \frac{1}{2}$.

When $x = \frac{1}{2}$, $y = 2(\frac{1}{2})^2 - 2(\frac{1}{2}) + 5$
 $= \frac{1}{2} - 1 + 5$
 $= \frac{9}{2}$

So, the vertex is $(\frac{1}{2}, \frac{9}{2})$.

$a > 0$, so the vertex is a minimum turning point.

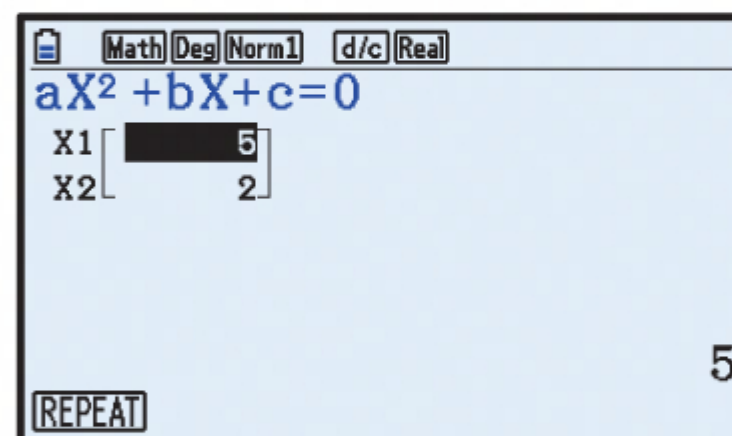
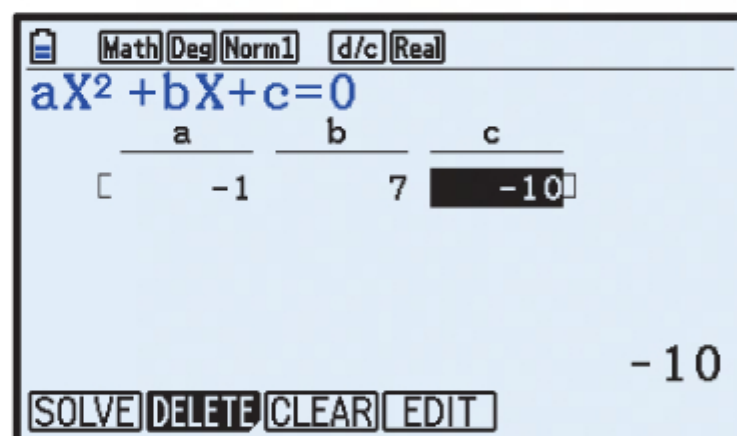
\therefore the minimum value of $y = 2x^2 - 2x + 5$ is $y = \frac{9}{2}$ when $x = \frac{1}{2}$.

11 $y = -x^2 + 7x - 10$

a i When $x = 0$, $y = -10$
 \therefore the y -intercept is -10 .

ii When $y = 0$,
 $-x^2 + 7x - 10 = 0$
 $\therefore x = 2$ or 5

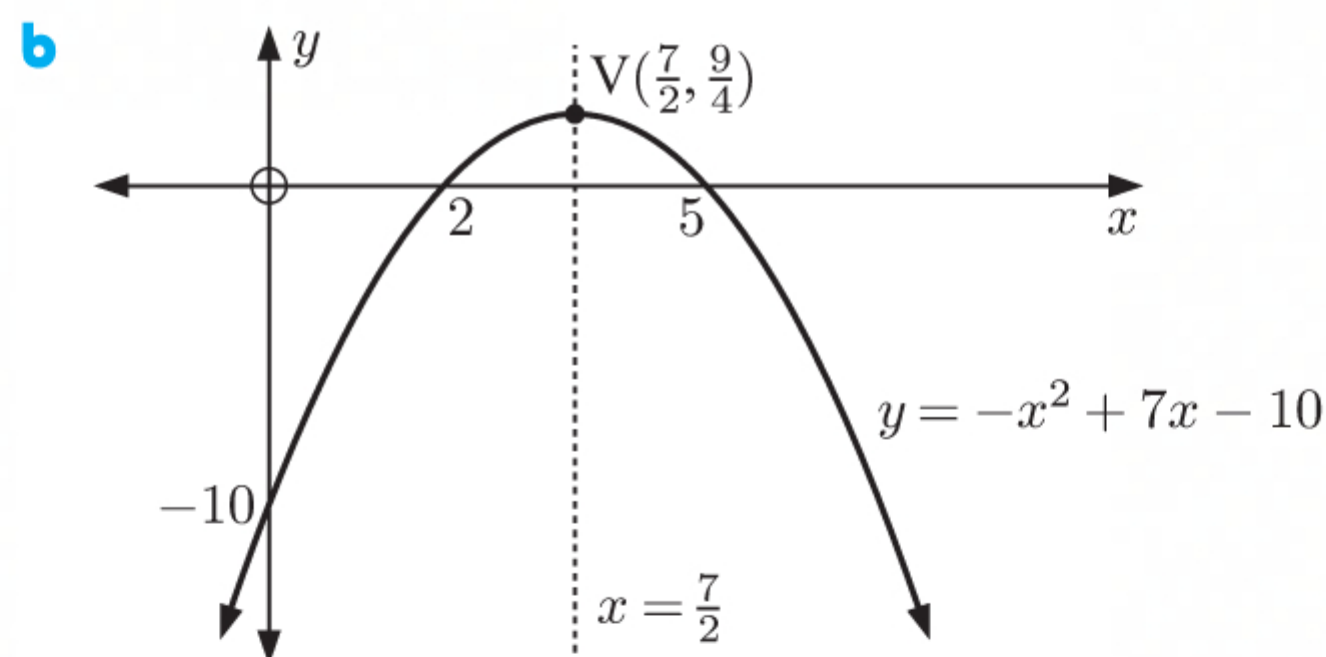
\therefore the x -intercepts are
 2 and 5 .



iii $\frac{7}{2}$ is halfway between 2 and 5 , so the axis of symmetry is $x = \frac{7}{2}$.

iv When $x = \frac{7}{2}$, $y = -(\frac{7}{2})^2 + 7(\frac{7}{2}) - 10$
 $= -\frac{49}{4} + \frac{49}{2} - 10$
 $= \frac{9}{4}$

\therefore the vertex is $(\frac{7}{2}, \frac{9}{4})$.



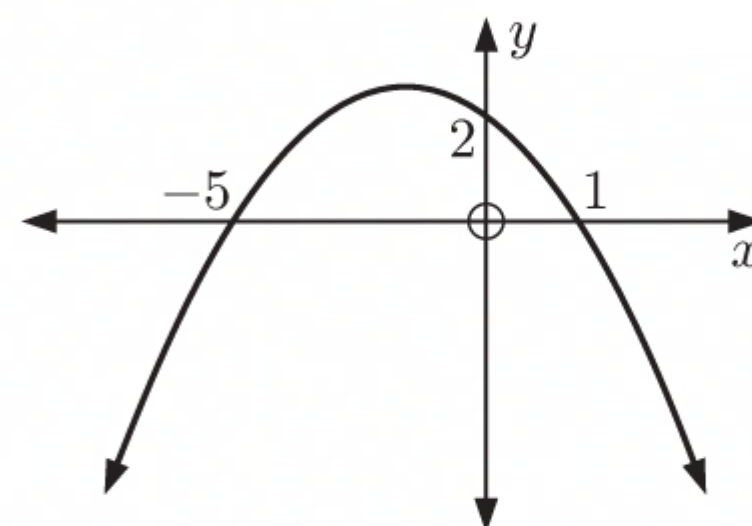
c The domain is $\{x \mid x \in \mathbb{R}\}$.

The range is $\{y \mid y \leq \frac{9}{4}\}$.

12 a Since the x -intercepts are -5 and 1 , $y = a(x + 5)(x - 1)$.

When $x = 0$, $y = 2$
 $\therefore 2 = a(5)(-1)$
 $\therefore a = -\frac{2}{5}$

The quadratic is $y = -\frac{2}{5}(x + 5)(x - 1)$.



- b** -2 is halfway between -5 and 1 , so the axis of symmetry is $x = -2$.

$$\begin{aligned}\text{When } x = -2, \quad y &= -\frac{2}{5}(-2+5)(-2-1) \\ &= -\frac{2}{5}(3)(-3) \\ &= \frac{18}{5}\end{aligned}$$

\therefore the vertex is $(-2, \frac{18}{5})$.

- 13 a** Since the graph touches the x -axis at 3 , the quadratic has the form $y = a(x-3)^2$, $a \neq 0$.

$$\begin{aligned}\text{When } x = 2, \quad y &= 2 \\ \therefore 2 &= a(2-3)^2 \\ \therefore 2 &= a(-1)^2 \\ \therefore a &= 2\end{aligned}$$

$$\begin{aligned}\text{The quadratic is } y &= 2(x-3)^2 \\ &= 2(x^2 - 6x + 9) \\ &= 2x^2 - 12x + 18\end{aligned}$$

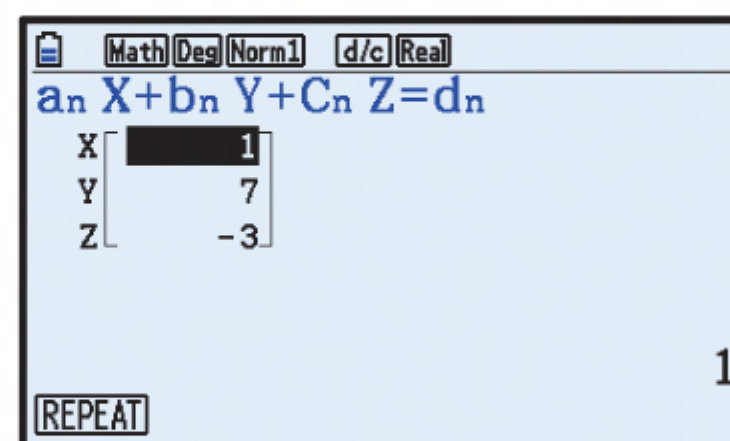
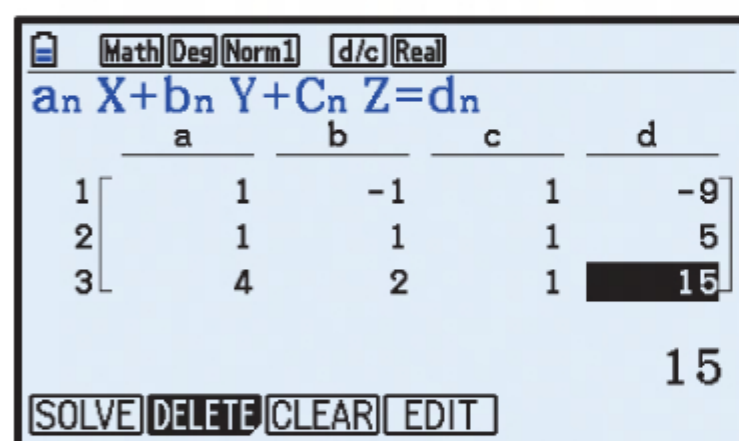
- b** Since the x -intercepts are 3 and -2 , the quadratic has the form $y = a(x-3)(x+2)$, $a \neq 0$.

$$\begin{aligned}\text{When } x = 0, \quad y &= 3 \\ \therefore 3 &= a(-3)(2) \\ \therefore a &= -\frac{1}{2}\end{aligned}$$

$$\begin{aligned}\text{The quadratic is } y &= -\frac{1}{2}(x-3)(x+2) \\ &= -\frac{1}{2}(x^2 - x - 6) \\ &= -\frac{1}{2}x^2 + \frac{1}{2}x + 3\end{aligned}$$

- c** When $x = -1$, $y = -9$ $\therefore -9 = a(-1)^2 + b(-1) + c$ or $a - b + c = -9$
 When $x = 1$, $y = 5$ $\therefore 5 = a(1)^2 + b(1) + c$ or $a + b + c = 5$
 When $x = 2$, $y = 15$ $\therefore 15 = a(2)^2 + b(2) + c$ or $4a + 2b + c = 15$

We solve the system of equations $\begin{cases} a - b + c = -9 \\ a + b + c = 5 \\ 4a + 2b + c = 15 \end{cases}$ simultaneously using technology.



We find that $a = 1$, $b = 7$, and $c = -3$.

So, the quadratic is $y = x^2 + 7x - 3$.

d The vertex has x -coordinate 3 $\therefore -\frac{b}{2a} = 3$
 $\therefore -b = 6a$
 $\therefore 6a + b = 0$

When $x = 3$, $y = 15$ $\therefore 15 = a(3)^2 + b(3) + c$ or $9a + 3b + c = 15$

When $x = 1$, $y = 7$ $\therefore 7 = a(1)^2 + b(1) + c$ or $a + b + c = 7$

We solve the system of equations $\begin{cases} 6a + b = 0 \\ 9a + 3b + c = 15 \\ a + b + c = 7 \end{cases}$ simultaneously using technology.

	a	b	c	d
1	6	1	0	0
2	9	3	1	15
3	1	1	1	7

	X	Y	Z
	-2	12	-3

We find that $a = -2$, $b = 12$, and $c = -3$.

So, the quadratic is $y = -2x^2 + 12x - 3$.

14 Let the quadratic function be $y = ax^2 + bx + c$.

When $x = 2$, $y = 5$ $\therefore 5 = a(2)^2 + b(2) + c$ or $4a + 2b + c = 5$

When $x = 6$, $y = -1$ $\therefore -1 = a(6)^2 + b(6) + c$ or $36a + 6b + c = -1$

When $x = 2$, $y = -3$ $\therefore -3 = a(2)^2 + b(2) + c$ or $4a + 2b + c = -3$

We solve the system of equations $\begin{cases} 4a + 2b + c = 5 \\ 36a + 6b + c = -1 \\ 4a + 2b + c = -3 \end{cases}$ simultaneously using technology.

	a	b	c	d
1	4	2	1	5
2	36	6	1	-1
3	4	2	1	-3

	No Solution
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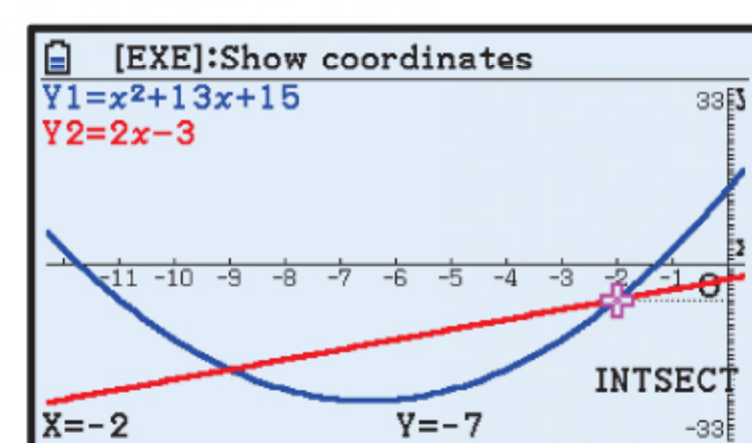
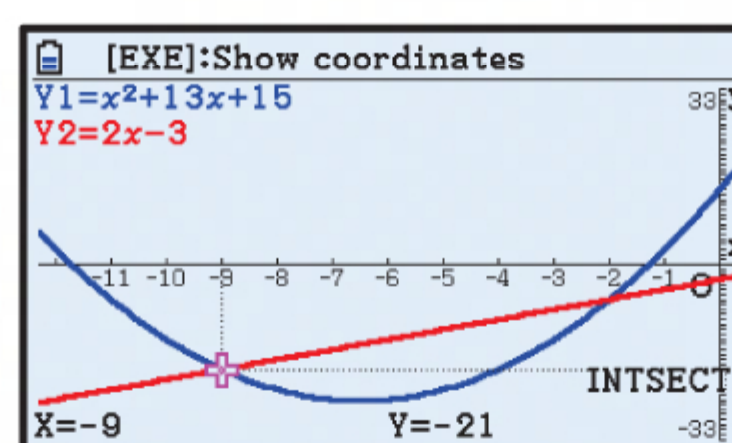
We find that there are no solutions for a , b , and c .

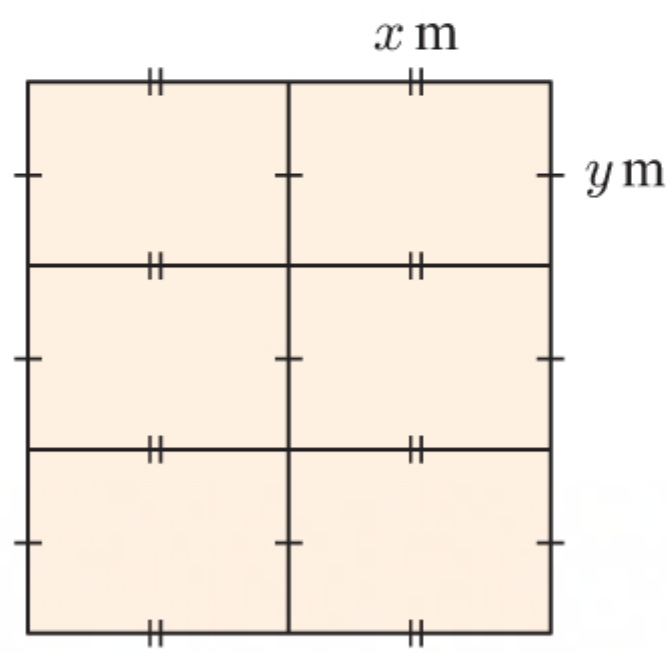
So, there is no such quadratic.

The x -coordinate 2 corresponds to two different points, which is not possible for a quadratic function.

15 We graph $Y_1 = X^2 + 13X + 15$ and $Y_2 = 2X - 3$ on the same set of axes.

The graphs intersect at $(-9, -21)$ and $(-2, -7)$.




16 a

There are 8 sections of fence with length x m, and 9 sections of fence with length y m, with total length 600 m.

$$\therefore 8x + 9y = 600$$

- c** The area A is a quadratic with $a = -\frac{8}{9}$, $b = \frac{200}{3}$, and $c = 0$.

Since $a < 0$, its shape is .

The maximum area occurs when $x = -\frac{b}{2a} = -\frac{\frac{200}{3}}{2(-\frac{8}{9})} = \frac{75}{2}$.

$$\begin{aligned} \text{When } x = \frac{75}{2}, \quad y &= \frac{600 - 8(\frac{75}{2})}{9} \\ &= \frac{600 - 300}{9} \\ &= \frac{300}{9} \\ &= \frac{100}{3} = 33\frac{1}{3} \end{aligned}$$

$$\begin{aligned} \text{and } A &= -\frac{8}{9}\left(\frac{75}{2}\right)^2 + \frac{200}{3}\left(\frac{75}{2}\right) \\ &= -1250 + 2500 \\ &= 1250 \end{aligned}$$

\therefore the maximum possible area for each pen is 1250 m^2 when the dimensions are $37\frac{1}{2} \text{ m}$ by $33\frac{1}{3} \text{ m}$.


- 17 a** If $\$x$ is the price increase of the sunglasses, then the retailer will lose $\frac{x}{1.5}$ customers.

So, the revenue $R = \text{cost of sunglasses} \times \text{number of customers}$

$$= (45 + x) \left(50 - \frac{x}{1.5}\right) \text{ dollars per day}$$

$$\begin{aligned} \text{b } R &= (45 + x) \left(50 - \frac{x}{1.5}\right) \\ &= 2250 - 30x + 50x - \frac{2}{3}x^2 \\ &= -\frac{2}{3}x^2 + 20x + 2250 \end{aligned}$$

- c** For the quadratic function R , $a = -\frac{2}{3}$, $b = 20$, and $c = 2250$.

Since $a < 0$, the shape is .

The maximum revenue occurs when $x = -\frac{b}{2a} = -\frac{20}{2(-\frac{2}{3})} = 15$.

$$\begin{aligned} \text{When } x = 15, \quad R &= -\frac{2}{3}(15)^2 + 20(15) + 2250 \\ &= -150 + 300 + 2250 \\ &= 2400 \end{aligned}$$

\therefore the maximum daily revenue is \$2400 when the sunglasses are sold for $\$45 + \$15 = \$60$ each.

- b** If $8x + 9y = 600$, then $y = \frac{600 - 8x}{9}$.

The area of each pen is $A = xy$.

Substituting $y = \frac{600 - 8x}{9}$ into A we get

$$A = x \left(\frac{600 - 8x}{9} \right)$$

$$\therefore A = \frac{600x}{9} - \frac{8x^2}{9}$$

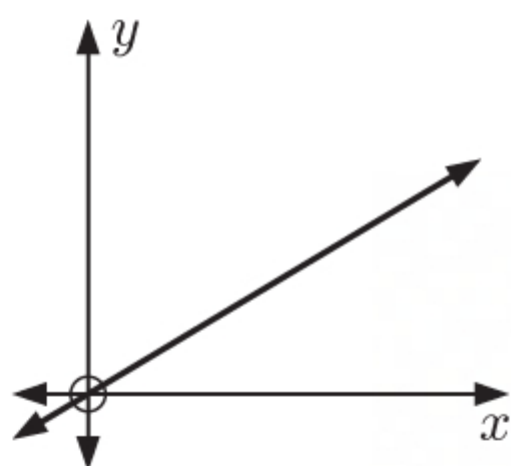
$$\therefore A = -\frac{8}{9}x^2 + \frac{200}{3}x \text{ m}^2$$

Chapter 7

DIRECT AND INVERSE VARIATION

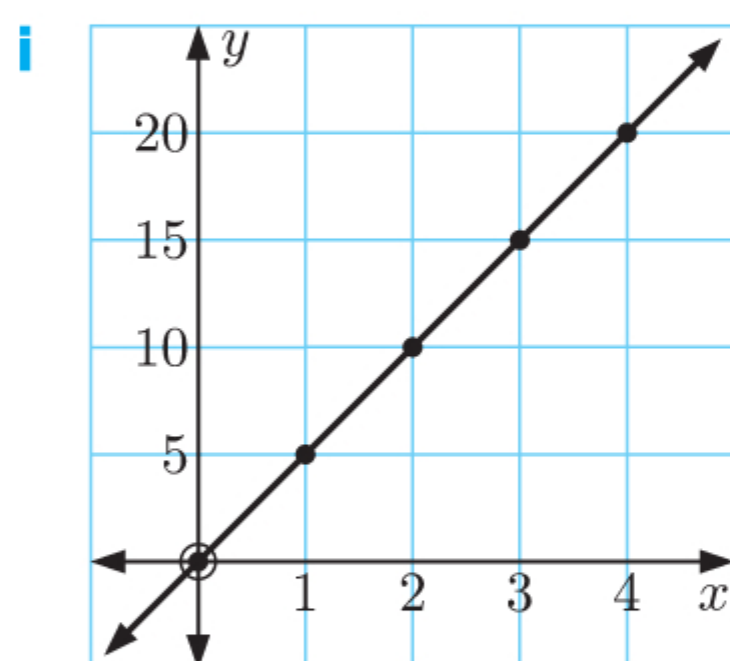
EXERCISE 7A

- 1** **D** indicates that y is directly proportional to x , as it is a straight line which passes through the origin.



2 a

x	0	1	2	3	4
y	0	5	10	15	20



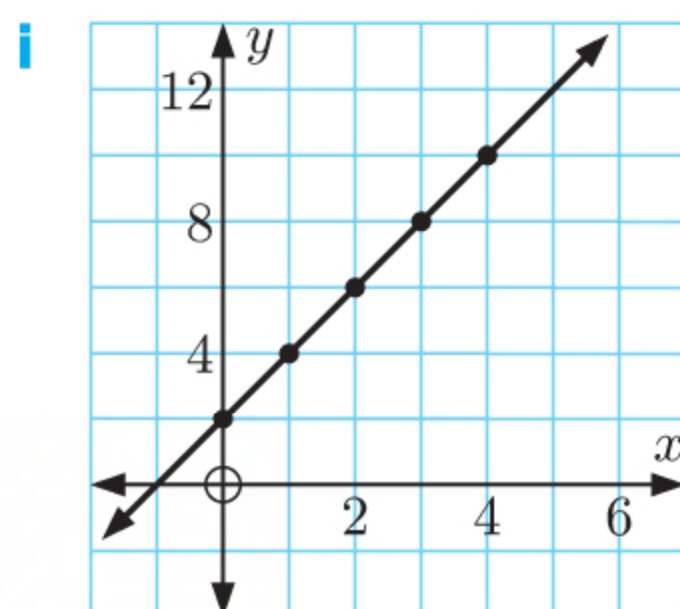
- ii** The points lie on a straight line which passes through the origin, so $y \propto x$.

$$\text{The gradient of the line} = \frac{5 - 0}{1 - 0} = 5$$

\therefore the proportionality constant $k = 5$.

b

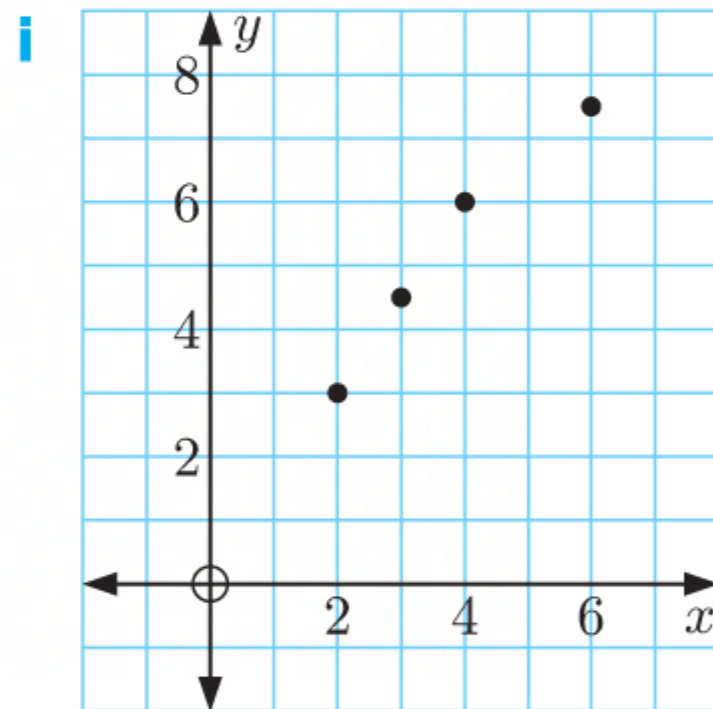
x	0	1	2	3	4
y	2	4	6	8	10



- ii** The points lie on a straight line which does not pass through the origin, so y is not directly proportional to x .

c

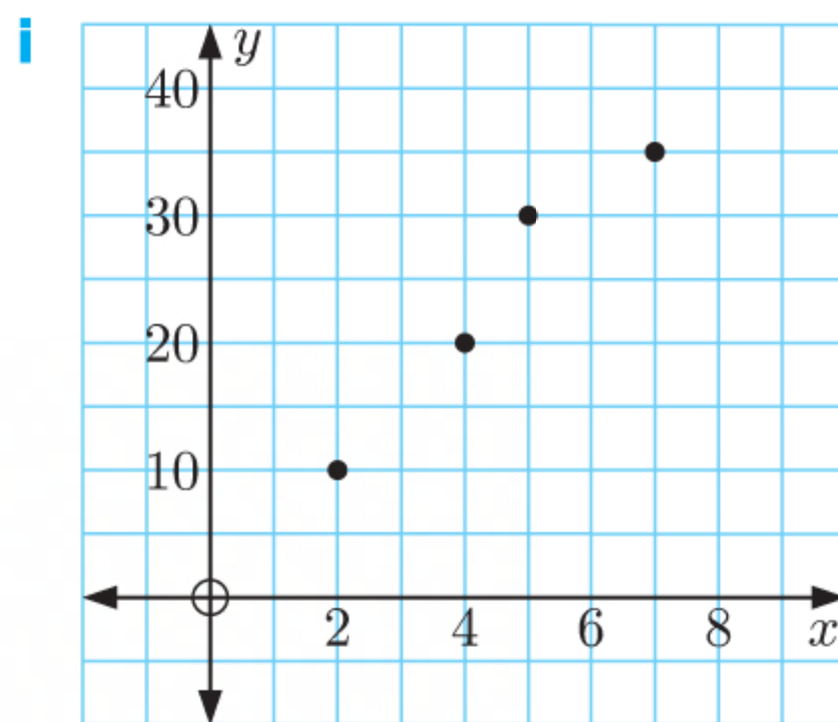
x	2	3	4	6
y	3	4.5	6	7.5



- ii** The points do not lie on a straight line, so y is not directly proportional to x .

d

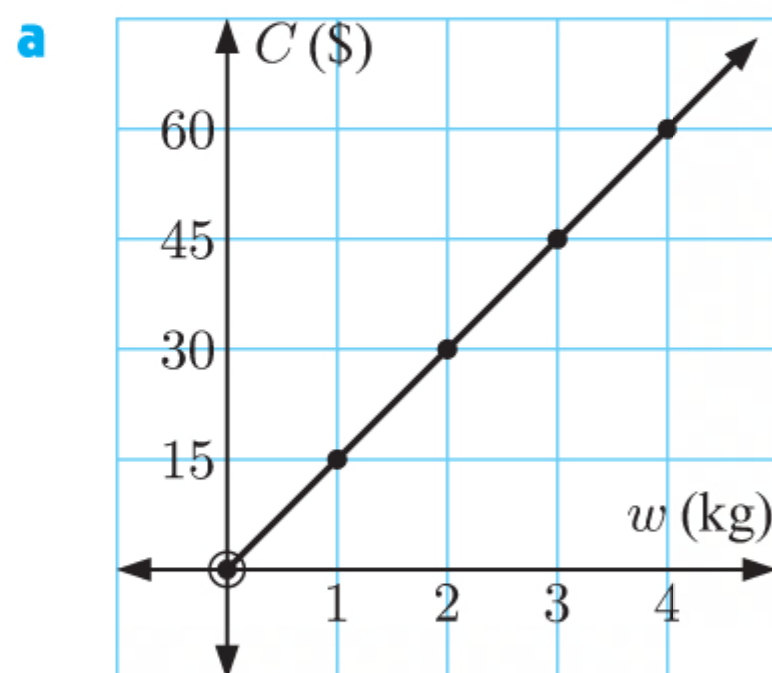
x	2	4	5	7
y	10	20	30	35



- ii** The points do not lie on a straight line, so y is not directly proportional to x .

3

Weight (w kg)	0	1	2	3	4
Cost ($\$C$)	0	15	30	45	60



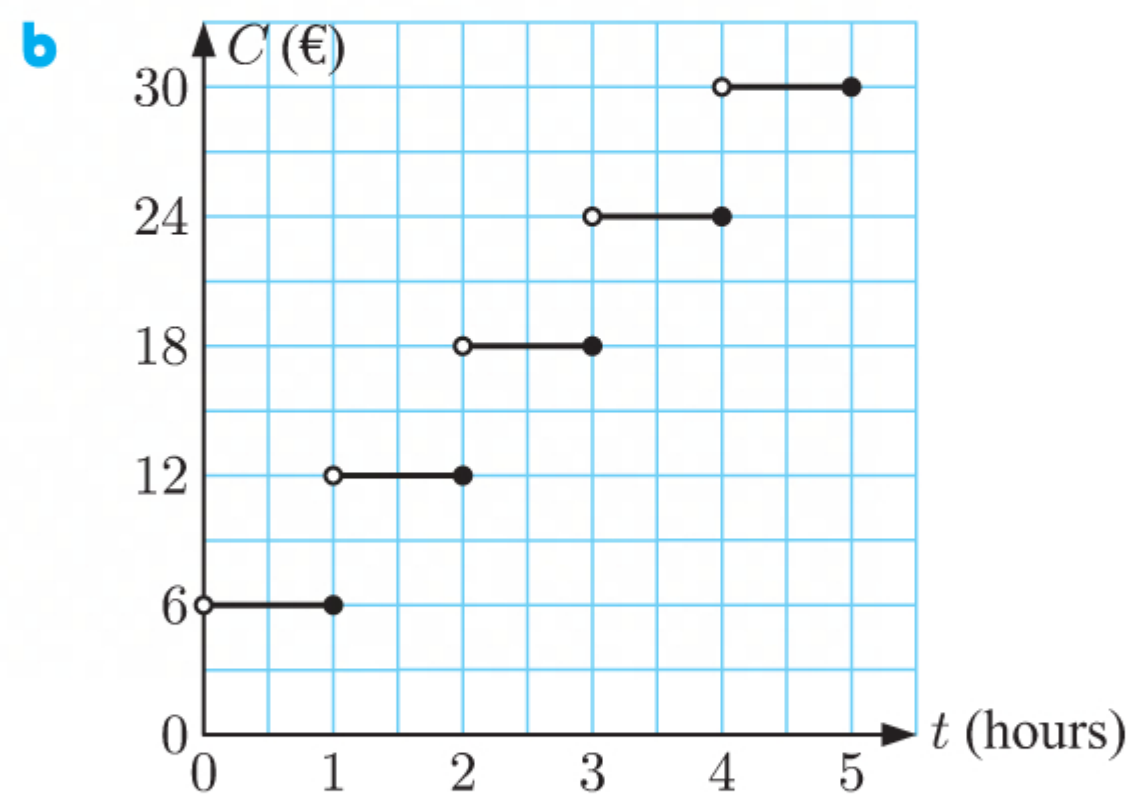
- b** The graph is a straight line which passes through the origin, so C is directly proportional to w .

c The gradient of the line $= \frac{15 - 0}{1 - 0} = 15$
 $\therefore C = 15w$

4

Time (t hours)	Cost ($\text{€}C$)
$0 < t \leq 1$	6
$1 < t \leq 2$	12
$2 < t \leq 3$	18
$3 < t \leq 4$	24
$4 < t \leq 5$	30

- a**
- i** When $t = 1$, the cost is €6.
 - ii** When $t = 2$, the cost is €12.
 - iii** When $t = 2.5$, the cost is €18.
 - iv** When $t = 4$, the cost is €24.



The graph of C against t is not a straight line through the origin.
 $\therefore C$ and t are not directly proportional.

5 $y \propto x$

a If x is doubled, then
 x is multiplied by 2
 $\therefore y$ is multiplied by 2
 $\therefore y$ is doubled.

c If x is divided by 3, then
 x is multiplied by $\frac{1}{3}$
 $\therefore y$ is multiplied by $\frac{1}{3}$
 $\therefore y$ is divided by 3.

e If x is decreased by 70%, then
 x is multiplied by $1 - 0.7 = 0.3$
 $\therefore y$ is multiplied by 0.3
 $\therefore y$ is decreased by 70%.

b If y is multiplied by 6, then
 x is multiplied by 6.

d If y is increased by 20%, then
 y is multiplied by 1.2
 $\therefore x$ is multiplied by 1.2
 $\therefore x$ is increased by 20%.

f Now $y = kx$ or $x = \frac{y}{k}$ where k is the proportionality constant.

If y is increased by 10, then

y becomes $y + 10$

$$\begin{aligned} \therefore x \text{ becomes } \frac{y + 10}{k} &= \frac{y}{k} + \frac{10}{k} \\ &= x + \frac{10}{k} \end{aligned}$$

$\therefore x$ is increased by $\frac{10}{k}$.

6 When the temperature in $^{\circ}\text{C}$ is 0, the temperature in $^{\circ}\text{F}$ is 32, so the graph of these variables does not pass through the origin.
 \therefore these variables are not directly proportional.

7 We assume the glass is a truncated cone, which is initially empty.

a The *volume of water* is increasing at a constant rate, and there is initially no water in the glass. So, *volume of water* and *time* are **directly proportional**.

b The *volume of water* and *height* are both initially 0, but adding a unit of volume will add less and less to the height as more water is added, as the radius is getting larger. So, *volume of water* and *height* are **not directly proportional**.

c *Height* and *slant height* are both initially 0, and form two sides of a right angled triangle.

$$\text{Now } \cos \theta = \frac{\text{height}}{\text{slant height}}$$

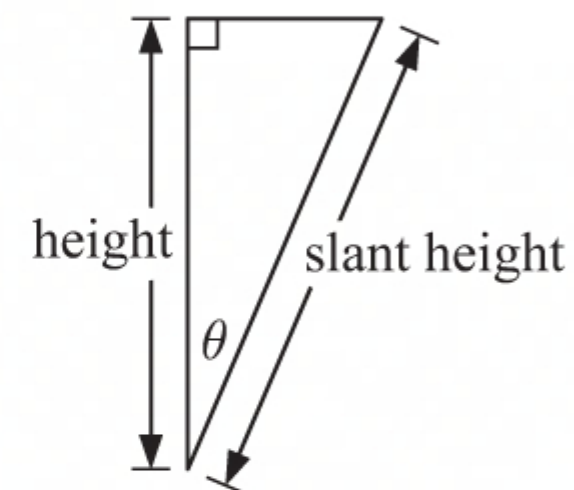
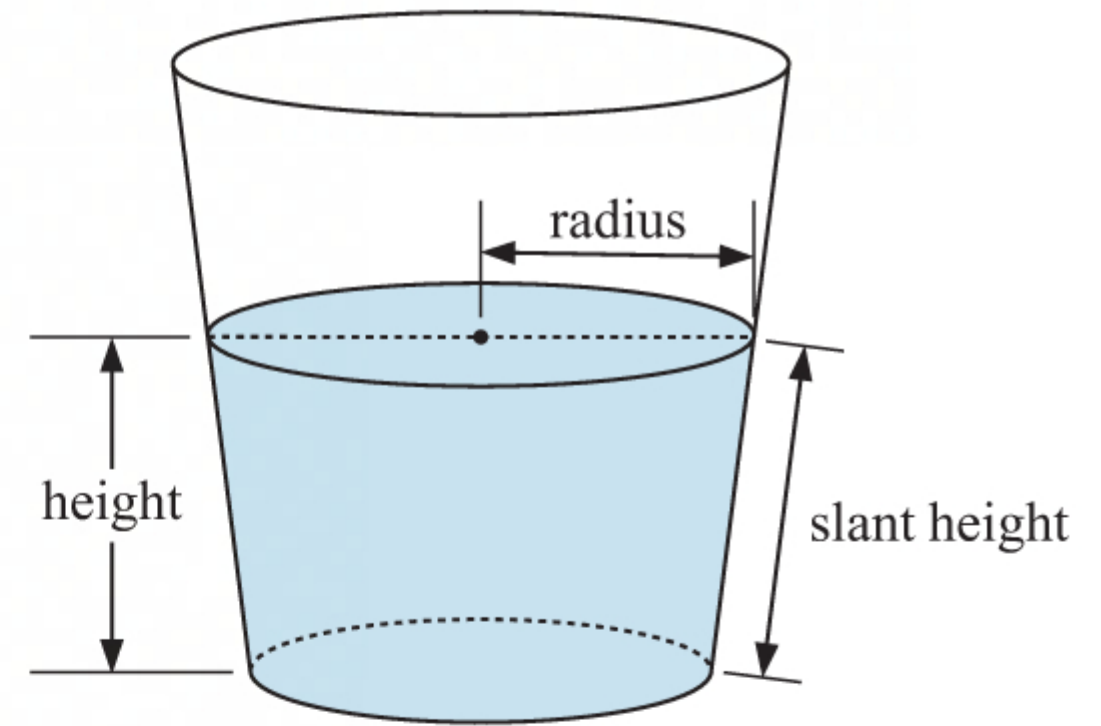
$$\therefore \text{height} = \text{slant height} \times \cos \theta$$

where $\cos \theta$ is a constant.

So, *height* and *slant height* are **directly proportional**.

d The *height* is initially 0, but the *radius* is not. So, *height* and *radius* are **not directly proportional**.

e *Slant height* is directly proportional to *height* (from c), and *weight of water* is directly proportional to *volume of water*. However *height* and *volume of water* are not directly proportional (from b). So, *slant height* and *weight of water* are **not directly proportional**.



8 $y \propto x$

a

x	7	21
y	24	?

$\times 3$

To change x from 7 to 21, we multiply by 3.

Since $y \propto x$, we must also multiply y by 3.

$$\therefore y = 24 \times 3 = 72$$

b

x	7	?
y	24	30

$\times \frac{30}{24}$

To change y from 24 to 30, we multiply by $\frac{30}{24} = \frac{5}{4}$.

Since $y \propto x$, we must also multiply x by $\frac{5}{4}$.

$$\begin{aligned} \therefore x &= 7 \times \frac{5}{4} \\ &= \frac{35}{4} \end{aligned}$$

9 $M \propto l$

a

l	9	45
M	40	?

$\times 5$

To change l from 9 to 45, we multiply by 5.

Since $M \propto l$, we must also multiply M by 5.

$$\therefore M = 40 \times 5 = 200$$

b

l	9	?
M	40	8

$\times \frac{8}{40}$

To change M from 40 to 8, we multiply by $\frac{1}{5}$.

Since $M \propto l$, we must also multiply l by $\frac{1}{5}$.

$$\therefore l = 9 \times \frac{1}{5} = \frac{9}{5}$$

- 10 a** Energy usage E and time t are directly proportional, so $E = kt$ where k is the proportionality constant.

When $t = 2$ hours, $E = 3.6$ kWh, so $3.6 = 2k$

$$\therefore k = 1.8$$

This means the heater uses 1.8 kWh of energy each hour.

- b** When $t = 1.5$ hours, $E = 1.8 \times 1.5 = 2.7$ kWh.

If the heater is on for 1.5 hours, it uses 2.7 kWh of energy.

- c** When $E = 6$ kWh, $6 = 1.8t$

$$\therefore t = \frac{6}{1.8} = 3\frac{1}{3} \text{ hours}$$

The heater uses 6 kWh of energy when it is on for $3\frac{1}{3}$ hours or 3 hours 20 minutes.

- 11** Let D m be represented as d cm on Isabella's diagram.

$\therefore D \propto d$ so $D = kd$ where k is the proportionality constant.

- a** When $D = 60$, $d = 3$, so $60 = k(3)$

$$\therefore k = 20$$

So, 1 cm on the diagram represents an actual length of 20 m.

- b** When $D = 110$, $110 = 20d$

$$\therefore d = 5.5$$

\therefore the football pitch is 5.5 cm long on Isabella's diagram.

- c** When $d = 15$, $D = 20 \times 15$

$$= 300$$

\therefore the actual length of the fence is 300 m.

- 12** The player's momentum p is directly proportional to his velocity v .

$\therefore p = kv$ where k is a constant.

When $v = 5 \text{ m s}^{-1}$, $p = 610 \text{ kg m s}^{-1}$, so $610 = k(5)$

$$\therefore k = 122$$

So, $p = 122v$.

- a** When $v = 3 \text{ m s}^{-1}$, $p = 122 \times 3$
 $= 366 \text{ kg m s}^{-1}$

- b** When $p = 420 \text{ kg m s}^{-1}$, $420 = 122v$

$$\therefore v = \frac{420}{122}$$

$$\approx 3.44 \text{ m s}^{-1}$$

- 13 a i** If the plumber is called out but spends no time doing labour ($t = 0$), then the charge $C = 30$.

\therefore the graph of C against t has C -intercept 30, so does not pass through the origin.

$\therefore C$ and t are not directly proportional.

- ii** When $t = 0$, $C = 30$

$$\therefore \text{when } t = 0, C - 30 = 30 - 30 = 0$$

The plumber charges a constant rate per hour, so the graph of $(C - 30)$ against t passes through the origin and is a straight line.

$\therefore (C - 30)$ and t are directly proportional.

b $(C - 30) \propto t$, so $C - 30 = kt$

When $t = 1.5$, $C = 120$

$$\therefore 120 - 30 = k \times 1.5$$

$$\therefore k = \frac{90}{1.5} = 60$$

So, $C = 60t + 30$

When $t = 5$, $C = 60 \times 5 + 30$
 $= 330$

\therefore the charge for a 5 hour job is £330.

EXERCISE 7B

1 a $A = \pi r^2$, so $A \propto r^2$ and $k = \pi$.

b $V = \frac{4}{3}\pi r^3$, so $V \propto r^3$ and $k = \frac{4}{3}\pi$.

c $T = \frac{3n^4}{4} = \frac{3}{4}n^4$, so $T \propto n^4$ and $k = \frac{3}{4}$.

2 $y \propto x^3$

a If x is doubled, then
 x is multiplied by 2
 $\therefore x^3$ is multiplied by $2^3 = 8$
 $\therefore y$ is multiplied by 8.

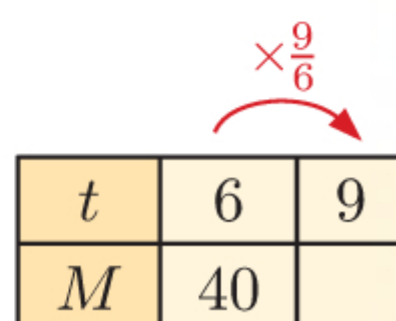
b If x is divided by 10, then
 x is multiplied by $\frac{1}{10}$
 $\therefore x^3$ is multiplied by $\left(\frac{1}{10}\right)^3 = \frac{1}{1000}$
 $\therefore y$ is multiplied by $\frac{1}{1000}$
 $\therefore y$ is divided by 1000.

c If x is increased by 20%, then
 x is multiplied by 1.2
 $\therefore x^3$ is multiplied by $(1.2)^3 = 1.728$
 $\therefore y$ is multiplied by 1.728
 $\therefore y$ is increased by 72.8%.

d If y is multiplied by 2.5, then
 x^3 is multiplied by 2.5
 $\therefore x$ is multiplied by $\sqrt[3]{2.5}$.

3 $M \propto t^2$, $t > 0$

a



t	6	9
M	40	

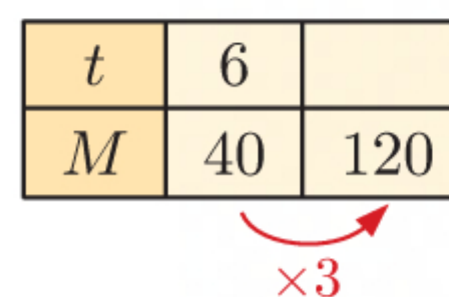
t is multiplied by $\frac{9}{6} = \frac{3}{2}$

$\therefore t^2$ is multiplied by $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$

$\therefore M$ is multiplied by $\frac{9}{4}$ {as $M \propto t^2$ }

$\therefore M = 40 \times \frac{9}{4} = 90$

b



t	6	
M	40	120

M is multiplied by 3

$\therefore t^2$ is multiplied by 3 {as $M \propto t^2$ }

$\therefore t$ is multiplied by $\sqrt{3}$ {as $t > 0$ }

$\therefore t = 6\sqrt{3} \approx 10.4$

4 $V \propto y^3$

a

y	3	12
V	30	

y is multiplied by 4
 $\therefore y^3$ is multiplied by $4^3 = 64$
 $\therefore V$ is multiplied by 64 {as $V \propto y^3$ }
 $\therefore V = 30 \times 64 = 1920$

b

y	3	
V	30	180

V is multiplied by 6
 $\therefore y^3$ is multiplied by 6 {as $V \propto y^3$ }
 $\therefore y$ is multiplied by $\sqrt[3]{6}$
 $\therefore y = 3\sqrt[3]{6} \approx 5.45$

5 The mass of glass m g is directly proportional to the square of the length l cm.

$$\therefore m = kl^2 \text{ where } k \text{ is a constant.}$$

When $m = 900$, $l = 30$, so $900 = k \times 30^2$

$$\therefore 900 = k \times 900$$

$$\therefore k = 1$$

So, $m = l^2$.

When $l = 50$, $m = 50^2 = 2500$

\therefore a 50 cm square sheet of glass has mass 2500 g.

6 The amount of medicine $V \propto d^3$

$$\therefore V = kd^3 \text{ where } k \text{ is a constant.}$$

When $d = 6$ cm, $V = 40$ mL, so $40 = k \times 6^3$

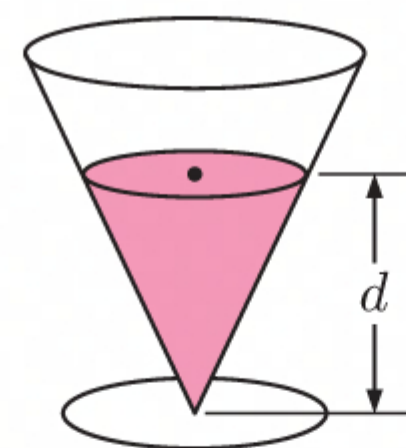
$$\therefore 40 = k \times 216$$

$$\therefore k = \frac{40}{216} = \frac{5}{27}$$

So $V = \frac{5}{27}d^3$.

a When $d = 4$ cm, $V = \frac{5}{27} \times 4^3$
 $= \frac{320}{27} \approx 11.9$ mL

b When $V = 30$ mL, $30 = \frac{5}{27}d^3$
 $\therefore d^3 = 27 \times 6$
 $\therefore d = 3\sqrt[3]{6} \approx 5.45$ cm



7 The volume V is directly proportional to the cube of its side lengths l , so $V \propto l^3$.

a If l is increased by 5%, then

l is multiplied by 1.05

$$\therefore l^3 \text{ is multiplied by } (1.05)^3 = 1.157625$$

$$\therefore V \text{ is multiplied by } 1.157625 \text{ {as } } V \propto l^3 \}$$

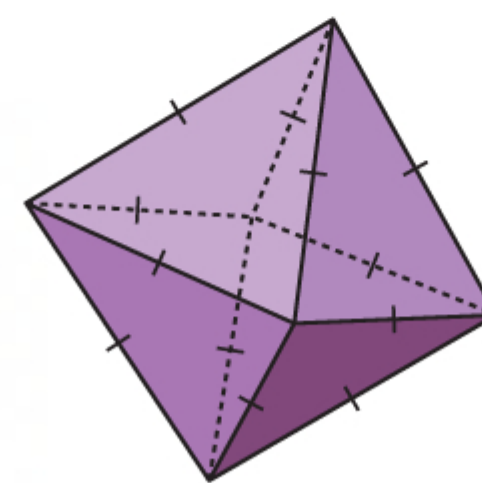
So, the volume is increased by about 15.8%.

b V is multiplied by 2

$$\therefore l^3 \text{ is multiplied by } 2 \text{ {as } } V \propto l^3 \}$$

$$\therefore l \text{ is multiplied by } \sqrt[3]{2} \approx 1.260$$

So, the side length is increased by about 26.0%.



8 a $E \propto m$ with proportionality constant $k = \frac{1}{2}v^2$.

b $E \propto v^2$ with proportionality constant $k = \frac{1}{2}m$.

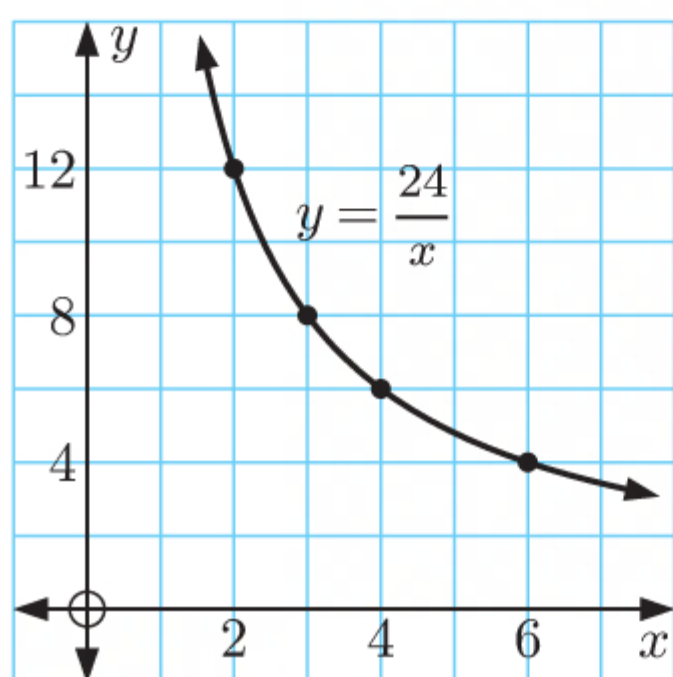
- c** If v is decreased by 10%, then
 v is multiplied by $1 - 0.1 = 0.9$
 $\therefore v^2$ is multiplied by $(0.9)^2$
 $\therefore E$ is multiplied by $(0.9)^2 = 0.81$ {as $E \propto v^2$ from **b**}
 So, the kinetic energy decreases by 19%.
- d** We assume that the amount of heat generated is directly proportional to the stopping distance d .
 So, $d \propto E$ and $E \propto v^2$ {from **b**}
 $\therefore d \propto v^2$

EXERCISE 7C

1 a

x	2	3	4	6
y	12	8	6	4
xy	24	24	24	24

$xy = 24$ for each point.
 $\therefore x$ and y are inversely proportional,
 and $y = \frac{24}{x}$.

**b**

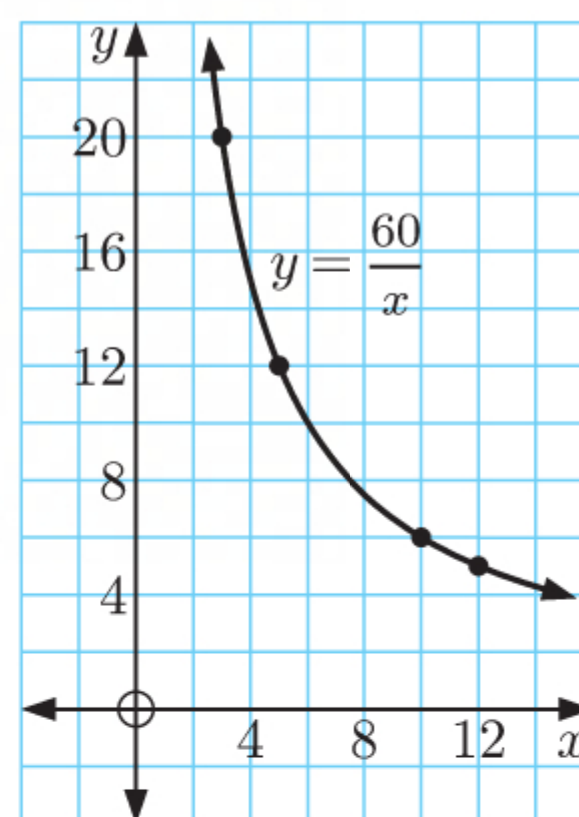
x	1	2	4	5
y	20	10	6	4
xy	20	20	24	20

xy is not the same value for each point.
 $\therefore x$ and y are not inversely proportional.

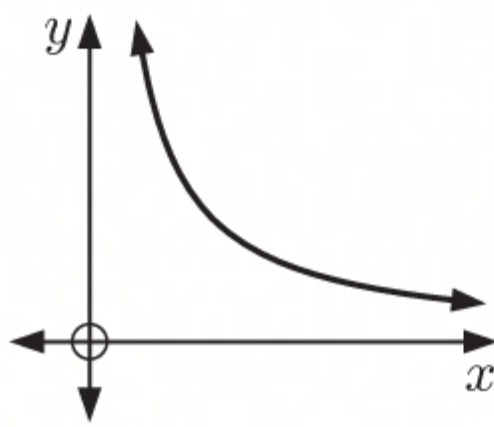
c

x	3	5	10	12
y	20	12	6	5
xy	60	60	60	60

$xy = 60$ for each point.
 $\therefore x$ and y are inversely proportional,
 and $y = \frac{60}{x}$.



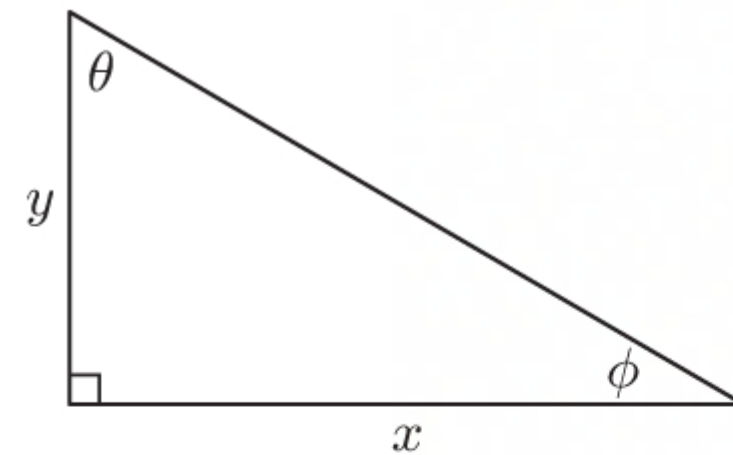
2



B indicates that y is inversely proportional to x , as the graph is a hyperbola.

3 **a** No, $\theta = 90 - \phi$. There is no constant k such that $\theta \times \phi = k$.
 $\therefore \theta$ and ϕ are not inversely proportional.

b $\tan \theta = \frac{x}{y}$ and $\tan \phi = \frac{y}{x}$
 $\therefore \tan \theta = \frac{1}{\tan \phi}$
 $\therefore \tan \theta$ and $\tan \phi$ are inversely proportional.



4 $y \propto \frac{1}{x}$

a If x is doubled, then
 x is multiplied by 2
 $\therefore y$ is multiplied by $\frac{1}{2}$
 $\therefore y$ is halved.

c If x is multiplied by $\frac{9}{5}$, then
 y is multiplied by $\frac{5}{9}$.

b If x is divided by 7, then
 x is multiplied by $\frac{1}{7}$
 $\therefore y$ is multiplied by 7.

d If x is increased by 30%, then
 x is multiplied by 1.3
 $\therefore y$ is multiplied by $\frac{1}{1.3} \approx 0.769$
 $\therefore y$ is decreased by about 23.1%.

5 $C \propto \frac{1}{t}$

a

t	6	18
C	15	

t is multiplied by 3
 $\therefore C$ is multiplied by $\frac{1}{3}$ {as $C \propto \frac{1}{t}$ }
 $\therefore C = \frac{15}{3} = 5$

b

t	6	
C	15	20

C is multiplied by $\frac{20}{15} = \frac{4}{3}$
 $\therefore t$ is multiplied by $\frac{3}{4}$ {as $C \propto \frac{1}{t}$ }
 $\therefore t = 6 \times \frac{3}{4} = \frac{9}{2} = 4.5$

6 The time taken t is inversely proportional to the number of gardeners n , so $t = \frac{k}{n}$.

$t = 6$ hours when $n = 5$, so $6 = \frac{k}{5}$
 $\therefore k = 30$

So when $n = 3$, $t = \frac{30}{3} = 10$ hours

\therefore it would take 3 gardeners 10 hours to do the task.

- 7** The object's acceleration a is inversely proportional to its mass m , so $a = \frac{k}{m}$.

$$a = 1.5 \text{ m s}^{-2} \text{ when } m = 5 \text{ kg, so } 1.5 = \frac{k}{5}$$

$$\therefore k = 7.5$$

So, $a = \frac{7.5}{m}$.

- a** When $m = 2 \text{ kg}$, $a = \frac{7.5}{2} = 3.75 \text{ m s}^{-2}$. **b** When $a = 10 \text{ m s}^{-2}$, $\frac{7.5}{m} = 10$
 $\therefore m = 0.75 \text{ kg}$

- 8 a** The *amount* in Wendy's savings account is constant, and the cost of the total number of shares is the *share price* multiplied by the *number of shares*.
 This is a relationship of the form $xy = k$, so *number of shares* is inversely proportional to *share price*.

- b i** If the share price drops by £0.15 from £6.25, then it has been multiplied by

$$\frac{6.25 - 0.15}{6.25} = \frac{122}{125}.$$

This means the number of shares Wendy can buy is multiplied by $\frac{125}{122}$, so she can now buy $\frac{125}{122}n$ shares (rounded down).

- ii** For Wendy to be able to buy 1.5 times as many shares, each share must be

$$\frac{1}{1.5} = \frac{2}{3} \text{ times the price.}$$

This means the price must decrease by $33\frac{1}{3}\%$.

EXERCISE 7D

1 $y \propto \frac{1}{x^3}$

- a** If x is doubled, then
 x is multiplied by 2
 $\therefore x^3$ is multiplied by $2^3 = 8$
 $\therefore \frac{1}{x^3}$ is multiplied by $\frac{1}{8}$
 $\therefore y$ is multiplied by $\frac{1}{8}$
 $\therefore y$ is divided by 8.

- c** If y is multiplied by 64, then
 $\frac{1}{x^3}$ is multiplied by 64
 $\therefore x^3$ is multiplied by $\frac{1}{64}$
 $\therefore x$ is multiplied by $\sqrt[3]{\frac{1}{64}} = \frac{1}{4}$
 $\therefore x$ is divided by 4.

- b** If x is multiplied by $\frac{3}{5}$, then
 x^3 is multiplied by $\left(\frac{3}{5}\right)^3 = \frac{27}{125}$
 $\therefore \frac{1}{x^3}$ is multiplied by $\frac{125}{27}$
 $\therefore y$ is multiplied by $\frac{125}{27}$.

2 $y \propto \frac{1}{x^2}$

a

x	8	24
y	27	

x is multiplied by 3

$\therefore x^2$ is multiplied by $3^2 = 9$

$\therefore \frac{1}{x^2}$ is multiplied by $\frac{1}{9}$

$\therefore y$ is multiplied by $\frac{1}{9}$ {as $y \propto \frac{1}{x^2}$ }

$\therefore y = 27 \times \frac{1}{9} = 3$

b

x	8	
y	27	75

y is multiplied by $\frac{75}{27} = \frac{25}{9}$

$\therefore \frac{1}{x^2}$ is multiplied by $\frac{25}{9}$ {as $y \propto \frac{1}{x^2}$ }

$\therefore x^2$ is multiplied by $\frac{9}{25}$

$\therefore x$ is multiplied by $\sqrt{\frac{9}{25}} = \frac{3}{5}$ {as $x > 0$ }

$\therefore x = 8 \times \frac{3}{5} = \frac{24}{5}$

3 $M \propto \frac{1}{c^3}$

a

c	12	8
M	64	

c is multiplied by $\frac{8}{12} = \frac{2}{3}$

$\therefore c^3$ is multiplied by $\left(\frac{2}{3}\right)^3 = \frac{8}{27}$

$\therefore \frac{1}{c^3}$ is multiplied by $\frac{27}{8}$

$\therefore M$ is multiplied by $\frac{27}{8}$ {as $M \propto \frac{1}{c^3}$ }

$\therefore M = 6.4 \times \frac{27}{8} = 216$

b

c	12	
M	64	1

M is multiplied by $\frac{1}{64}$

$\therefore \frac{1}{c^3}$ is multiplied by $\frac{1}{64}$ {as $M \propto \frac{1}{c^3}$ }

$\therefore c^3$ is multiplied by 64

$\therefore c$ is multiplied by $\sqrt[3]{64} = 4$

$\therefore c = 12 \times 4 = 48$

4 a The volume of a cylinder $V = \pi r^2 h$, where h is the height and r is the radius

$$\therefore h = \frac{V}{\pi} \times \frac{1}{r^2}$$

$$\therefore h \propto \frac{1}{r^2} \quad \{\text{since } V \text{ and } \pi \text{ are constants}\}$$

So, the height of the can is inversely proportional to the square of its radius.

- b** If r is multiplied by $\frac{3.39}{3.04}$, then

$$h \text{ is multiplied by } \left(\frac{3.04}{3.39}\right)^2 \quad \left\{\text{as } h \propto \frac{1}{r^2}\right\}$$

$$\therefore \text{ the height is } 12.9 \times \left(\frac{3.04}{3.39}\right)^2 \approx 10.4 \text{ cm.}$$

- c** If h is multiplied by $\frac{15.3}{12.9}$, then

$$\frac{1}{r^2} \text{ is multiplied by } \frac{15.3}{12.9} \quad \left\{\text{as } h \propto \frac{1}{r^2}\right\}$$

$$\therefore r \text{ is multiplied by } \sqrt{\frac{12.9}{15.3}} \quad \left\{\text{as } r > 0\right\}$$

$$\therefore \text{ the radius is } 3.04 \times \sqrt{\frac{12.9}{15.3}} \approx 2.79 \text{ cm.}$$

- d** The cans would otherwise be too narrow or too wide for practical use.

- 5** The tidal acceleration a is inversely proportional to the cube of the distance d , so $a \propto \frac{1}{d^3}$.

- a** If d is increased by 10%, then

$$d \text{ is multiplied by } 1.1$$

$$\therefore a \text{ is multiplied by } \frac{1}{(1.1)^3} \approx 0.751 \quad \left\{\text{as } a \propto \frac{1}{d^3}\right\}$$

So, the tidal acceleration is decreased by about 24.9%.

- b** If a is tripled, then

$$a \text{ is multiplied by } 3$$

$$\therefore \frac{1}{d^3} \text{ is multiplied by } 3 \quad \left\{\text{as } a \propto \frac{1}{d^3}\right\}$$

$$\therefore d^3 \text{ is multiplied by } \frac{1}{3}$$

$$\therefore d \text{ is multiplied by } \frac{1}{\sqrt[3]{3}} \approx 0.693$$

So, the distance is decreased by about 30.7%.

EXERCISE 7E

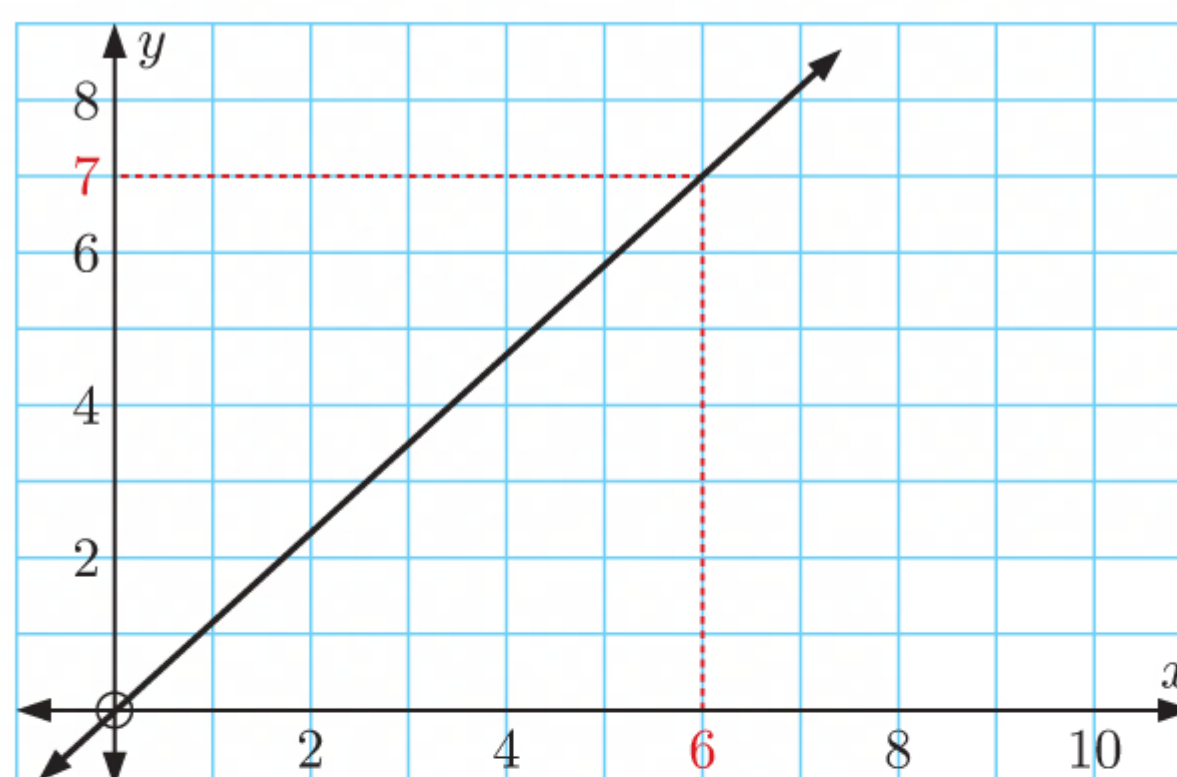
- 1 a** The graph of y against x is a straight line which passes through the origin.

- b** $y \propto x$, so $y = kx$ where k is a constant.

From the graph we see that $y = 7$ when $x = 6$.

$$\therefore 7 = k \times 6 \text{ and so } k = \frac{7}{6}.$$

So, the equation of the variation model is $y = \frac{7}{6}x$.



2 $H \propto d^2$, so $H = kd^2$ where k is a constant.

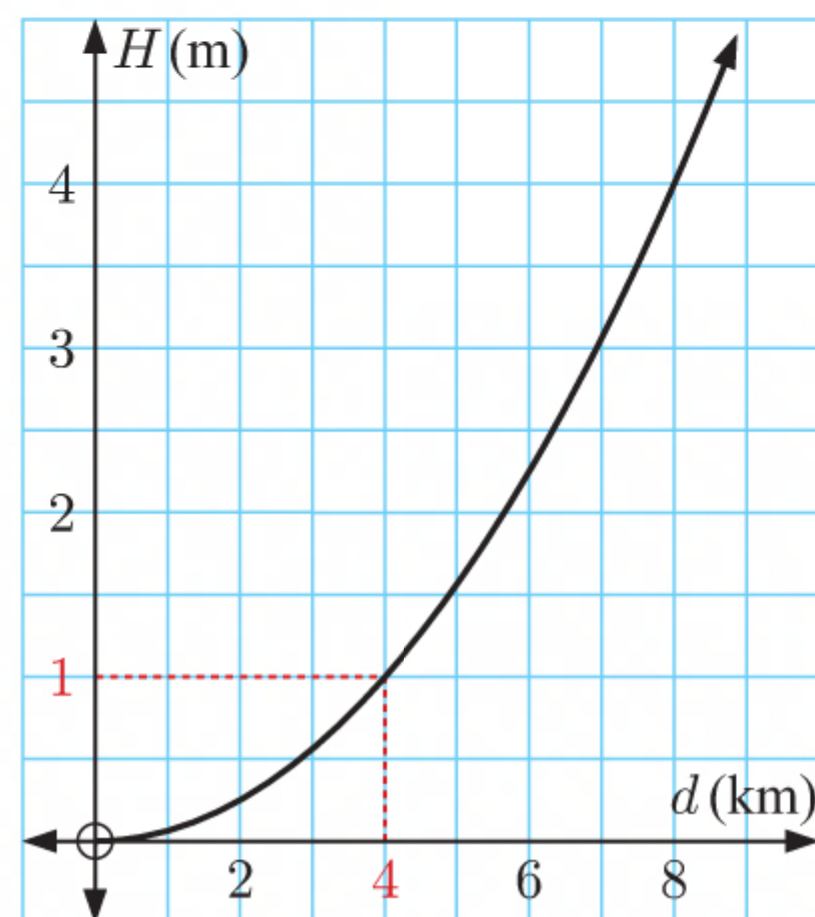
a From the graph, we see that $H = 1$ when $d = 4$.

$$\therefore 1 = k \times 4^2$$

$$\therefore k = \frac{1}{16}$$

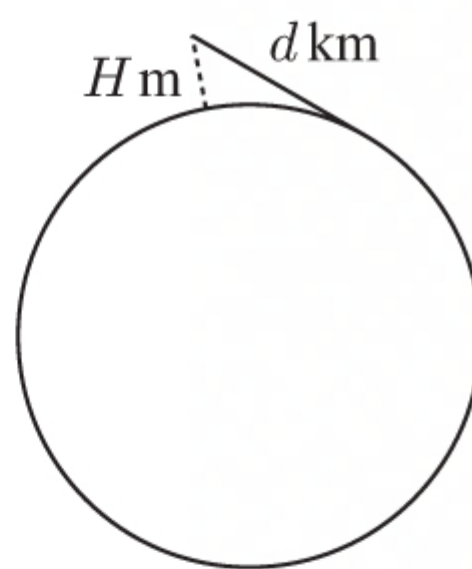
So, the equation of the variation model is

$$H = \frac{d^2}{16}.$$



b When $d = 16$, $H = \frac{1}{16} \times 16^2 = 16$.

So, Teresa is 16 m above sea level.



3 a $m \propto l^3$, so $m = kl^3$ where k is a constant.

$$m = 0.4 \text{ when } l = 20, \text{ so } 0.4 = k \times 20^3$$

$$\therefore 0.4 = 8000k$$

$$\therefore k = \frac{1}{20\,000}$$

$$\text{So, the variation model is } m = \frac{l^3}{20\,000}.$$

b When $l = 50$, $m = \frac{50^3}{20\,000}$

$$= \frac{125\,000}{20\,000} = 6.25$$

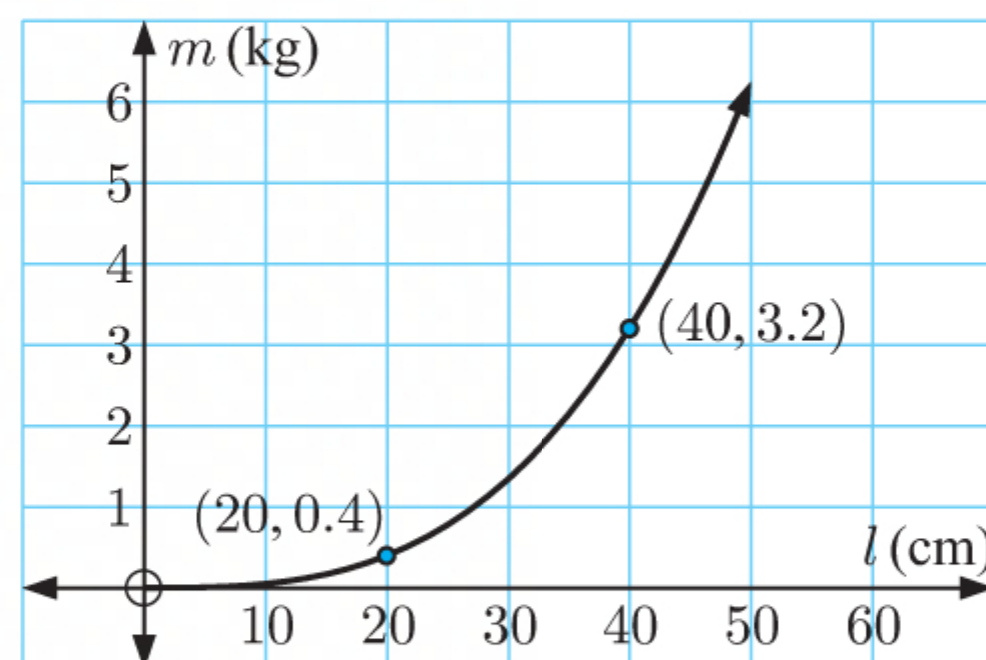
So, the mass of a 50 cm long model car is 6.25 kg.

c When $m = 1$, $\frac{l^3}{20\,000} = 1$

$$\therefore l^3 = 20\,000$$

$$\therefore l = \sqrt[3]{20\,000} \approx 27.1$$

So, the length of a model car with mass 1 kg is about 27.1 cm.

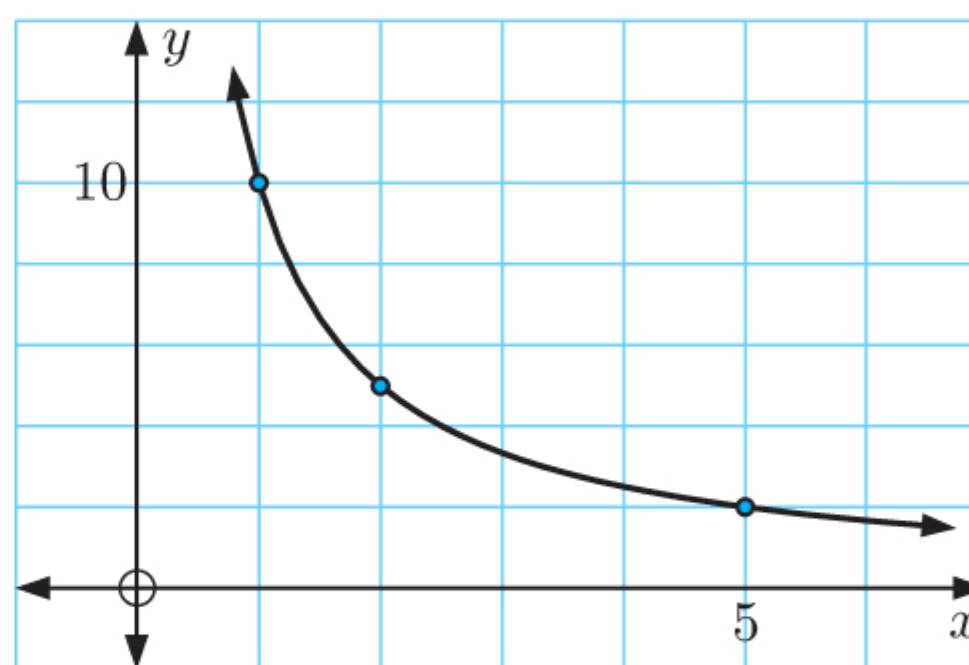


- 4 a** The points are $(1, 10)$, $(2, 5)$, and $(5, 2)$.
 $xy = 10$ for each point, so it is reasonable to assume that y varies inversely with x .

b $xy = 10$

$$\therefore y = \frac{10}{x}$$

c When $x = 8$, $y = \frac{10}{8}$
 $= \frac{5}{4}$



5 a

x	0.25	0.5	1	2
y	80	20	5	1.25
x^2y	5	5	5	5

$x^2y = 5$ for every data point

$$\therefore y = \frac{5}{x^2}$$

$$\therefore k = 5$$

b When $y = 0.5$, $\frac{5}{x^2} = 0.5$
 $\therefore x^2 = 10$
 $\therefore x = \sqrt{10}$ {as $x > 0$ }

6

v	10	20	30	40
R	0.5	4	13.5	32

- a i** $R \propto v^2$, so $R = kv^2$ where k is a constant

Using the first point, $0.5 = k \times 10^2$

$$\therefore k = \frac{1}{200}$$

$$\therefore R = \frac{v^2}{200}$$

ii When $v = 20$, $R = \frac{20^2}{200} = 2 \neq 4$

When $v = 30$, $R = \frac{30^2}{200} = 4.5 \neq 13.5$

When $v = 40$, $R = \frac{40^2}{200} = 8 \neq 32$

\therefore this model is incorrect.

b

v	10	20	30	40
R	0.5	4	13.5	32
$\frac{R}{v^3}$	$\frac{1}{2000}$	$\frac{1}{2000}$	$\frac{1}{2000}$	$\frac{1}{2000}$

$$\frac{R}{v^3} = \frac{1}{2000} \text{ for every data point, so } R = \frac{1}{2000} v^3.$$

c When $v = 50$, $R = \frac{1}{2000} \times 50^3$

$$= \frac{125\,000}{2000} = 62.5$$

So, when the car's velocity is 50 km h^{-1} , its air resistance is 62.5 units.

EXERCISE 7F

1 a

x	1	2	3	4
y	0.6	9.7	48.8	153.5

The correlation coefficient r is very close to 1, so the fit is excellent.

The power is very close to 4, so it is reasonable to conclude that y is directly proportional to x^4 .

The model is $y \approx 0.602x^4$.

PowerReg
a = 0.60215882
b = 3.99982755
r = 0.99999778
r ² = 0.99999556
MSe = 3.8454E-05
y = a · x ^b
[COPY]

b

x	2	3	6	9
y	100	29.6	3.7	1.1

The correlation coefficient r is very close to -1 , so the fit is excellent.

The power is very close to -3 , so it is reasonable to conclude that y is inversely proportional to x^3 .

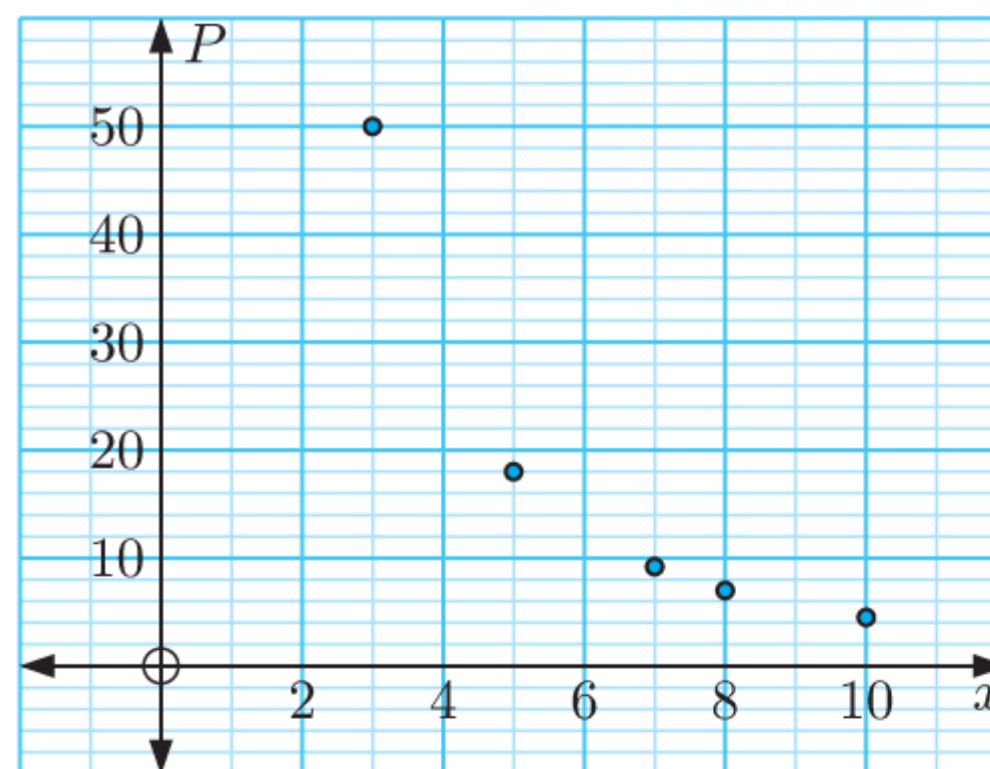
The model is $y \approx \frac{799}{x^3}$.

PowerReg
a = 798.571493
b = -2.9986991
r = -0.9999997
r ² = 0.99999957
MSe = 2.637E-06
y = a · x ^b
[COPY]

2 a

x	3	5	7	8	10
P	50	18	9.2	7	4.5

We expect inverse variation between the variables, as the points appear to lie on a curve which is asymptotic to both axes.



b The correlation coefficient r is very close to -1 , so the fit is excellent.

The power is very close to -2 , so it is reasonable to conclude that P is inversely proportional to x^2 .

The model is $P \approx \frac{451}{x^2}$.

c When $x = 4$, $P \approx \frac{451}{4^2}$

$$\approx 28.2$$

PowerReg
a = 450.652452
b = -2.001098
r = -0.999997
r ² = 0.99999418
MSe = 6.8335E-06
y = a · x ^b
[COPY]

3

Time (t minutes)	8	25	32	45
Percentage charge ($C\%$)	10	31	40	56

- a** We would expect direct variation between C and t because:
- Zach only charges his phone when the battery has completely run out, so when $t = 0$, $C = 0$. Hence the graph of C against t should pass through the origin.
 - We expect the charge to increase as time increases.

- b** The correlation coefficient r is very close to 1, so the fit is excellent.

The power is very close to 1, so it is reasonable to conclude that C is directly proportional to t .

The model is $C \approx 1.25t$.

- c** When $t = 56$, $C \approx 1.25 \times 56$
 ≈ 70

So, after 56 minutes, the phone will receive about 70% charge.

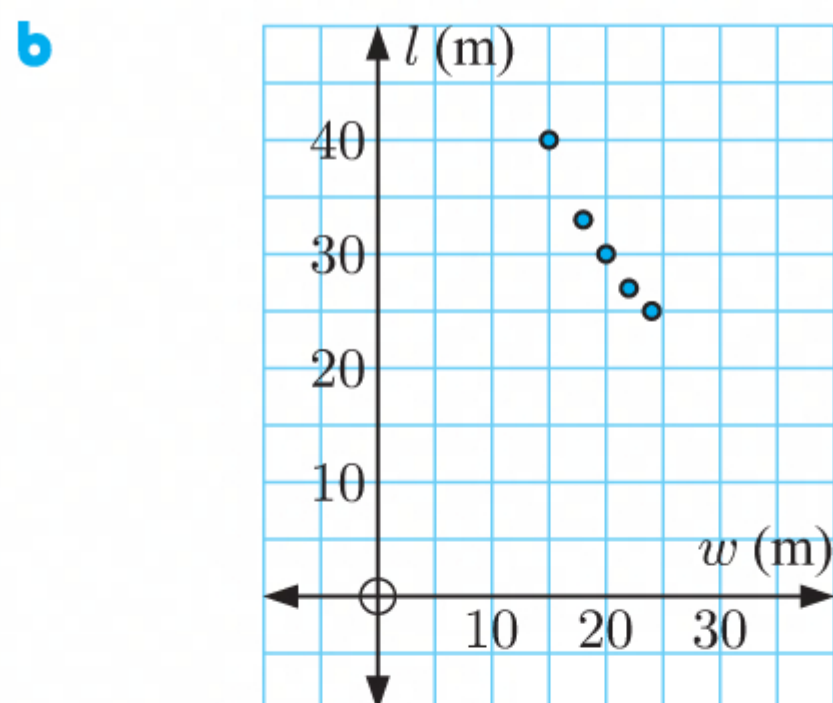
- d** The percentage charge $C\%$ is always between 0% and 100%, and $C \approx 1.25t$,
 so $0 \leq 1.25t \leq 100$
 $\therefore 0 \leq t \leq 80$

Deg Norm1 d/c Real PowerReg $a = 1.2544452$ $b = 0.99787713$ $r = 0.99998871$ $r^2 = 0.99997742$ $MSe = 1.89 \times 10^{-5}$ $y = a \cdot x^b$
COPY DRAW

4

Width (w metres)	15	18	20	22	24
Length (l metres)	40	33	30	27	25
lw	600	594	600	594	600

- a** $lw \approx 600$ for all l and w
 \therefore we expect inverse variation between l and w .



Yes, this diagram is consistent with the answer to **a**.
 The points appear to lie on a curve which is asymptotic to both axes, as expected for inverse variation.

- c** The correlation coefficient r is very close to -1 , so the fit is excellent.
 The power is very close to -1 , so it is reasonable to conclude that l and w are inversely proportional.

The model is $l \approx \frac{602}{w}$.

- d** When $w = 23$, $l \approx \frac{602}{23}$
 ≈ 26.2

So a 23 m wide block will be about 26.2 m long.

Deg Norm1 d/c Real PowerReg $a = 602.441233$ $b = -1.0027183$ $r = -0.9995525$ $r^2 = 0.99910527$ $MSe = 4.0074E-05$ $y = a \cdot x^b$
COPY

- e Most houses would require a block at least 15 m wide, and most blocks would be longer than they are wide.

If $l = w$ then $w \approx \sqrt{602} \approx 24.5$, so $15 \leq w \leq 24.5$.

5

Speed ($s \text{ m s}^{-1}$)	5.56	6.00	6.41	6.80	7.17	7.52
Turning radius ($R \text{ m}$)	1.0	1.2	1.4	1.6	1.8	2.0

- a The correlation coefficient r is very close to -1 , so the fit is excellent.

The model is $R \approx 0.0197s^{2.29}$.

- b When $s = 10$, $R \approx 0.0197 \times 10^{2.29} \approx 3.84$

So the turning radius of a car travelling at 10 m s^{-1} is about 3.84 m.

- c When $R = 4$, $0.0197s^{2.29} \approx 4$

$$\therefore s^{2.29} \approx \frac{4}{0.0197}$$

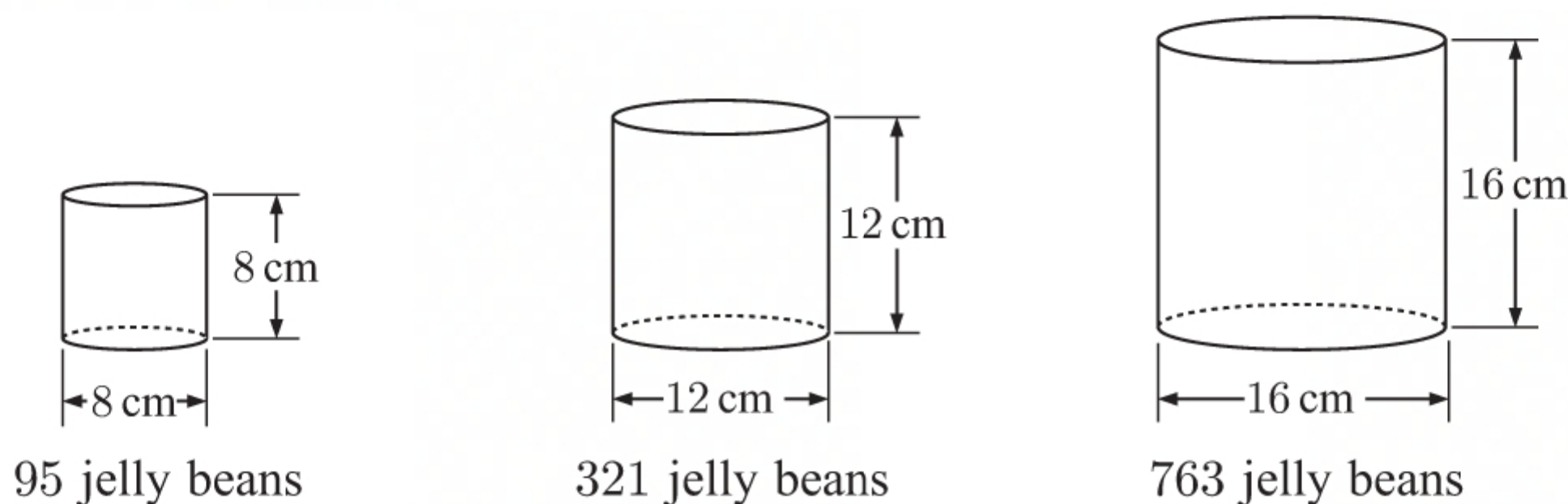
Using technology, $s \approx 10.2$

So a car with a turning radius of 4 m is travelling at about 10.2 m s^{-1} .

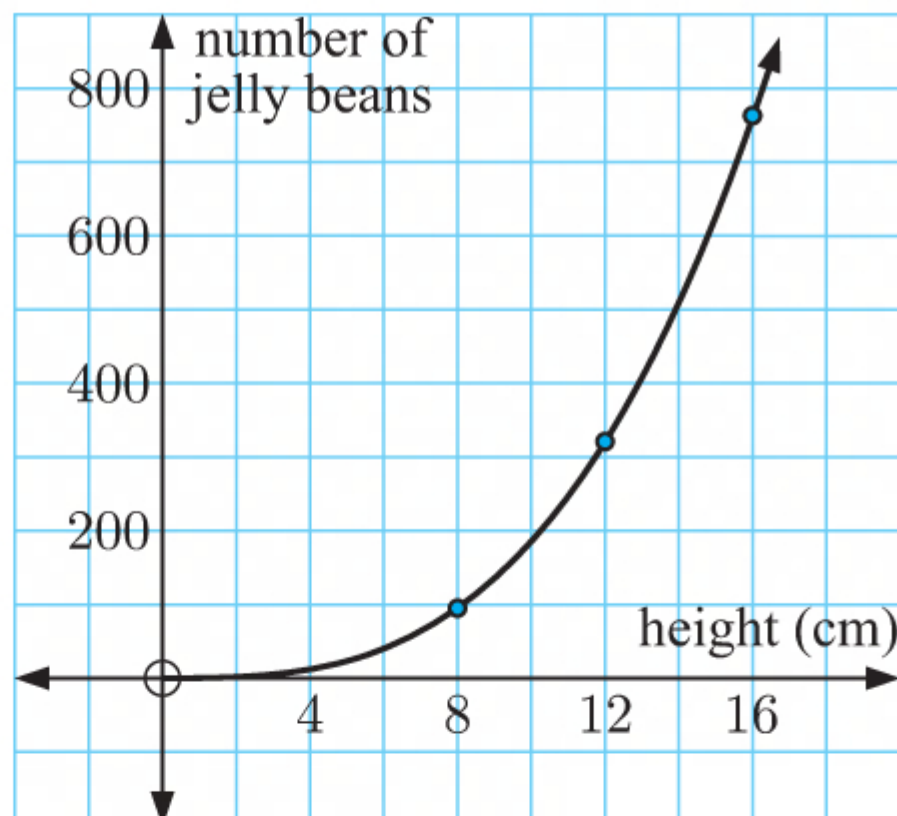
PowerReg
a = 0.01968407
b = 2.2926614
r = 0.99986688
r ² = 0.99973379
MSe = 2.2273E-05
y = a · x ^b
COPY

Eq: $x^{2.29} = \frac{4}{0.0197}$
x = 10.17847858
Lft = 203.0456853
Rgt = 203.0456853
REPEAT

6



a



- b No, the *number of jelly beans* does not increase *in proportion* with the *jar height* as the graph between these variables is not a straight line.

- c The volume of the jar $V = \pi r^2 h$, and its radius $r = \frac{h}{2}$ where h is the height of the jar.

$$\therefore V = \frac{\pi}{4} h^3$$

$$\therefore V \propto h^3 \quad \left\{ \frac{\pi}{4} \text{ is a constant} \right\}$$

The number of jelly beans N which a jar can hold is directly proportional to its volume V , and its volume is proportional to the cube of its height.

So, we expect the *number of jelly beans* to increase in proportion to the cube of the *jar height*.

- d We find the power model which best fits the points $(8, 95)$, $(12, 321)$, and $(16, 763)$.

The correlation coefficient r is very close to 1, so the fit is excellent.

The power is very close to 3, so it is reasonable to conclude that N is directly proportional to h^3 .

The model is $N \approx 0.183h^3$.

Deg	Norm1	d/c	Real
PowerReg			
a	=	0.18337986	
b	=	3.00549981	
r	=	0.9999998	
r ²	=	0.99999961	
MSe	=	8.5154E-07	
y=a · x^b			
[COPY]			

- e When $h = 20$, $N \approx 0.183 \times 20^3$
 ≈ 1464

Jill should guess 1464 jelly beans.

7

Distance (d m)	0.1	0.25	0.5	0.75	1	1.5
Force (F N)	563	90.0	22.5	10.0	5.63	2.50

- a The correlation coefficient r is very close to -1 , so the fit is excellent.

The power is very close to -2 , so it is reasonable to conclude that F is inversely proportional to d^2 .

The model is $F \approx \frac{5.63}{d^2}$.

Deg	Norm1	d/c	Real
PowerReg			
a	=	5.62602995	
b	=	-2.0001591	
r	=	-0.9999999	
r ²	=	0.99999995	
MSe	=	2.3199E-07	
y=a · x^b			
[COPY]			

- b When $d = 0.4$, $F \approx \frac{5.63}{0.4^2}$
 ≈ 35.2

So when the spheres are 0.4 m apart, the force will be about 35.2 N.

- c When $F = 650$, $\frac{5.63}{d^2} \approx 650$

$$\therefore d \approx \sqrt{\frac{5.63}{650}} \quad \{ \text{as } d > 0 \}$$

$$\approx 0.0931$$

So the spheres are about 0.0931 m apart when the force is 650 N.

8	Temperature (T °C)	5	10	15	20	25	30	35
	Pressure ($P \times 10^5$ Pa)	3.22	3.28	3.33	3.39	3.45	3.51	3.57

- a** The correlation coefficient $r \approx 0.962$, so the fit is reasonable.
 The model is $P \approx 2.93 \times T^{0.0521}$.
 The power of T is not close to 1, so P is not directly proportional to T .

<div> Deg Norm1 d/c Real </div> <div> PowerReg </div> <div> a = 2.92643977 </div> <div> b = 0.05209766 </div> <div> r = 0.96226598 </div> <div> r² = 0.92595583 </div> <div> MSe = 1.2214E-04 </div> <div> y = a · x^b </div> <div> COPY </div>

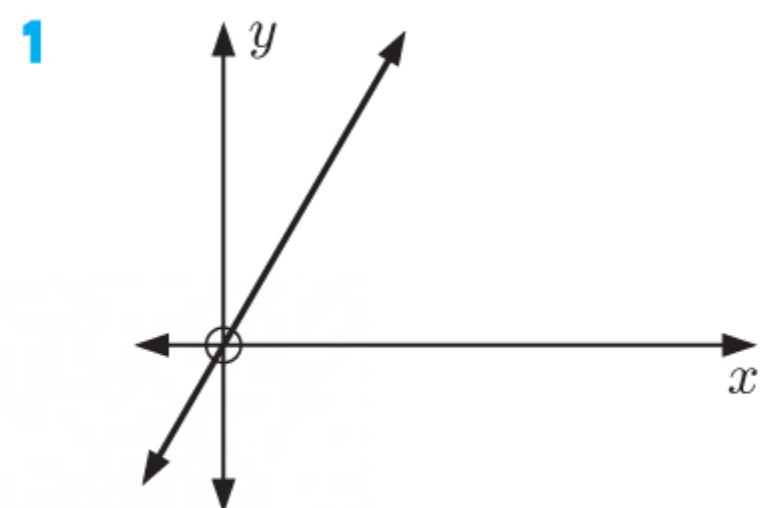
b	i	Temperature (T K)	278.15	283.15	288.15	293.15	298.15	303.15	308.15
		Pressure ($P \times 10^5$ Pa)	3.22	3.28	3.33	3.39	3.45	3.51	3.57

- ii** The correlation coefficient for the modified data is $r \approx 0.9997$, which is much closer to 1 than the value of r in **a**. This means the new power model is a much better fit for the data.
 The model is $P \approx 0.0112T$ where T is measured in kelvin.

<div> Deg Norm1 d/c Real </div> <div> PowerReg </div> <div> a = 0.0112377 </div> <div> b = 1.00518688 </div> <div> r = 0.99971241 </div> <div> r² = 0.99942491 </div> <div> MSe = 9.4865E-07 </div> <div> y = a · x^b </div> <div> COPY </div>
--

- iii** Yes, it is reasonable to assume that pressure and temperature are directly proportional when temperature is measured in kelvin, as our power model suggests that $P \propto T$.

REVIEW SET 7A



C indicates that y is directly proportional to x , as the graph is a straight line which passes through the origin.

2 $A \propto t$

- a** If t is multiplied by 4, then
 A is multiplied by 4.

- b** If A is increased by 5%, then
 A is multiplied by 1.05
 $\therefore t$ is multiplied by 1.05
 $\therefore t$ is increased by 5%.

- 3** A person's weight on the Moon M is directly proportional to their weight on Earth E , so $M = kE$ where k is a constant.

When $M = 124$ N, $E = 750$ N, so $124 = k(750)$

$$\therefore k = \frac{124}{750} = \frac{62}{375}$$

So, $M = \frac{62}{375}E$.

When $E = 640$ N, $M = \frac{62}{375} \times 640 \approx 106$ N.

So, if John weighs 640 N on Earth, then he weighs about 106 N on the Moon.

- 4** **a** $y = 5x^2$, so $y \propto x^2$ and $k = 5$ **b** $P = \frac{2n^4}{3} = \frac{2}{3}n^4$, so $P \propto n^4$ and $k = \frac{2}{3}$
- c** $V = \frac{\sqrt{5}}{4}a^3$, so $V \propto a^3$ and $k = \frac{\sqrt{5}}{4}$

- 5** **a** $y \propto x^2$, so $y = kx^2$ where k is a constant.

When $x = 8$ and $y = 30$, $30 = k \times 8^2$

$$\therefore k = \frac{30}{64} = \frac{15}{32}$$

So the model is $y = \frac{15}{32}x^2$.

- b** **i** When $x = 4$, $y = \frac{15}{32} \times 4^2$
 $= \frac{15}{2}$
 $= 7.5$

- ii** When $y = 150$, $\frac{15}{32}x^2 = 150$
 $\therefore x^2 = 320$
 $\therefore x = 8\sqrt{5}$ {as $x > 0$ }
 ≈ 17.9

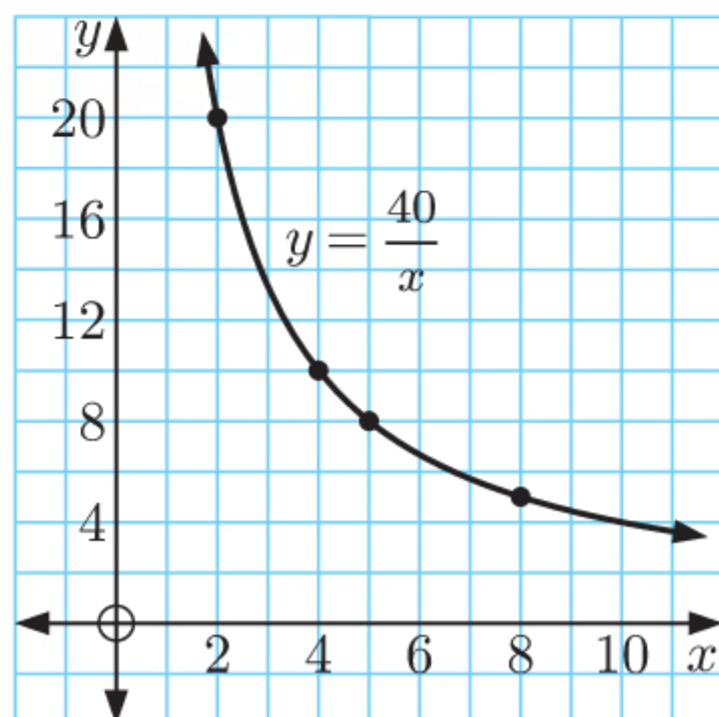
6 **a**

x	2	4	5	8
y	20	10	8	5
xy	40	40	40	40

$xy = 40$ for each point.

$\therefore x$ and y are inversely proportional,

$$\text{and } y = \frac{40}{x}.$$



b

x	3	5	8	10
y	20	12	8	6
xy	60	60	64	60

xy is not the same value for each point.

$\therefore x$ and y are not inversely proportional.

- 7** The frequency f is inversely proportional to wavelength λ , so $f = \frac{k}{\lambda}$ where k is a constant.

$$\text{When } f = 500 \text{ THz, } \lambda = 600 \text{ nm, so } 500 = \frac{k}{600}$$

$$\therefore k = 300\,000$$

$$\text{So, } f = \frac{300\,000}{\lambda}.$$

$$\text{When } \lambda = 480 \text{ nm, } f = \frac{300\,000}{480}$$

$$= 625 \text{ THz}$$

\therefore the frequency of the blue light wave is 625 THz.

- 8** The resistance r is inversely proportional to the square of the diameter d , so $r = \frac{k}{d^2}$ where k is a constant.

$$r = 0.24 \text{ ohms when } d = 0.44 \text{ cm, so } 0.24 = \frac{k}{(0.44)^2}$$

$$\therefore k = 0.24(0.44)^2$$

$$= 0.046\,464$$

a When $d = 0.3 \text{ cm}$, $r = \frac{0.046\,464}{(0.3)^2}$

$$\approx 0.516 \text{ ohms}$$

b When $r = 0.45 \text{ ohms}$, $\frac{0.046\,464}{d^2} = 0.45$

$$\therefore d^2 = \frac{0.046\,464}{0.45}$$

$$\therefore d \approx 0.321 \text{ cm}$$

- 9 a** $y \propto x^2$, so $y = kx^2$ where k is a constant.

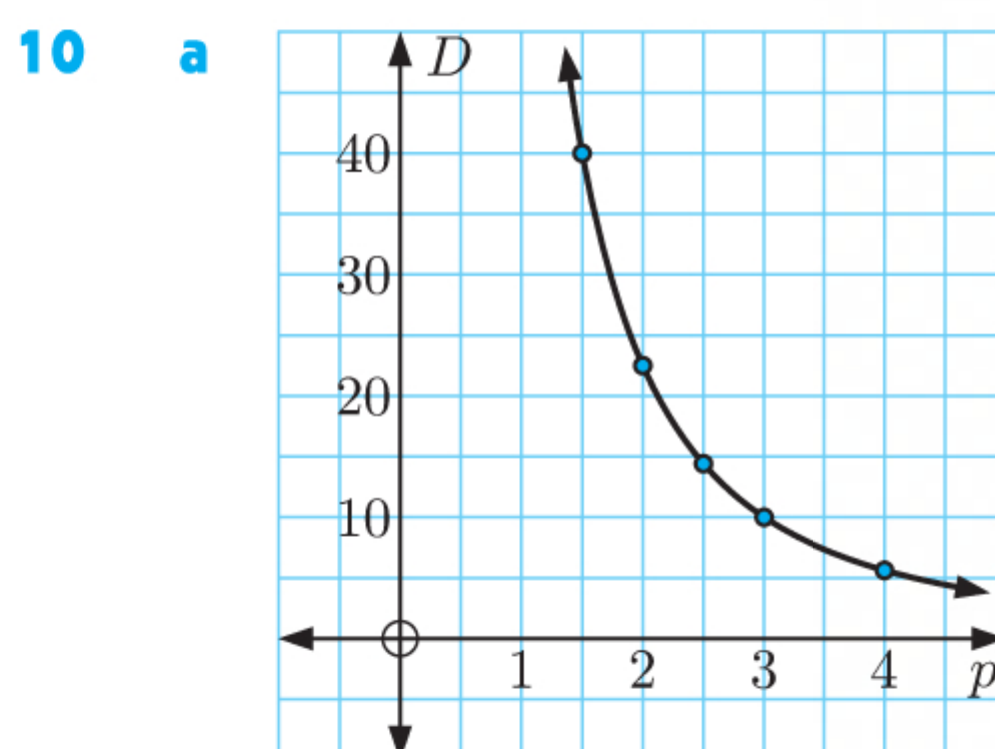
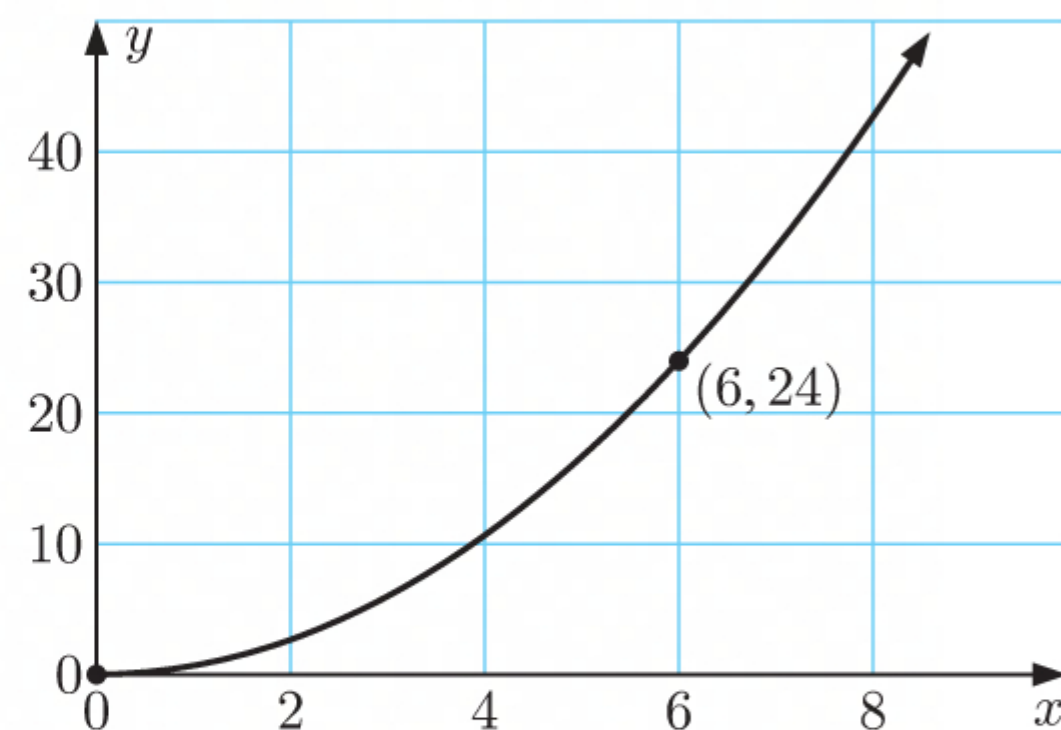
$$y = 24 \text{ when } x = 6, \text{ so } 24 = k \times 6^2$$

$$\therefore k = \frac{24}{36} = \frac{2}{3}$$

The model is $y = \frac{2}{3}x^2$.

b When $x = 11$, $y = \frac{2}{3} \times 11^2$

$$= \frac{242}{3} = 80\frac{2}{3}$$



- b** The points appear to lie on a curve which is asymptotic to both axes, which is what we would expect to see if $D \propto \frac{1}{p^2}$.

c

p	1.5	2	2.5	3	4
D	40	22.5	14.4	10	5.625
$p^2 D$	90	90	90	90	90

$p^2 D = 90$ at every point
 $\therefore k = 90$

d When $p = 5$, $D = \frac{90}{5^2}$
 $= \frac{18}{5} = 3.6$


11 a

x	2	4	5	7
y	12	96	188	514

The correlation coefficient r is very close to 1, so the fit is excellent.

The power is very close to 3, so it is reasonable to assume that y is directly proportional to x^3 .

The model is $y \approx 1.50x^3$.

	Deg	Norm1	d/c	Real
PowerReg				
a = 1.50060337				
b = 3.00001458				
r = 0.99999951				
r ² = 0.99999903				
MSe = 3.6613E-06				
y = a · x ^b				
				COPY


b

x	0.9	1.4	2.2	2.5
y	762	130	21.3	12.8

The correlation coefficient r is very close to -1 , so the fit is excellent.

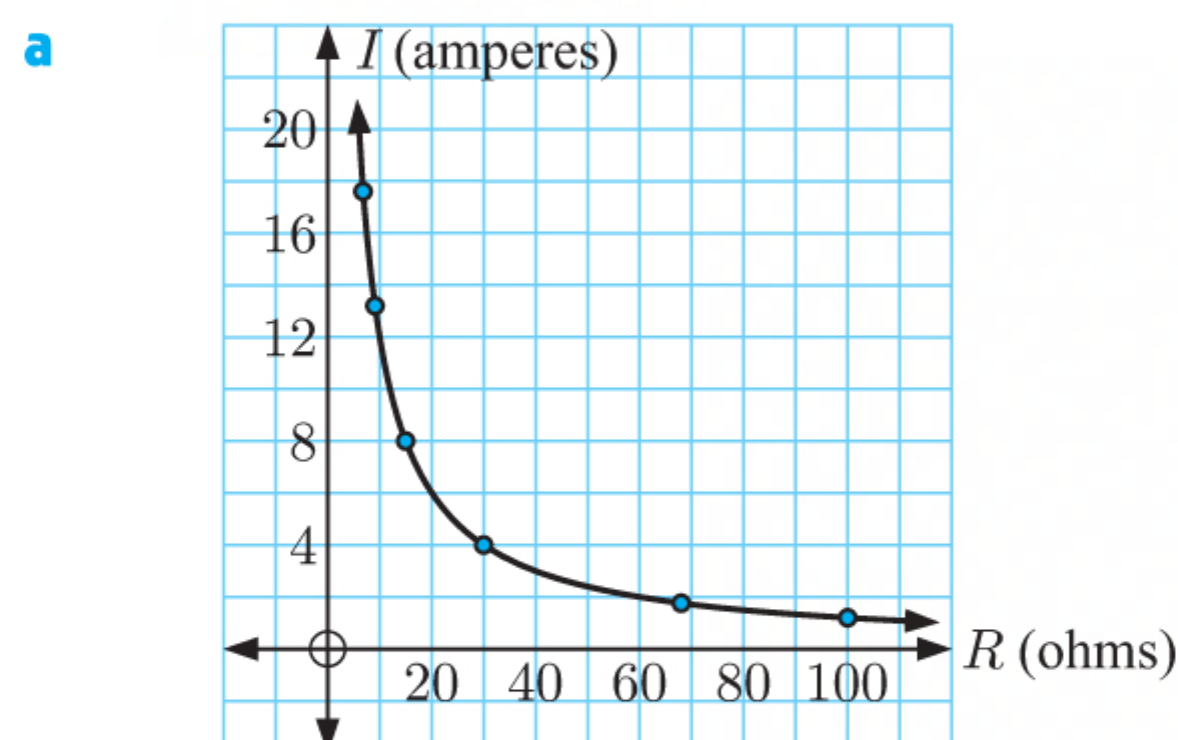
The power is very close to -4 , so it is reasonable to assume that y is inversely proportional to x^4 .

The model is $y \approx \frac{500}{x^4}$.

	Deg	Norm1	d/c	Real
PowerReg				
a =499.727434				
b =-4.0006089				
r =-0.9999998				
r ² =0.99999974				
MSe=1.3214E-06				
y=a · x^b				
				COPY

12

Resistance (R ohms)	6.8	9.1	15	30	68	100
Current (I amperes)	17.6	13.2	8.00	4.00	1.76	1.20
RI	119.68	120.12	120	120	119.68	120




b We expect inverse variation between the variables, as $RI \approx 120$ for each point and the points appear to lie on a line which is asymptotic to both axes.

c The correlation coefficient r is very close to -1 , so the fit is excellent.

The power is very close to -1 , so it is reasonable to assume that I and R are inversely proportional.

The model is $I \approx \frac{120}{R}$.

	Deg	Norm1	d/c	Real
PowerReg				
a = 119.938255				
b = -1.0000657				
r = -0.9999989				
r ² = 0.99999795				
MSe = 3.0232 × 10 ⁻⁶				
y = a · x ^b				
			COPY	DRAW

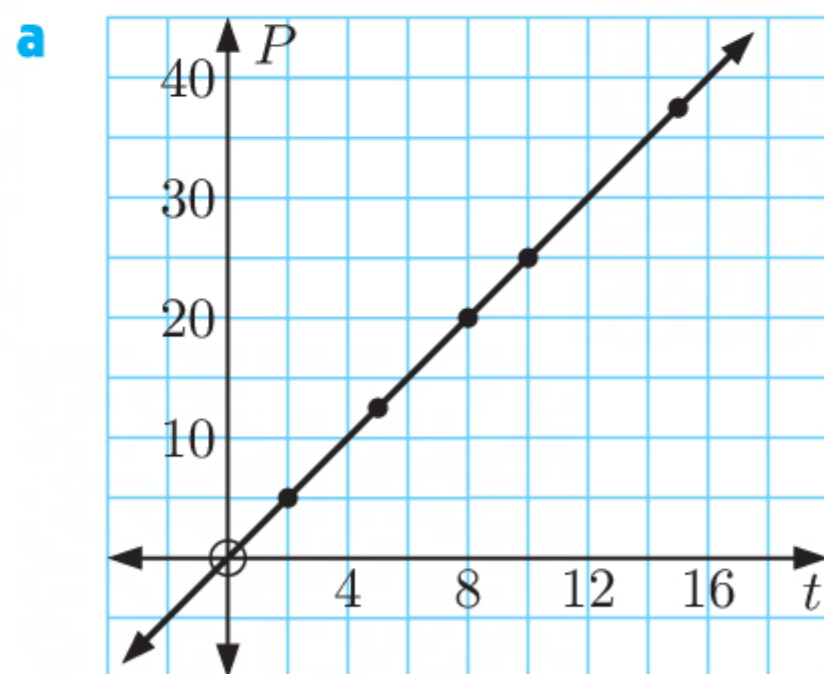
d When $R = 250$, $I \approx \frac{120}{250}$
 ≈ 0.48

So the current passing through a 250 ohm resistor will be about 0.48 amperes.

REVIEW SET 7B

1

t	2	5	8	10	15
P	5	12.5	20	25	37.5



b The graph of P against t is a straight line which passes through the origin.

c The gradient of the line is $\frac{12.5 - 5}{5 - 2} = 2.5$
 $\therefore P = 2.5t$

2 The current C and the force F are directly proportional, so $F = kC$ where k is a constant.

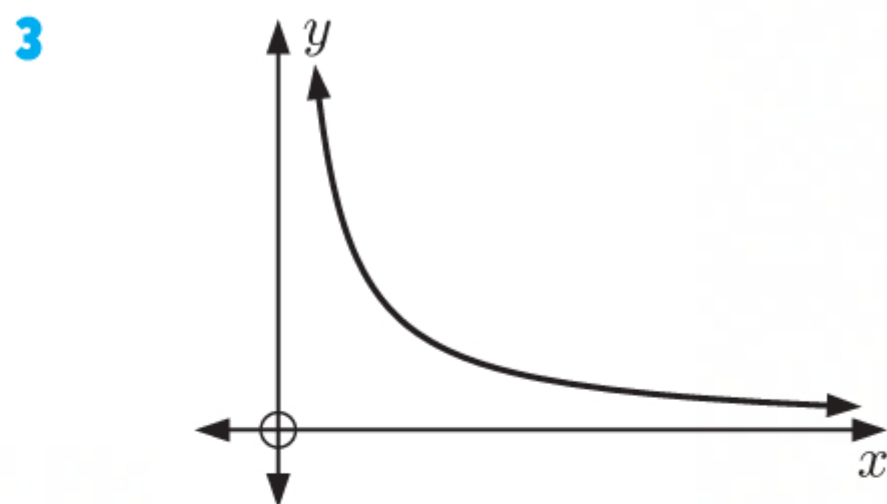
When $C = 1.09$ A, $F = 2.18$ N, so $2.18 = k \times 1.09$

$$\therefore k = 2$$

$$\therefore F = 2t$$

a When $C = 1.45$ A, $F = 2 \times 1.45$
 $= 2.9$ N

b When $F = 3.6$ N, $2C = 3.6$
 $\therefore C = 1.8$ A



B indicates that y is inversely proportional to x , as the graph is asymptotic to both axes.

4 a The area of the garden bed is
 $A = \pi r^2$, where r is the radius
 $\therefore A \propto r^2$

The amount of compost is directly proportional to A .

\therefore the amount of compost is directly proportional to the square of the radius.

b Let C be the amount of compost used.

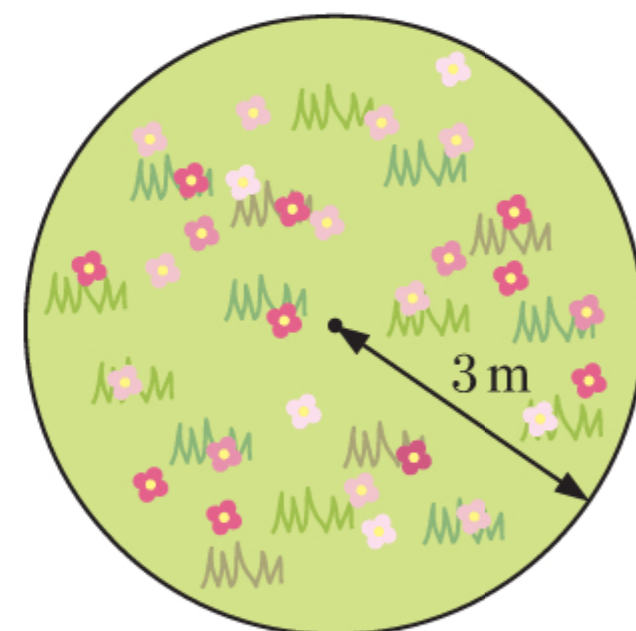
If r is increased by 15%, then

r is multiplied by 1.15

$\therefore C$ is multiplied by $(1.15)^2 = 1.3225$ {as $C \propto r^2$ from **a**}

$\therefore C$ is increased by 32.25%.

So, Tamzin needs $250 \times 0.3225 = 80.625$ kg extra compost.



- c** If C is increased by 40, then

$$C \text{ is multiplied by } \frac{250 + 40}{250} = \frac{29}{25}$$

$$\therefore r^2 \text{ is multiplied by } \frac{29}{25} \quad \{\text{as } C \propto r^2 \text{ from a}\}$$

$$\therefore r \text{ is multiplied by } \sqrt{\frac{29}{25}} = \frac{\sqrt{29}}{5} \approx 1.0770 \quad \{\text{as } r > 0\}$$

$$\therefore r \text{ is increased by about } 7.70\%.$$

So, Tamzin can extend the radius by about $3 \times 0.0770 \approx 0.231$ m which is about 23.1 cm.

- 5 a** $y \propto \frac{1}{x^3}$, so $y = \frac{k}{x^3}$ where k is a constant.

$$y = 16 \text{ when } x = 6, \text{ so } 16 = \frac{k}{6^3}$$

$$\therefore k = 16 \times 216 = 3456$$

$$\text{So the model is } y = \frac{3456}{x^3}.$$

b i When $x = 4$, $y = \frac{3456}{4^3}$
 $= 54$

ii When $y = -2$, $\frac{3456}{x^3} = -2$
 $\therefore x^3 = -1728$
 $\therefore x = -12$

- 6** The number of workers n is inversely proportional to the number of days t to paint the silo, so $n = \frac{k}{t}$ where k is a constant.

$$\text{When } n = 3, t = 18, \text{ so } 3 = \frac{k}{18}$$

$$\therefore k = 54$$

$$\text{So } n = \frac{54}{t}.$$

$$\text{When } n = 8, 8 = \frac{54}{t}$$

$$\therefore t = \frac{54}{8} = 6.75$$

\therefore it will take 6.75 days for 8 people to paint the silo.

- 7** The radius r is inversely proportional to the square of the orbital speed s , so $r \propto \frac{1}{s^2}$.

If r is increased by 20%, then

r is multiplied by 1.2

$$\therefore \frac{1}{s^2} \text{ is multiplied by } 1.2 \quad \{\text{as } r \propto \frac{1}{s^2}\}$$

$$\therefore s^2 \text{ is multiplied by } \frac{1}{1.2} = \frac{5}{6}$$

$$\therefore s \text{ is multiplied by } \sqrt{\frac{5}{6}} \approx 0.9129 \quad \{\text{as } s > 0\}$$

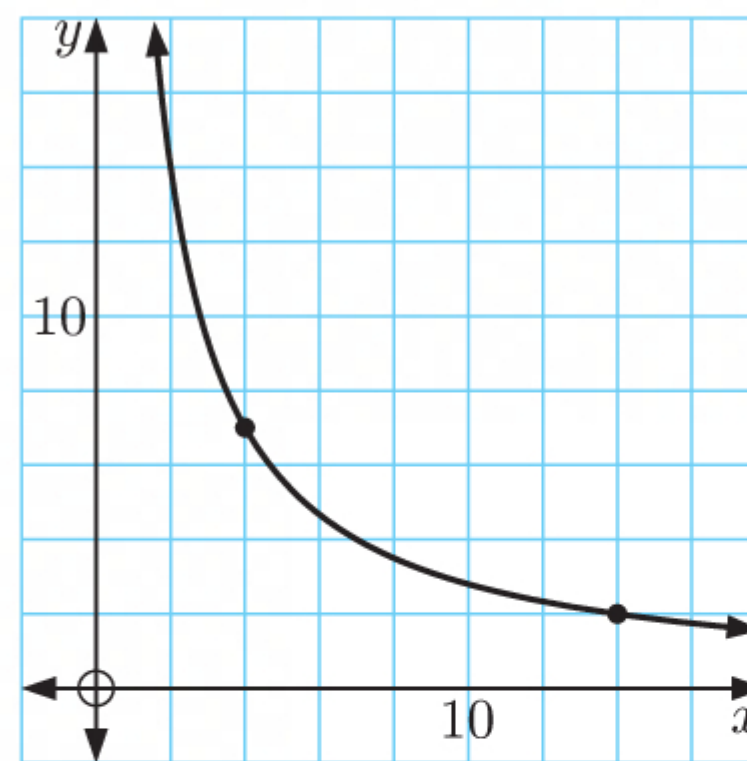
So, the orbital speed decreases by about 8.71%.

- 8 a** $y \propto \frac{1}{x}$, so $y = \frac{k}{x}$ where k is a constant.

The points $(4, 7)$ and $(14, 2)$ are marked on the graph.

$$\begin{aligned} \text{Using } (4, 7), y = 7 \text{ when } x = 4, \text{ so } 7 &= \frac{k}{4} \\ \therefore k &= 28 \\ \therefore y &= \frac{28}{x} \end{aligned}$$

Check: When $x = 14$, $y = \frac{28}{14} = 2$. ✓



- b** When $x = 0.1$, $y = \frac{28}{0.1} = 280$.

9

h	2	4	6
V	3.2	25.6	86.4

- a** If the height of a regular pyramid increases, then each side length also increases. This means the pyramid gets larger in all 3 dimensions as the height increases, so we should expect that V is directly proportional to h^3 .

- b** $V \propto h^3$, so $V = kh^3$ where k is a constant.

$$V = 3.2 \text{ when } h = 2, \text{ so } 3.2 = k \times 2^3$$

$$\therefore k = \frac{3.2}{8} = 0.4$$

$$\therefore V = \frac{2}{5}h^3$$

c When $h = 4$, $V = \frac{2}{5} \times 4^3$
 $= \frac{2}{5} \times 64$
 $= \frac{128}{5} = 25.6$ ✓

When $h = 6$, $V = \frac{2}{5} \times 6^3$
 $= \frac{2}{5} \times 216$
 $= \frac{432}{5} = 86.4$ ✓

So the model $V = \frac{2}{5}h^3$ satisfies every data point.

d i When $h = 8$, $V = \frac{2}{5} \times 8^3$
 $= \frac{2}{5} \times 512$
 $= \frac{1024}{5} = 204.8$

ii When $V = 50$, $\frac{2}{5}h^3 = 50$
 $\therefore h^3 = 125$
 $\therefore h = 5$

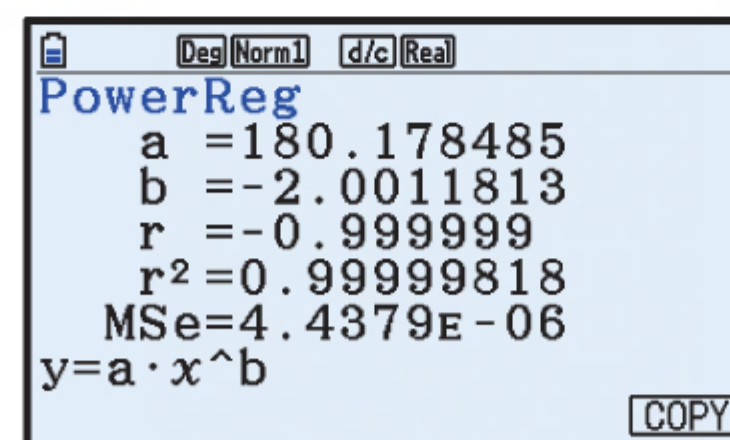
10

x	2	3	6	8	10
y	45	20	5	2.8	1.8

- a** The correlation coefficient r is very close to -1 , so the fit is excellent.

The power is very close to 2, so it is reasonable to assume that y is inversely proportional to x^2 .

The model is $y \approx \frac{180}{x^2}$.

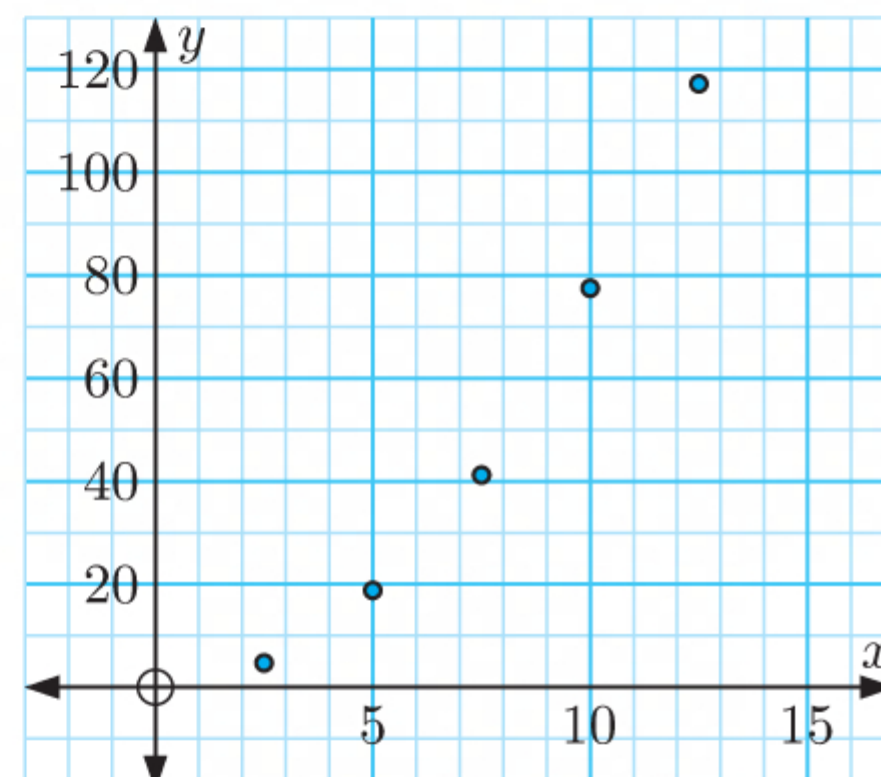


- b** When $x = 5$, $y \approx \frac{180}{5^2}$
 ≈ 7.2

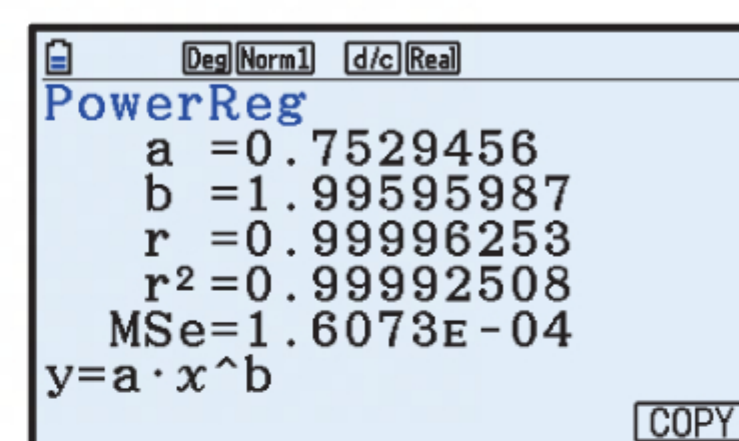
11 a

x	2.5	5	7.5	10	12.5
y	4.7	18.8	41.2	75	117.2

We expect direct variation between the variables, as the graph of y against x appears to be a curve which passes through the origin.



- b** The correlation coefficient r is very close to 1, so the fit is excellent.
 The power is very close to 2, so it is reasonable to assume that y is directly proportional to x^2 .
 The model is $y \approx 0.753x^2$.

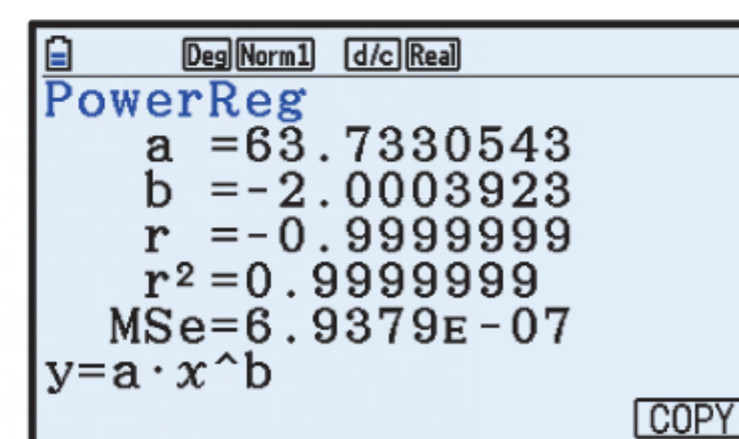


12 a

Distance (d m)	1	5	10	15	20
Sound intensity (I W m $^{-2}$)	63.7	2.55	0.637	0.283	0.159
$I \times d$	63.7	12.75	6.37	4.245	3.18

The values of $I \times d$ are not constant, so I and d are *not* inversely proportional, and Abbas is incorrect.

- b** The correlation coefficient is very close to -1 , so the fit is excellent.
 The power is very close to -2 , so it is reasonable to assume that I is inversely proportional to d^2 .
 The model is $I \approx \frac{63.7}{d^2}$.



- c** If d is increased by 40%, then
 d is multiplied by 1.4

$\therefore I$ is multiplied by $\frac{1}{(1.4)^2} \approx 0.510$ {as $I \propto \frac{1}{d^2}$ from **b**}

$\therefore I$ is decreased by about 49%.

Chapter 8

EXPONENTIALS AND LOGARITHMS

EXERCISE 8A

- 1
- a $y = 3^x$ is an exponential function as the variable x appears in the exponent.
 - b $f(x) = 4^{2x} - 1$ is an exponential function as the variable x appears in the exponent.
 - c $y = x^3 - 2$ is not an exponential function as the variable x does not appear in the exponent.
 - d $f(x) = 7 - 2^{-x}$ is an exponential function as the variable x appears in the exponent.
 - e $g(x) = \sqrt{x} + 5$ is not an exponential function as the variable x does not appear in the exponent.
 - f $f(x) = -3 \times 5^{\frac{x}{2}}$ is an exponential function as the variable x appears in the exponent.

2 $f(x) = 2^x - 3$

a $f(2) = 2^2 - 3$
 $= 4 - 3$
 $= 1$

b $f(1) = 2^1 - 3$
 $= 2 - 3$
 $= -1$

c $f(0) = 2^0 - 3$
 $= 1 - 3$
 $= -2$

d $f(-1) = 2^{-1} - 3$
 $= \frac{1}{2} - 3$
 $= -\frac{5}{2}$ or $-2\frac{1}{2}$

e $f(-2) = 2^{-2} - 3$
 $= \frac{1}{2^2} - 3$
 $= -\frac{11}{4}$ or $-2\frac{3}{4}$

3 $f(x) = 5 \times 3^x$

a $f(1) = 5 \times 3^1$
 $= 5 \times 3$
 $= 15$

b $f(3) = 5 \times 3^3$
 $= 5 \times 27$
 $= 135$

c $f(0) = 5 \times 3^0$
 $= 5 \times 1$
 $= 5$

d $f(-4) = 5 \times 3^{-4}$
 $= 5 \times \frac{1}{3^4}$
 $= \frac{5}{81}$

e $f(-1) = 5 \times 3^{-1}$
 $= 5 \times \frac{1}{3}$
 $= \frac{5}{3}$ or $1\frac{2}{3}$

4 $f(x) = 2 \times 2^x$

a $f(4) = 2 \times 2^4$
 $= 2^5$
 $= 32$

b $f(0) = 2 \times 2^0$
 $= 2 \times 1$
 $= 2$

c $f(1) = 2 \times 2^1$
 $= 2^2$
 $= 4$

d $f(-1) = 2 \times 2^{-1}$
 $= 2^0$
 $= 1$

e $f(-5) = 2 \times 2^{-5}$
 $= 2^{-4}$
 $= \frac{1}{16}$

5 $g(x) = 5^{-x}$

a $g(1) = 5^{-1}$
 $= \frac{1}{5}$

b $g(3) = 5^{-3}$
 $= \frac{1}{5^3}$
 $= \frac{1}{125}$

c $g(0) = 5^0$
 $= 1$

d $g(-2) = 5^{-(-2)}$
 $= 5^2$
 $= 25$

e $g(-3) = 5^{-(-3)}$
 $= 5^3$
 $= 125$

6 $h(x) = 3 \times (1.1)^x$

a $h(0) = 3 \times (1.1)^0$
 $= 3$

b $h(1) = 3 \times (1.1)^1$
 $= 3.3$

c $h(5) = 3 \times (1.1)^5$
 ≈ 4.83

d $h(-2) = 3 \times (1.1)^{-2}$
 ≈ 2.48

e $h(3.8) = 3 \times (1.1)^{3.8}$
 ≈ 4.31

7 **a** $y = 2^x + 1$

When $x = 3$, $y = 2^3 + 1$
 $= 8 + 1$
 $= 9$

\therefore the point $(3, 9)$ satisfies $y = 2^x + 1$.

c $f(x) = 3^{-x} - 2$
 $\therefore f(0) = 3^0 - 2$
 $= 1 - 2$
 $= -1$

\therefore the point $(0, -1)$ satisfies
 $f(x) = 3^{-x} - 2$.

e $f(x) = -4 \times 3^x + 1$
 $\therefore f(2) = -4 \times 3^2 + 1$
 $= -4 \times 9 + 1$
 $= -35$

\therefore the point $(2, -13)$ does not satisfy
 $f(x) = -4 \times 3^x + 1$.

b $f(x) = 5^{2x}$

$\therefore f(1) = 5^{2 \times 1}$
 $= 25$

\therefore the point $(1, 5)$ does not satisfy
 $f(x) = 5^{2x}$.

d $y = 6 \times 2^x$

When $x = -1$, $y = 6 \times 2^{-1}$
 $= 6 \times \frac{1}{2}$
 $= 3$

\therefore the point $(-1, 3)$ satisfies
 $y = 6 \times 2^x$.

f $y = 4^{-3x} + 2$

When $x = -1$, $y = 4^{-3(-1)} + 2$
 $= 4^3 + 2$
 $= 64 + 2$
 $= 66$

\therefore the point $(-1, 66)$ satisfies
 $y = 4^{-3x} + 2$.

8 $f(x) = 3^x - 1$

$\therefore f(0) = 3^0 - 1$
 $= 1 - 1$
 $= 0$

The function passes through the origin $(0, 0)$, so the axes intercepts are both zero.

9 $f(x) = 2^{-x} - 8$

a $f(0) = 2^0 - 8$
 $= 1 - 8$
 $= -7$

\therefore the y -intercept is -7 .

b $f(-3) = 2^{-(-3)} - 8$
 $= 2^3 - 8$
 $= 8 - 8$
 $= 0$

\therefore the x -intercept is -3 .

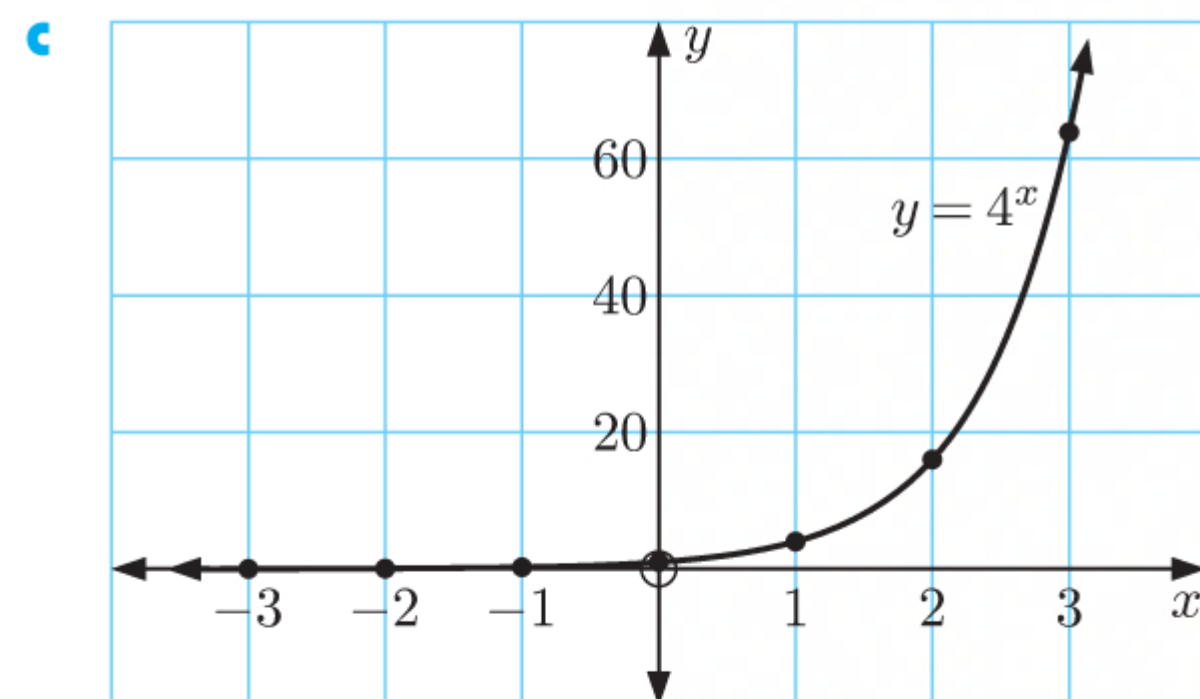
EXERCISE 8B

1 $y = 4^x$

a

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{64}$	$\frac{1}{16}$	$\frac{1}{4}$	1	4	16	64

- b i If x is increased by 1, the value of y is quadrupled.
 ii If x is decreased by 1, the value of y is divided by 4.

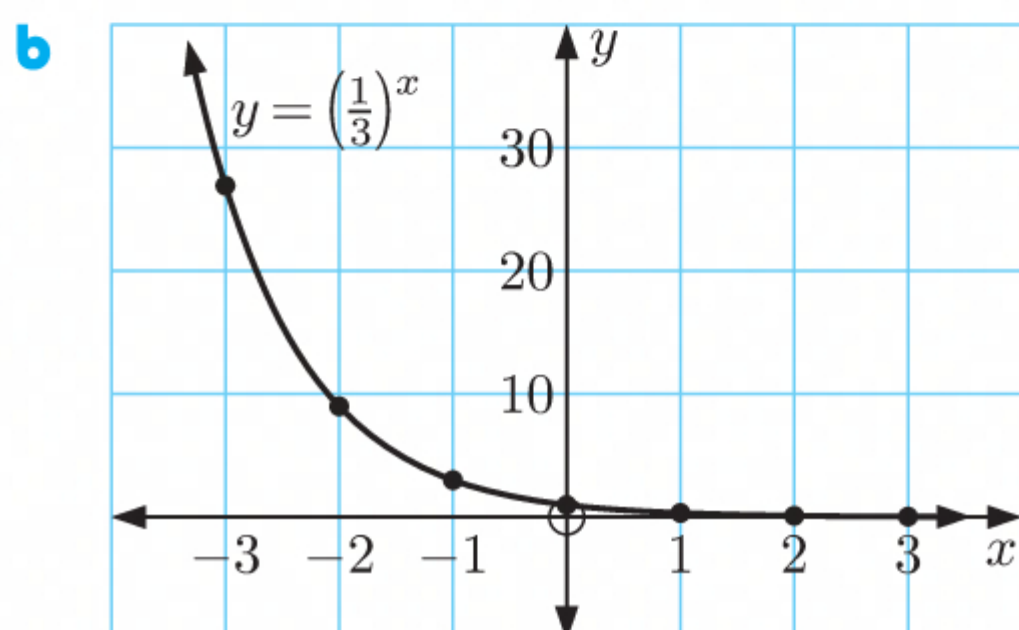


- d i As $x \rightarrow \infty$, $y \rightarrow \infty$.
 ii As $x \rightarrow -\infty$, $y \rightarrow 0^+$.
 e From d ii, as $x \rightarrow -\infty$, $y \rightarrow 0^+$.
 \therefore the horizontal asymptote is $y = 0$.

2 $y = \left(\frac{1}{3}\right)^x$

a

x	-3	-2	-1	0	1	2	3
y	27	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$



- c The graph of $y = \left(\frac{1}{3}\right)^x$ is decreasing.

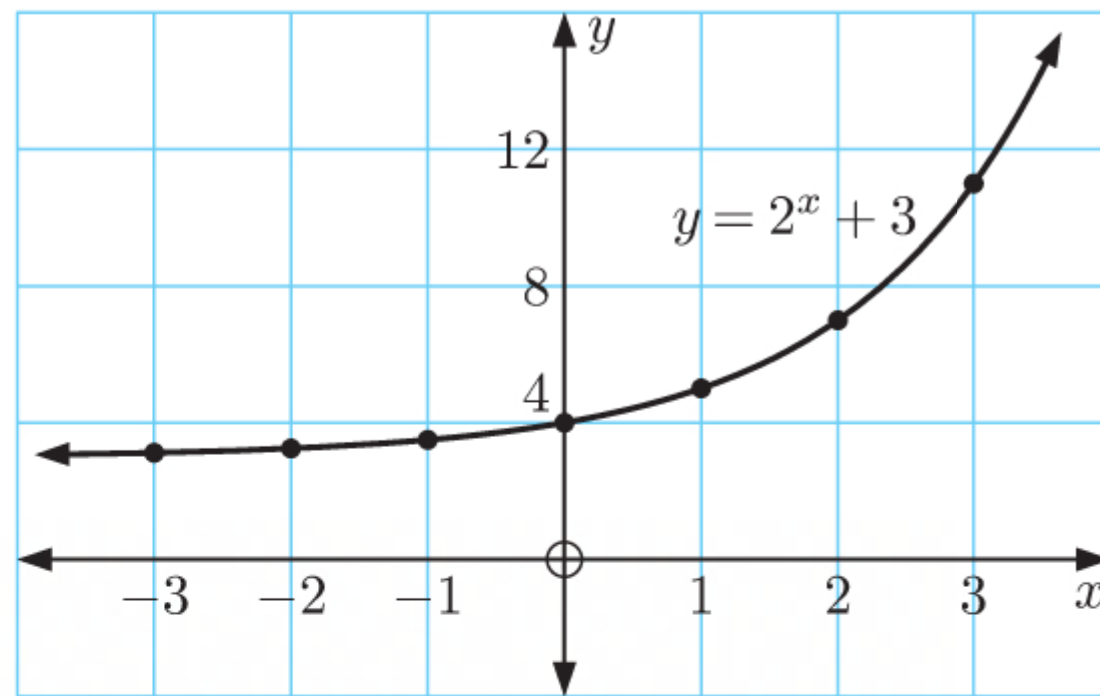
d i As $x \rightarrow \infty$, $y \rightarrow 0^+$.

ii As $x \rightarrow -\infty$, $y \rightarrow \infty$.

e From **d i**, as $x \rightarrow -\infty$, $y \rightarrow 0^+$.
 \therefore the horizontal asymptote is $y = 0$.

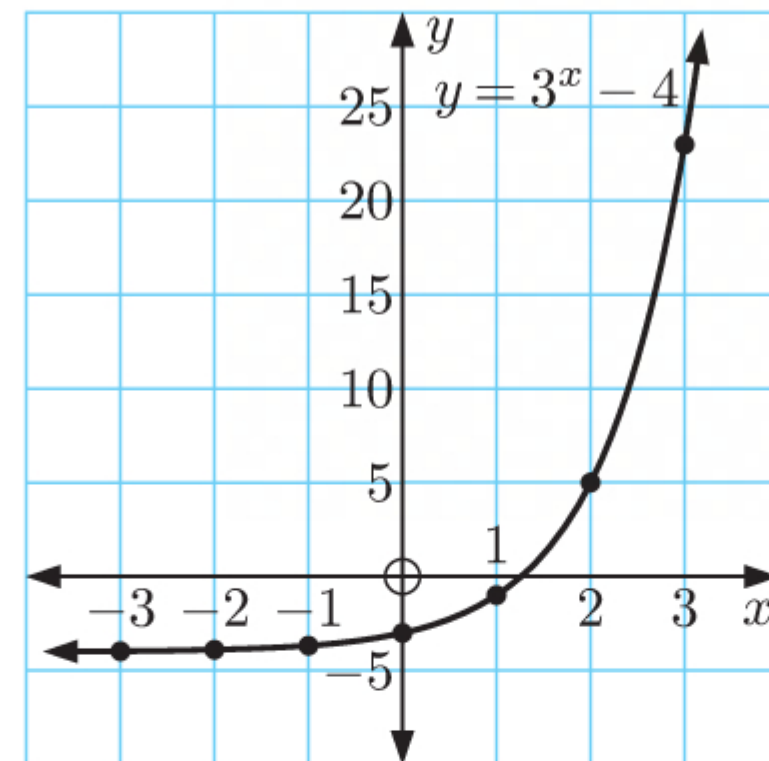
3 a $y = 2^x + 3$

x	-3	-2	-1	0	1	2	3
y	$3\frac{1}{8}$	$3\frac{1}{4}$	$3\frac{1}{2}$	4	5	7	11



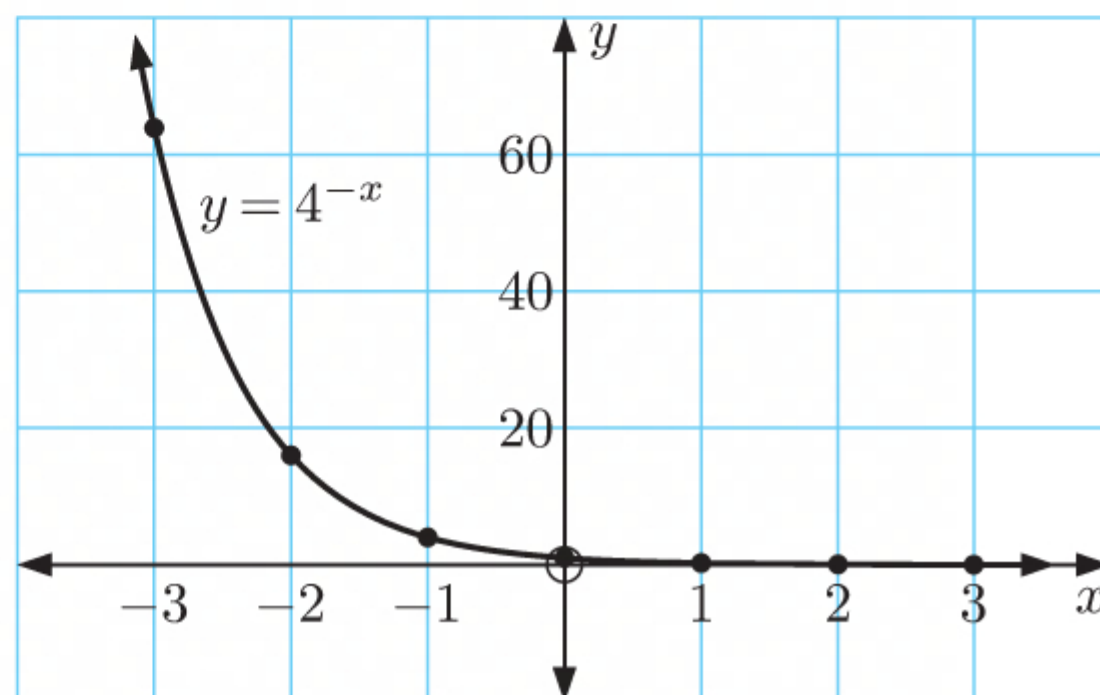
b $y = 3^x - 4$

x	-3	-2	-1	0	1	2	3
y	$-3\frac{26}{27}$	$-3\frac{8}{9}$	$-3\frac{2}{3}$	-3	-1	5	23



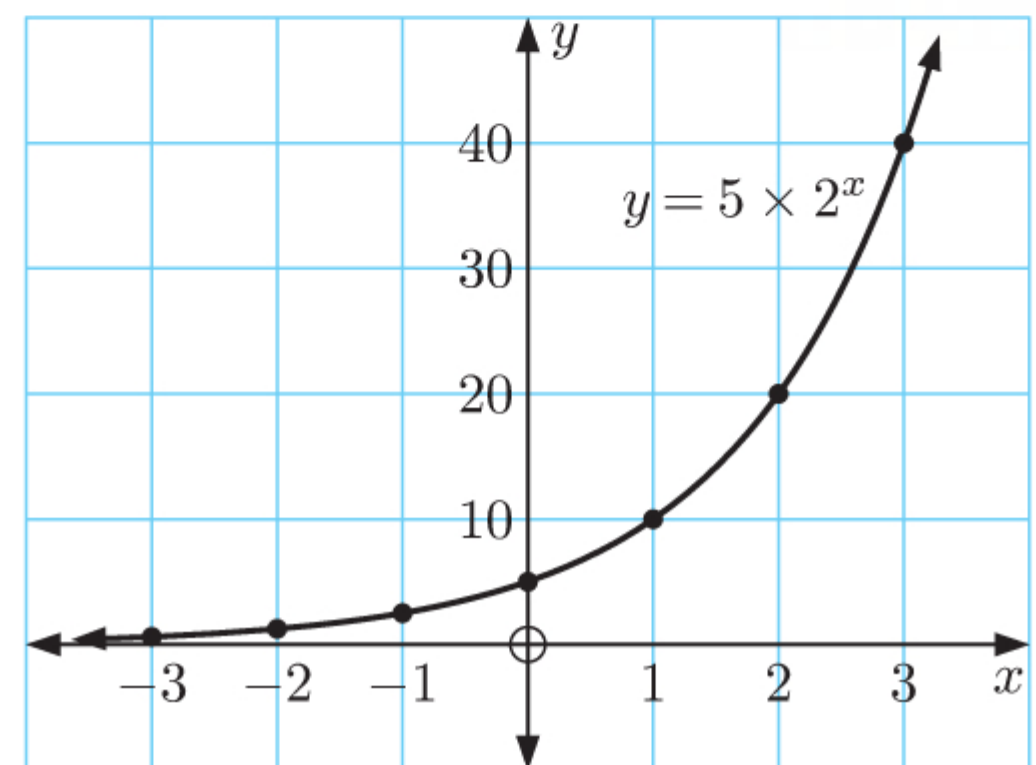
c $y = 4^{-x}$

x	-3	-2	-1	0	1	2	3
y	64	16	4	1	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$



d $y = 5 \times 2^x$

x	-3	-2	-1	0	1	2	3
y	$\frac{5}{8}$	$\frac{5}{4}$	$\frac{5}{2}$	5	10	20	40



4 a i $y = 2^{-x}$

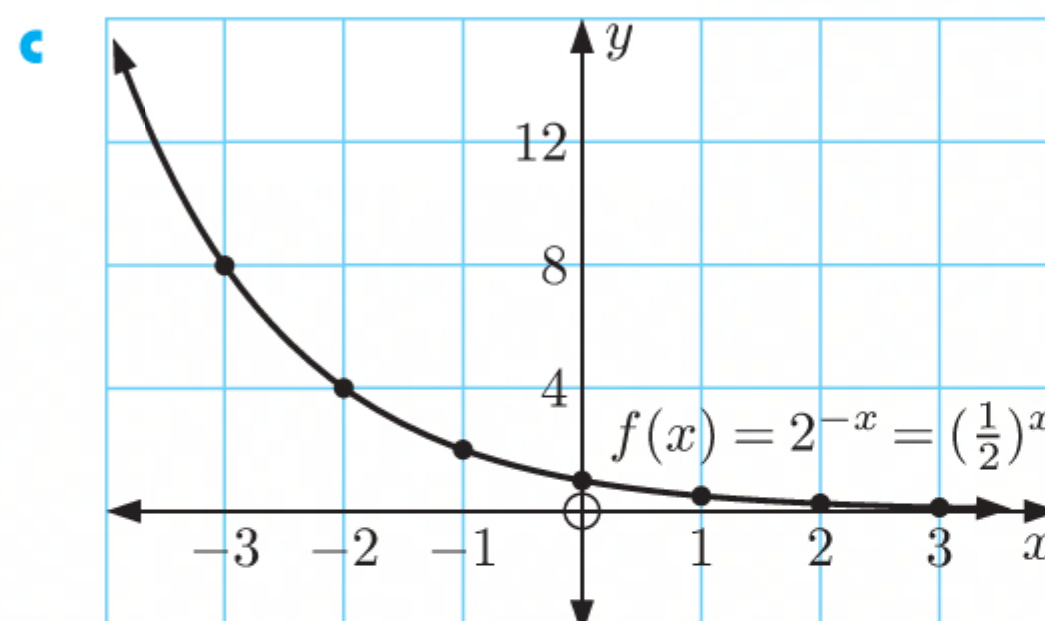
x	-3	-2	-1	0	1	2	3
y	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

ii $y = \left(\frac{1}{2}\right)^x$

x	-3	-2	-1	0	1	2	3
y	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

The corresponding y -values are the same for each function.

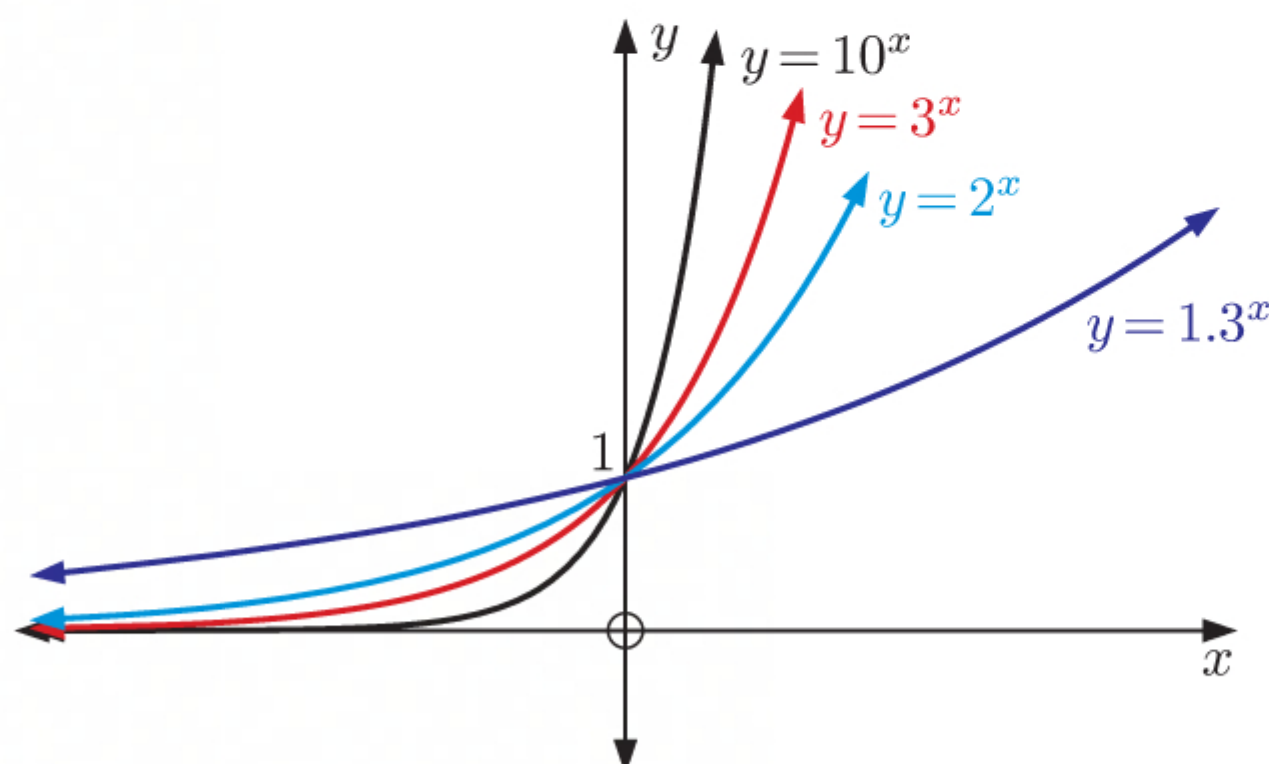
b $2^{-x} = (2^{-1})^x$
 $= \left(\frac{1}{2}\right)^x$



INVESTIGATION 1

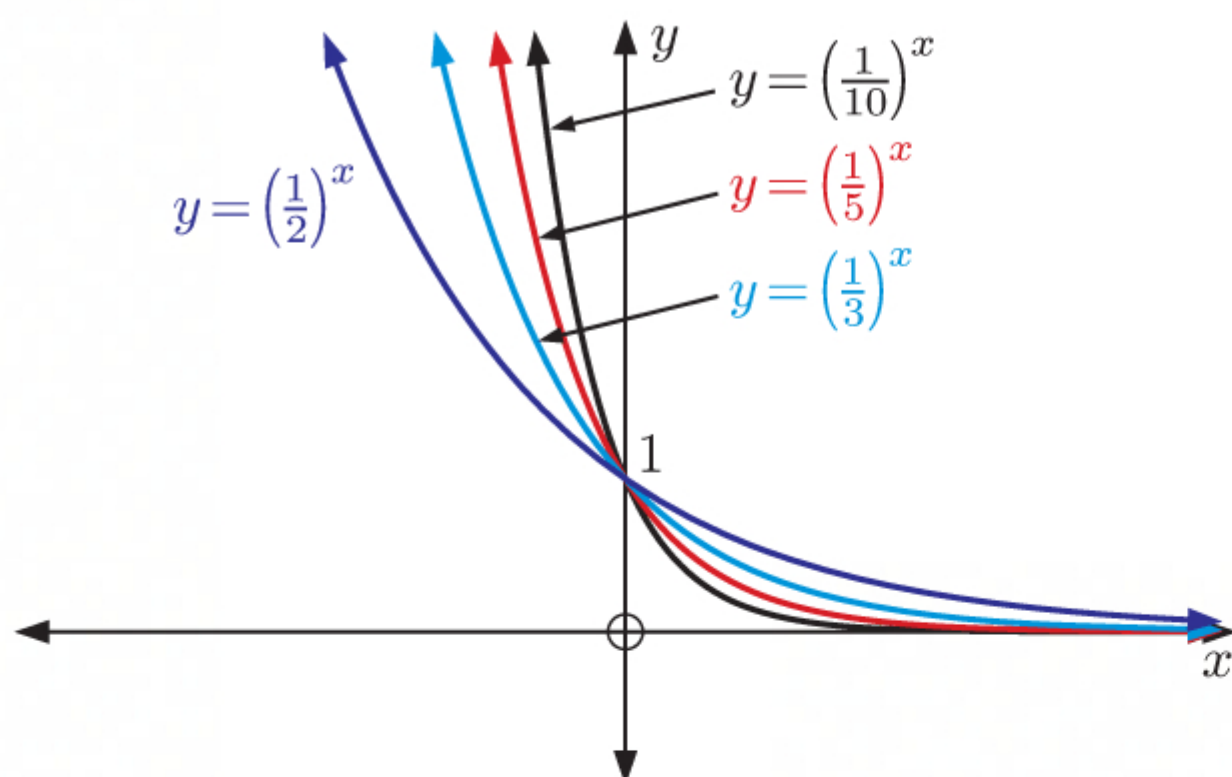
EXPONENTIAL GRAPHS

1 a

b For $y = a^x$ where $a > 1$:

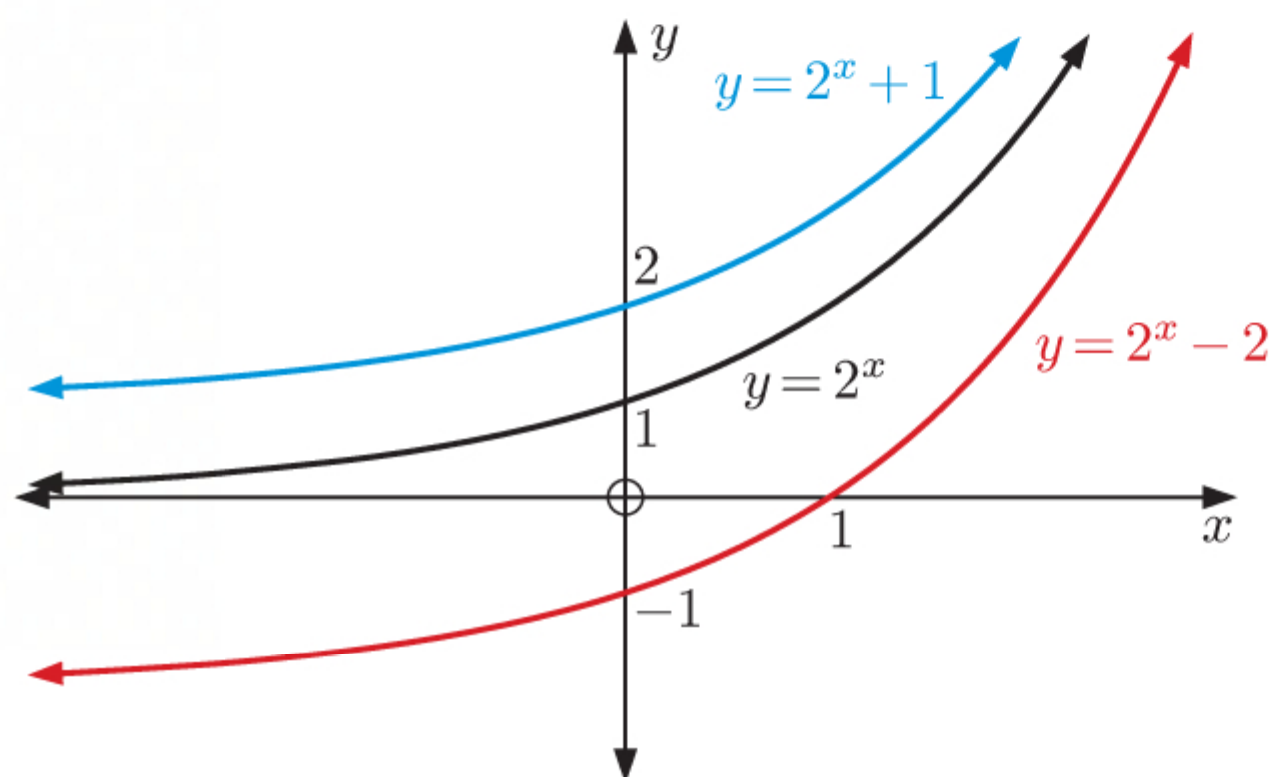
- i The function is increasing.
- ii As $x \rightarrow -\infty$, $y \rightarrow 0^+$, so the horizontal asymptote is $y = 0$.
- iii a affects how steeply the graph increases or decreases. Increasing a makes the graph increase more steeply, and decreasing a makes the graph increase less steeply.

2 a

b For $y = a^x$ where $0 < a < 1$:

- i The function is decreasing.
- ii As $x \rightarrow \infty$, $y \rightarrow 0^+$, so the horizontal asymptote is $y = 0$.
- iii a affects how steeply the graph increases or decreases. Increasing a makes the graph decrease less steeply, and decreasing a makes the graph decrease more steeply.

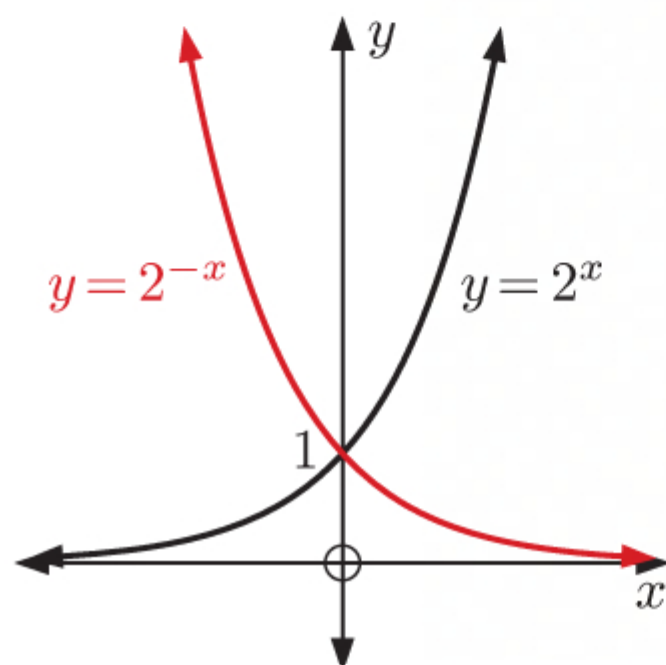
3 a



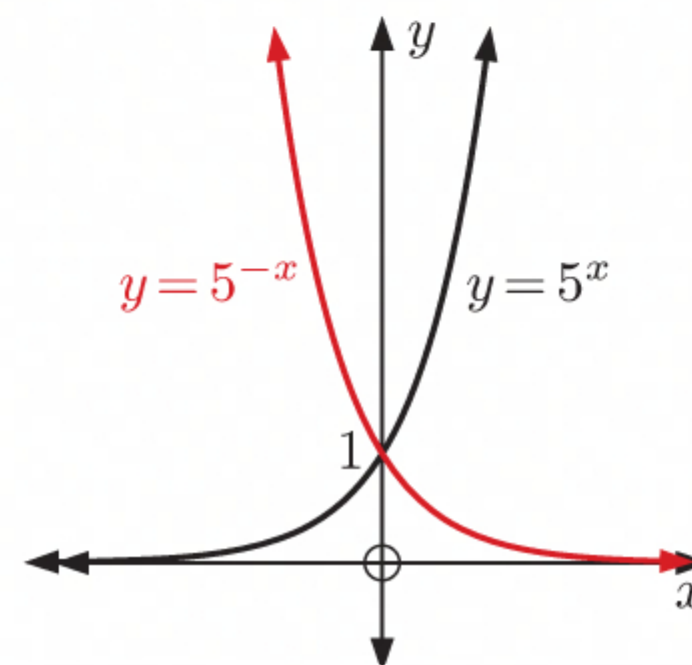
b For $y = 2^x + c$ where c is a constant:

- i** The value of c controls the vertical position of the graph.
- ii** The value of c has no effect on the shape of the graph.
- iii** As $x \rightarrow -\infty$, $2^x \rightarrow 0^+$, so $y = 2^x + c \rightarrow c^+$.
 \therefore the horizontal asymptote of $y = 2^x + c$ is $y = c$.

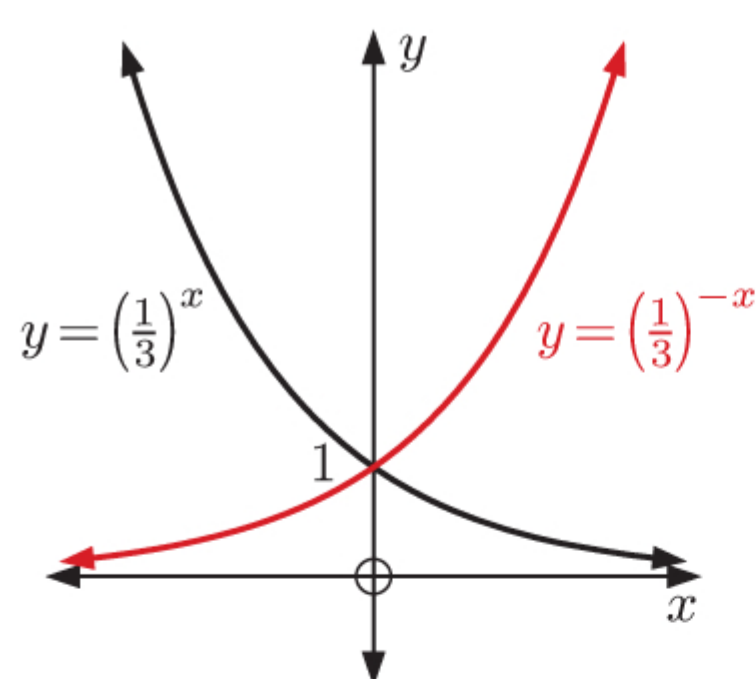
4 a i



ii



iii



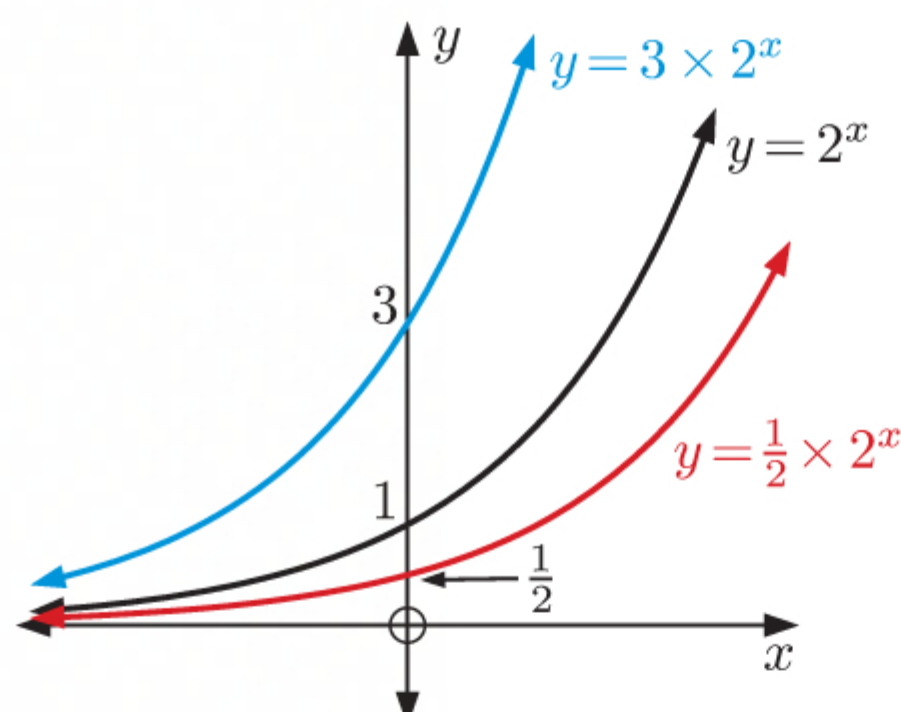
b For $y = a^x$ or $y = a^{-x}$ where $a > 0$, $a \neq 1$:

- i** When $x = 0$, $y = a^0 = 1$ or $y = a^{-0} = 1$.
 \therefore the y -intercept of each graph is 1.
- ii** For $a > 1$, as $x \rightarrow \infty$, $y = a^{-x} \rightarrow 0^+$, and as $x \rightarrow -\infty$, $y = a^x \rightarrow 0^+$.
 For $0 < a < 1$, as $x \rightarrow -\infty$, $y = a^{-x} \rightarrow 0^+$, and as $x \rightarrow \infty$, $y = a^x \rightarrow 0^+$.
 \therefore the horizontal asymptote of each graph is $y = 0$.
- iii** $y = a^{-x}$ is a reflection of $y = a^x$ in the y -axis.

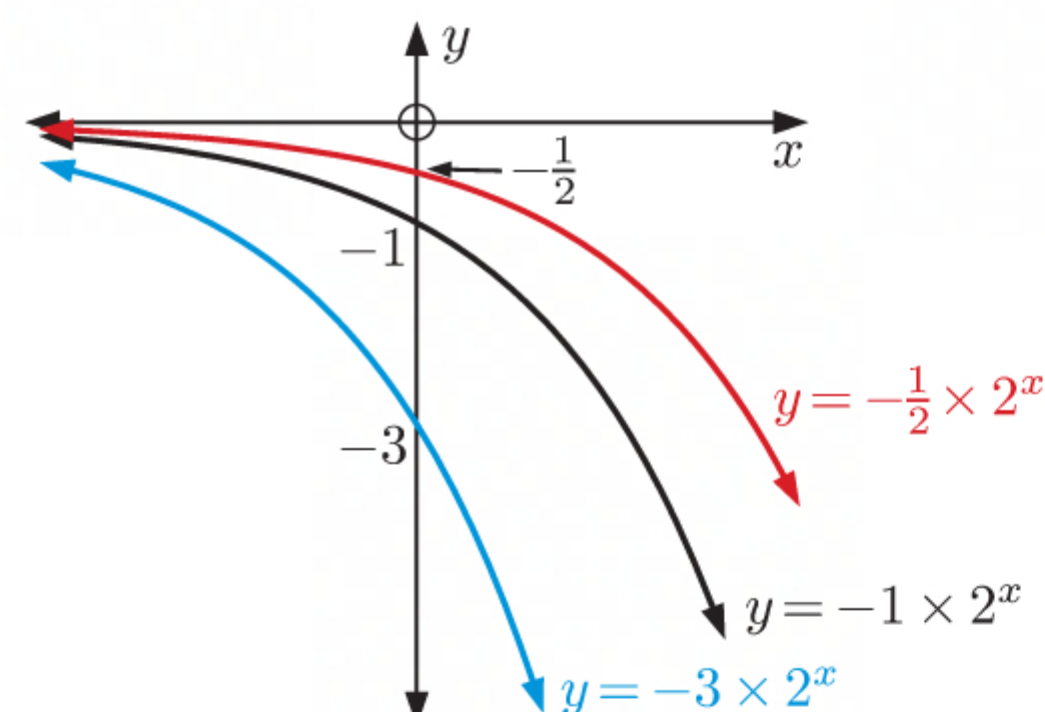
$$\begin{aligned} \mathbf{c} \quad a^{-x} &= (a^{-1})^x \\ &= \left(\frac{1}{a}\right)^x, \quad a > 0, \quad a \neq 1 \end{aligned}$$

Since $a^{-x} = \left(\frac{1}{a}\right)^x$ for all $a > 0$, $a \neq 1$, the graphs of $y = a^{-x}$ and $y = \left(\frac{1}{a}\right)^x$ are identical.

5 a i



ii



b For $y = k \times 2^x$ where k is a constant:

- i** The *sign* of k determines whether the graph lies above or below the asymptote.
- ii** The *size* of k affects how steeply the graph increases or decreases.
- iii** For $k > 0$, as $x \rightarrow -\infty$, $y = k \times 2^x \rightarrow 0^+$.
For $k < 0$, as $x \rightarrow -\infty$, $y = k \times 2^x \rightarrow 0^-$.
 \therefore the horizontal asymptote of each graph is $y = 0$.

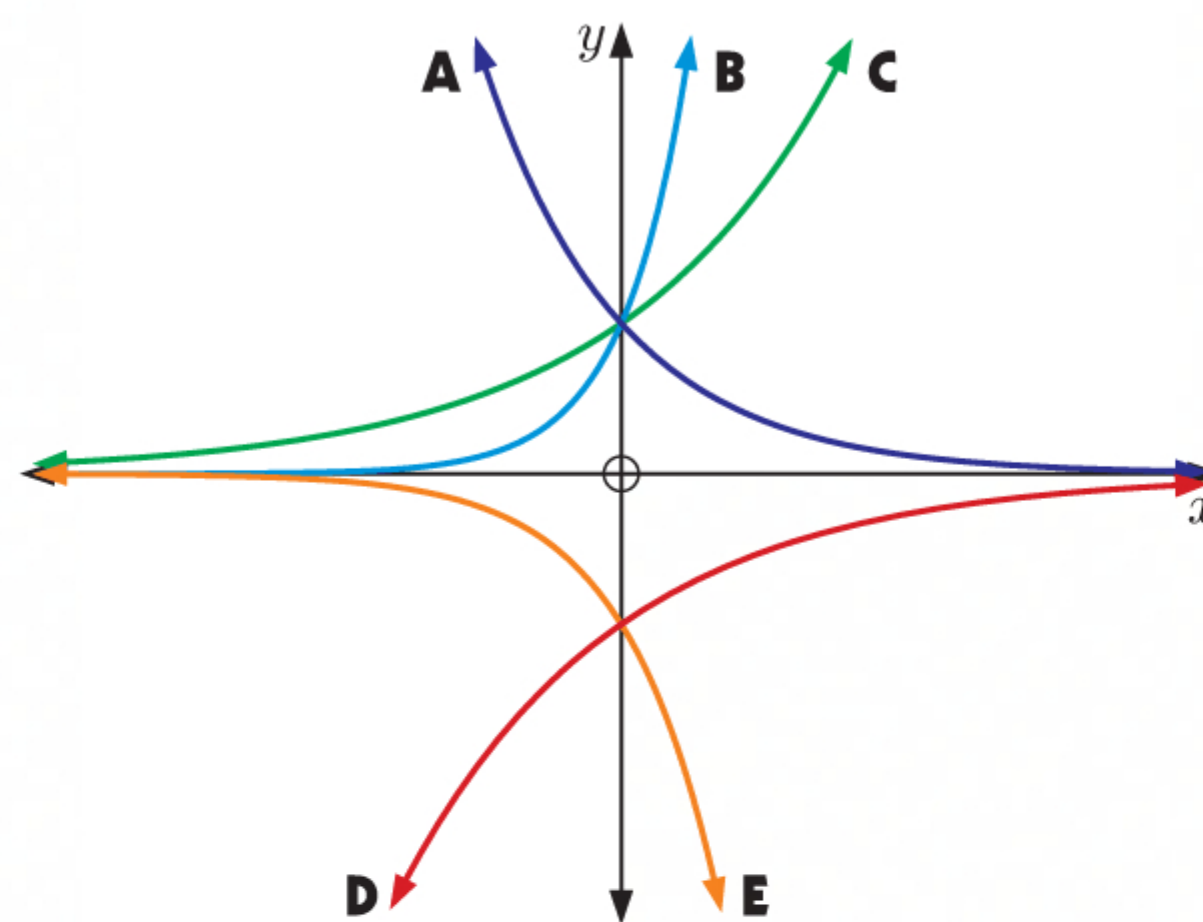
EXERCISE 8C

1 a, b Both $y = 2^x$ and $y = 10^x$ have $k > 0$ and $a > 1$.

\therefore both graphs lie above the horizontal asymptote $y = 0$ and are increasing.

$y = 10^x$ is steeper than $y = 2^x$ as $10 > 2$.

$\therefore y = 2^x$ corresponds to **C**, and $y = 10^x$ corresponds to **B**.



c $y = -5^x$ has $k < 0$ and $a > 1$.

\therefore the graph lies below the horizontal asymptote $y = 0$ and is decreasing.

$\therefore y = -5^x$ corresponds to **E**.

d $y = \left(\frac{1}{3}\right)^x$ has $k > 0$ and $0 < a < 1$.

\therefore the graph lies above the horizontal asymptote $y = 0$ and is decreasing.

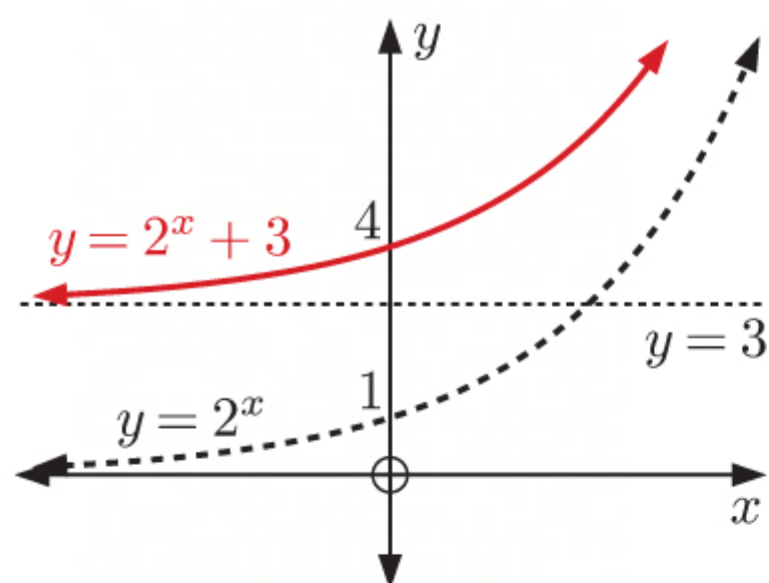
$\therefore y = \left(\frac{1}{3}\right)^x$ corresponds to **A**.

e $y = -\left(\frac{1}{2}\right)^x$ has $k < 0$ and $0 < a < 1$.

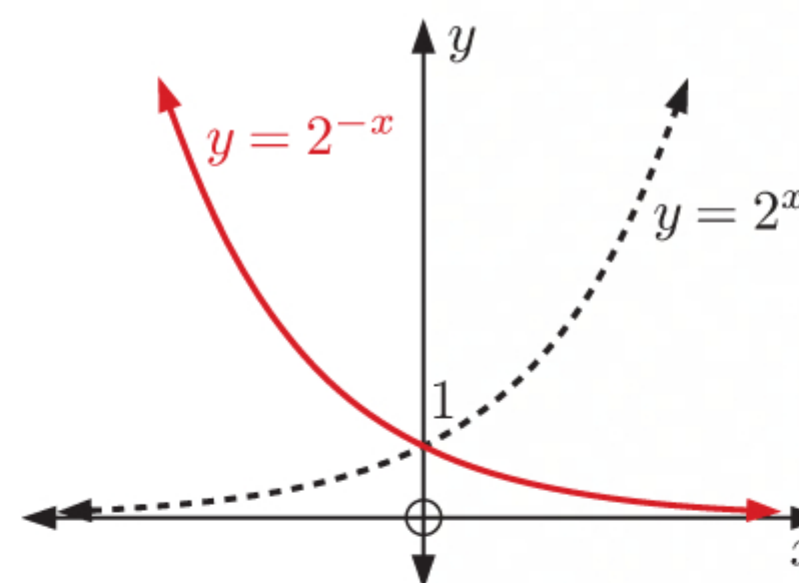
\therefore the graph lies below the horizontal asymptote $y = 0$ and is increasing.

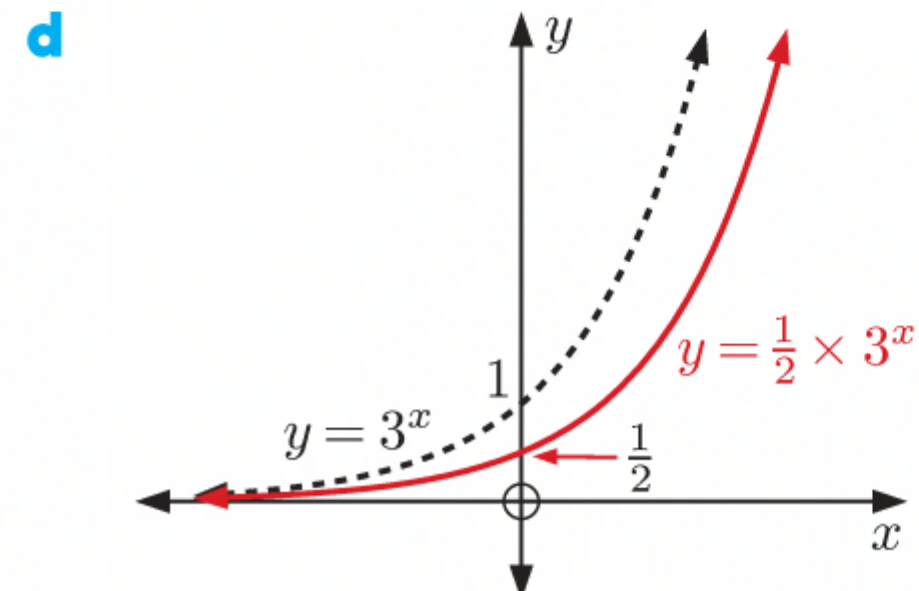
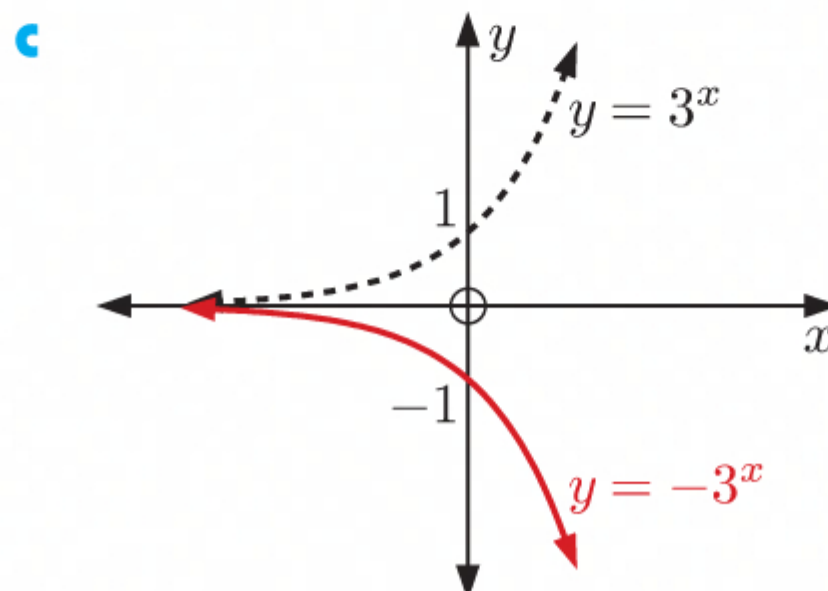
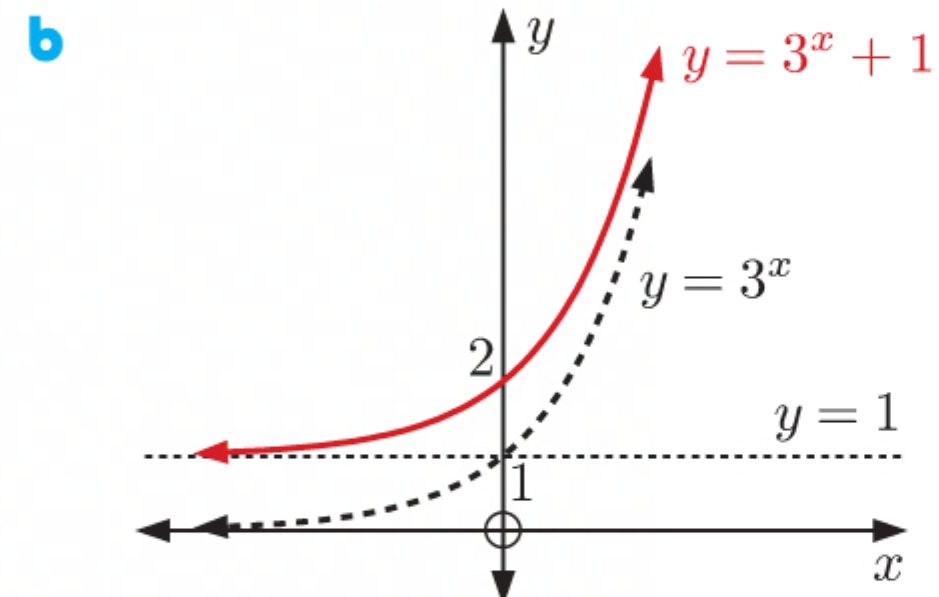
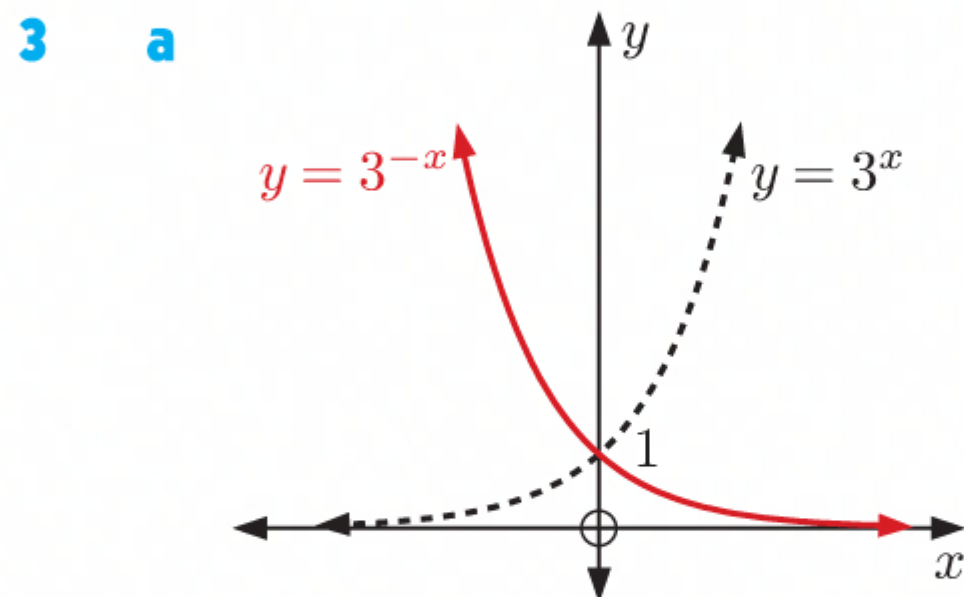
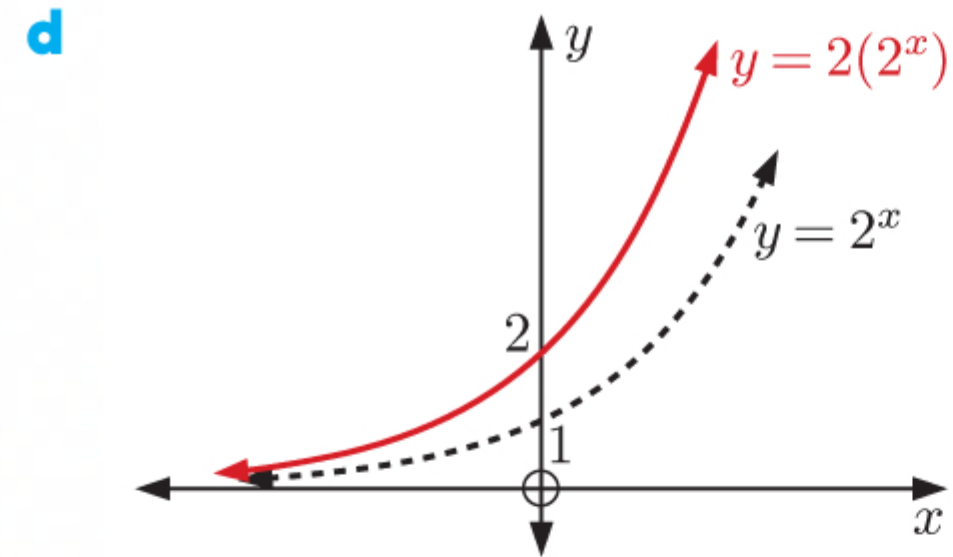
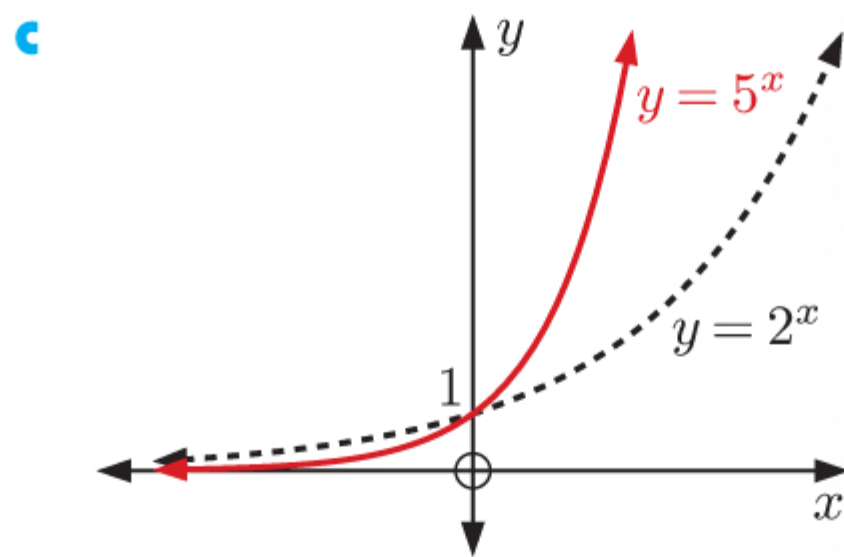
$\therefore y = -\left(\frac{1}{2}\right)^x$ corresponds to **D**.

2 a



b





- 4 a** The graph of $y = 5^x - 1$ has horizontal asymptote $y = -1$.
b The graph of $y = 2^{-x} + 4$ has horizontal asymptote $y = 4$.
c The graph of $y = 3 \times 4^x + 1$ has horizontal asymptote $y = 1$.
d The graph of $y = -\left(\frac{1}{2}\right)^x - 5$ has horizontal asymptote $y = -5$.

5 a

$$\begin{aligned} f(x) &= 3^x + 4 \\ f(0) &= 3^0 + 4 \\ &= 1 + 4 \\ &= 5 \end{aligned}$$

\therefore the y -intercept is 5.

c

$$\begin{aligned} f(x) &= 3 \times 2^x + 7 \\ f(0) &= 3 \times 2^0 + 7 \\ &= 3 \times 1 + 7 \\ &= 10 \end{aligned}$$

\therefore the y -intercept is 10.

b

$$\begin{aligned} f(x) &= 6^{-x} - 2 \\ f(0) &= 6^0 - 2 \\ &= 1 - 2 \\ &= -1 \end{aligned}$$

\therefore the y -intercept is -1 .

d

$$\begin{aligned} f(x) &= -\frac{1}{2} \times 5^x + 6 \\ f(0) &= -\frac{1}{2} \times 5^0 + 6 \\ &= -\frac{1}{2} \times 1 + 6 \\ &= 5\frac{1}{2} \end{aligned}$$

\therefore the y -intercept is $5\frac{1}{2}$.

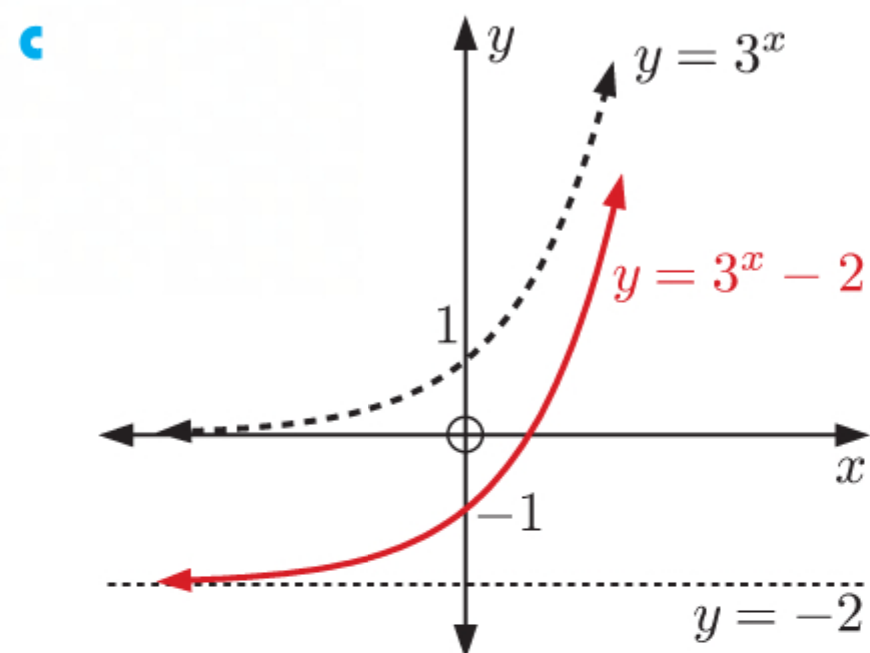
6 $f(x) = 3^x - 2$

a i $f(0) = 3^0 - 2$
 $= 1 - 2$
 $= -1$

ii $f(2) = 3^2 - 2$
 $= 9 - 2$
 $= 7$

iii $f(-2) = 3^{-2} - 2$
 $= \frac{1}{9} - 2$
 $= -\frac{17}{9} = -1\frac{8}{9}$

b The graph of $y = 3^x - 2$ has horizontal asymptote $y = -2$.



d The domain is $\{x \mid x \in \mathbb{R}\}$.
The range is $\{y \mid y > -2\}$.

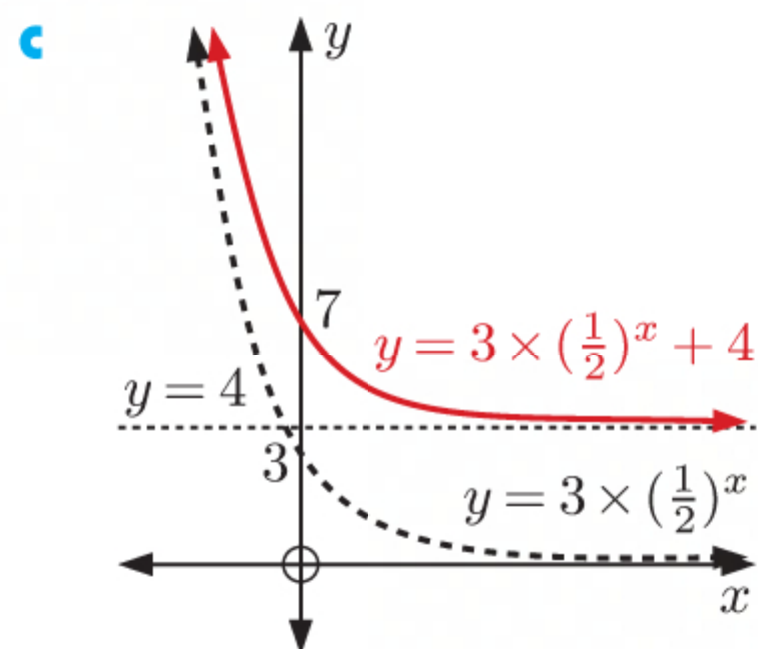
7 $g(x) = 3 \times \left(\frac{1}{2}\right)^x + 4$

a i $g(0) = 3 \times \left(\frac{1}{2}\right)^0 + 4$
 $= 3 \times 1 + 4$
 $= 7$

ii $g(2) = 3 \times \left(\frac{1}{2}\right)^2 + 4$
 $= 3 \times \frac{1}{4} + 4$
 $= \frac{19}{4} = 4\frac{3}{4}$

iii $g(-2) = 3 \times \left(\frac{1}{2}\right)^{-2} + 4$
 $= 3 \times 2^2 + 4$
 $= 3 \times 4 + 4$
 $= 16$

b The graph of $y = 3 \times \left(\frac{1}{2}\right)^x + 4$ has horizontal asymptote $y = 4$.



d The domain is $\{x \mid x \in \mathbb{R}\}$.
The range is $\{y \mid y > 4\}$.

8 a $y = 2^x + 1$

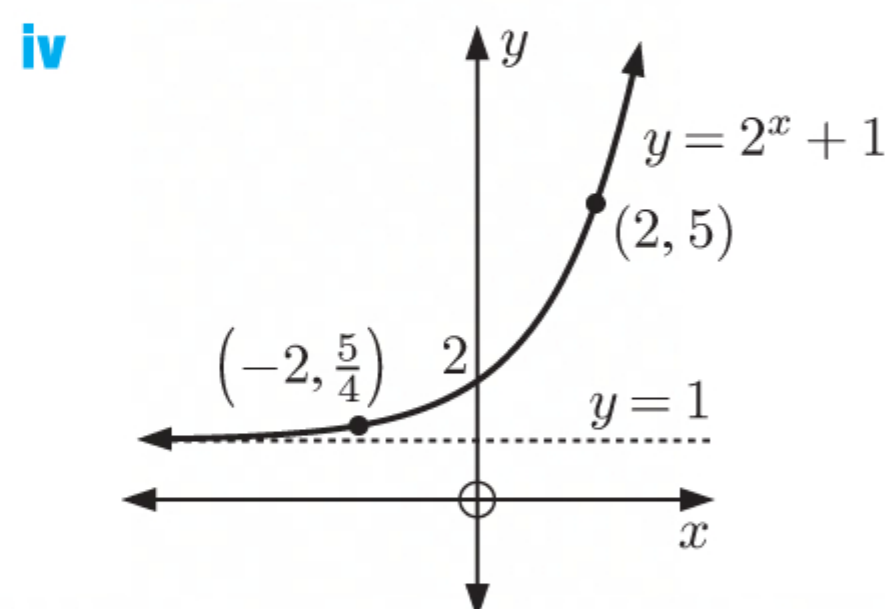
i When $x = 0$, $y = 2^0 + 1$
 $= 1 + 1$
 $= 2$

\therefore the y -intercept is 2.

iii When $x = 2$, $y = 2^2 + 1$
 $= 4 + 1$
 $= 5$

When $x = -2$, $y = 2^{-2} + 1$
 $= \frac{1}{4} + \frac{4}{4}$
 $= \frac{5}{4}$

ii The horizontal asymptote is $y = 1$.



v The domain is $\{x \mid x \in \mathbb{R}\}$. The range is $\{y \mid y > 1\}$.

b $y = 3^{-x} + 4$

i When $x = 0$, $y = 3^0 + 4$
 $= 1 + 4$
 $= 5$

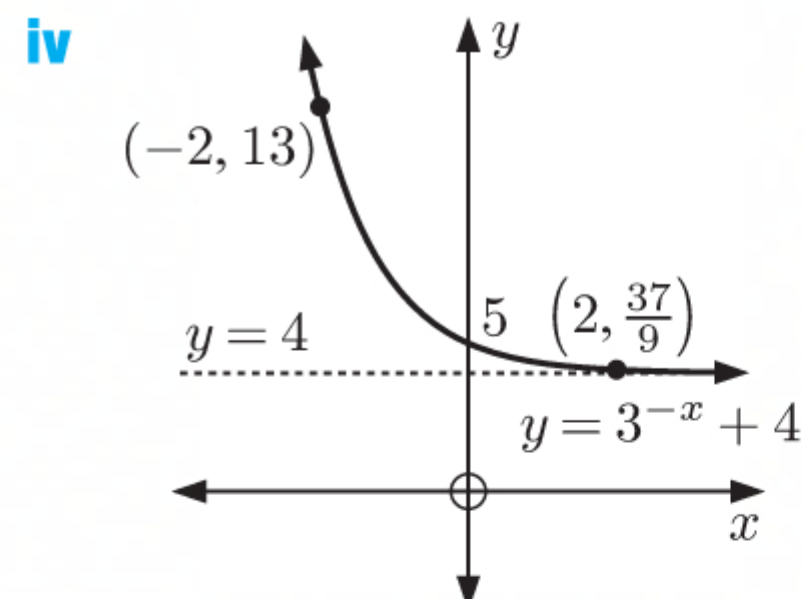
\therefore the y -intercept is 5.

iii When $x = 2$, $y = 3^{-2} + 4$
 $= \frac{1}{3^2} + 4$
 $= \frac{1}{9} + \frac{36}{9}$
 $= \frac{37}{9}$

When $x = -2$, $y = 3^{-(-2)} + 4$
 $= 3^2 + 4$
 $= 9 + 4$
 $= 13$

v The domain is $\{x \mid x \in \mathbb{R}\}$. The range is $\{y \mid y > 4\}$.

ii The horizontal asymptote is $y = 4$.



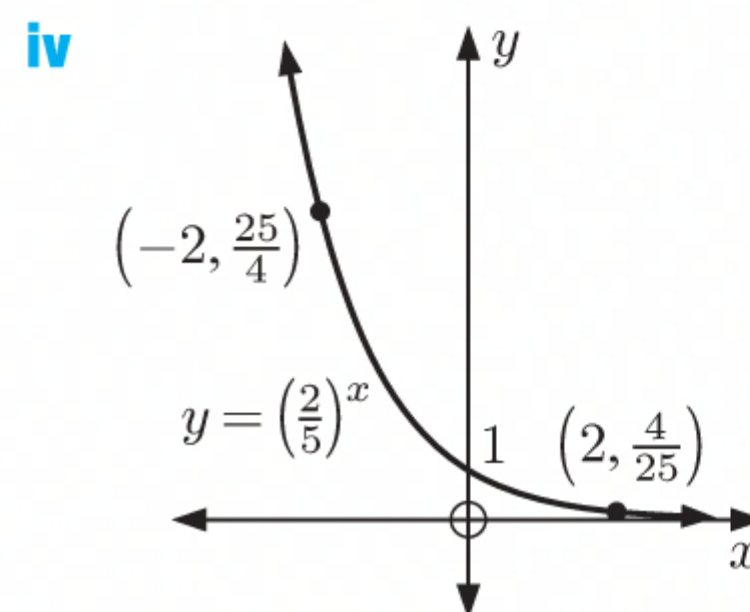
c $y = \left(\frac{2}{5}\right)^x$

i When $x = 0$, $y = \left(\frac{2}{5}\right)^0 = 1$
 \therefore the y -intercept is 1.

iii When $x = 2$, $y = \left(\frac{2}{5}\right)^2$
 $= \frac{2^2}{5^2}$
 $= \frac{4}{25}$

When $x = -2$, $y = \left(\frac{2}{5}\right)^{-2}$
 $= \frac{2^{-2}}{5^{-2}}$
 $= \frac{5^2}{2^2}$
 $= \frac{25}{4}$

ii The horizontal asymptote is $y = 0$.



v The domain is $\{x \mid x \in \mathbb{R}\}$. The range is $\{y \mid y > 0\}$.

d $y = \left(\frac{1}{2}\right)^x - 3$

i When $x = 0$, $y = \left(\frac{1}{2}\right)^0 - 3$
 $= 1 - 3$
 $= -2$

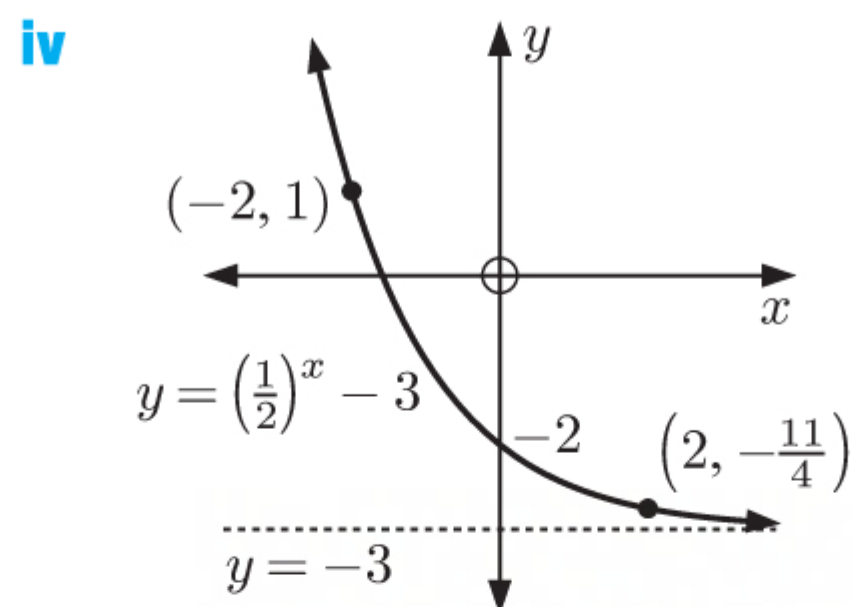
\therefore the y -intercept is -2 .

iii When $x = 2$, $y = \left(\frac{1}{2}\right)^2 - 3$
 $= \frac{1^2}{2^2} - 3$
 $= \frac{1}{4} - \frac{12}{4}$
 $= -\frac{11}{4}$

When $x = -2$, $y = \left(\frac{1}{2}\right)^{-2} - 3$
 $= \frac{1^{-2}}{2^{-2}} - 3$
 $= \frac{2^2}{1^2} - 3$
 $= 4 - 3$
 $= 1$

v The domain is $\{x \mid x \in \mathbb{R}\}$. The range is $\{y \mid y > -3\}$.

ii The horizontal asymptote is $y = -3$.



e $y = 2 - 2^x$

i When $x = 0$, $y = 2 - 2^0$
 $= 2 - 1$
 $= 1$

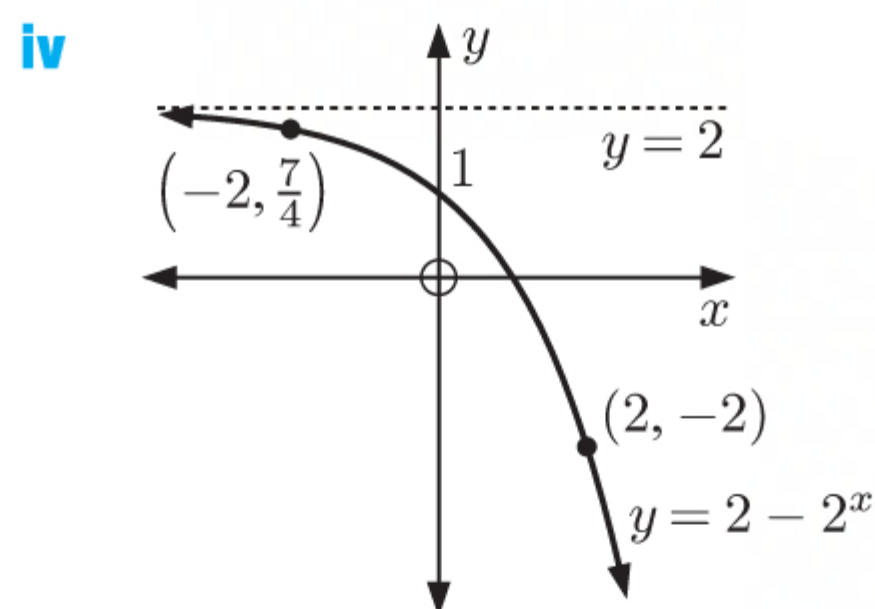
\therefore the y -intercept is 1 .

iii When $x = 2$, $y = 2 - 2^2$
 $= 2 - 4$
 $= -2$

When $x = -2$, $y = 2 - 2^{-2}$
 $= 2 - \frac{1}{2^2}$
 $= \frac{8}{4} - \frac{1}{4}$
 $= \frac{7}{4}$

v The domain is $\{x \mid x \in \mathbb{R}\}$. The range is $\{y \mid y < 2\}$.

ii The horizontal asymptote is $y = 2$.



f $y = 4^{-x} + 3$

i When $x = 0$, $y = 4^0 + 3$
 $= 1 + 3$
 $= 4$

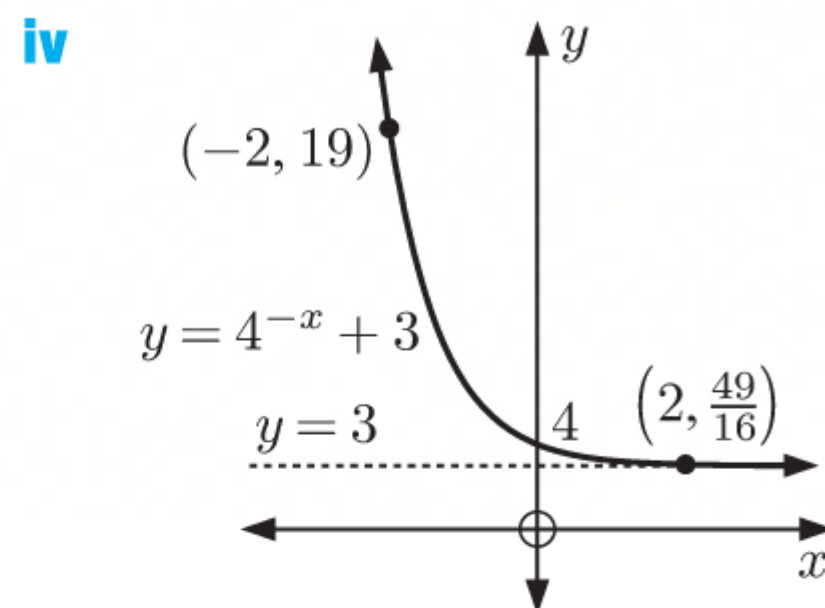
\therefore the y -intercept is 4.

iii When $x = 2$, $y = 4^{-2} + 3$
 $= \frac{1}{4^2} + 3$
 $= \frac{1}{16} + \frac{48}{16}$
 $= \frac{49}{16}$

When $x = -2$, $y = 4^{-(-2)} + 3$
 $= 4^2 + 3$
 $= 16 + 3$
 $= 19$

v The domain is $\{x \mid x \in \mathbb{R}\}$. The range is $\{y \mid y > 3\}$.

ii The horizontal asymptote is $y = 3$.



g $y = 3 - 2^{-x}$

i When $x = 0$, $y = 3 - 2^0$
 $= 3 - 1$
 $= 2$

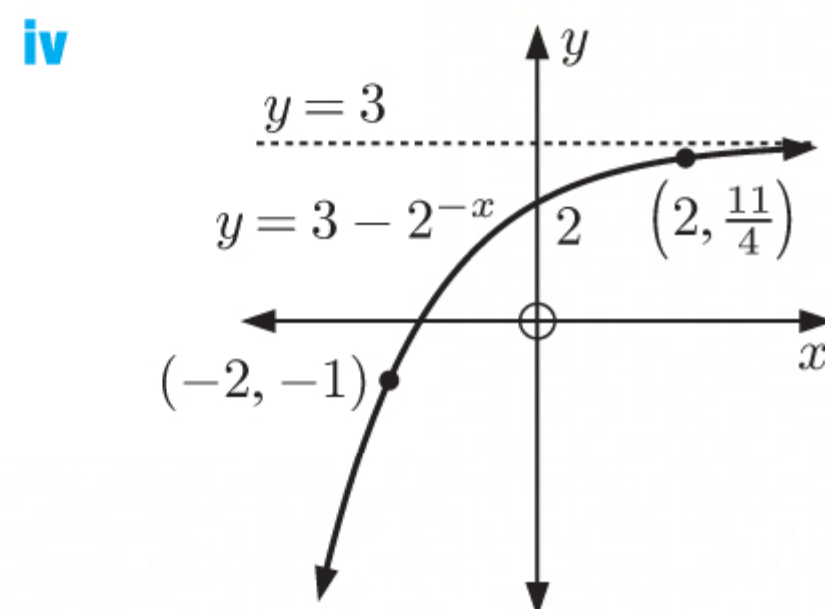
\therefore the y -intercept is 2.

iii When $x = 2$, $y = 3 - 2^{-2}$
 $= 3 - \frac{1}{2^2}$
 $= \frac{12}{4} - \frac{1}{4}$
 $= \frac{11}{4}$

When $x = -2$, $y = 3 - 2^{-(-2)}$
 $= 3 - 2^2$
 $= 3 - 4$
 $= -1$

v The domain is $\{x \mid x \in \mathbb{R}\}$. The range is $\{y \mid y < 3\}$.

ii The horizontal asymptote is $y = 3$.



h $y = -\frac{1}{2} \times 3^{-x} + 1$

i When $x = 0$, $y = -\frac{1}{2} \times 3^0 + 1$
 $= -\frac{1}{2} \times 1 + 1$
 $= \frac{1}{2}$

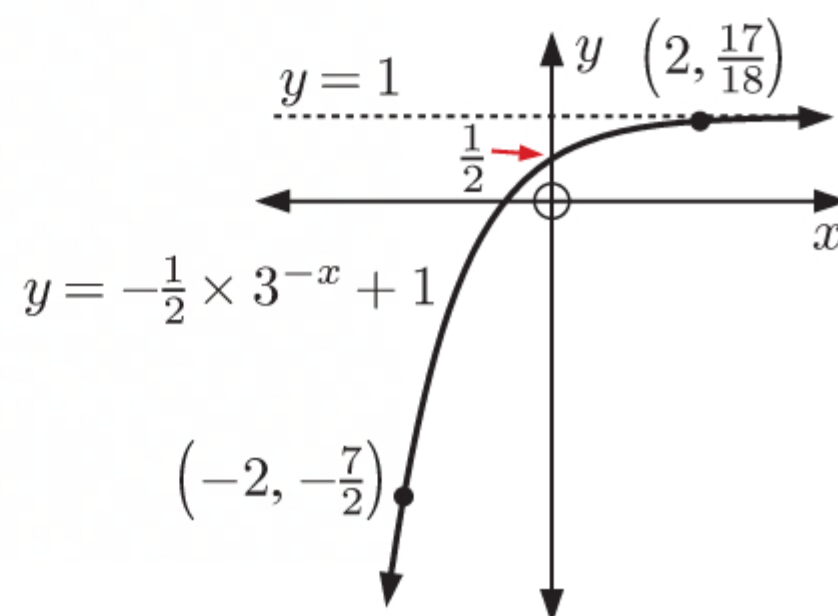
\therefore the y -intercept is $\frac{1}{2}$.

iii When $x = 2$, $y = -\frac{1}{2} \times 3^{-2} + 1$
 $= -\frac{1}{2} \times \frac{1}{3^2} + 1$
 $= -\frac{1}{2} \times \frac{1}{9} + 1$
 $= -\frac{1}{18} + \frac{18}{18}$
 $= \frac{17}{18}$

When $x = -2$, $y = -\frac{1}{2} \times 3^{-(-2)} + 1$
 $= -\frac{1}{2} \times 3^2 + 1$
 $= -\frac{1}{2} \times 9 + 1$
 $= -\frac{9}{2} + \frac{2}{2}$
 $= -\frac{7}{2}$

ii The horizontal asymptote is $y = 1$.

iv



v The domain is $\{x \mid x \in \mathbb{R}\}$. The range is $\{y \mid y < 1\}$.

9 a $y = k \times 2^x + c$

Substituting $(0, -5)$ into the equation gives

$$-5 = k \times 2^0 + c$$

$$\therefore k + c = -5$$

Substituting $(2, 10)$ into the equation gives

$$10 = k \times 2^2 + c$$

$$\therefore 4k + c = 10$$

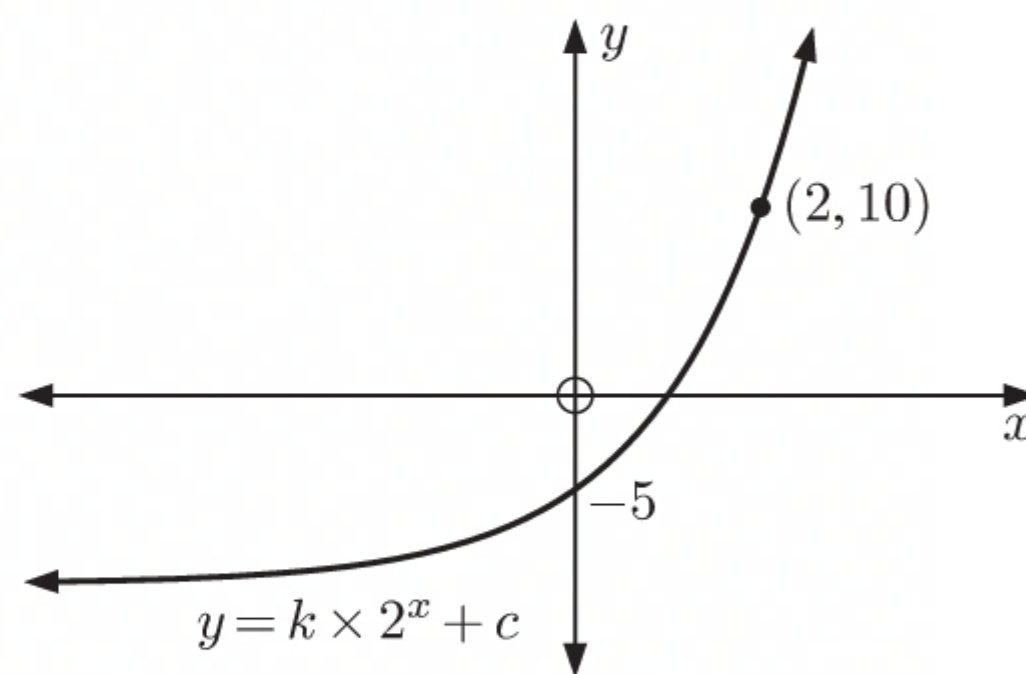
Solving the system of equations

$$\begin{cases} k + c = -5 \\ 4k + c = 10 \end{cases}$$

simultaneously gives $k = 5$,
 $c = -10$.

b $y = 5 \times 2^x - 10$

When $x = 6$, $y = 5 \times 2^6 - 10$
 $= 5 \times 64 - 10$
 $= 310$



Calculator screen showing the solution of the system of equations using the matrix method. The matrix is $\begin{bmatrix} 1 & 1 & -5 \\ 4 & 1 & 10 \end{bmatrix}$. The result for k is 5 and for c is -10.

Calculator screen showing the solution of the system of equations using the elimination method. The result for k is 5 and for c is -10.

10 a $f(0) = 3.5 - a^0$
 $= 3.5 - 1$
 $= 2.5$

\therefore the y -intercept is 2.5.

\therefore P is (0, 2.5).

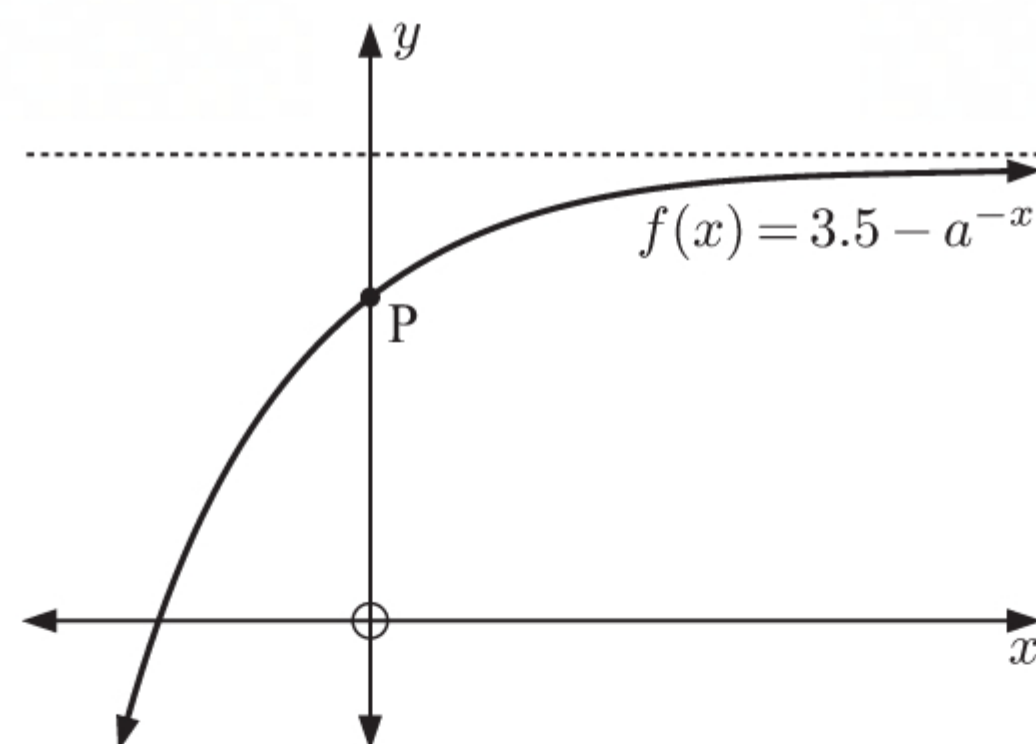
b The point $(-1, 2)$ lies on the graph.

$\therefore f(-1) = 2$

$\therefore 3.5 - a^1 = 2$

$\therefore a = 1.5$

c $f(x) = 3.5 - 1.5^{-x}$ has horizontal asymptote $y = 3.5$.



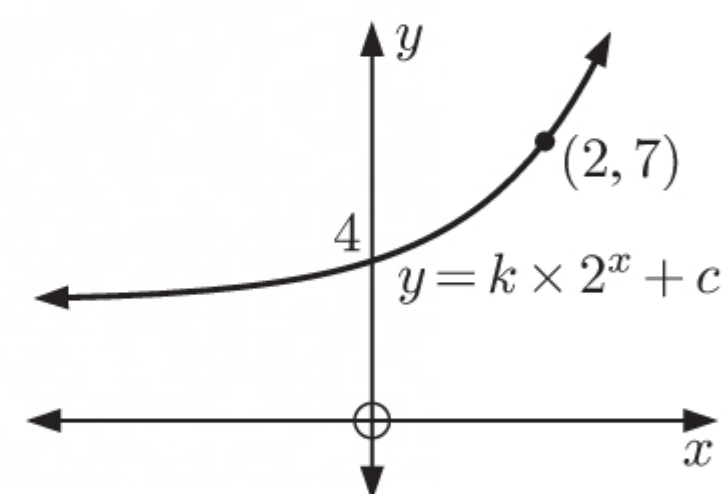
11 a $y = k \times 2^x + c$

Substituting $(0, 4)$ into the equation gives $4 = k \times 2^0 + c$

$\therefore k + c = 4$

Substituting $(2, 7)$ into the equation gives $7 = k \times 2^2 + c$

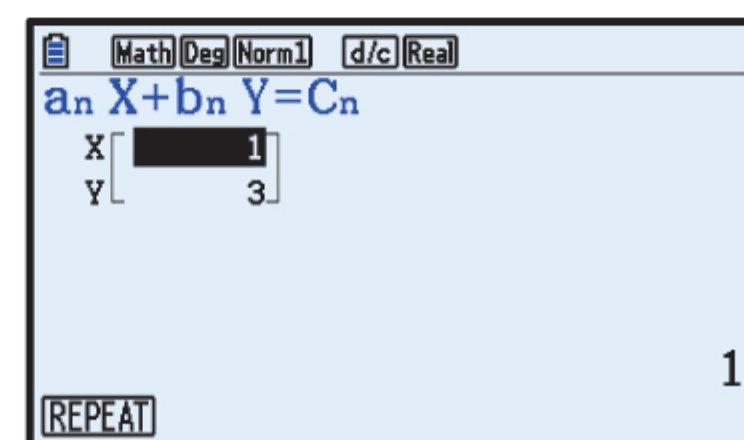
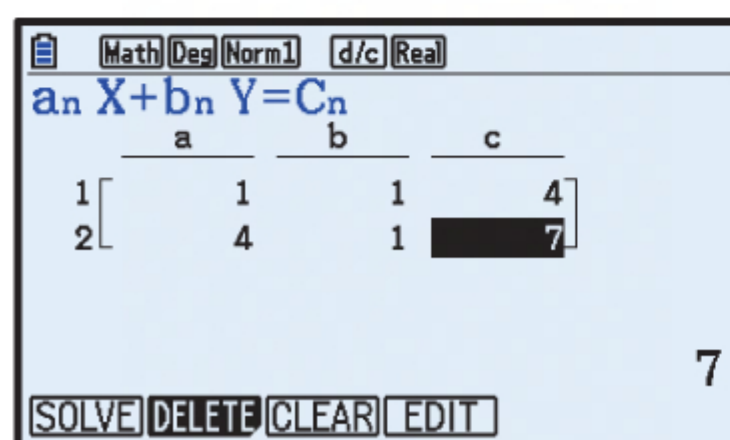
$\therefore 4k + c = 7$



Solving the system of equations

$$\begin{cases} k + c = 4 \\ 4k + c = 7 \end{cases} \text{ simultaneously}$$

gives $k = 1$, $c = 3$.



\therefore the exponential model is $y = 2^x + 3$.

b $y = k \times 3^x + c$

Substituting $(-1, 3)$ into the equation gives

$3 = k \times 3^{-1} + c$

$\therefore \frac{1}{3}k + c = 3$

Substituting $(1, -13)$ into the equation gives

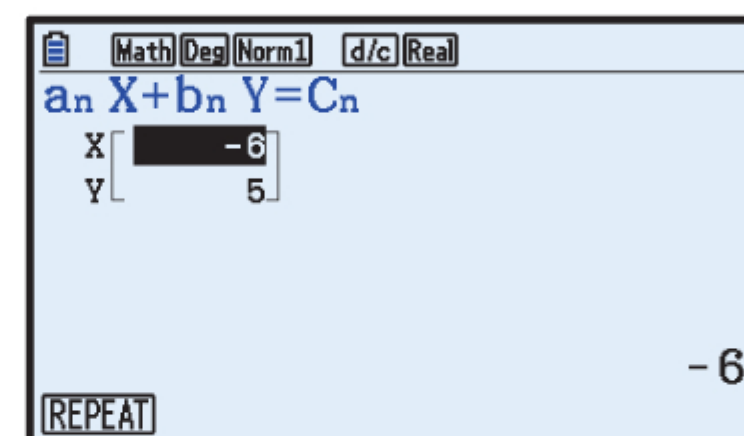
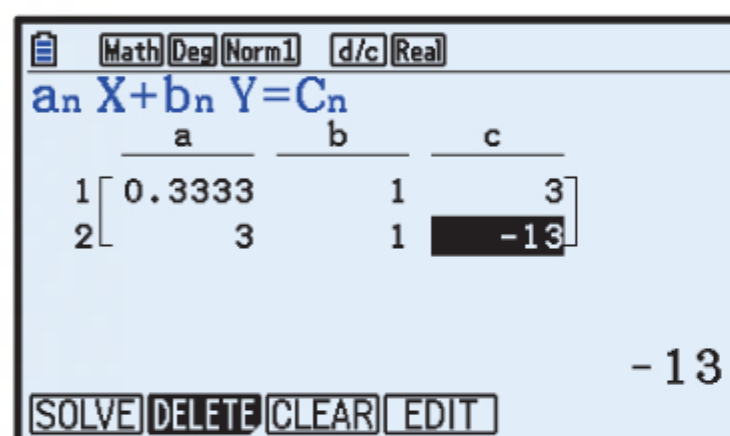
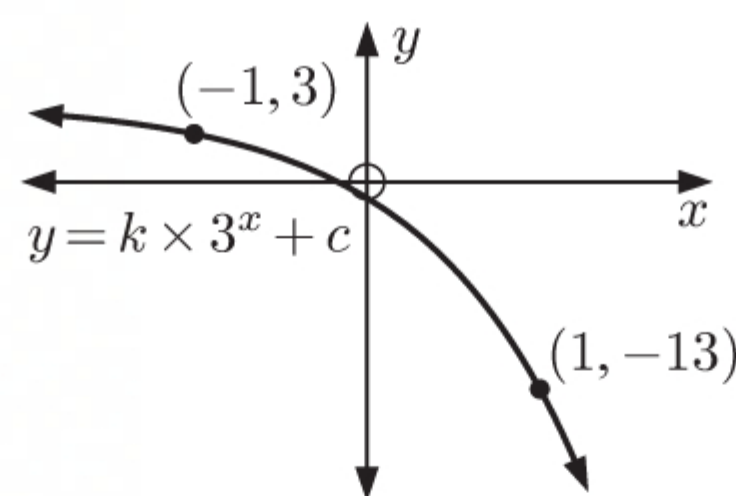
$-13 = k \times 3^1 + c$

$\therefore 3k + c = -13$

Solving the system of equations

$$\begin{cases} \frac{1}{3}k + c = 3 \\ 3k + c = -13 \end{cases}$$

simultaneously gives $k = -6$,
 $c = 5$.



\therefore the exponential model is $y = -6 \times 3^x + 5$.

c $y = k \times 2^{-x} + c$

Substituting $(-2, 9)$ into the equation gives

$$\begin{aligned} 9 &= k \times 2^{-(-2)} + c \\ &= k \times 2^2 + c \end{aligned}$$

$$\therefore 4k + c = 9$$

Substituting $(1, -5)$ into the equation gives

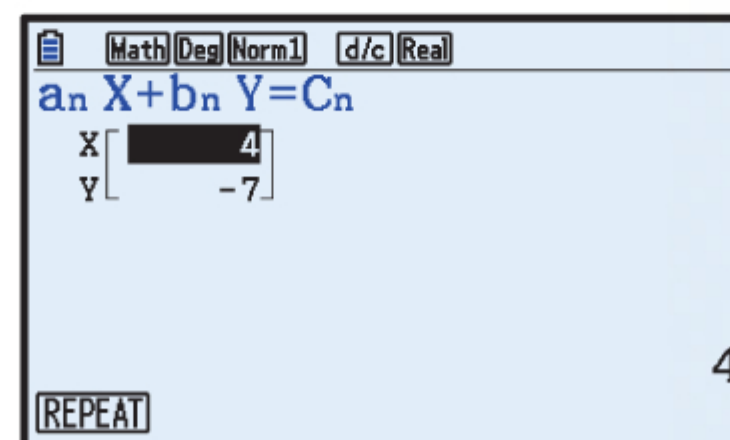
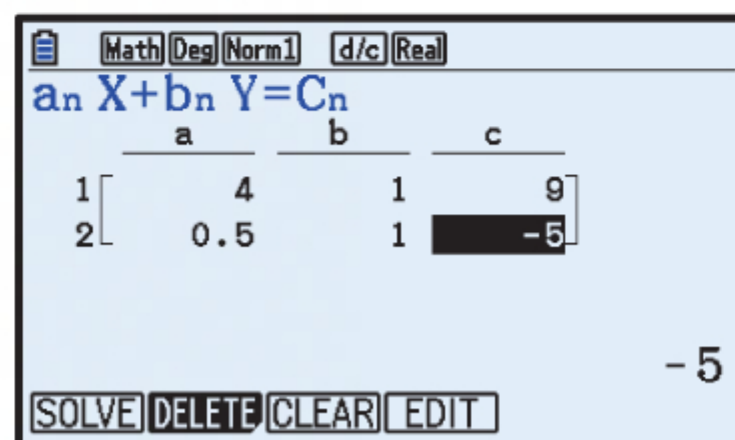
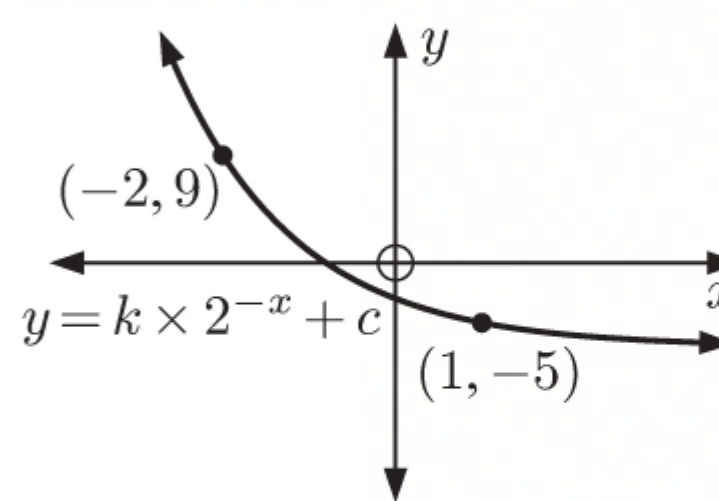
$$-5 = k \times 2^{-1} + c$$

$$\therefore \frac{1}{2}k + c = -5$$

Solving the system of equations

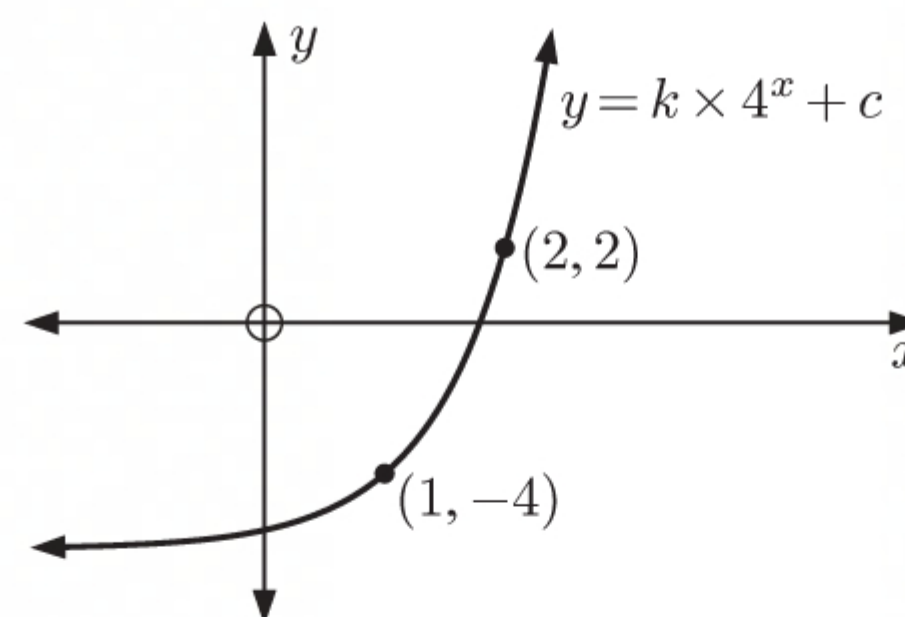
$$\begin{cases} 4k + c = 9 \\ \frac{1}{2}k + c = -5 \end{cases}$$

simultaneously gives $k = 4$,
 $c = -7$.



\therefore the exponential model is $y = 4 \times 2^{-x} - 7$.

- 12 a i The graph is increasing.
 $\therefore k$ must be positive.
- ii The horizontal asymptote lies below the x -axis.
 $\therefore c$ must be negative.



b $y = k \times 4^x + c$

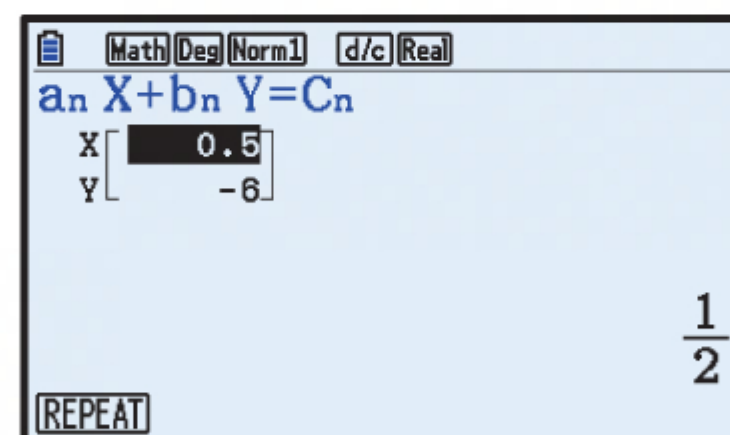
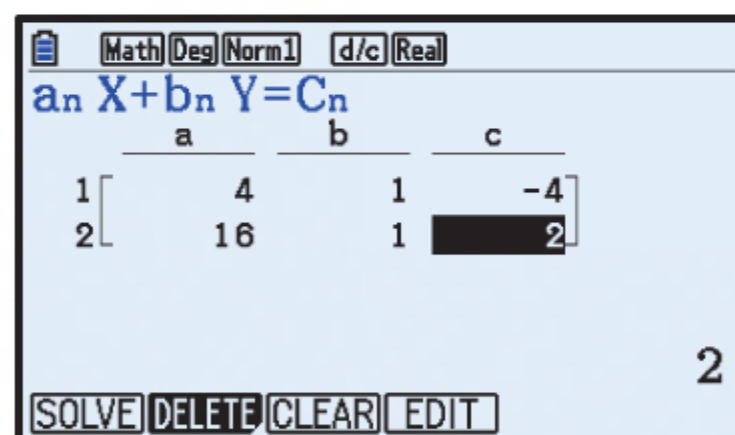
Substituting $(1, -4)$ into the equation gives $-4 = k \times 4^1 + c$
 $\therefore 4k + c = -4$

Substituting $(2, 2)$ into the equation gives $2 = k \times 4^2 + c$
 $\therefore 16k + c = 2$

Solving the system of equations

$$\begin{cases} 4k + c = -4 \\ 16k + c = 2 \end{cases}$$

simultaneously gives $k = \frac{1}{2}$,
 $c = -6$.



\therefore the exponential function is $y = \frac{1}{2} \times 4^x - 6$.

c When $x = 0$, $y = \frac{1}{2} \times 4^0 - 6$
 $= \frac{1}{2} \times 1 - 6$
 $= -\frac{11}{2}$

\therefore the y -intercept is $-\frac{11}{2}$.

d The horizontal asymptote of $y = \frac{1}{2} \times 4^x - 6$ is $y = -6$.

13 $f(x) = k \times a^x + c$

The graph has horizontal asymptote $y = 2$, so $c = 2$.

The y -intercept is 10, so $f(0) = 10$

$$\therefore k \times a^0 + 2 = 10$$

$$\therefore k = 8$$

The graph passes through $(5, 258)$, so $f(5) = 258$

$$\therefore 8 \times a^5 + 2 = 258$$

$$\therefore 8a^5 = 256$$

$$\therefore a^5 = 32$$

$$\therefore a = 2$$

So, the exponential function is $f(x) = 8 \times 2^x + 2$.

14 $f(x) = k \times a^{-x} + c$

The graph has horizontal asymptote $y = 4$, so $c = 4$.

The y -intercept is 1, so $f(0) = 1$

$$\therefore k \times a^0 + 4 = 1$$

$$\therefore k = -3$$

The graph passes through $(1, 2)$, so $f(1) = 2$

$$\therefore -3 \times a^{-1} + 4 = 2$$

$$\therefore -\frac{3}{a} = -2$$

$$\therefore a = \frac{3}{2}$$

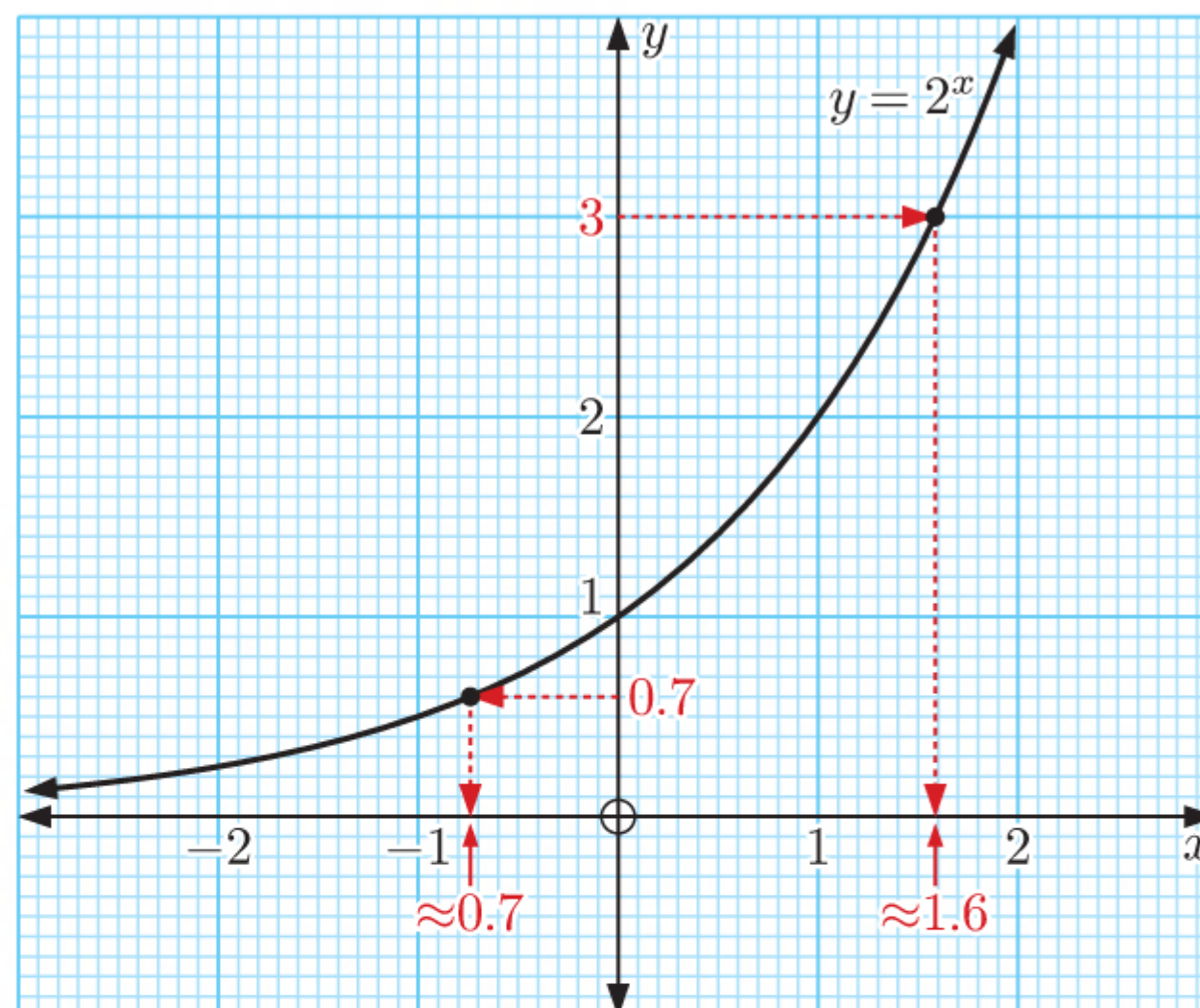
So, the exponential function is $f(x) = -3 \times \left(\frac{3}{2}\right)^{-x} + 4$.

EXERCISE 8D

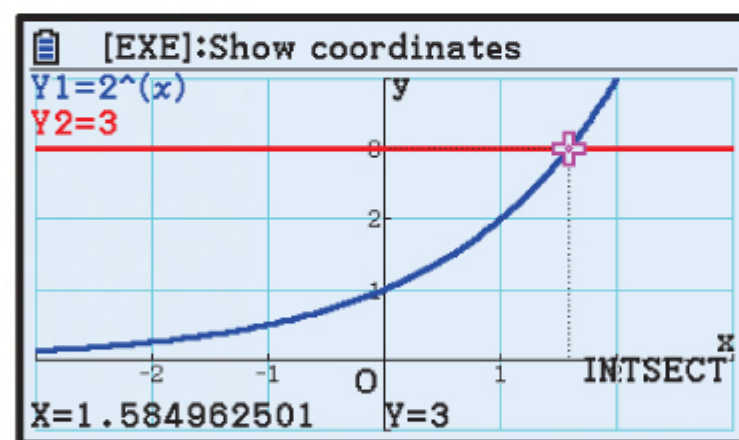
1 a From the graph:

i $2^x = 3$ when $x \approx 1.6$.

ii $2^x = 0.6$ when $x \approx -0.7$.

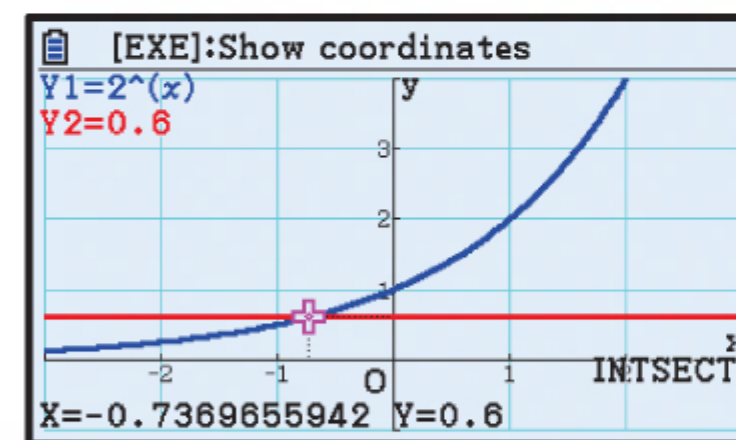


- b i** We graph $Y_1 = 2^x$ and $Y_2 = 3$ on the same set of axes, and find their point of intersection.



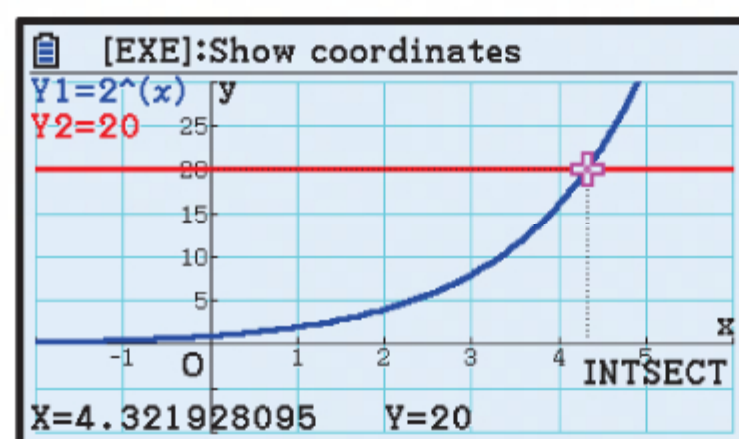
The solution is $x \approx 1.58$. ✓

- ii** We graph $Y_1 = 2^x$ and $Y_2 = 0.6$ on the same set of axes, and find their point of intersection.



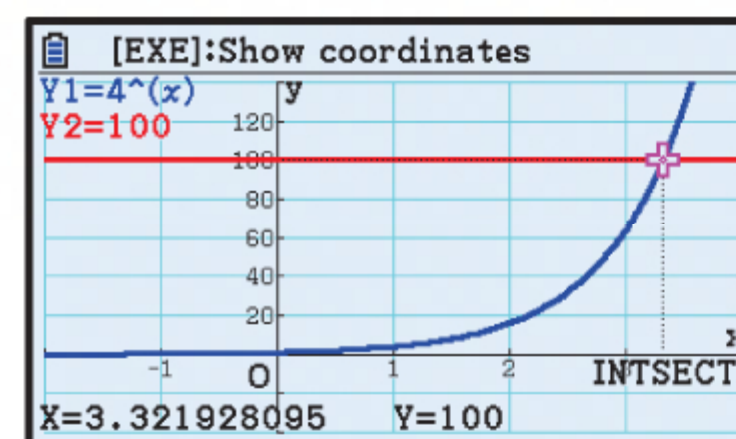
The solution is $x \approx -0.737$. ✓

- 2 a** We graph $Y_1 = 2^x$ and $Y_2 = 20$ on the same set of axes, and find their point of intersection.



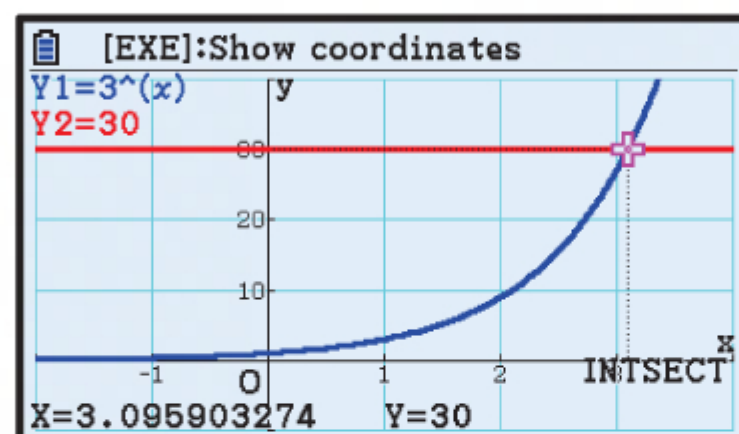
The solution is $x \approx 4.32$.

- b** We graph $Y_1 = 4^x$ and $Y_2 = 100$ on the same set of axes, and find their point of intersection.



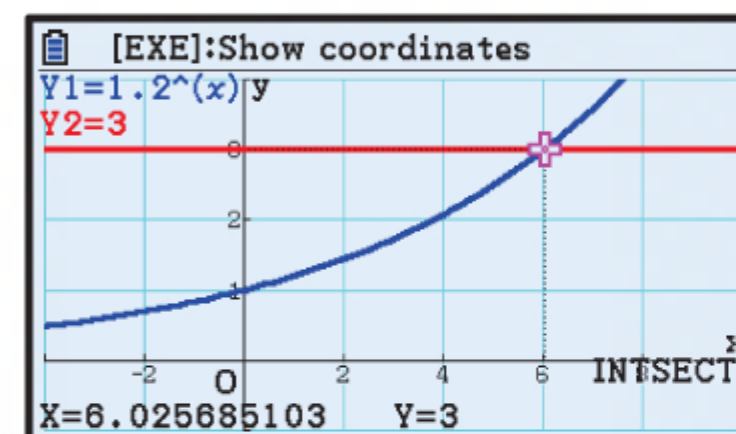
The solution is $x \approx 3.32$.

- c** We graph $Y_1 = 3^x$ and $Y_2 = 30$ on the same set of axes, and find their point of intersection.



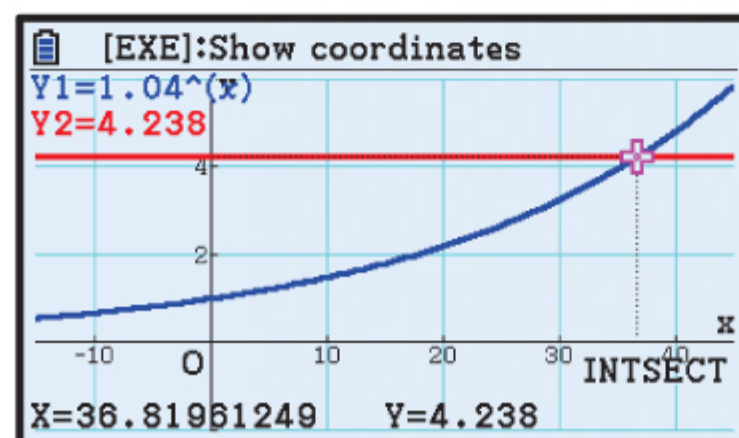
The solution is $x \approx 3.10$.

- d** We graph $Y_1 = (1.2)^x$ and $Y_2 = 3$ on the same set of axes, and find their point of intersection.



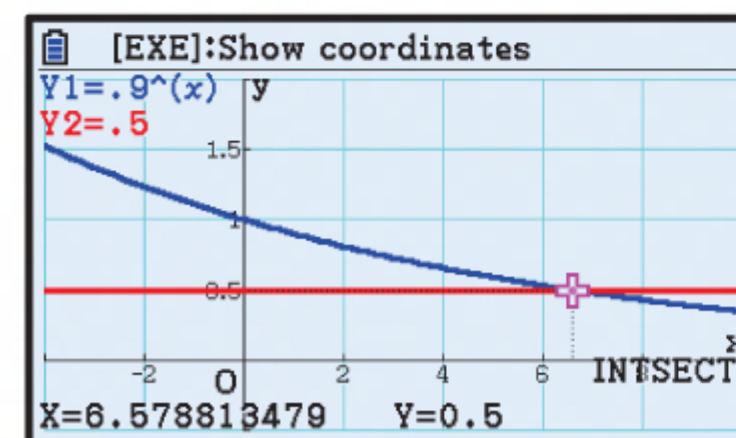
The solution is $x \approx 6.03$.

- e** We graph $Y_1 = (1.04)^x$ and $Y_2 = 4.238$ on the same set of axes, and find their point of intersection.



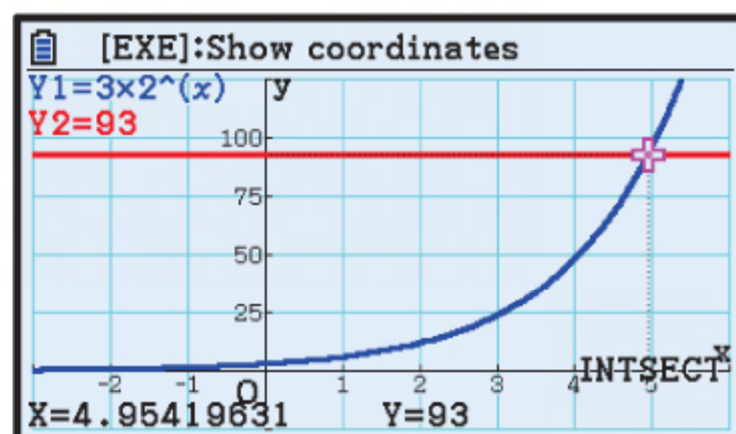
The solution is $x \approx 36.8$.

- f** We graph $Y_1 = (0.9)^x$ and $Y_2 = 0.5$ on the same set of axes, and find their point of intersection.



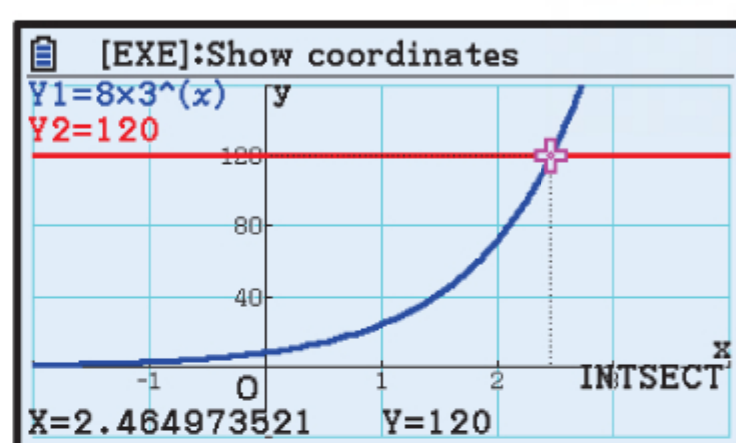
The solution is $x \approx 6.58$.

- 3 a** We graph $Y_1 = 3 \times 2^x$ and $Y_2 = 93$ on the same set of axes, and find their point of intersection.



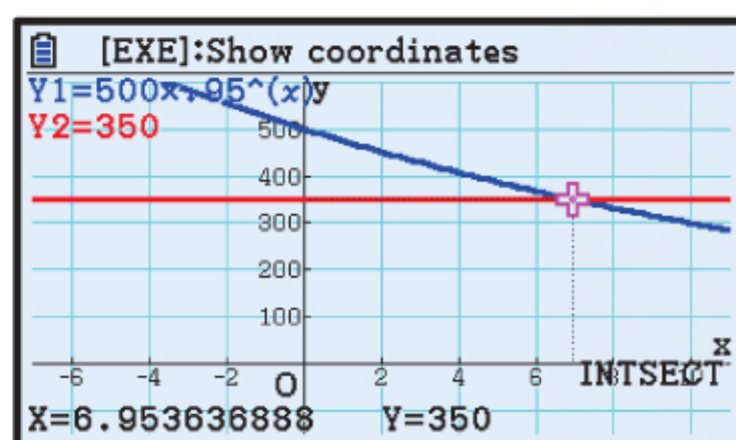
The solution is $x \approx 4.95$.

- c** We graph $Y_1 = 8 \times 3^x$ and $Y_2 = 120$ on the same set of axes, and find their point of intersection.



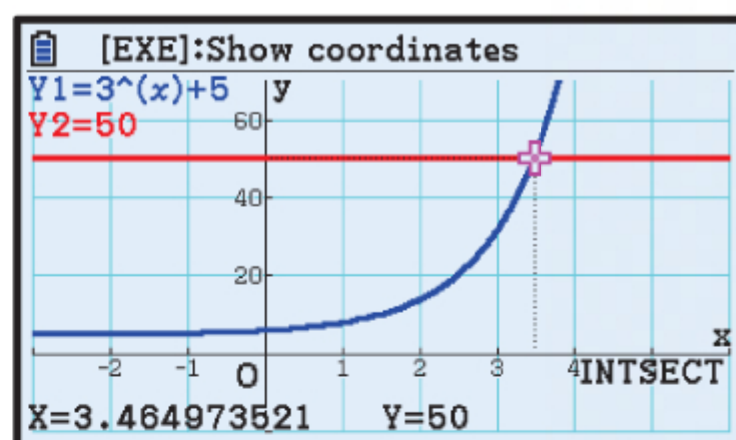
The solution is $x \approx 2.46$.

- e** We graph $Y_1 = 500 \times (0.95)^x$ and $Y_2 = 350$ on the same set of axes, and find their point of intersection.



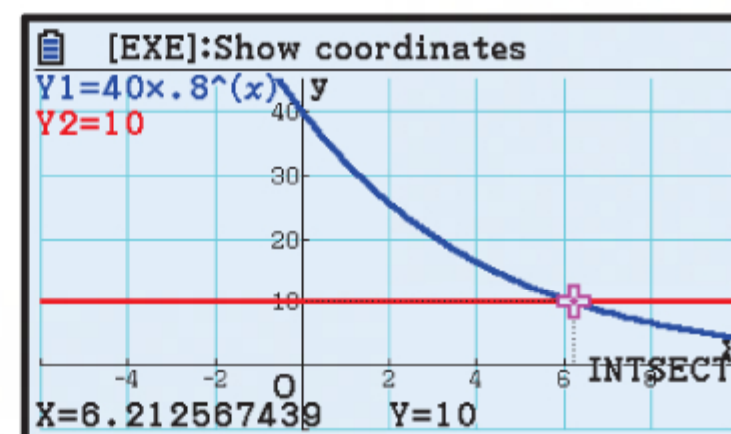
The solution is $x \approx 6.95$.

- g** We graph $Y_1 = 3^x + 5$ and $Y_2 = 50$ on the same set of axes, and find their point of intersection.



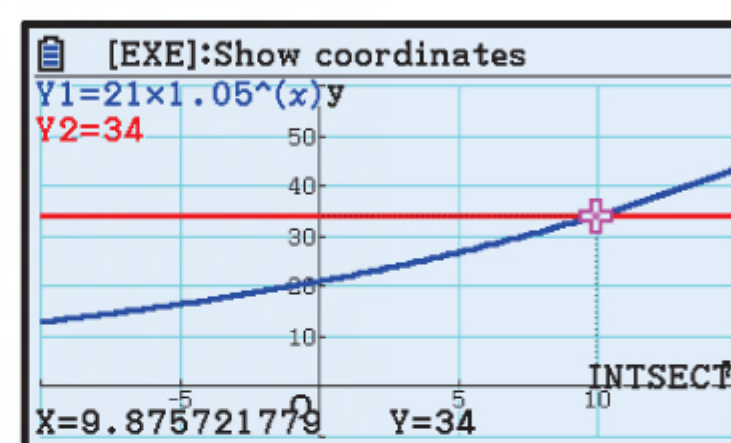
The solution is $x \approx 3.46$.

- b** We graph $Y_1 = 40 \times (0.8)^x$ and $Y_2 = 10$ on the same set of axes, and find their point of intersection.



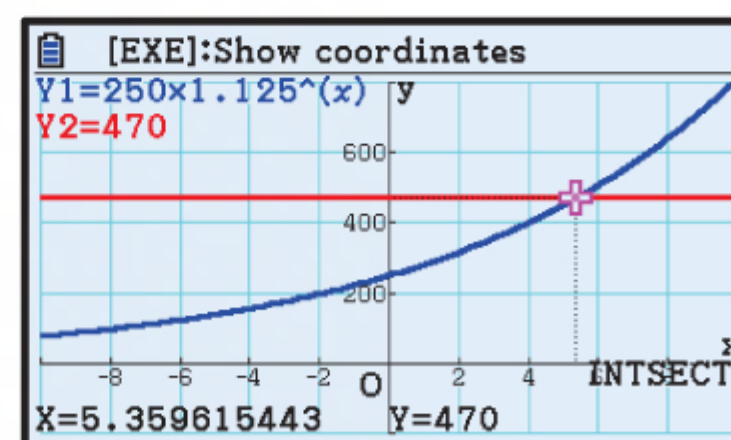
The solution is $x \approx 6.21$.

- d** We graph $Y_1 = 21 \times (1.05)^x$ and $Y_2 = 34$ on the same set of axes, and find their point of intersection.



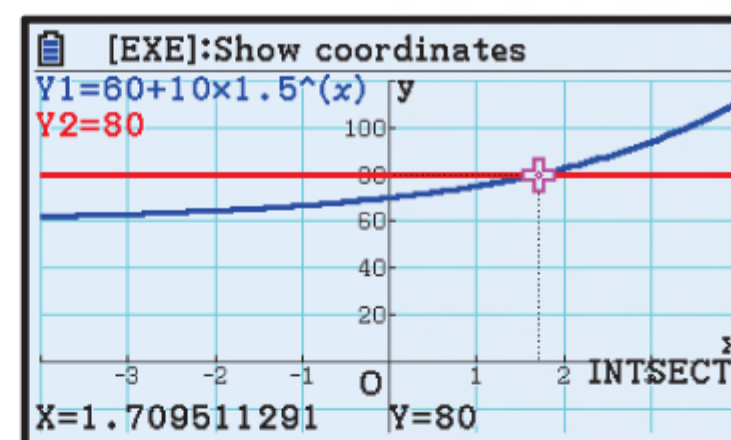
The solution is $x \approx 9.88$.

- f** We graph $Y_1 = 250 \times (1.125)^x$ and $Y_2 = 470$ on the same set of axes, and find their point of intersection.



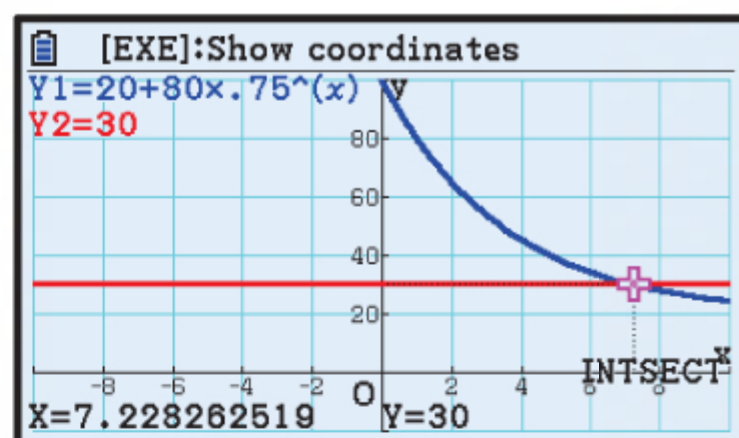
The solution is $x \approx 5.36$.

- h** We graph $Y_1 = 60 + 10 \times (1.5)^x$ and $Y_2 = 80$ on the same set of axes, and find their point of intersection.



The solution is $x \approx 1.71$.

- i** We graph $Y_1 = 20 + 80 \times (0.75)^x$ and $Y_2 = 30$ on the same set of axes, and find their point of intersection.



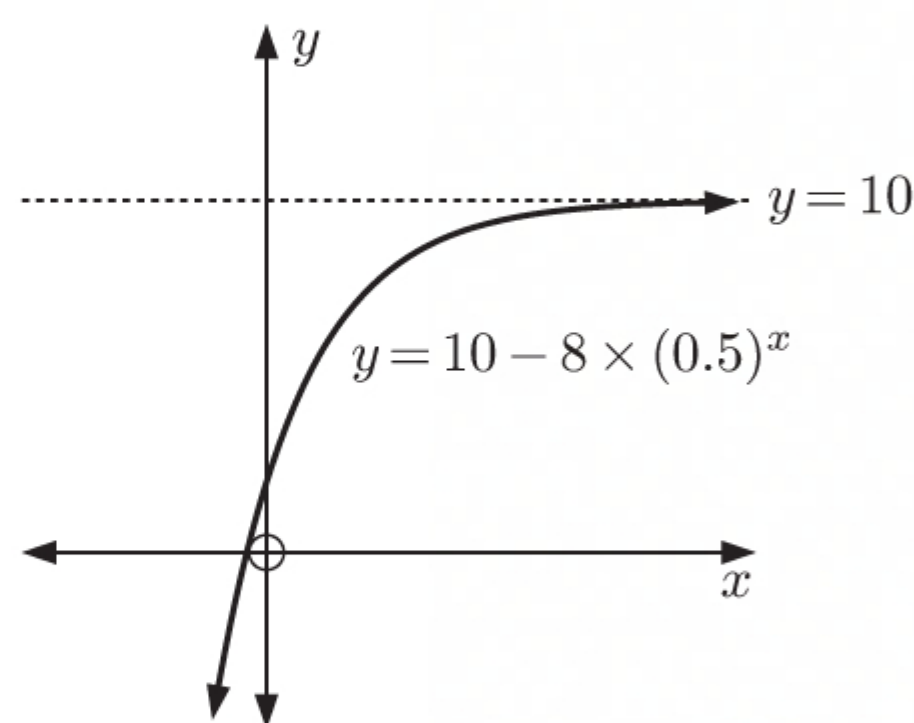
The solution is $x \approx 7.23$.

- 4** The graph of $y = 10 - 8 \times (0.5)^x$ lies below its horizontal asymptote $y = 10$.

$\therefore y < 10$ for all x .

So, the equation $10 - 8 \times (0.5)^x = k$ has:

- a** 1 solution for $k < 10$
- b** no solutions for $k \geq 10$.



EXERCISE 8E.1

- 1** $A(t) = 3 \times (1.08)^t$ square metres

a $A(0) = 3 \times (1.08)^0$
 $= 3$

The initial area covered by the weed was 3 m^2 .

- b** The multiplier is 1.08, so the area increases by 8% each day.

c i $A(2) = 3 \times (1.08)^2$
 ≈ 3.50

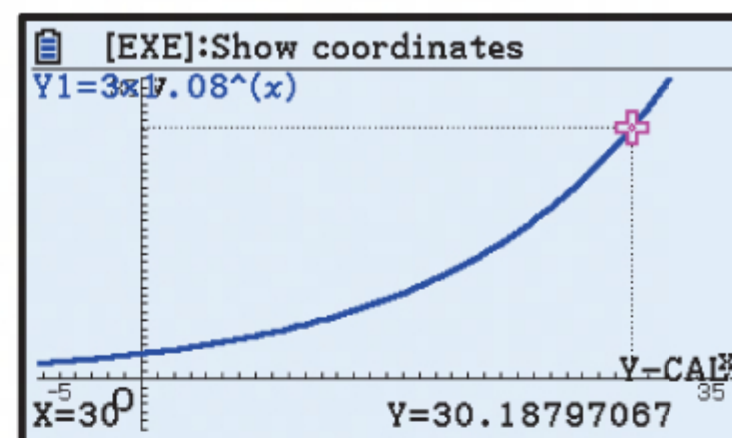
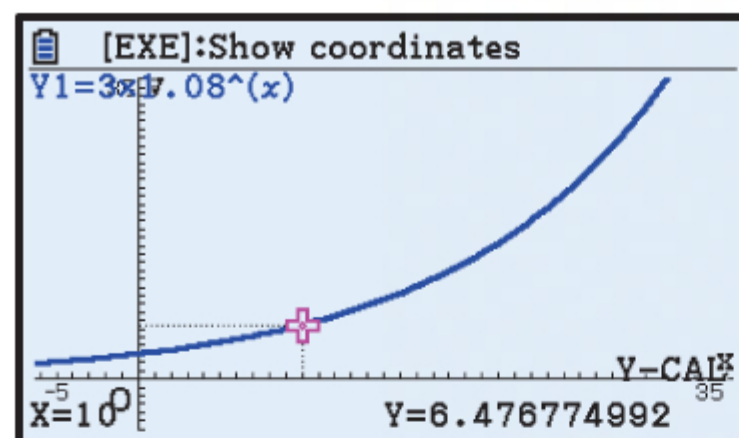
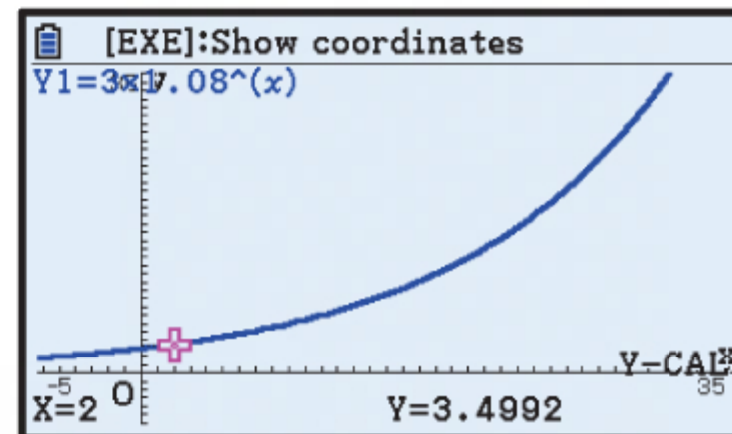
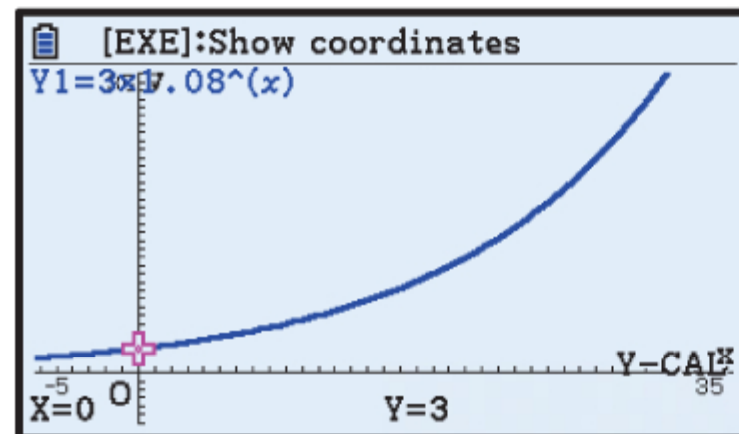
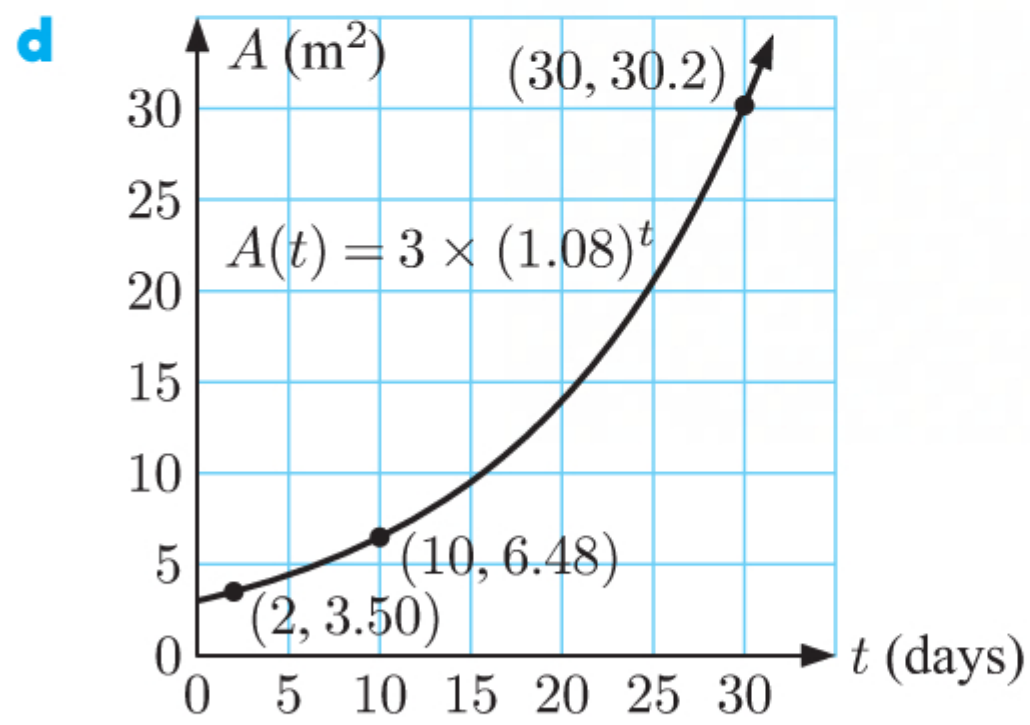
The area covered after 2 days is about 3.50 m^2 .

iii $A(30) = 3 \times (1.08)^{30}$
 ≈ 30.2

The area covered after 30 days is about 30.2 m^2 .

ii $A(10) = 3 \times (1.08)^{10}$
 ≈ 6.48

The area covered after 10 days is about 6.48 m^2 .



2 $W(t) = 100 \times (1.07)^t$ grams

a $W(0) = 100 \times (1.07)^0$
 $= 100 \times 1$
 $= 100$

\therefore the initial weight was 100 grams.

b The value 1.07 means that the weight is increasing by 7% every hour.

c i $W(4) = 100 \times (1.07)^4$
 ≈ 131

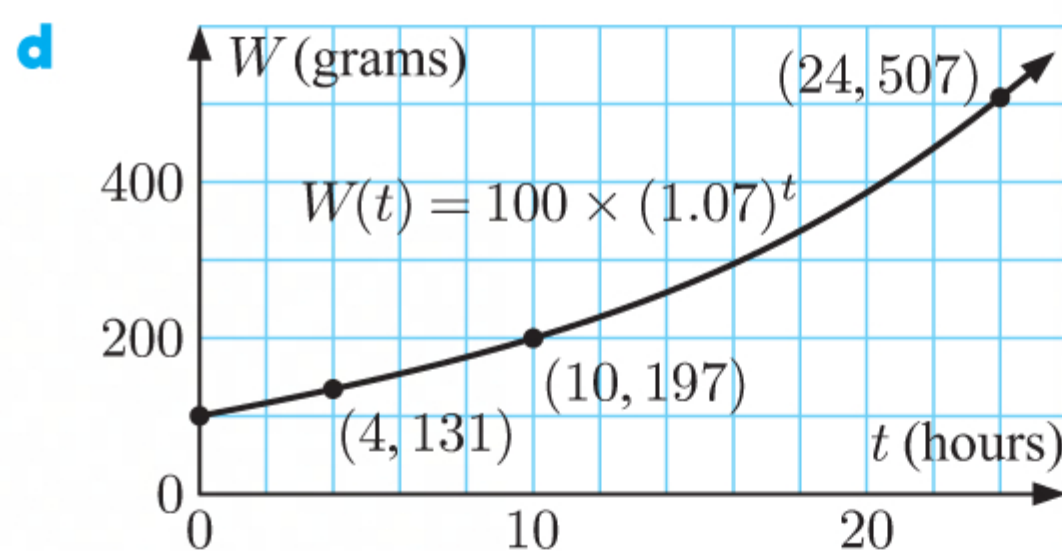
The weight after 4 hours is about 131 grams.

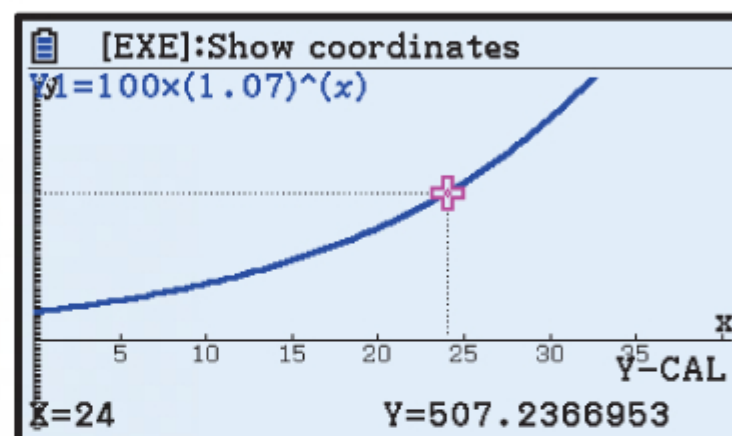
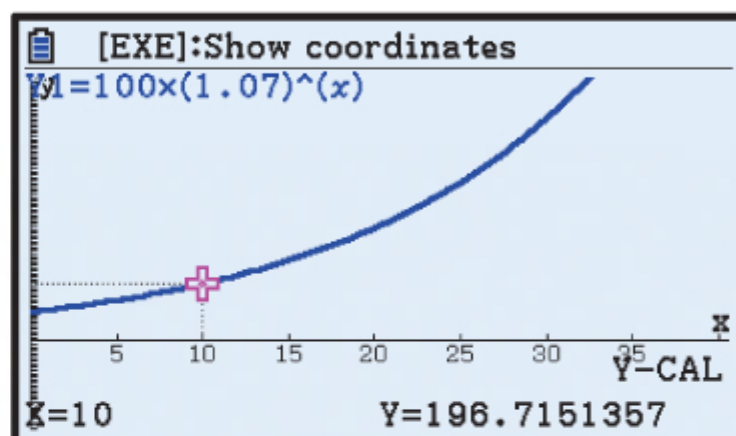
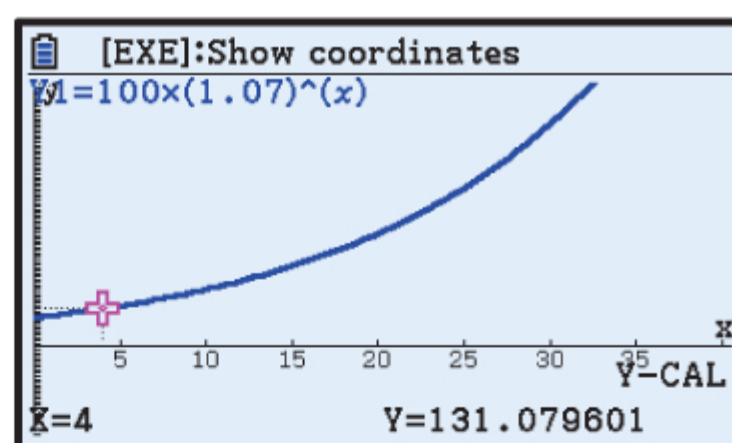
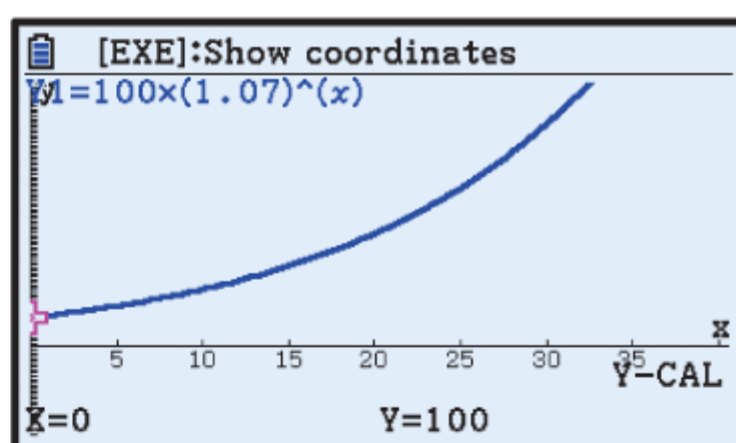
ii $W(10) = 100 \times (1.07)^{10}$
 ≈ 197

The weight after 10 hours is about 197 grams.

iii $W(24) = 100 \times (1.07)^{24}$
 ≈ 507

The weight after 24 hours is about 507 grams.





3 $P(n) = P_0 \times (1.23)^n$ possums

a The initial population is 50 possums, so $P(0) = 50$

$$\therefore P_0 \times (1.23)^0 = 50$$

$$\therefore P_0 = 50$$

b $P(n) = 50 \times (1.23)^n$

i $P(2) = 50 \times (1.23)^2$
 ≈ 75.6

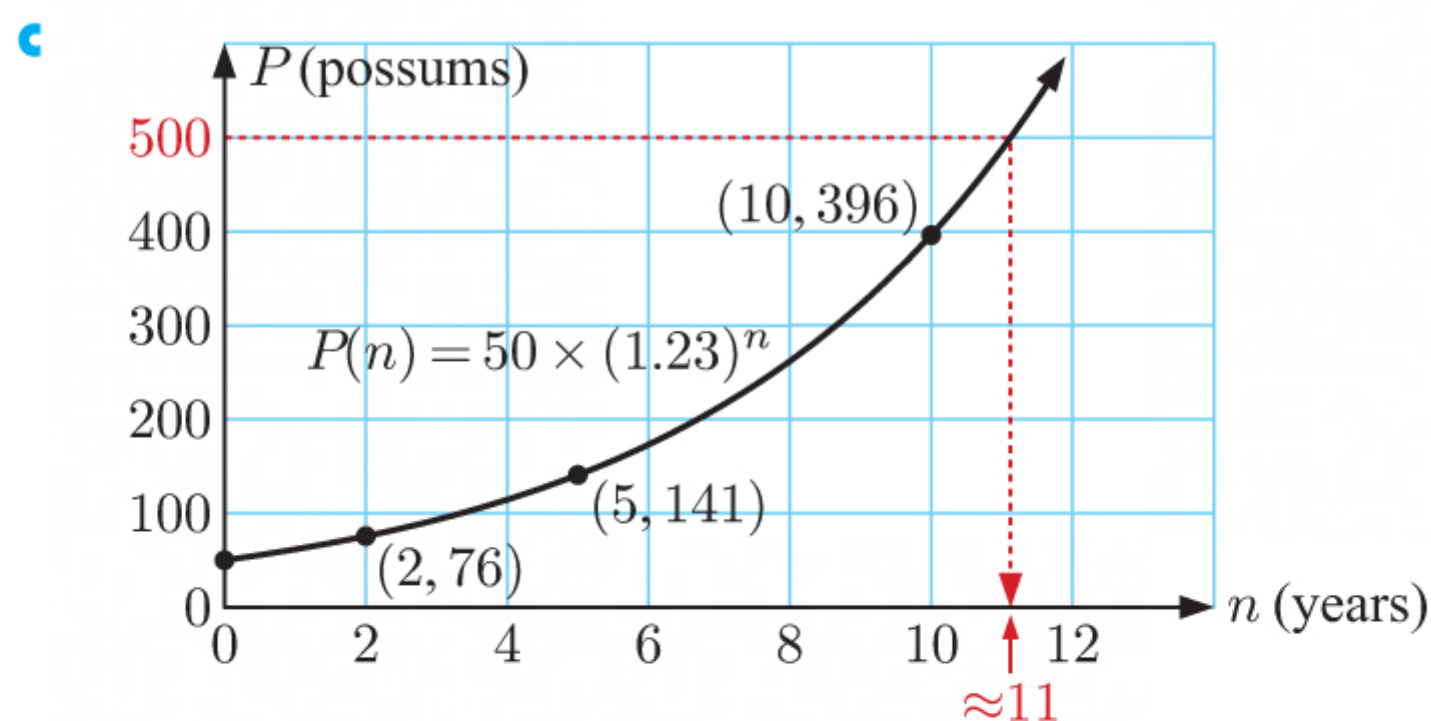
After 2 years, the expected population is about 76 possums.

ii $P(5) = 50 \times (1.23)^5$
 ≈ 141

After 5 years, the expected population is about 141 possums.

iii $P(10) = 50 \times (1.23)^{10}$
 ≈ 396

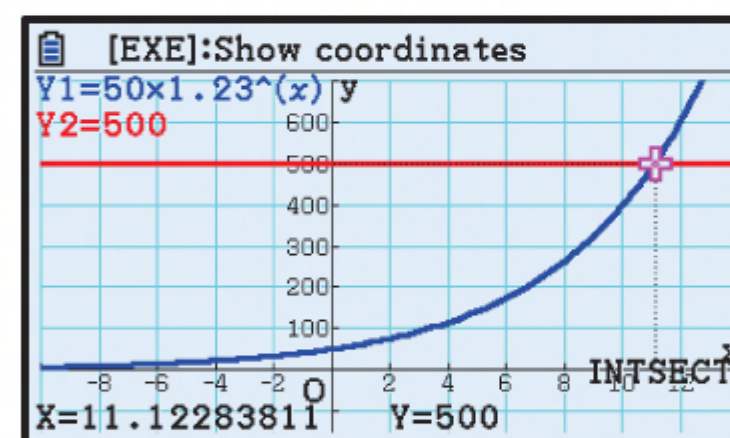
After 10 years, the expected population is about 396 possums.



d From the graph in c, it appears that it would take about 11 years for the population to reach 500.

When $P = 500$, $500 = 50 \times (1.23)^n$.

Using technology, $n \approx 11.1$.



It will take about 11.1 years for the population to reach 500. ✓

4 $V(t) = 5 \times (1.03)^t$ units

a i $V(0) = 5 \times (1.03)^0$
 $= 5$

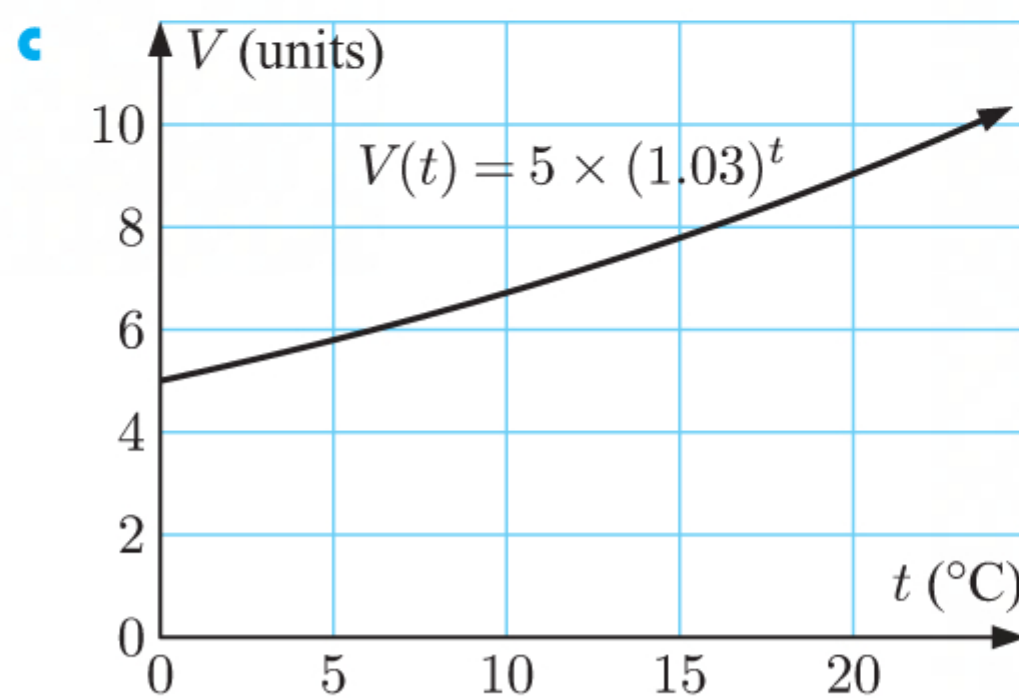
The speed of the reaction at 0°C is 5 units.

ii $V(20) = 5 \times (1.03)^{20}$
 ≈ 9.03

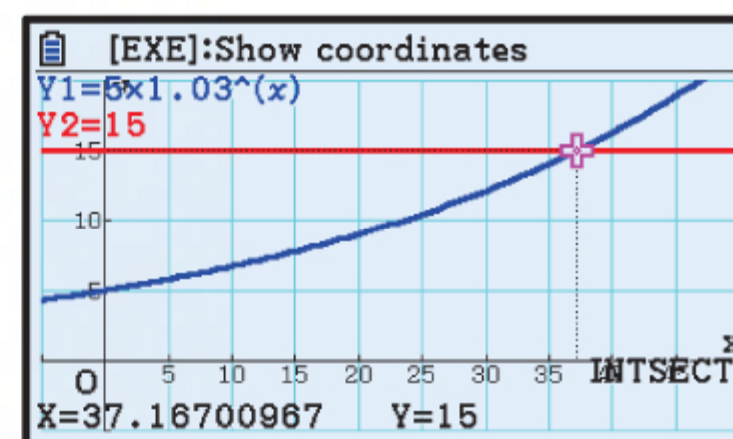
The speed of the reaction at 20°C is about 9.03 units.

b Percentage increase in speed from 0°C to $20^\circ\text{C} = \left(\frac{V(20) - V(0)}{V(0)} \right) \times 100\%$
 $\approx \left(\frac{9.03 - 5}{5} \right) \times 100\%$
 $\approx 80.6\%$

So, the percentage increase in reaction speed from 0°C to 20°C is about 80.6%.



d When $V(t) = 15$, $15 = 5 \times (1.03)^t$.
 Using technology, $t \approx 37.2$.



The speed of the reaction will reach 15 units when the temperature is about 37.2°C .

5 $N = 4 \times 1.332^t$, $t \geq 0$

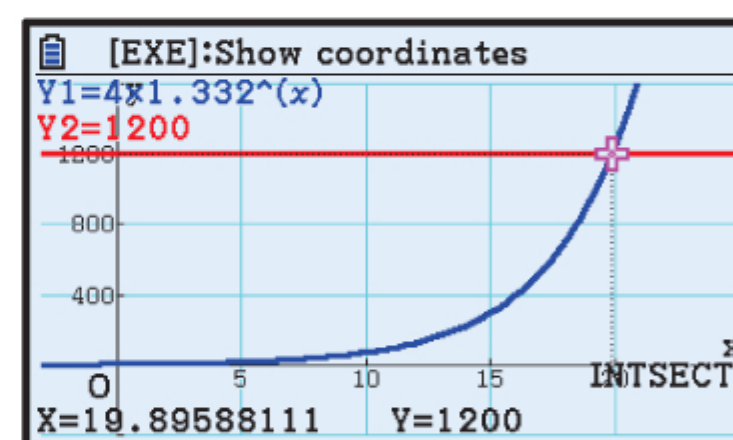
a When $t = 0$, $N = 4 \times 1.332^0$
 $= 4 \times 1$
 $= 4$

\therefore the number of people initially infected was 4.

b When $t = 16$, $N = 4 \times 1.332^{16}$
 ≈ 393

\therefore the number of people infected after 16 days was about 393.

c When $N = 1200$, $1200 = 4 \times 1.332^t$.
 Using technology, $t \approx 19.9$.



It will take about 19.9 days for everybody in the school to catch the flu.

d Since there are 1200 people in the school and it takes about 19.9 days for all of them to catch the flu (from c), it is only reasonable to use this model for $0 \leq t \leq 19.9$.

6 $B(t) = B_0 \times a^t$ bears

a There were initially 200 bears introduced to the island, so $B(0) = 200$

$$\therefore B_0 \times a^0 = 200$$

$$\therefore B_0 = 200$$

b 2000 is 2 years after 1998, so $t = 2$.

$$B(t) = 200 \times a^t$$

$$B(2) = 200 \times a^2$$

$$\therefore 242 = 200 \times a^2$$

$$\therefore \frac{242}{200} = a^2$$

$$\therefore a = \sqrt{\frac{242}{200}} = 1.1 \quad \{a > 0\}$$

The bear population is increasing by 10% every year.

c 2018 is 20 years after 1998, so $t = 20$.

$$B(20) = 200 \times (1.1)^{20}$$

$$\approx 1350$$

The expected bear population in 2018 is about 1350 bears.

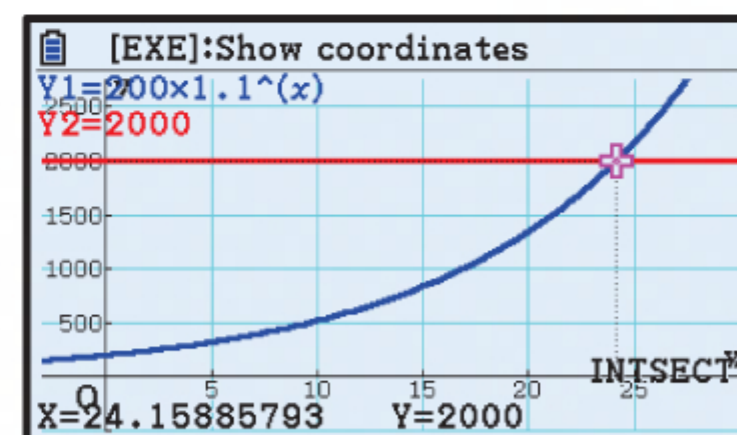
d 2008 is 10 years after 1998, so $t = 10$.

$$B(10) = 200 \times (1.1)^{10}$$

$$\begin{aligned} \text{Percentage increase from 2008 to 2018} &= \left(\frac{B(20) - B(10)}{B(10)} \right) \times 100\% \\ &= \left(\frac{200 \times (1.1)^{20} - 200 \times (1.1)^{10}}{200 \times (1.1)^{10}} \right) \times 100\% \\ &\approx 159\% \end{aligned}$$

e When $B(t) = 2000$, $2000 = 200 \times (1.1)^t$.

Using technology, $t \approx 24.2$.



It will take about 24.2 years for the population to reach 2000.

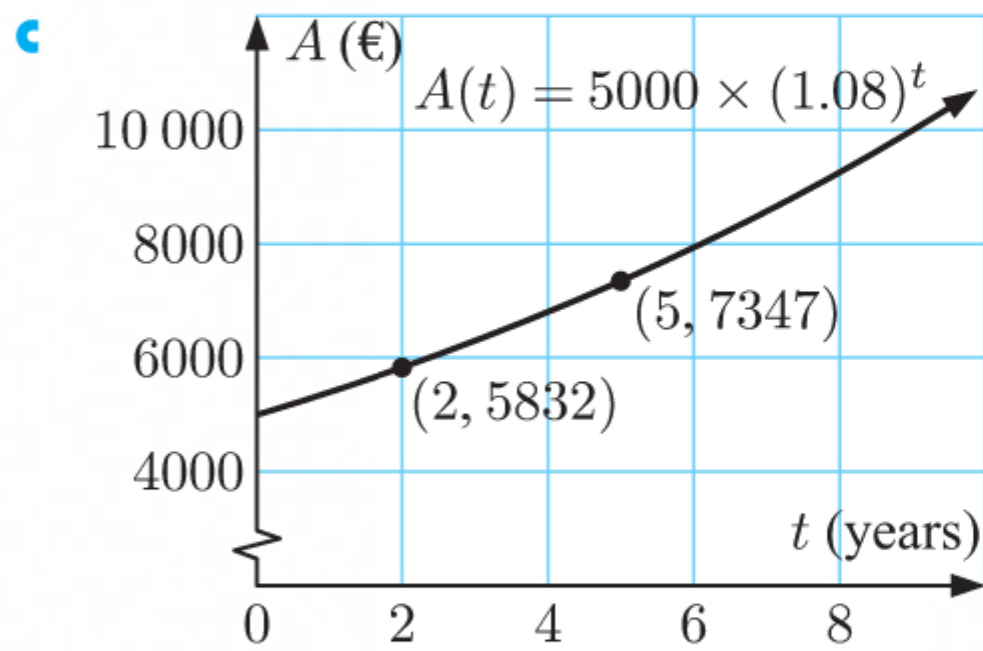
7 a $A(t) = 5000 \times (1.08)^t$ euros, where t is the number of years since Kayla deposited €5000.

b i $A(2) = 5000 \times (1.08)^2$
 $= 5832$

So there was €5832 in the account after 2 years.

ii $A(5) = 5000 \times (1.08)^5$
 ≈ 7346.64

So there was €7346.64 in the account after 5 years.

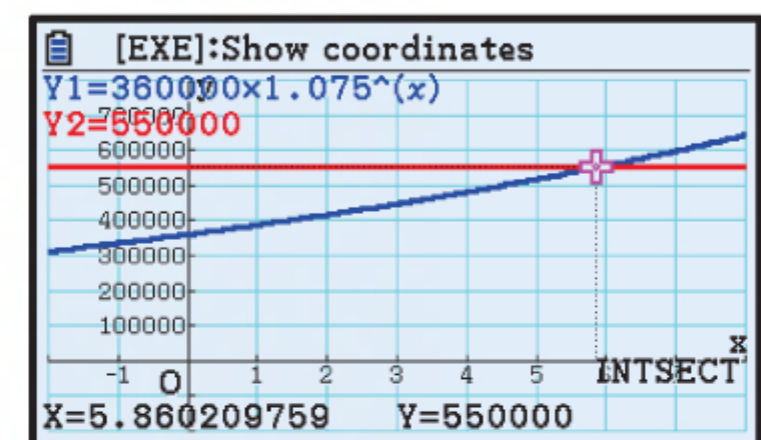
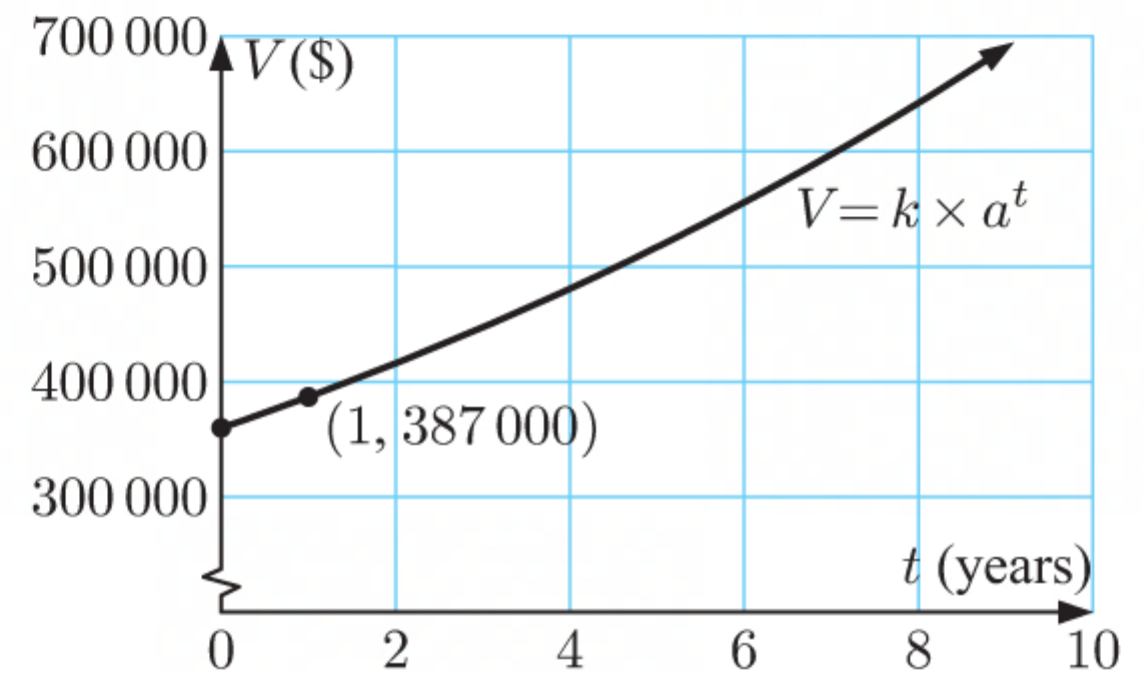


- 8 a** $V = k \times a^t$ dollars, $t \geq 0$
 The value increases by 7.5% each year,
 so the multiplier $a = 1.075$.

b $V = k \times (1.075)^t$
 When $t = 1$, $V = 387\,000$
 $\therefore k \times (1.075)^1 = 387\,000$
 $\therefore k = \frac{387\,000}{1.075}$
 $\therefore k = 360\,000$

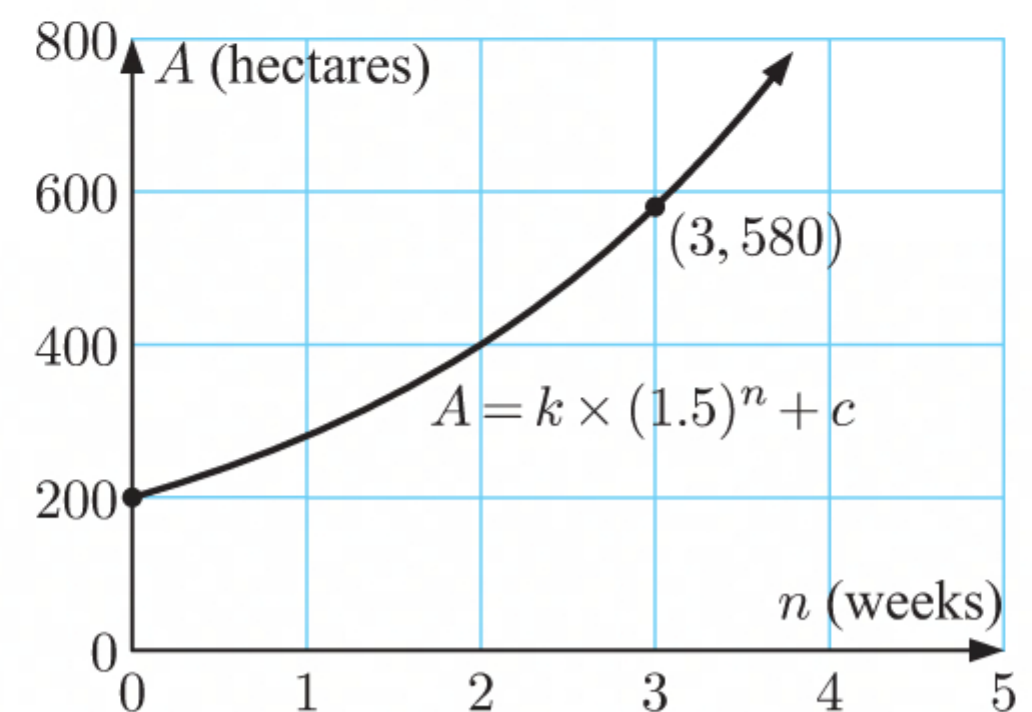
The original value of the house was \$360 000.

- c** $V = 360\,000 \times (1.075)^t$
 When $V = 550\,000$, $550\,000 = 360\,000 \times (1.075)^t$.
 Using technology, $t \approx 5.86$.



It will take about 5.86 years for the house's value to reach \$550 000.

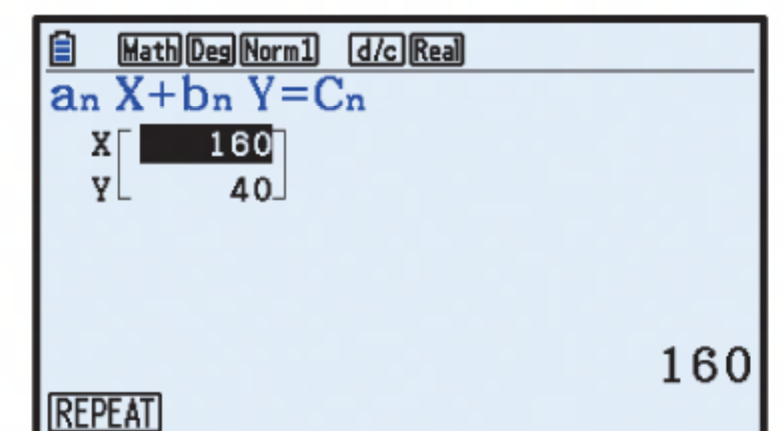
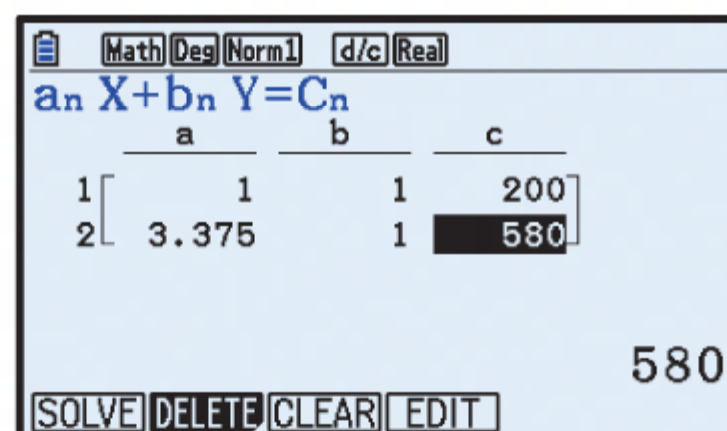
- 9** $A(n) = k \times (1.5)^n + c$ hectares
- a** From the graph, when $n = 0$, $A = 200$
 $\therefore k \times (1.5)^0 + c = 200$
 $\therefore k + c = 200$
 Also, when $n = 3$, $A = 580$
 $\therefore k \times (1.5)^3 + c = 580$
 $\therefore 3.375k + c = 580$



Solving the system of equations

$$\begin{cases} k + c = 200 \\ 3.375k + c = 580 \end{cases}$$

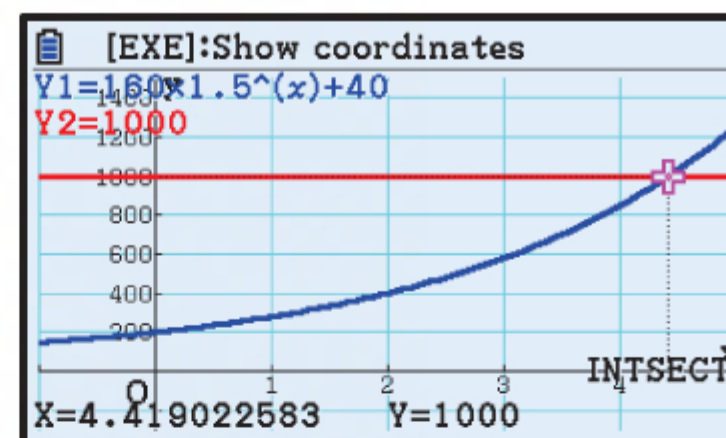
simultaneously gives
 $k = 160$, $c = 40$.



b $A(n) = 160 \times (1.5)^n + 40$

When $A(n) = 1000$, $1000 = 160 \times (1.5)^n + 40$.

Using technology, $n \approx 4.42$.



It will take about 4.42 weeks for the infested area to reach 1000 hectares.

10 $V = c - k \times (0.8)^t \text{ m s}^{-1}$

a When $t = 0$, $V = c - k \times (0.8)^0$
 $= c - k$

Now, $V = 0$ at time $t = 0$

$$\therefore 0 = c - k$$

$$\therefore c = k$$

b $V = k - k \times (0.8)^t$

When $t = 2$, $V = k - k \times (0.8)^2$

$$\therefore 21.6 = k - 0.64k$$

$$\therefore 21.6 = 0.36k$$

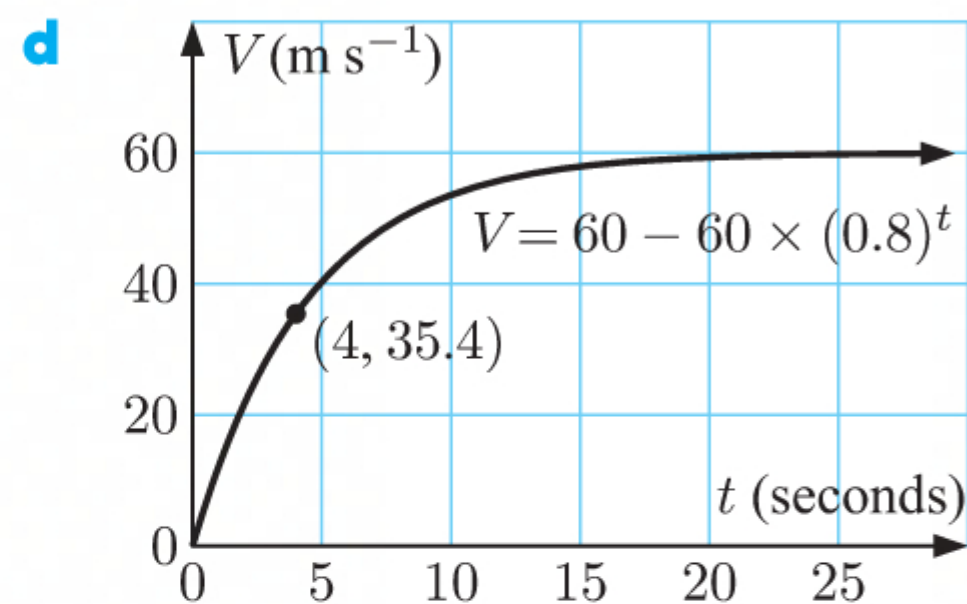
$$\therefore k = \frac{21.6}{0.36}$$

$$\therefore k = 60$$

So, the exponential model is $V = 60 - 60 \times (0.8)^t \text{ m s}^{-1}$.

c When $t = 4$, $V = 60 - 60 \times (0.8)^4$
 $= 60 - 60 \times 0.4096$
 $= 35.424$

After 4 seconds, the speed of the parachutist is about 35.4 m s^{-1} .



e From the graph, the parachutist accelerates rapidly until he approaches his terminal velocity of 60 m s^{-1} .

EXERCISE 8E.2

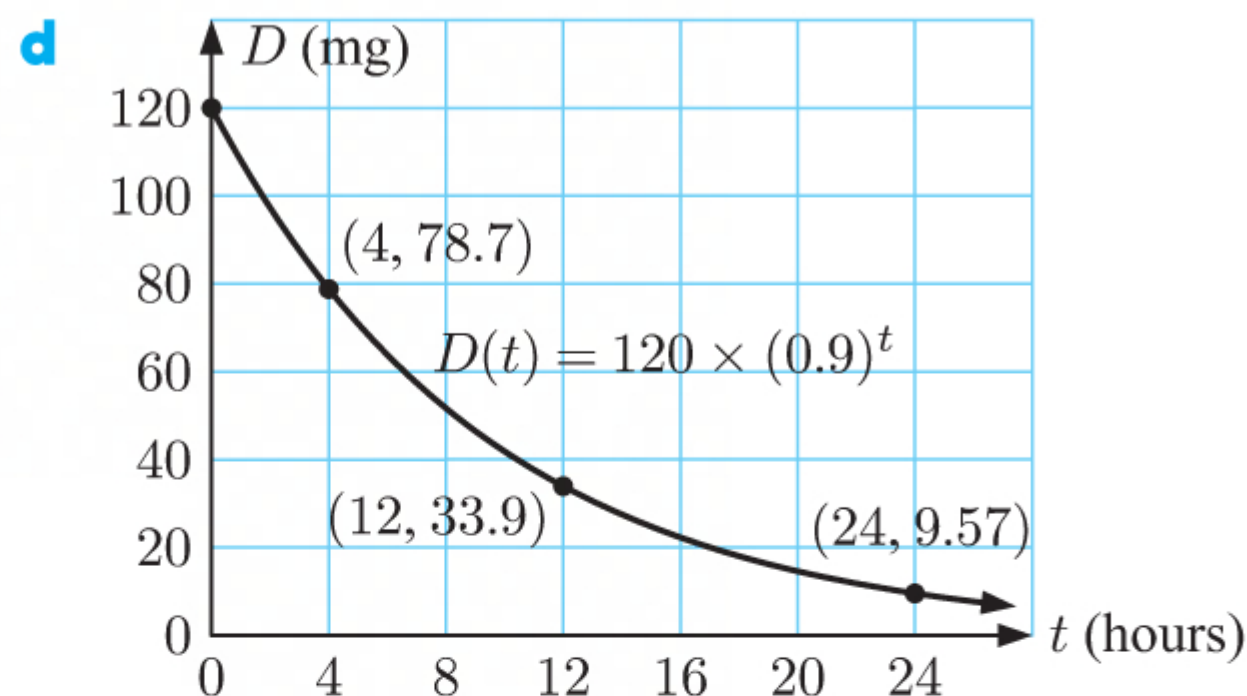
1 $D(t) = 120 \times (0.9)^t$ mg

a The value 0.9 means the amount of drug in the body decreases by 10% each hour.

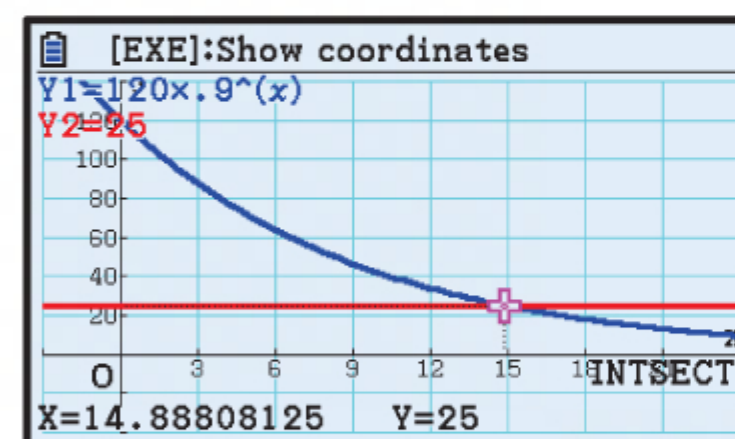
b $D(0) = 120 \times (0.9)^0 = 120$ mg $D(4) = 120 \times (0.9)^4 \approx 78.7$ mg $D(12) = 120 \times (0.9)^{12} \approx 33.9$ mg

$D(24) = 120 \times (0.9)^{24} \approx 9.57$ mg

c $D(0) = 120$ mg, so the original drug dose was 120 mg.



e When $D(t) = 25$, $25 = 120 \times (0.9)^t$.
Using technology, $t \approx 14.9$.



It will take about 14.9 hours for there to be only 25 mg of the drug left in the body.

2 $W(t) = 250 \times (0.998)^t$ grams

a $W(0) = 250 \times (0.998)^0 = 250$

\therefore there was initially 250 grams of radioactive substance set aside.

b i $W(400) = 250 \times (0.998)^{400} \approx 112$

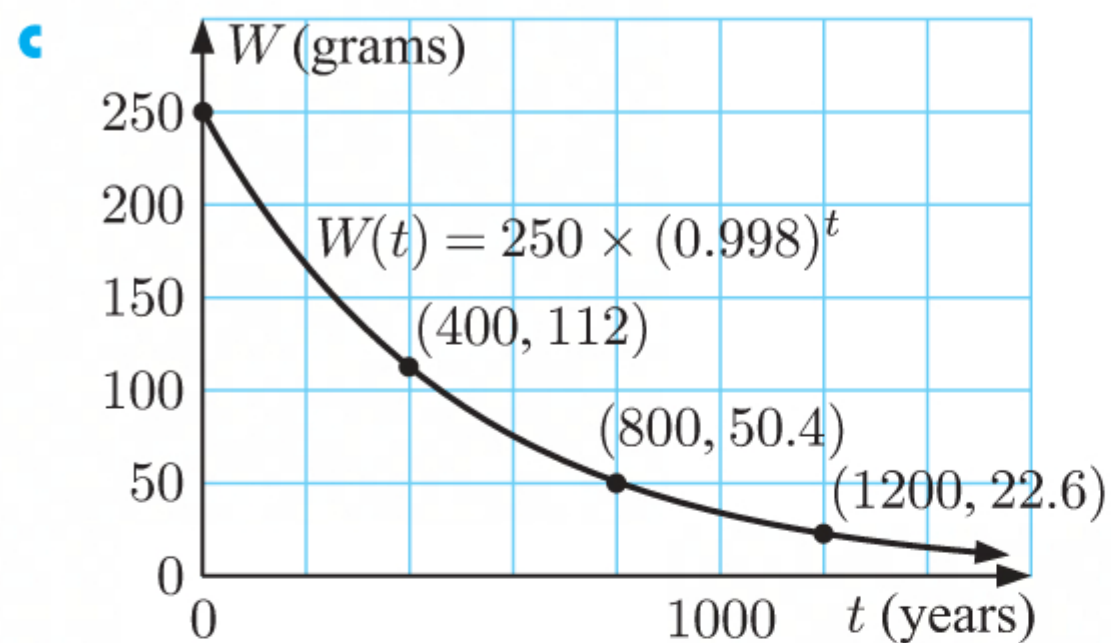
The weight was about 112 grams after 400 years.

ii $W(800) = 250 \times (0.998)^{800} \approx 50.4$

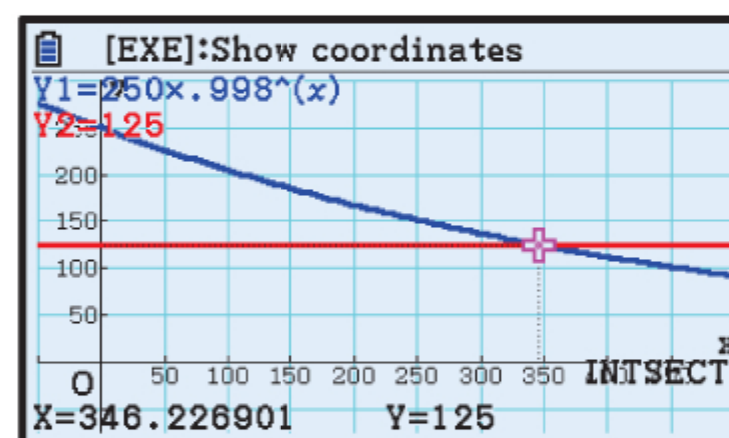
The weight was about 50.4 grams after 800 years.

iii $W(1200) = 250 \times (0.998)^{1200} \approx 22.6$

The weight was about 22.6 grams after 1200 years.



- d** When $W(t) = 125$, $125 = 250 \times (0.998)^t$.
Using technology, $t \approx 346.2$.



It will take about 346 years for the substance to decay to 125 grams.

3 $I(t) = 0.6 \times (0.03)^t$ amps

a $I(0) = 0.6 \times (0.03)^0$
 $= 0.6$

\therefore the initial current was 0.6 amps.

b **i** $I(0.1) = 0.6 \times (0.03)^{0.1}$
 ≈ 0.423

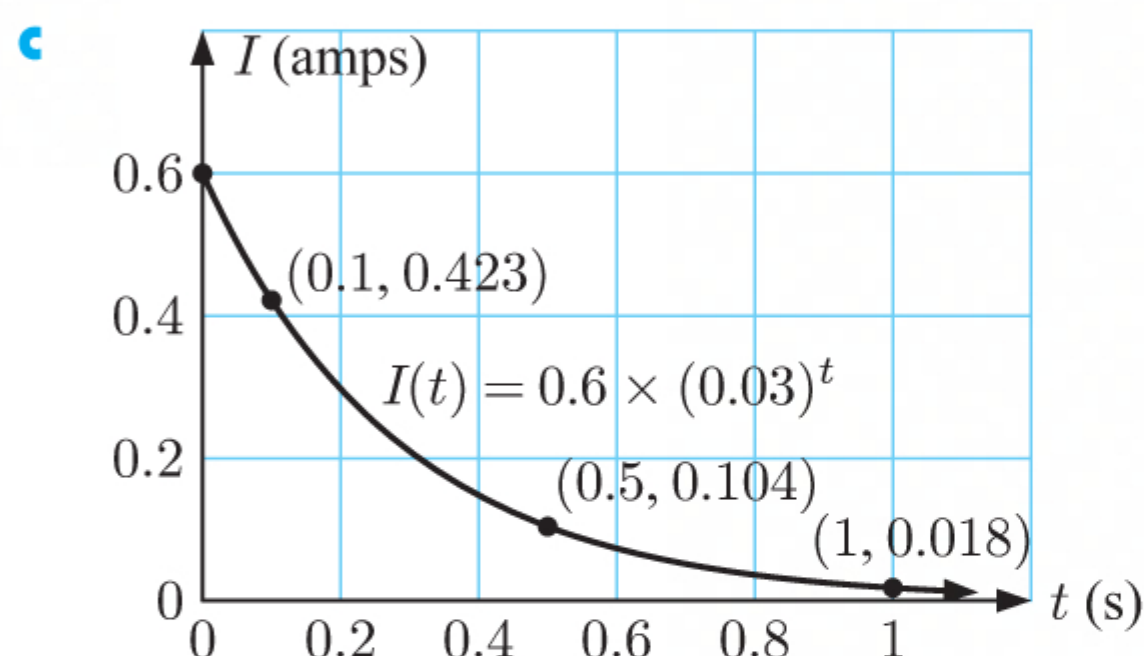
The current after 0.1 seconds is about 0.423 amps.

iii $I(1) = 0.6 \times (0.03)^1$
 $= 0.018$

The current after 1 second is 0.018 amps.

ii $I(0.5) = 0.6 \times (0.03)^{0.5}$
 ≈ 0.104

The current after 0.5 seconds is about 0.104 amps.



4 $T(t) = 100 \times (1.02)^{-t}$ °C

a $T(0) = 100 \times (1.02)^0$
 $= 100$

The initial temperature of the liquid was 100°C.

b i $T(15) = 100 \times (1.02)^{-15}$
 ≈ 74.3

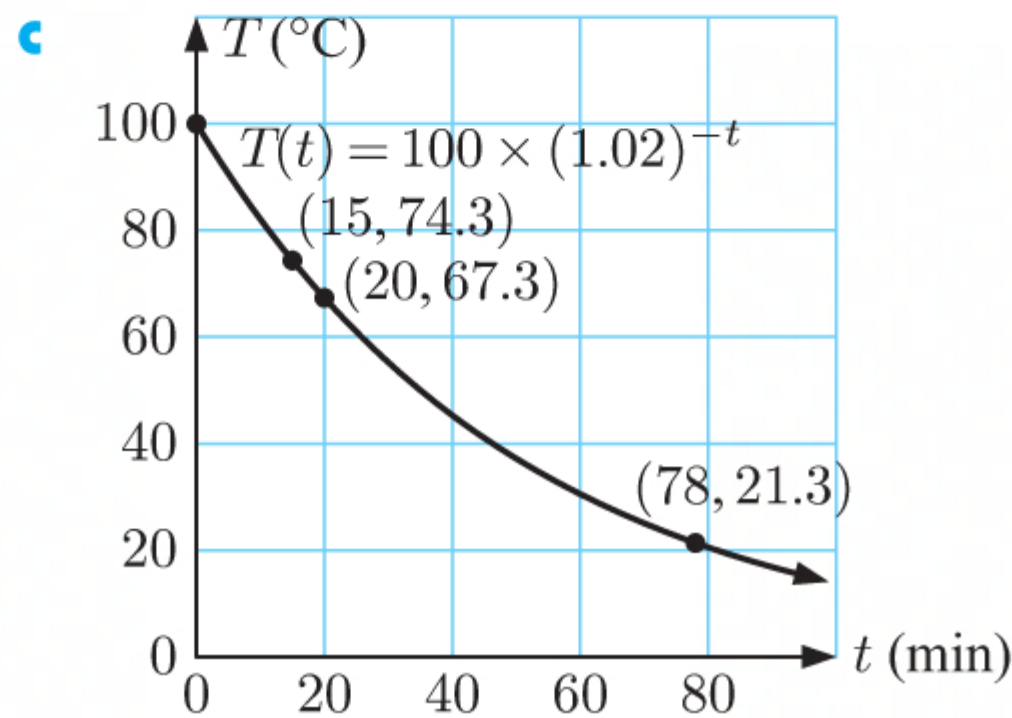
The temperature was about 74.3°C after 15 minutes.

iii $T(78) = 100 \times (1.02)^{-78}$
 ≈ 21.3

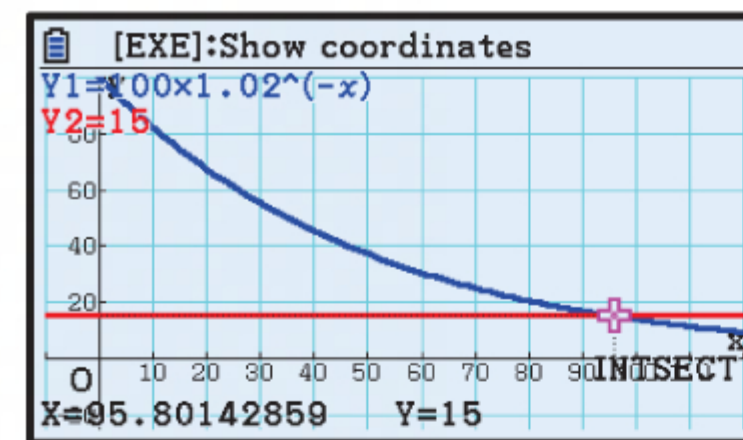
The temperature was about 21.3°C after 78 minutes.

ii $T(20) = 100 \times (1.02)^{-20}$
 ≈ 67.3

The temperature was about 67.3°C after 20 minutes.



d When $T(t) = 15$, $15 = 100 \times (1.02)^{-t}$.
 Using technology, $t \approx 95.8$.



It will take about 95.8 minutes for the temperature of the liquid to fall to 15°C .

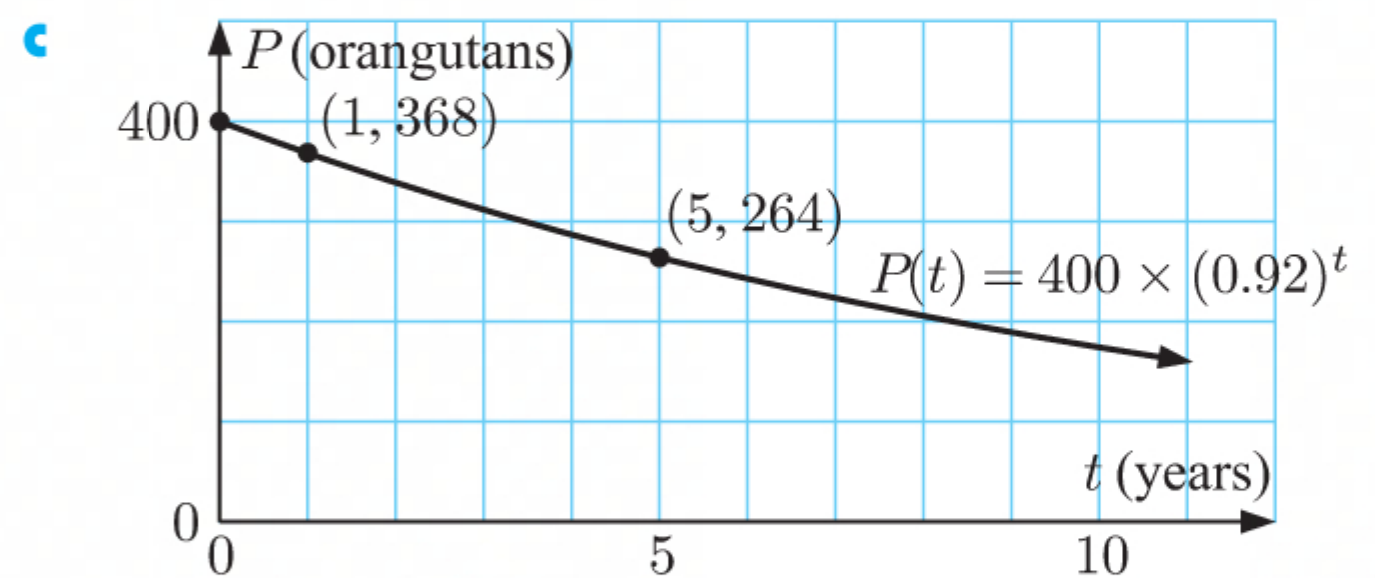
5 a $P(t) = 400 \times (0.92)^t$ orangutans

b i $P(1) = 400 \times (0.92)^1$
 $= 368$

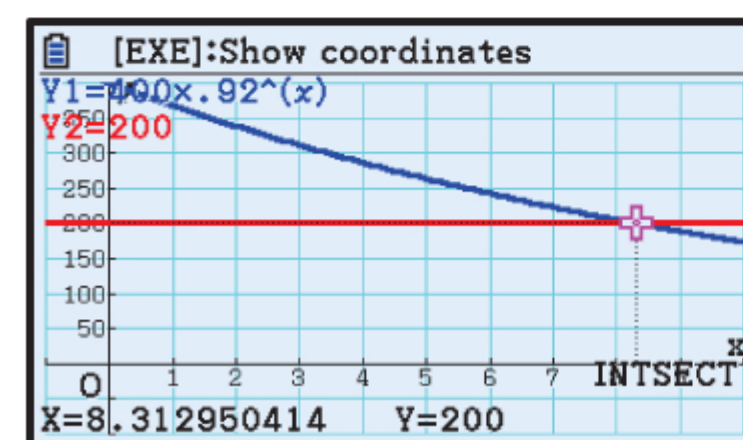
There were 368 orangutans after 1 year.

ii $P(5) = 400 \times (0.92)^5$
 ≈ 264

There were about 264 orangutans after 5 years.



d When $P(t) = 200$, $200 = 400 \times (0.92)^t$.
 Using technology, $t \approx 8.31$.



The population will fall to 200 after about 8.31 years, or about 8 years and 114 days.

6 $L = 10 \times a^d$ units, $a > 0$

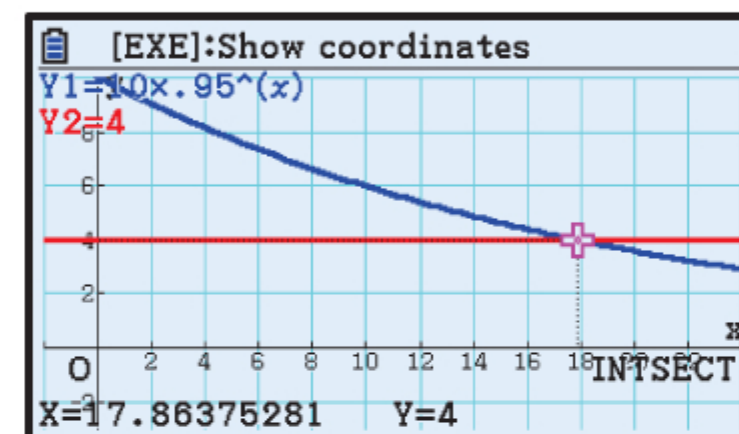
a As the intensity of light diminishes as the depth increases, we would expect that $0 < a < 1$.

b When $d = 1$, $L = 9.5$
 $\therefore 10 \times a^1 = 9.5$
 $\therefore 10a = 9.5$
 $\therefore a = 0.95$

c $L = 10 \times (0.95)^d$
 When $d = 25$, $L = 10 \times (0.95)^{25}$
 ≈ 2.77

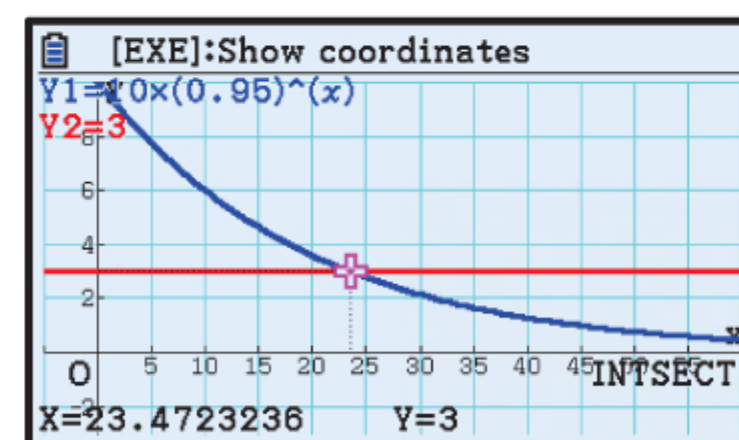
\therefore the intensity of light 25 m below the surface is about 2.77 units.

d When $L = 4$, $4 = 10 \times (0.95)^d$.
 Using technology, $d \approx 17.9$.



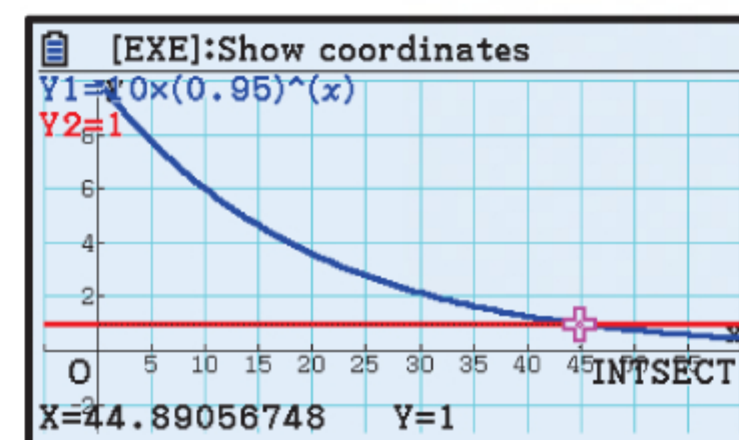
A depth of about 17.9 m has a light intensity of 4 units.

e When $L = 3$, $3 = 10 \times (0.95)^d$.
 Using technology, $d \approx 23.5$.



A depth of about 23.5 m has a light intensity of 3 units.

When $L = 1$, $1 = 10 \times (0.95)^d$.
 Using technology, $d \approx 44.9$.



A depth of about 44.9 m has a light intensity of 1 unit.

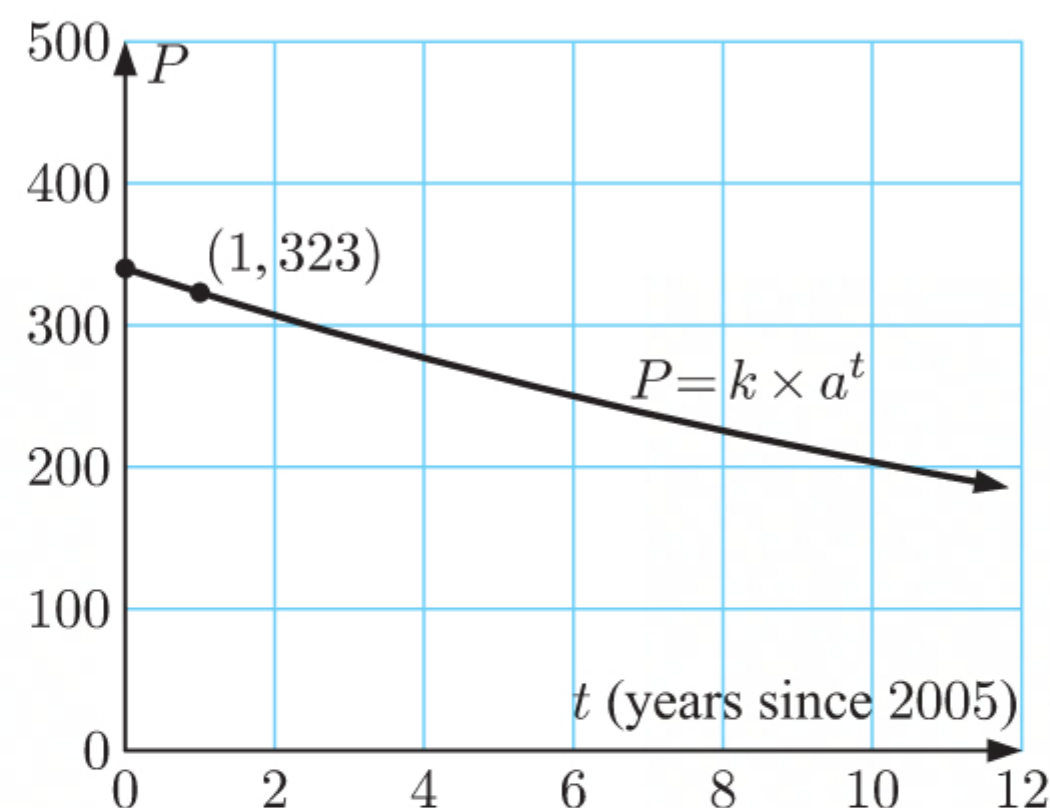
A depth between approximately 23.5 m and 44.9 m will have a light intensity between 1 and 3 units.

7 $P = k \times a^t$ turtles

a The population decreased by 5% each year, so the multiplier $a = 0.95$.

b $P = k \times (0.95)^t$
 When $t = 1$, $P = 323$
 $\therefore k \times (0.95)^1 = 323$
 $\therefore k = \frac{323}{0.95}$
 $\therefore k = 340$

The initial population of turtles was 340.



- c 2015 is 10 years after 2005, so $t = 10$.

$$P = 340 \times (0.95)^t$$

$$\begin{aligned} \text{When } t = 10, \quad P &= 340 \times (0.95)^{10} \\ &\approx 204 \end{aligned}$$

The population of turtles in 2015 was about 204.

- d No, it would not be reasonable to apply this model for negative values of t . The model is based on data collected since 2005. Before 2005 would be an extrapolation and therefore unreliable.

8 $V = k \times 0.7^t + c$ dollars, $t \geq 0$

- a The horizontal asymptote is $V = 1000$, so $c = 1000$.

The value of the car approaches a minimum “salvage value” of \$1000.

b $V = k \times 0.7^t + 1000$

$$\text{When } t = 3, \quad V = 8889$$

$$\therefore k \times 0.7^3 + 1000 = 8889$$

$$\therefore 0.343k = 7889$$

$$\therefore k = \frac{7889}{0.343}$$

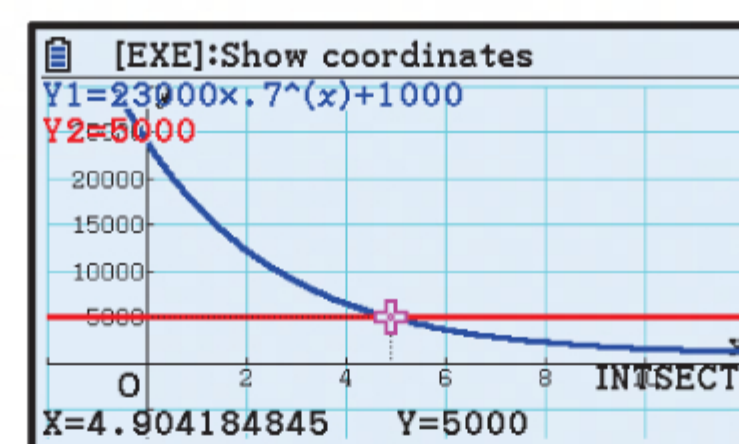
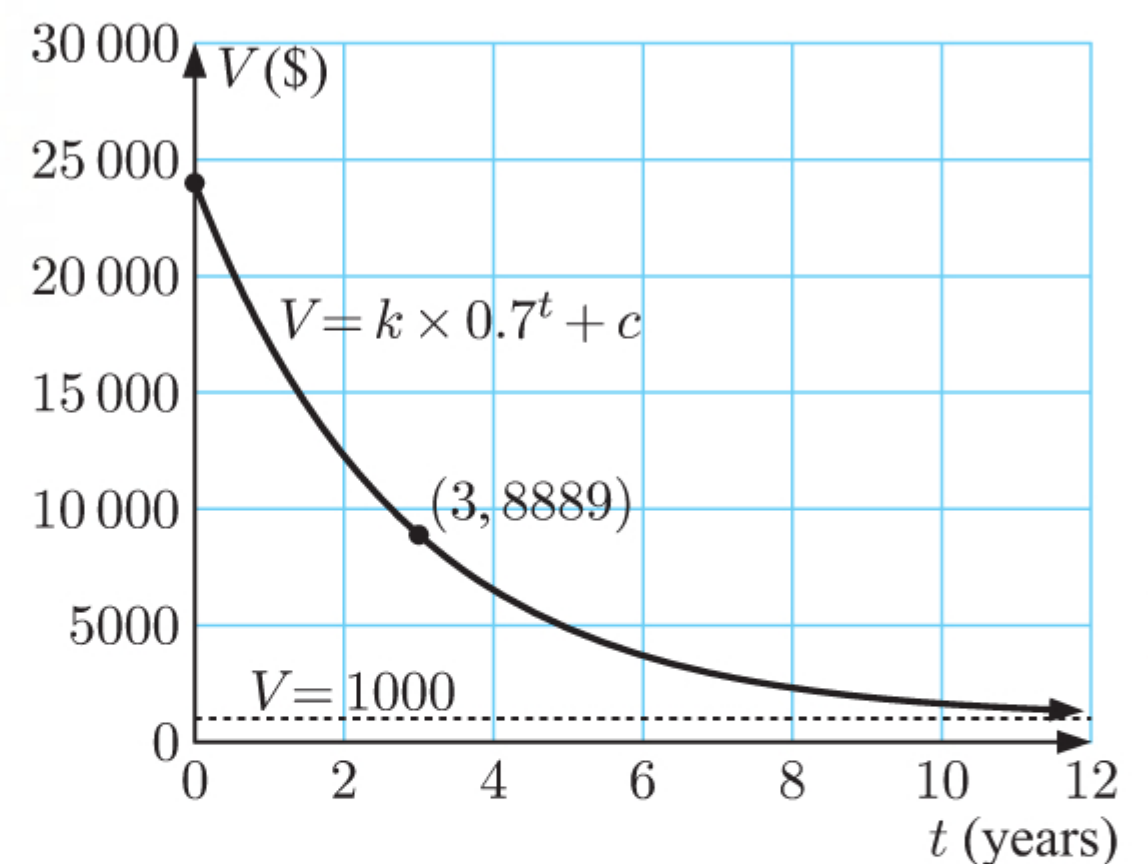
$$\therefore k = 23\,000$$

c $V = 23\,000 \times 0.7^t + 1000$

$$\begin{aligned} \text{When } t = 0, \quad V &= 23\,000 \times 0.7^0 + 1000 \\ &= 23\,000 + 1000 \\ &= 24\,000 \end{aligned}$$

The initial value of the car was \$24 000.

- d When $V = 5000$, $5000 = 23\,000 \times 0.7^t$.
Using technology, $t \approx 4.90$.



It will take about 5 years for the value of the car to reduce to \$5000.

- e No, the car depreciates by the same percentage each year *above* the salvage value of \$1000.

9 $T(t) = c + k \times 0.875^t \text{ } ^\circ\text{C}$

a $T(1) = 18$

$$\therefore c + k \times 0.875^1 = 18$$

$$\therefore c + 0.875k = 18$$

$$T(2) = 14.5$$

$$\therefore c + k \times 0.875^2 = 14.5$$

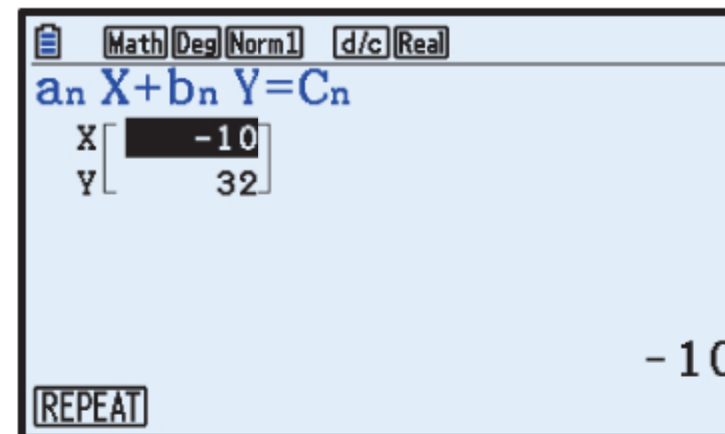
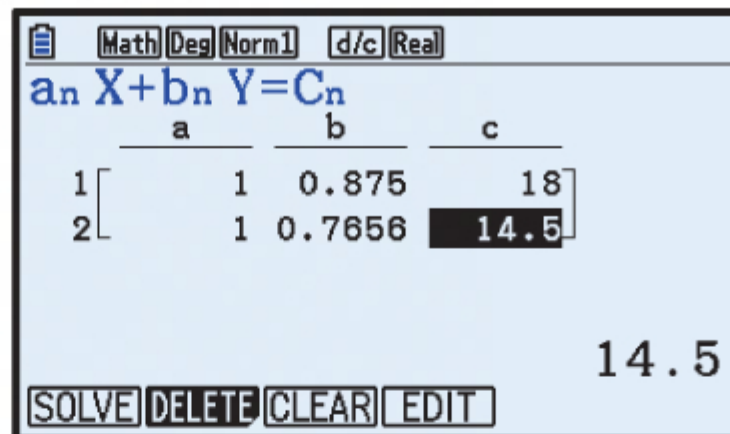
$$\therefore c + 0.765625k = 14.5$$

Solving the system of equations

$$\begin{cases} c + 0.875k = 18 \\ c + 0.765625k = 14.5 \end{cases}$$

simultaneously gives

$$c = -10, k = 32.$$



b $T(t) = -10 + 32 \times 0.875^t$

The horizontal asymptote is $T = -10$ which suggests that the temperature inside the freezer is -10°C .

c i $T(0) = -10 + 32 \times 0.875^0$
 $= -10 + 32$
 $= 22$

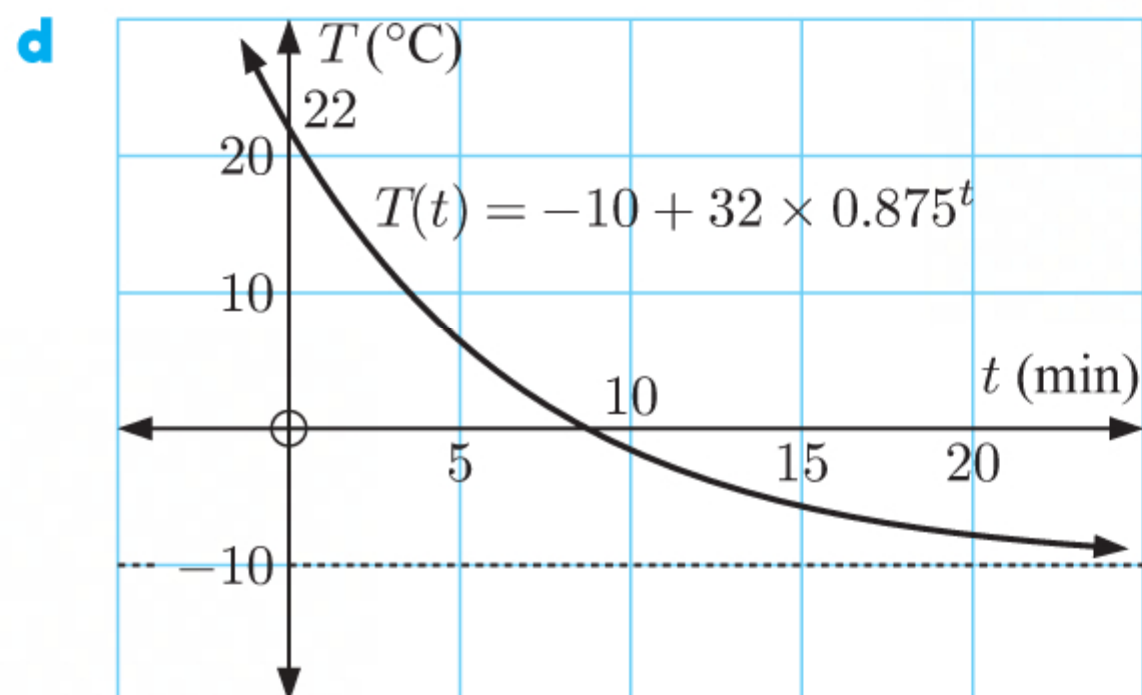
The temperature of the peas was 22°C when first placed in the freezer.

ii $T(5) = -10 + 32 \times 0.875^5$
 ≈ 6.41

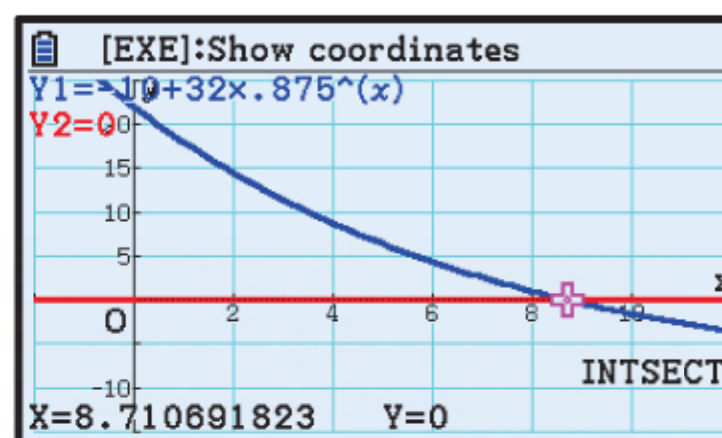
The temperature of the peas was about 6.41°C after 5 minutes.

iii $T(10) = -10 + 32 \times 0.875^{10}$
 ≈ -1.58

The temperature of the peas was about -1.58°C after 10 minutes.



e When $T(t) = 0$, $0 = -10 + 32 \times 0.875^t$.
 Using technology, $t \approx 8.71$.



It takes about 8.71 minutes for the temperature of the peas to fall to 0°C .

10 $W(t) = 10 \times a^t$ mg

a The value 10 indicates that the initial weight of the isotope is 10 mg.

b Fermium-253 has a half-life of 3 days.

So, after 3 days, its weight is $\frac{1}{2} \times 10 = 5$ mg.

Now, $W(3) = 10 \times a^3$

$$\therefore 5 = 10 \times a^3$$

$$\therefore \frac{5}{10} = a^3$$

$$\therefore a = \sqrt[3]{\frac{1}{2}}$$

$$\therefore a \approx 0.7937$$

Each day the isotope's weight decreases by about $(1 - 0.7937) \times 100\% \approx 20.63\%$.

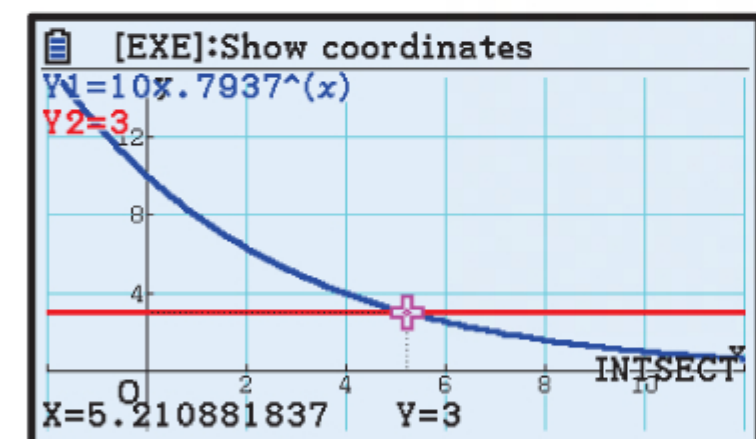
c $W(t) \approx 10 \times (0.7937)^t$

$$\therefore W(2) \approx 10 \times (0.7937)^2 \\ \approx 6.30$$

The weight of fermium-253 after 2 days is about 6.30 mg.

d i When $W(t) = 3$, $3 = 10 \times (0.7937)^t$.

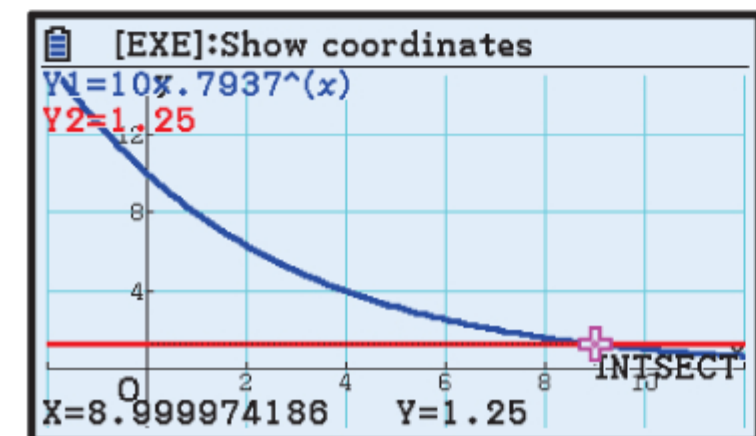
Using technology, $t \approx 5.21$.



It will take about 5.21 days for the weight of fermium-253 to fall to 3 mg.

ii When $W(t) = 1.25$, $1.25 = 10 \times (0.7937)^t$.

Using technology, $t \approx 9.00$.



It will take about 9 days for the weight of fermium-253 to fall to 1.25 mg.

INVESTIGATION 2

CONTINUOUS COMPOUND INTEREST

1 $u_n = u_0(1 + i)^n$, $u_0 = 1000$

a Interest paid annually:

$$n = 1, i = 6\% = 0.06$$

$$\therefore u_1 = 1000(1 + 0.06)^1 \\ = 1060$$

The final amount is \$1060.

b Interest paid quarterly:

$$n = 4, i = \frac{6\%}{4} = 0.015$$

$$\therefore u_4 = 1000(1 + 0.015)^4 \\ \approx 1061.36$$

The final amount is \$1061.36.

c Interest paid monthly:

$$n = 12, \quad i = \frac{6\%}{12} = 0.005$$

$$\therefore u_{12} = 1000(1 + 0.005)^{12} \\ \approx 1061.68$$

The final amount is \$1061.68.

d Interest paid daily:

$$n = 365.25, \quad i = \frac{6\%}{365.25}$$

$$\therefore u_{365.25} = 1000 \left(1 + \frac{0.06}{365.25}\right)^{365.25} \\ \approx 1061.83$$

The final amount is \$1061.83.

e Interest paid by the second:

$$n = 365.25 \times 24 \times 60 \times 60 = 31\,557\,600, \quad i = \frac{6\%}{31\,557\,600}$$

$$\therefore u_{31\,557\,600} = 1000 \left(1 + \frac{0.06}{31\,557\,600}\right)^{31\,557\,600} \\ \approx 1061.84$$

The final amount is \$1061.84.

f Interest paid by the millisecond:

$$n = 31\,557\,600 \times 1000 = 31\,557\,600\,000, \quad i = \frac{6\%}{31\,557\,600\,000}$$

$$\therefore u_{31\,557\,600\,000} = 1000 \left(1 + \frac{0.06}{31\,557\,600\,000}\right)^{31\,557\,600\,000} \\ \approx 1061.84$$

The final amount is \$1061.84.

Given a fixed interest rate per annum, paying out the interest more frequently results in a higher final amount, but seems to approach a particular value.

2 $u_n = u_0(1 + i)^n, \quad u_0 = 1000$

Interest is compounded N times during the year, so $i = \frac{6\%}{N} = \frac{0.06}{N}$.

a
$$u_1 = 1000 \left(1 + \frac{0.06}{N}\right)^1 \\ = 1000 \left(1 + \frac{0.06}{N}\right)$$

\therefore the value of the investment after the first period is $u_1 = \$1000 \left(1 + \frac{0.06}{N}\right)$.

b At the end of the year there have been N payments made, so $n = N$.

\therefore the value of the investment at the end of the year is $u_N = \$1000 \left(1 + \frac{0.06}{N}\right)^N$.

3 $u_n = u_0(1 + i)^n$

Interest is compounded N times during the year, so $i = \frac{r}{N}$.

a i $u_1 = u_0 \left(1 + \frac{r}{N}\right)^1$
 $= u_0 \left(1 + \frac{r}{N}\right)$

\therefore the value of the investment after the first period is $u_1 = u_0 \left(1 + \frac{r}{N}\right)$.

ii At the end of the first year there have been N payments made, so $n = N$.

\therefore the value of the investment after the first year is $u_N = u_0 \left(1 + \frac{r}{N}\right)^N$.

iii At the end of t years there have been $N \times t = Nt$ payments made, so $n = Nt$.

\therefore the value of the investment after t years is $u_n = u_0 \left(1 + \frac{r}{N}\right)^{Nt}$.

b $u_n = u_0 \left(1 + \frac{r}{N}\right)^{Nt}$ {from **a iii**}

$$= u_0 \left(1 + \frac{1}{a}\right)^{Nt}$$

{letting $a = \frac{N}{r}$ }

$$= u_0 \left(1 + \frac{1}{a}\right)^{art}$$

{since $N = ar$ }

$$\therefore u_n = u_0 \left[\left(1 + \frac{1}{a}\right)^a\right]^{rt}$$

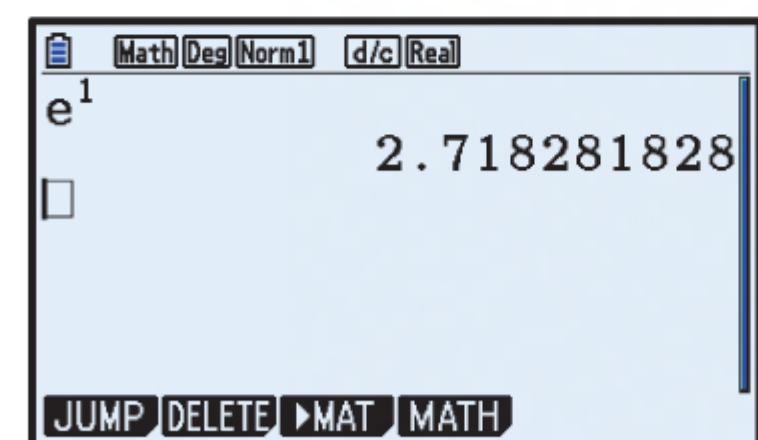
4 a $a = \frac{N}{r}$, so as $N \rightarrow \infty$, $a \rightarrow \infty$

b

a	$\left(1 + \frac{1}{a}\right)^a$
10	2.593 724 46
100	2.704 813 829
1000	2.716 923 932
10 000	2.718 145 927
100 000	2.718 268 237
1 000 000	2.718 280 469
10 000 000	2.718 281 693

5 $e^1 \approx 2.718 281 828$

This appears to be the value of $\left(1 + \frac{1}{a}\right)^a$ as $a \rightarrow \infty$.



6 $u_n = u_0 e^{rt}$, $u_0 = 1000$, $r = 0.06$, $t = 1$

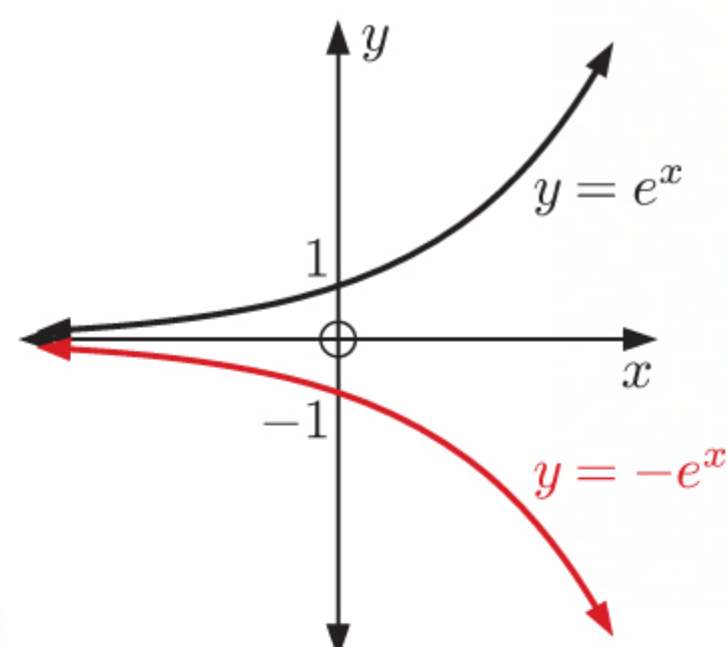
$\therefore u_n = 1000 \times e^{0.06 \times 1} \approx 1061.84$

The final amount is \$1061.84.

This is the same (to the nearest cent) as the final amount when interest was compounded by the second (in **1 e**) or by the millisecond (in **1 f**).

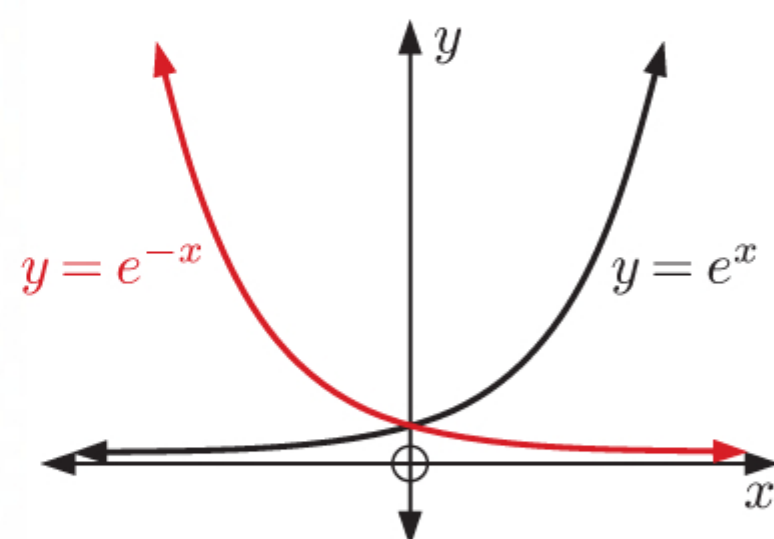
EXERCISE 8F

1



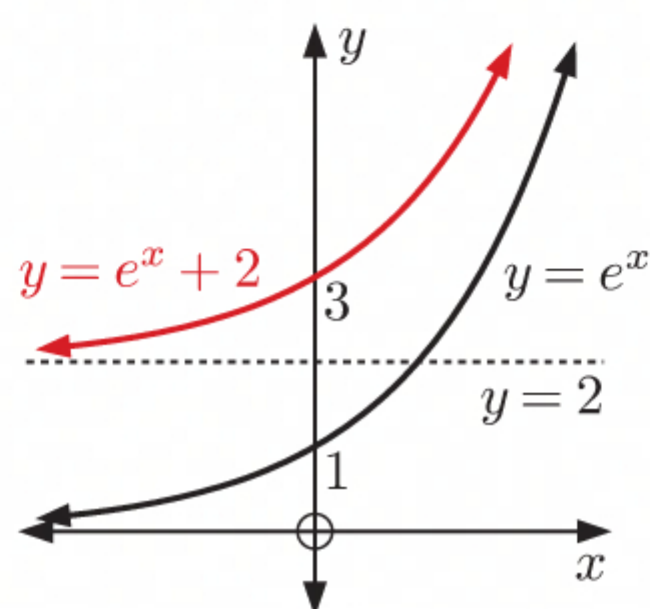
$y = -e^x$ is the reflection of $y = e^x$ in the x -axis.

2



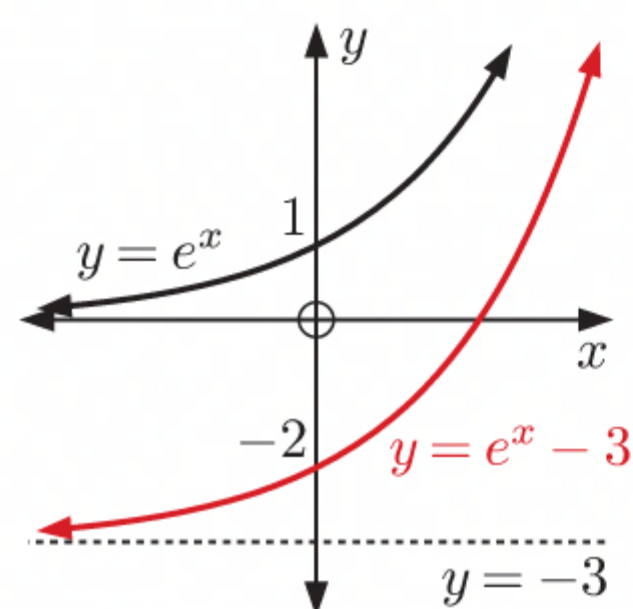
$y = e^{-x}$ is the reflection of $y = e^x$ in the y -axis.

3 a



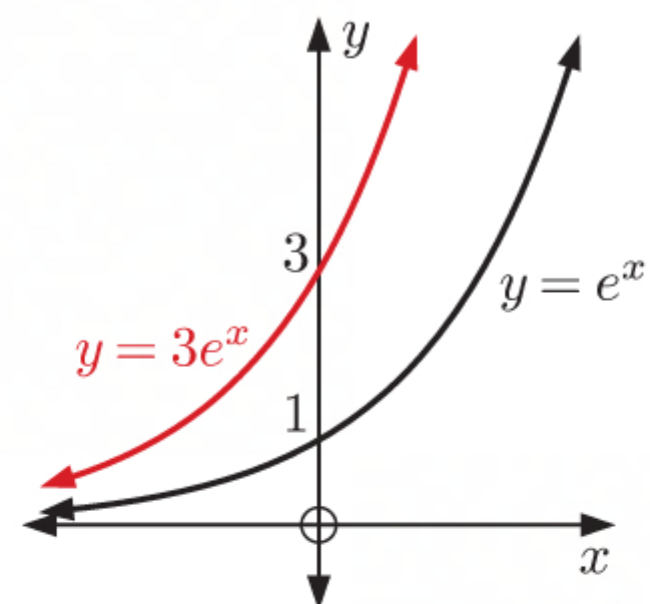
The y -intercept is 3.
The horizontal asymptote is $y = 2$.

b



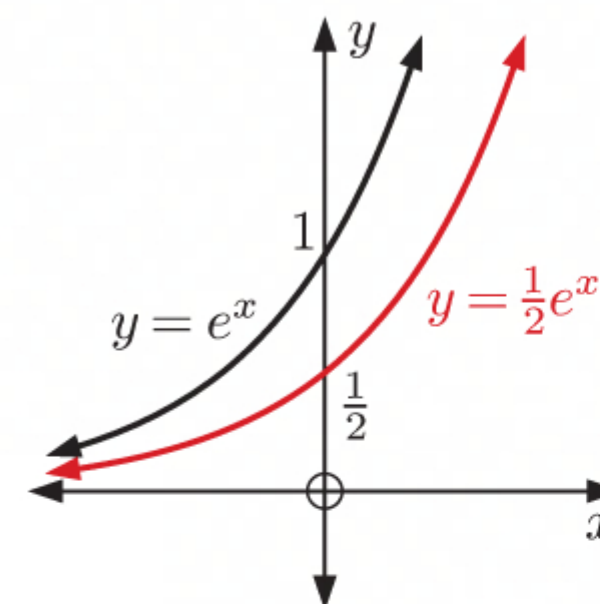
The y -intercept is -2 .
The horizontal asymptote is $y = -3$.

c

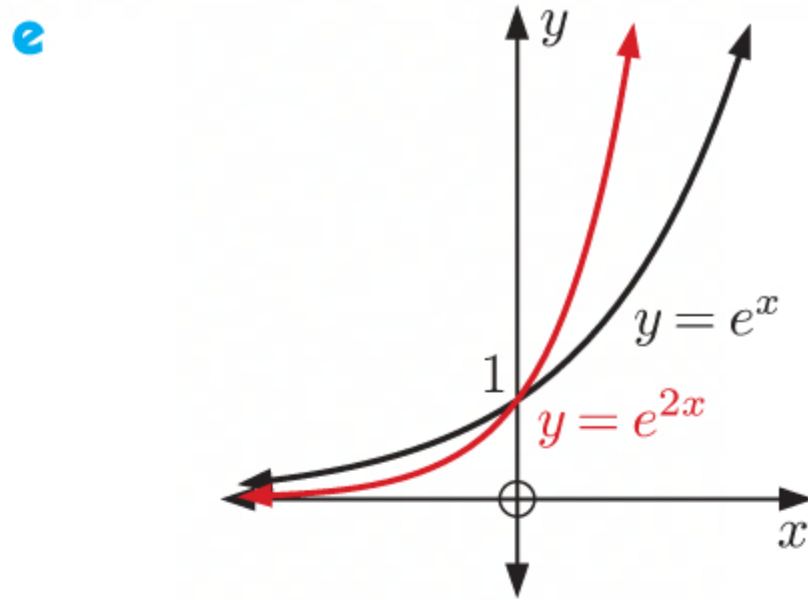


The y -intercept is 3.
The horizontal asymptote is $y = 0$.

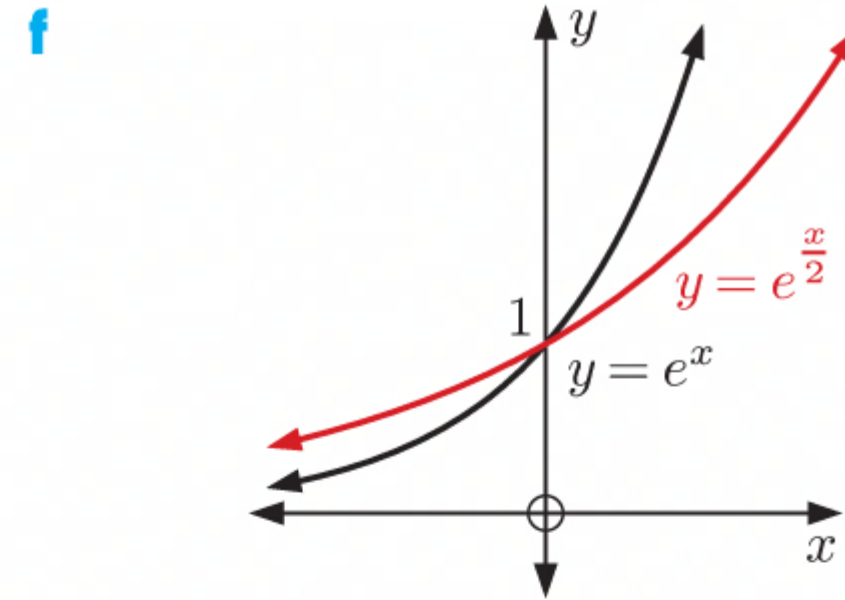
d



The y -intercept is $\frac{1}{2}$.
The horizontal asymptote is $y = 0$.



The y -intercept is 1.
The horizontal asymptote is $y = 0$.

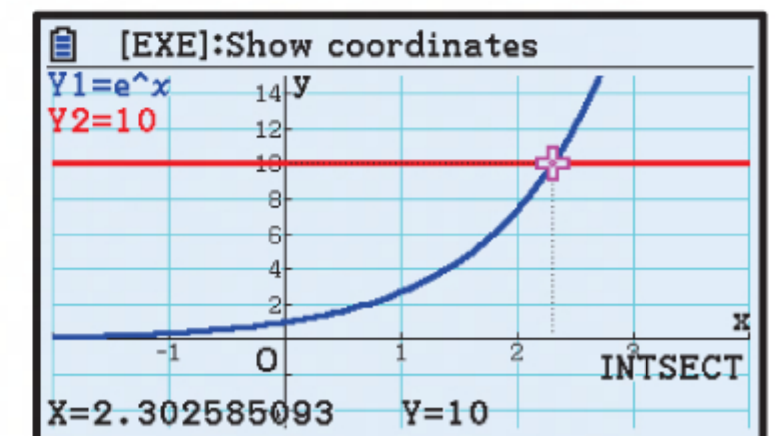


The y -intercept is 1.
The horizontal asymptote is $y = 0$.

- 4**
- | | | |
|---|--|--|
| a $e^2 \approx 7.3891$ | b $e^3 \approx 20.086$ | c $e^{0.5} \approx 1.6487$ |
| d $e^{-1} \approx 0.36788$ | e $e^{2.31} \approx 10.074$ | f $e^{-2.31} \approx 0.099261$ |
| g $e^{4.829} \approx 125.09$ | h $e^{-4.829} \approx 0.0079945$ | i $50e^{-0.1764} \approx 41.914$ |
| j $80e^{-0.6342} \approx 42.429$ | k $1000e^{1.2642} \approx 3540.3$ | l $0.25e^{-3.6742} \approx 0.0063424$ |

5 a $e^x = 10$

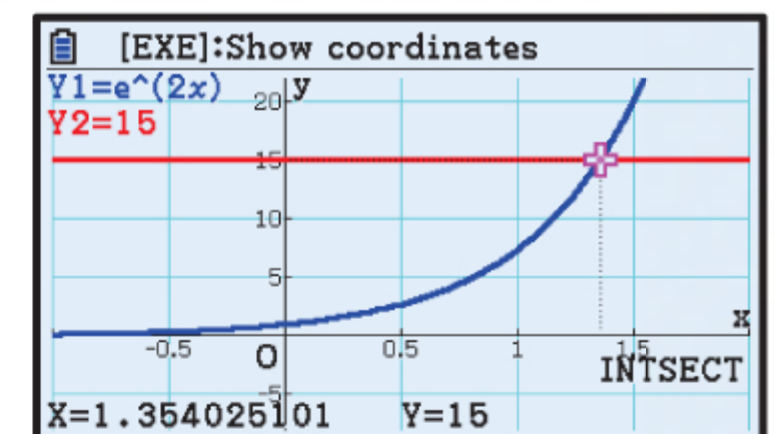
We graph $Y_1 = e^x$ and $Y_2 = 10$ on the same set of axes and find their point of intersection.



The solution is $x \approx 2.30$.

b $e^{2x} = 15$

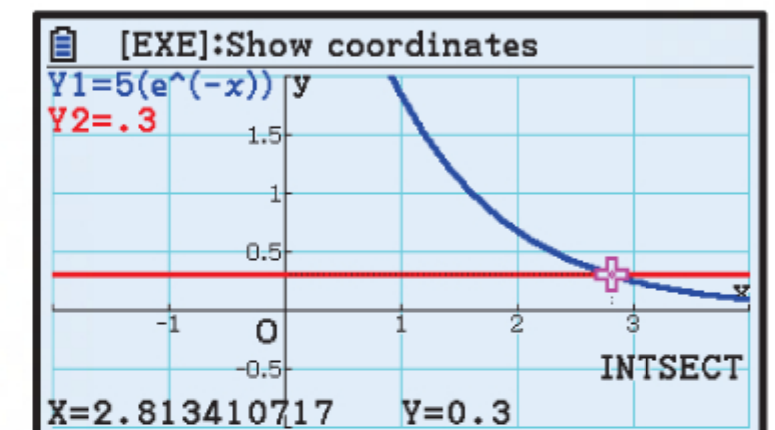
We graph $Y_1 = e^{2x}$ and $Y_2 = 15$ on the same set of axes and find their point of intersection.



The solution is $x \approx 1.35$.

c $5e^{-x} = 0.3$

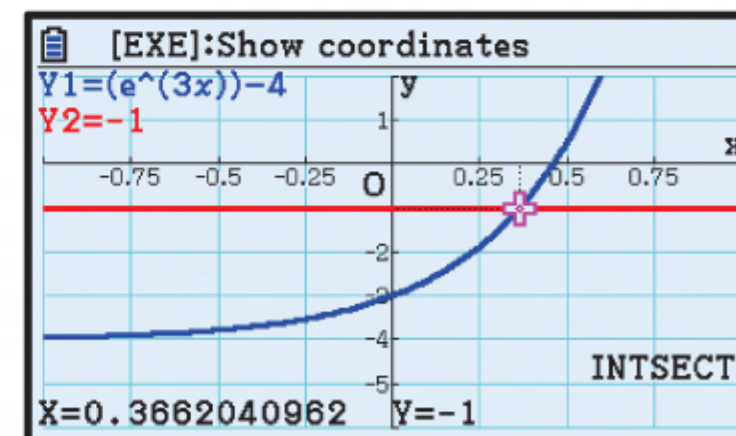
We graph $Y_1 = 5e^{-x}$ and $Y_2 = 0.3$ on the same set of axes and find their point of intersection.



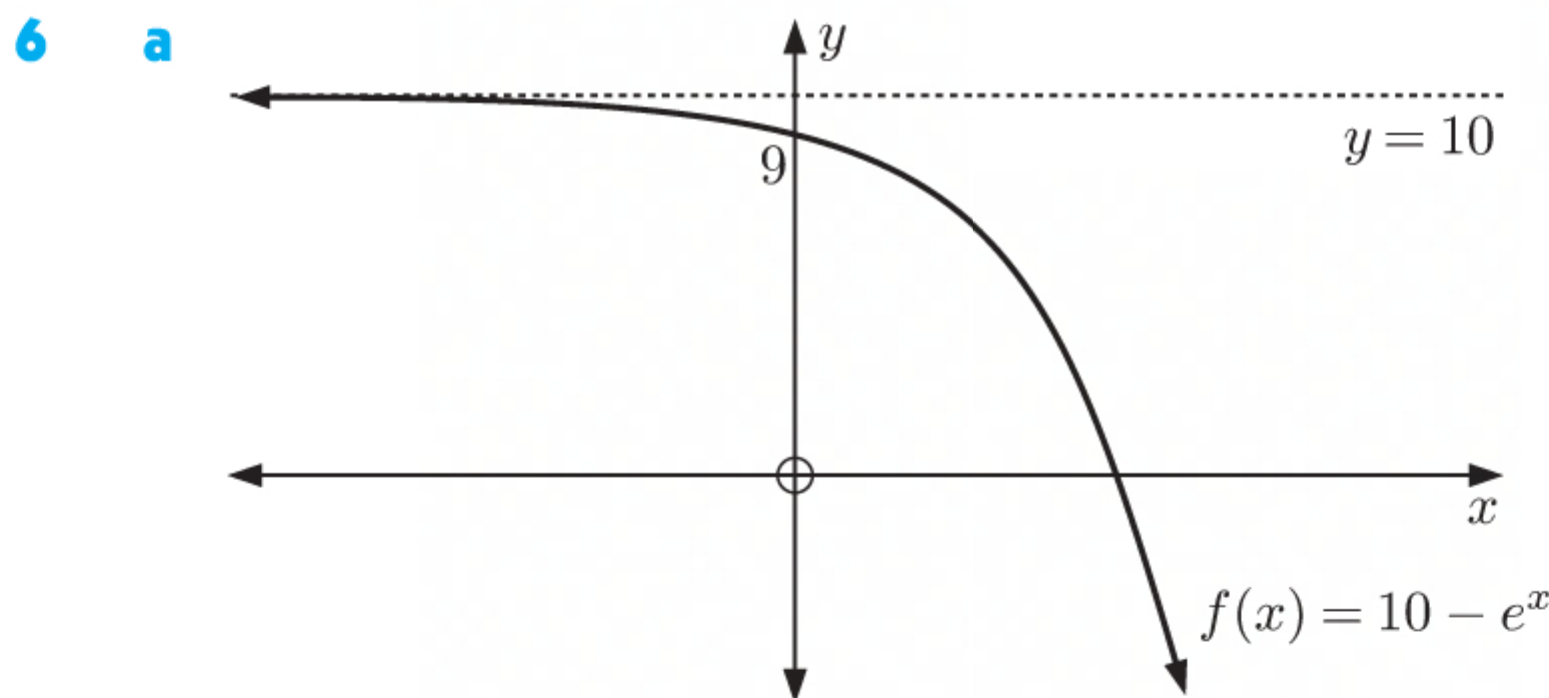
The solution is $x \approx 2.81$.

d $e^{3x} - 4 = -1$

We graph $Y_1 = e^{3x} - 4$ and $Y_2 = -1$ on the same set of axes and find their point of intersection.



The solution is $x \approx 0.366$.



b The domain of $f(x)$ is $\{x \mid x \in \mathbb{R}\}$.

The range of $f(x)$ is $\{y \mid y < 10\}$.

c As $x \rightarrow \infty$, $y \rightarrow -\infty$, and as $x \rightarrow -\infty$, $y \rightarrow 10^-$.

7 $W(t) = 2e^{\frac{t}{2}}$ grams

a i $W(0) = 2e^0$
 $= 2 \times 1$
 $= 2$

The weight of the culture is 2 grams initially.

iii $W(1\frac{1}{2}) = 2e^{\frac{3}{4}}$
 ≈ 4.23

The weight of the culture is about 4.23 grams after $1\frac{1}{2}$ hours.

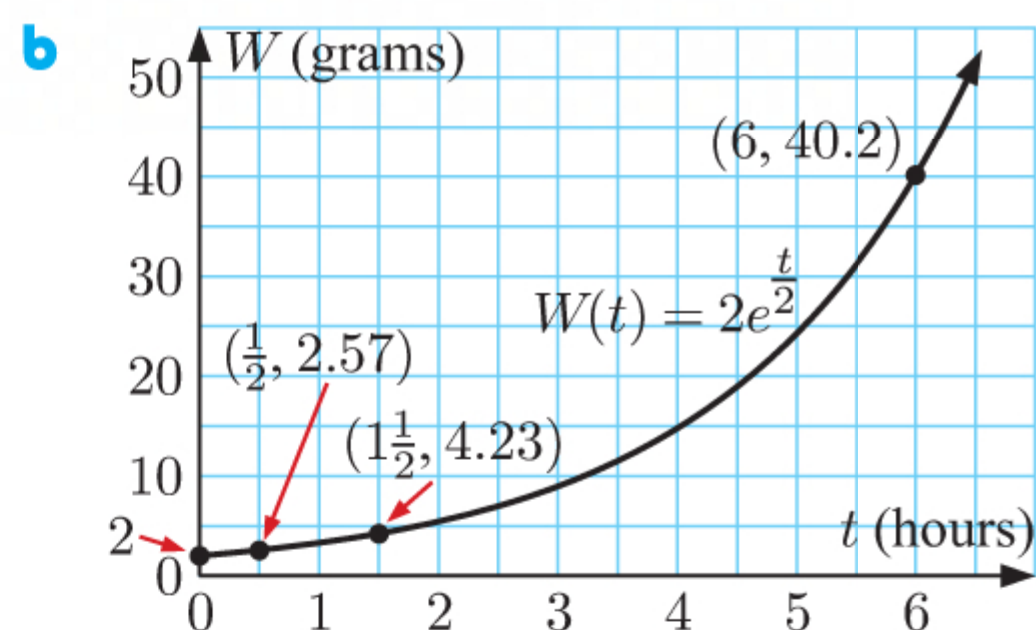
ii $t = 30 \text{ min} = \frac{1}{2} \text{ hour}$

$W(\frac{1}{2}) = 2e^{\frac{1}{4}}$
 ≈ 2.57

The weight of the culture is about 2.57 grams after 30 minutes.

iv $W(6) = 2e^3$
 ≈ 40.2

The weight of the culture is about 40.2 grams after 6 hours.



8 $I(t) = 75e^{-0.15t}$ amps

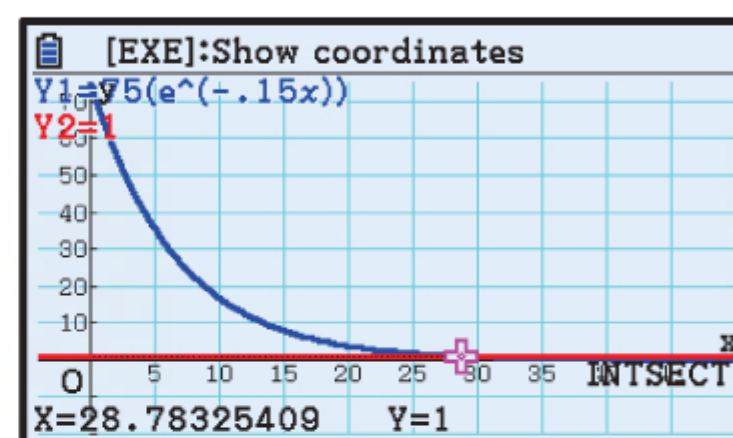
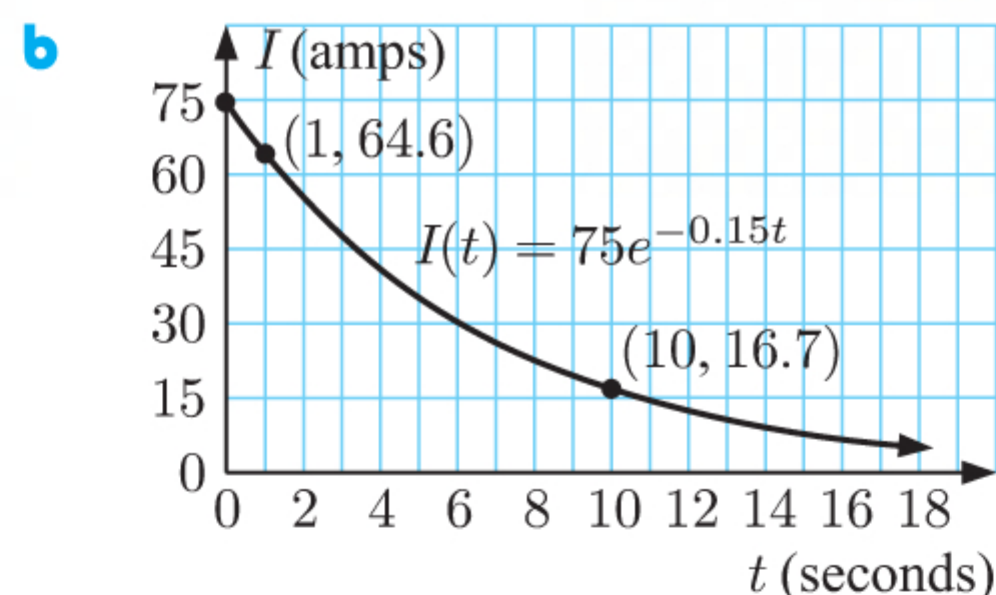
a i $I(1) = 75e^{-0.15}$
 ≈ 64.6

About 64.6 amps of current is still flowing after 1 second.

ii $I(10) = 75e^{-1.5}$
 ≈ 16.7

About 16.7 amps of current is still flowing after 10 seconds.

c When $I(t) = 1$, $1 = 75e^{-0.15t}$.
 Using technology, $t \approx 28.8$.



It will take about 28.8 seconds for the current to fall to 1 amp.

9 $A = k \times e^{-0.5t} + c$ kL

a The horizontal asymptote is $A = 3$, so $c = 3$.

$$A = k \times e^{-0.5t} + 3$$

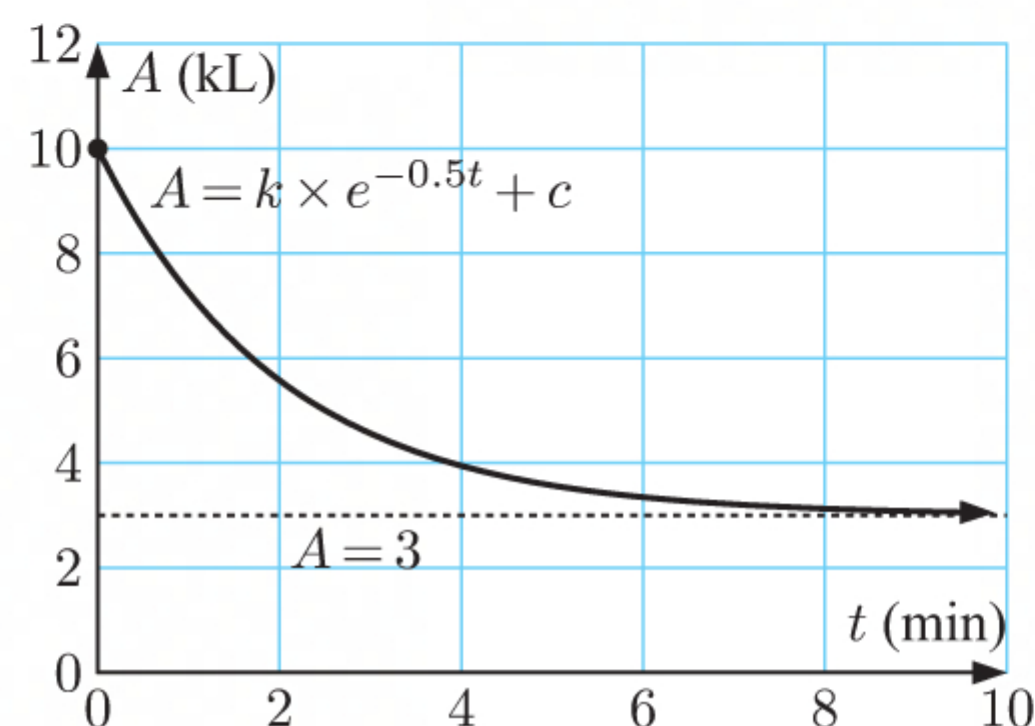
When $t = 0$, $A = 10$

$$\therefore k \times e^{-0.5(0)} + 3 = 10$$

$$\therefore k + 3 = 10$$

$$\therefore k = 7$$

So, the exponential model is $A = 7e^{-0.5t} + 3$.



b The hole is in the side of the tank as the tank never completely empties.

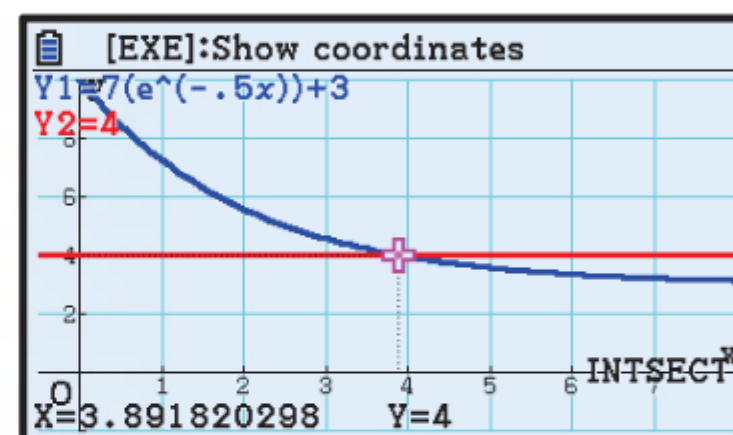
c When $t = 2$, $A = 7e^{-0.5(2)} + 3$
 $= 7e^{-1} + 3$
 ≈ 5.58

There is about 5.58 kL of water in the tank after 2 minutes.

d There is initially 10 kL of water in the tank, so the amount of water in the tank after losing 6 kL is $10 - 6 = 4$ kL.

When $A = 4$, $4 = 7e^{-0.5t} + 3$.

Using technology, $t \approx 3.89$.



It will take about 3.89 minutes for the tank to lose 6 kL of water.

10 $V(t) = 650(4 + 2 \times e^{-0.1t})$

a As $t \rightarrow \infty$, $e^{-0.1t} \rightarrow 0^+$
 \therefore the speed of the meteor is decreasing.

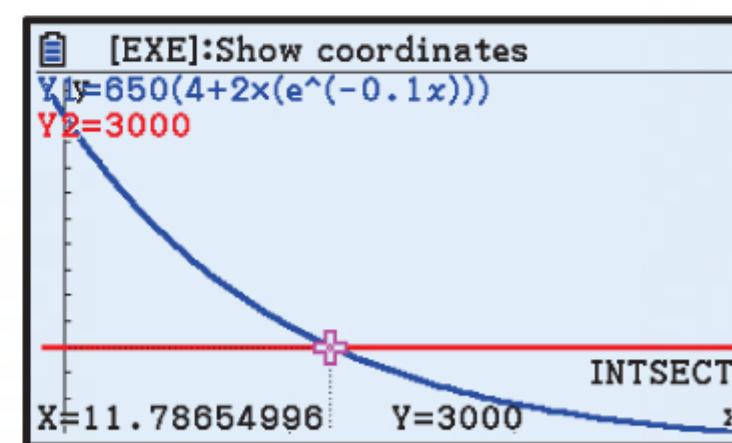
b i $V(0) = 650(4 + 2 \times e^{-0.1 \times 0})$
 $= 650(4 + 2 \times 1)$
 $= 650(6)$
 $= 3900$

The speed of the meteor when it was first sighted was 3900 m s^{-1} .

c When $V(t) = 3000$, $3000 = 650(4 + 2 \times e^{-0.1t})$.
 Using technology, $t \approx 11.8$.

ii $V(120) = 650(4 + 2 \times e^{-0.1 \times 120})$
 $= 650(4 + 2 \times e^{-12})$
 $\approx 2600 \text{ m s}^{-1}$

The speed of the meteor after 2 minutes was about 2600 m s^{-1} .



It will take about 11.8 seconds for the meteor's speed to reach 3000 m s^{-1} .

EXERCISE 8G

1 a $\log 10\,000 = \log(10^4)$
 $= 4$

c $\log 10 = \log(10^1)$
 $= 1$

2 a $\log(10^n) = n$

c $\log\left(\frac{10}{10^m}\right) = \log(10^{1-m})$
 $= 1 - m$

3 a $100 < 237 < 1000$
 $\therefore \log 100 < \log 237 < \log 1000$
 $\therefore \log(10^2) < \log 237 < \log(10^3)$
 $\therefore 2 < \log 237 < 3$

4 a We know that $\log 1 = \log(10^0) = 0$ and $\log(0.1) = \log(10^{-1}) = -1$.
 Also, $0.1 < 0.6 < 1 \therefore \log(0.1) < \log(0.6) < \log 1$
 $\therefore -1 < \log(0.6) < 0$

b $\log(0.6) \approx -0.22$ which is between -1 and 0 . ✓

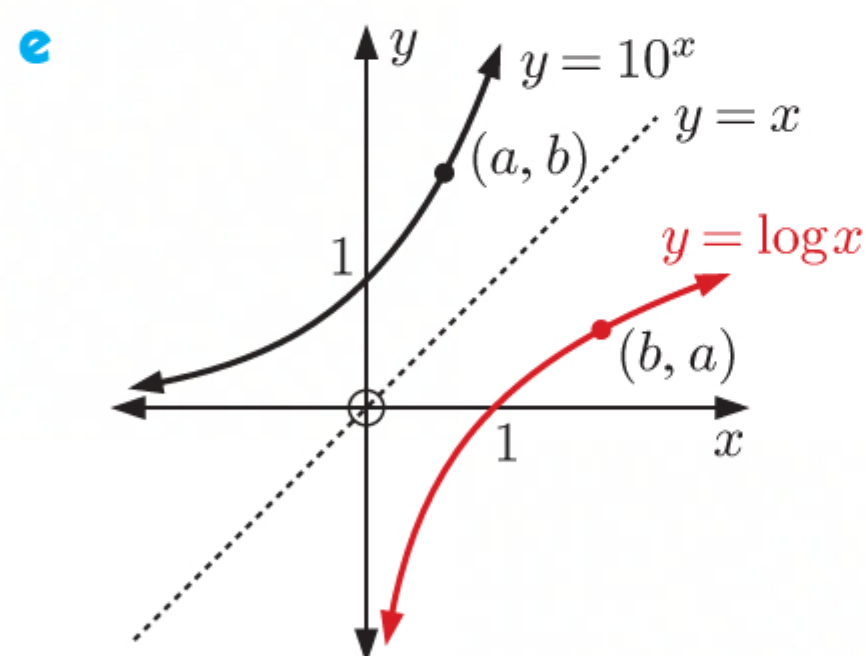
b $\log(0.001) = \log(10^{-3})$
 $= -3$

d $\log 1 = \log(10^0)$
 $= 0$

b $\log(10^a \times 100) = \log(10^a \times 10^2)$
 $= \log(10^{a+2})$
 $= a + 2$

d $\log\left(\frac{10^a}{10^b}\right) = \log(10^{a-b})$
 $= a - b$

b $\log 237 \approx 2.37$



f The y -intercept of $y = 10^x$ is 1.
 \therefore the x -intercept of $y = \log x$ is 1.

g The domain is $\{x \mid x > 0\}$. The range is $\{y \mid y \in \mathbb{R}\}$.

12 $M = \frac{2}{3} \log \left(\frac{E}{10^{4.8}} \right)$

a When $E = 6.2 \times 10^{13}$, $M = \frac{2}{3} \log \left(\frac{6.2 \times 10^{13}}{10^{4.8}} \right)$
 ≈ 5.99

b When $M = 5.1$, $5.1 = \frac{2}{3} \log \left(\frac{E}{10^{4.8}} \right)$

Using technology, $E \approx 2.82 \times 10^{12}$ joules.

13 $\text{pH} = -\log C$

a When $C = 0.000\,234$, $\text{pH} = -\log(0.000\,234)$
 ≈ 3.63

b When $\text{pH} = 6.2$, $6.2 = -\log C$
 $\therefore \log C = -6.2$
 $\therefore C = 10^{-6.2}$
 $\therefore C \approx 6.31 \times 10^{-7} \text{ mol L}^{-1}$

14 $n \approx 3.322 \log \left(\frac{f}{261.6} \right)$

a “Middle C” is 0 octaves above “Middle C”, so $n = 0$

$$\therefore 3.322 \log \left(\frac{f}{261.6} \right) \approx 0$$

$$\therefore \log \left(\frac{f}{261.6} \right) \approx 0$$

$$\therefore \frac{f}{261.6} \approx 1 \quad \{\text{since } \log 1 = 0\}$$

$$\therefore f \approx 261.6$$

So, the frequency of “Middle C” is about 261.6 Hz.

b When $f = 784$, $n \approx 3.322 \log \left(\frac{784}{261.6} \right)$
 ≈ 1.58

So, a note with frequency 784 Hz is about 1.58 octaves above “Middle C”.

c i When $n = 3$, $3 \approx 3.322 \log \left(\frac{f}{261.6} \right)$

Using technology, $f \approx 2090$

The note which is 3 octaves above “Middle C” has frequency of about 2090 Hz.

```

Math Deg Norm1 d/c Real
Eq: 3=3.322log (x/261.6)
x=2092.705806
Lft=3
Rgt=3
REPEAT

```

ii When $n = -1$, $-1 \approx 3.322 \log \left(\frac{f}{261.6} \right)$

Using technology, $f \approx 131$

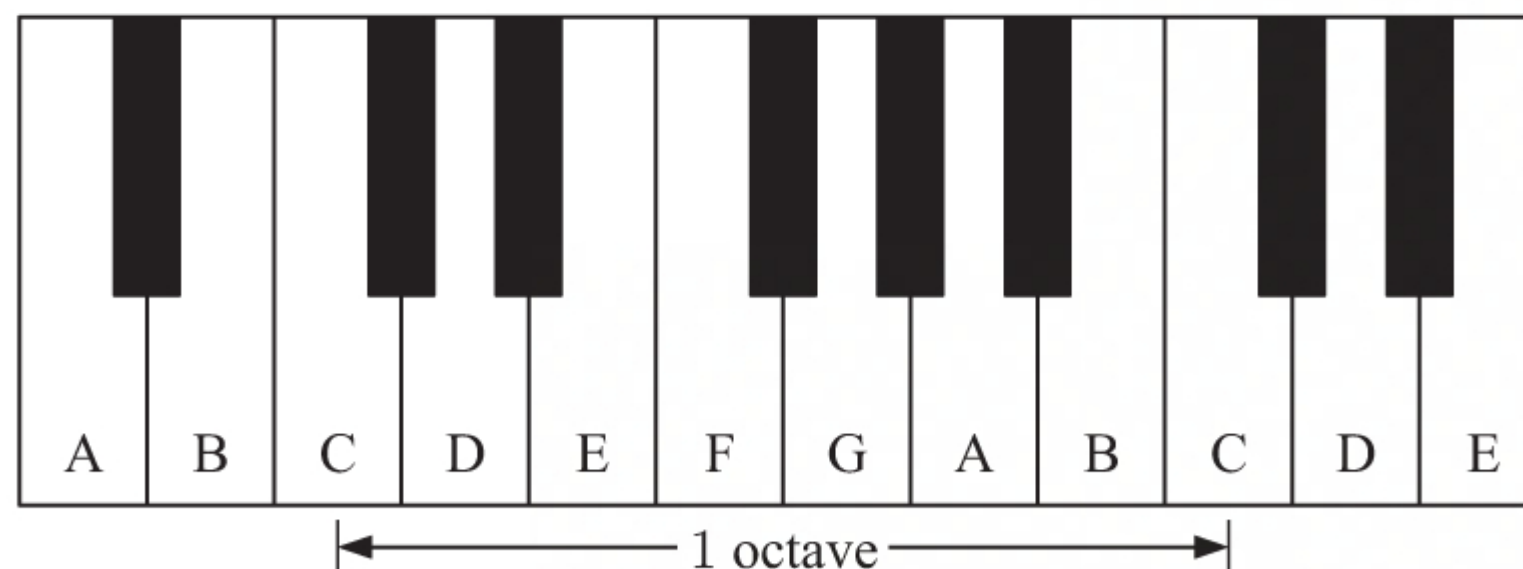
The note which is 1 octave below “Middle C” has frequency of about 131 Hz.

```

Math Deg Norm1 d/c Real
Eq: -1=3.322log (x/261.6)
x=130.8019624
Lft=-1
Rgt=-1
REPEAT

```

d



There are 12 notes in an octave, so each note is $\frac{1}{12}$ of an octave above the note before it.

The note which is adjacent to “Middle C” is $\frac{1}{12}$ of an octave above “Middle C”, so $n = \frac{1}{12}$.

When $n = \frac{1}{12}$, $\frac{1}{12} \approx 3.322 \log \left(\frac{f}{261.6} \right)$

Using technology, $f \approx 277.2$

So the note adjacent to “Middle C” has frequency of about 277.2 Hz.

```

Math Deg Norm1 d/c Real
Eq: 1/12=3.322log (x/261.6)
x=277.155199
Lft=0.08333333333
Rgt=0.08333333333
REPEAT

```

Since the frequency of “Middle C” is 261.6 Hz, the ratio of frequencies between two adjacent notes is $277.2 : 261.6 \approx 1.06 : 1$.

EXERCISE 8H

1 a $\ln(e^2)$
 $= 2$

b $\ln(e^4)$
 $= 4$

c $\ln 1$
 $= \ln(e^0)$
 $= 0$

d $\ln\left(\frac{1}{e^2}\right)$
 $= \ln(e^{-2})$
 $= -2$

2 a	$\ln(e^a)$ $= a$	b	$\ln(e \times e^a)$ $= \ln(e^{1+a})$ $= 1 + a$	c	$\ln(e^a \times e^b)$ $= \ln(e^{a+b})$ $= a + b$	d	$\ln((e^a)^b)$ $= \ln(e^{ab})$ $= ab$
e	$e^{\ln 3}$ $= 3$	f	$e^{2 \ln 3}$ $= (e^{\ln 3})^2$ $= 3^2$ $= 9$	g	$e^{-\ln 5}$ $= (e^{\ln 5})^{-1}$ $= 5^{-1}$ $= \frac{1}{5}$	h	$e^{-2 \ln 2}$ $= (e^{\ln 2})^{-2}$ $= 2^{-2}$ $= \frac{1}{2^2}$ $= \frac{1}{4}$

3 a $e \approx 2.718, \quad e^2 \approx 7.389$
 $\therefore e < 5 < e^2$
 $\therefore \ln e < \ln 5 < \ln(e^2)$
 $\therefore 1 < \ln 5 < 2$

b $\ln 5 \approx 1.609$

4 a $e^3 \approx 20.0855, \quad e^4 \approx 54.5982$

b $\therefore e^3 < 40 < e^4$
 $\therefore \ln(e^3) < \ln 40 < \ln(e^4)$
 $\therefore 3 < \ln 40 < 4$

So, $\ln 40$ lies between 3 and 4.

c $\ln 40 \approx 3.689$

5 a $\ln 12 \approx 2.485$

b $\ln 68 \approx 4.220$

c $\ln(1.4) \approx 0.336$

d $\ln(0.7) \approx -0.357$

e $\ln 500 \approx 6.215$

f $\ln 850 \approx 6.745$

g $\ln(0.02) \approx -3.912$

h $\ln(10^5) \approx 11.513$

i $\ln(0.006) \approx -5.116$

j $\ln 7500 \approx 8.923$

6 x does not exist such that $e^x = -2$ or 0 since $e^x > 0$ for all $x \in \mathbb{R}$.
 $\therefore \ln(-2)$ and $\ln 0$ do not exist.

7 a $\ln 24 \approx 3.1781$

b $24 = e^{\ln 24}$
 $\approx e^{3.1781}$

8 a $6 = e^{\ln 6}$
 $\approx e^{1.7918}$

b $60 = e^{\ln 60}$
 $\approx e^{4.0943}$

c $6000 = e^{\ln 6000}$
 $\approx e^{8.6995}$

d $0.6 = e^{\ln(0.6)}$
 $\approx e^{-0.5108}$

e $0.006 = e^{\ln(0.006)}$
 $\approx e^{-5.1160}$

f $15 = e^{\ln 15}$
 $\approx e^{2.7081}$

g $1500 = e^{\ln 1500}$
 $\approx e^{7.3132}$

h $1.5 = e^{\ln(1.5)}$
 $\approx e^{0.4055}$

i $0.15 = e^{\ln(0.15)}$
 $\approx e^{-1.8971}$

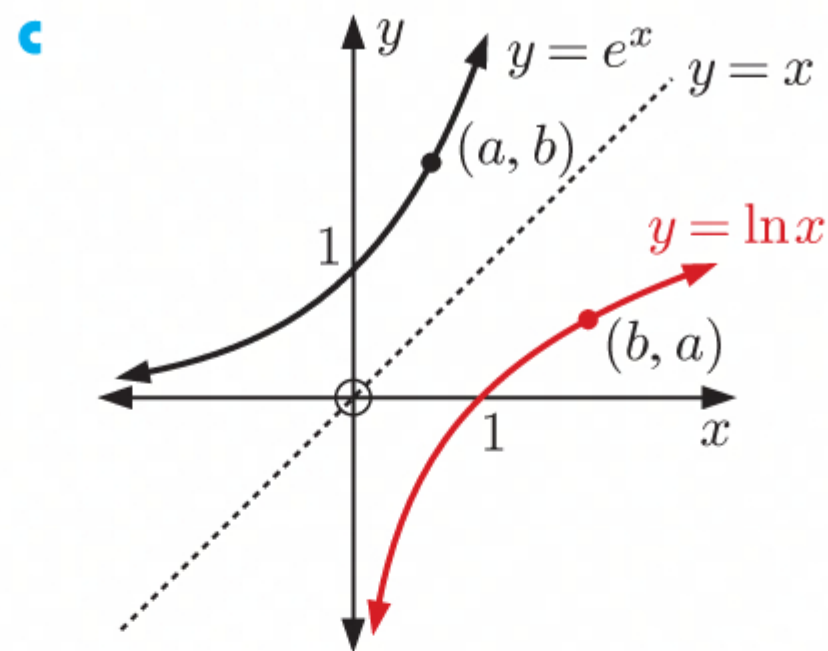
j $0.00015 = e^{\ln(0.00015)}$
 $\approx e^{-8.8049}$

- 9 a If (a, b) lies on the graph of $y = e^x$, then $b = e^a$
 $\therefore \ln b = \ln(e^a)$
 $\therefore \ln b = a$

For $y = \ln x$, when $x = b$, $y = \ln b$
 $\therefore y = a$

$\therefore (b, a)$ lies on the graph of $y = \ln x$.

- b $y = e^x$ and $y = \ln x$ are inverse functions.



- d The domain is $\{x \mid x > 0\}$. The range is $\{y \mid y \in \mathbb{R}\}$.

- 10 a $T = 2 \ln(n + 1)$ seconds

- i When $n = 5$, $T = 2 \ln 6 \approx 3.58$ seconds

It will take this person about 3.58 seconds to choose between 5 possible choices.

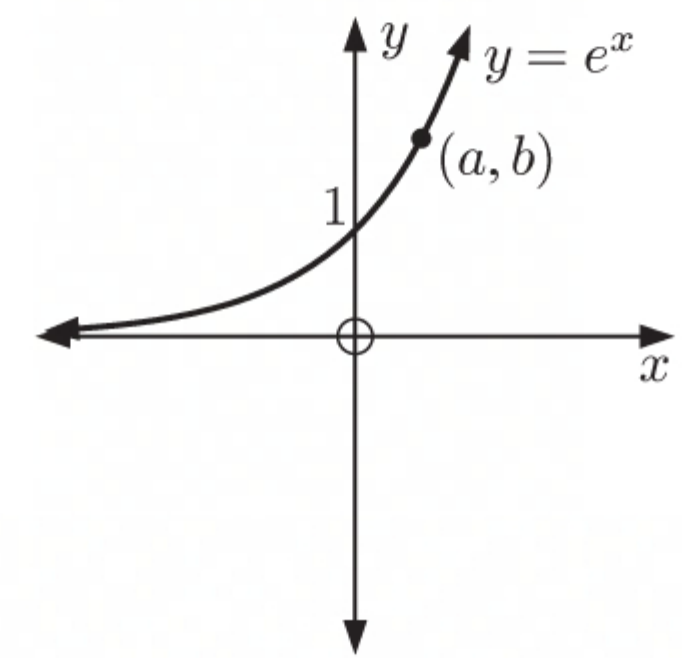
- ii When $n = 15$, $T = 2 \ln 16 \approx 5.55$ seconds

It will take this person about 5.55 seconds to choose between 15 possible choices.

- b When $n = 20$, $T = 2 \ln 21 \approx 6.09$ seconds

When $n = 40$, $T = 2 \ln 41 \approx 7.43$ seconds

The person will take $2 \ln 41 - 2 \ln 21 \approx 1.34$ seconds longer to make a selection.



REVIEW SET 8A

1 $f(x) = 3 \times 2^x$

a $f(0) = 3 \times 2^0$
 $= 3 \times 1$
 $= 3$

b $f(3) = 3 \times 2^3$
 $= 3 \times 8$
 $= 24$

c $f(-2) = 3 \times 2^{-2}$
 $= 3 \times \frac{1}{2^2}$
 $= 3 \times \frac{1}{4}$
 $= \frac{3}{4}$

2 a $y = 4^x - 1$

When $x = 2$, $y = 4^2 - 1$
 $= 16 - 1$
 $= 15$

\therefore the point $(2, 15)$ satisfies $y = 4^x - 1$.

b $f(x) = 5 \times 3^{-x}$

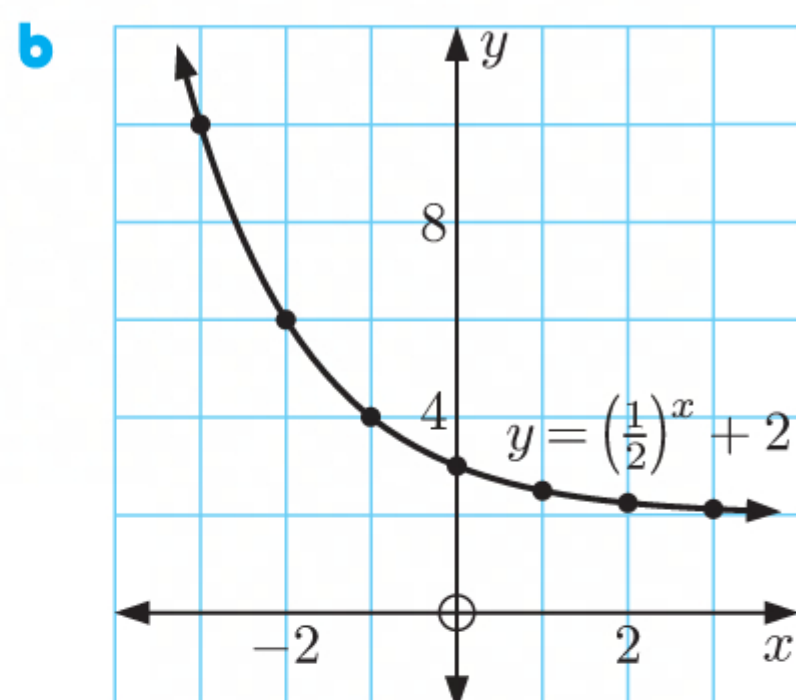
$\therefore f(-1) = 5 \times 3^{-(-1)}$
 $= 5 \times 3$
 $= 15$

\therefore the point $(-1, \frac{5}{3})$ does not satisfy $f(x) = 5 \times 3^{-x}$.

3 $y = \left(\frac{1}{2}\right)^x + 2$

a

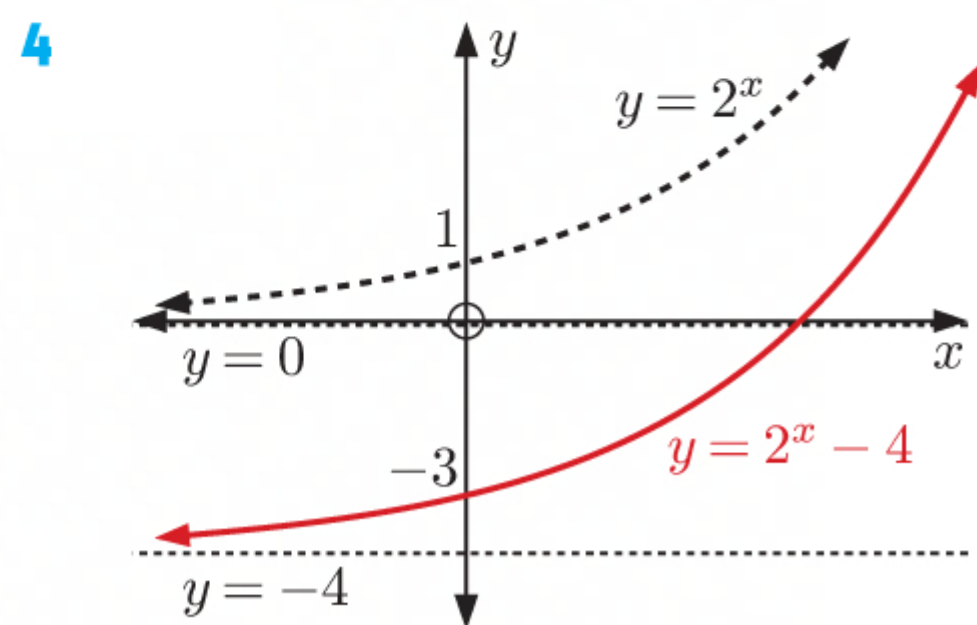
x	-3	-2	-1	0	1	2	3
y	10	6	4	3	$2\frac{1}{2}$	$2\frac{1}{4}$	$2\frac{1}{8}$



c **i** As $x \rightarrow \infty$, $y \rightarrow 2^+$.

ii As $x \rightarrow -\infty$, $y \rightarrow \infty$.

d The horizontal asymptote is $y = 2$.



$y = 2^x$ has y -intercept 1 and horizontal asymptote $y = 0$.

$y = 2^x - 4$ has y -intercept -3 and horizontal asymptote $y = -4$.

5 a $f(x) = 4^x + 2$
 $\therefore f(0) = 4^0 + 2$
 $= 1 + 2$
 $= 3$
 \therefore the y -intercept is 3.

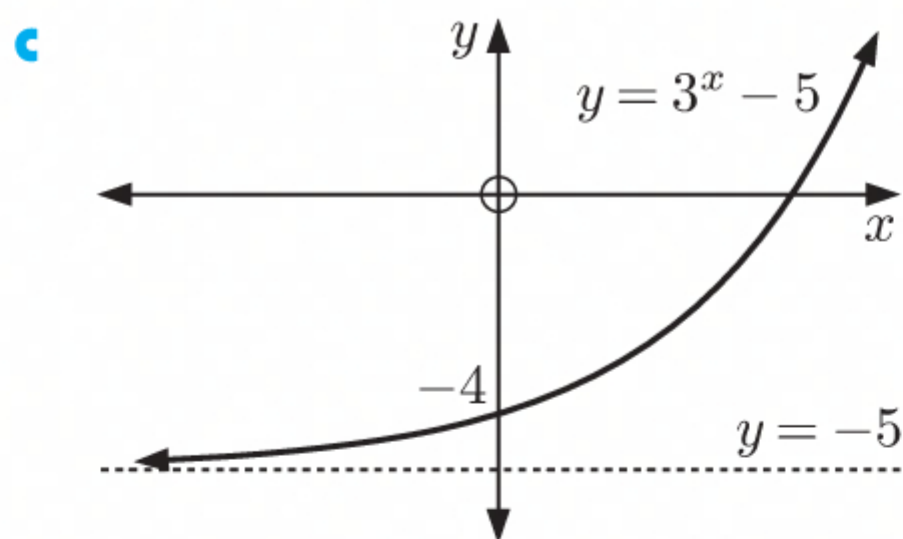
b $f(x) = 2 \times 5^{-x} - 6$
 $\therefore f(0) = 2 \times 5^0 - 6$
 $= 2 \times 1 - 6$
 $= -4$
 \therefore the y -intercept is -4 .

c $f(x) = -\frac{1}{3} \times 2^x + 5$
 $\therefore f(0) = -\frac{1}{3} \times 2^0 + 5$
 $= -\frac{1}{3} \times 1 + 5$
 $= 4\frac{2}{3}$
 \therefore the y -intercept is $4\frac{2}{3}$.

6 $y = 3^x - 5$

a When $x = 0$, $y = 3^0 - 5 = 1 - 5 = -4$
 When $x = 1$, $y = 3^1 - 5 = 3 - 5 = -2$
 When $x = 2$, $y = 3^2 - 5 = 9 - 5 = 4$
 When $x = -1$, $y = 3^{-1} - 5 = \frac{1}{3} - 5 = -4\frac{2}{3}$
 When $x = -2$, $y = 3^{-2} - 5 = \frac{1}{9} - 5 = -4\frac{8}{9}$

b As $x \rightarrow \infty$, $3^x \rightarrow \infty$
 and so $y \rightarrow \infty$
 As $x \rightarrow -\infty$, $3^x \rightarrow 0^+$
 and so $y \rightarrow -5^+$



d $y = -5$ is the horizontal asymptote.

7 a $y = k \times a^x$

Substituting $(0, 8)$ into the equation gives

$$8 = k \times a^0$$

$$\therefore k = 8$$

Substituting $(-2, \frac{9}{2})$ into the equation gives

$$\frac{9}{2} = k \times a^{-2}$$

$$\therefore \frac{9}{2} = \frac{8}{a^2} \quad \{\text{since } k = 8\}$$

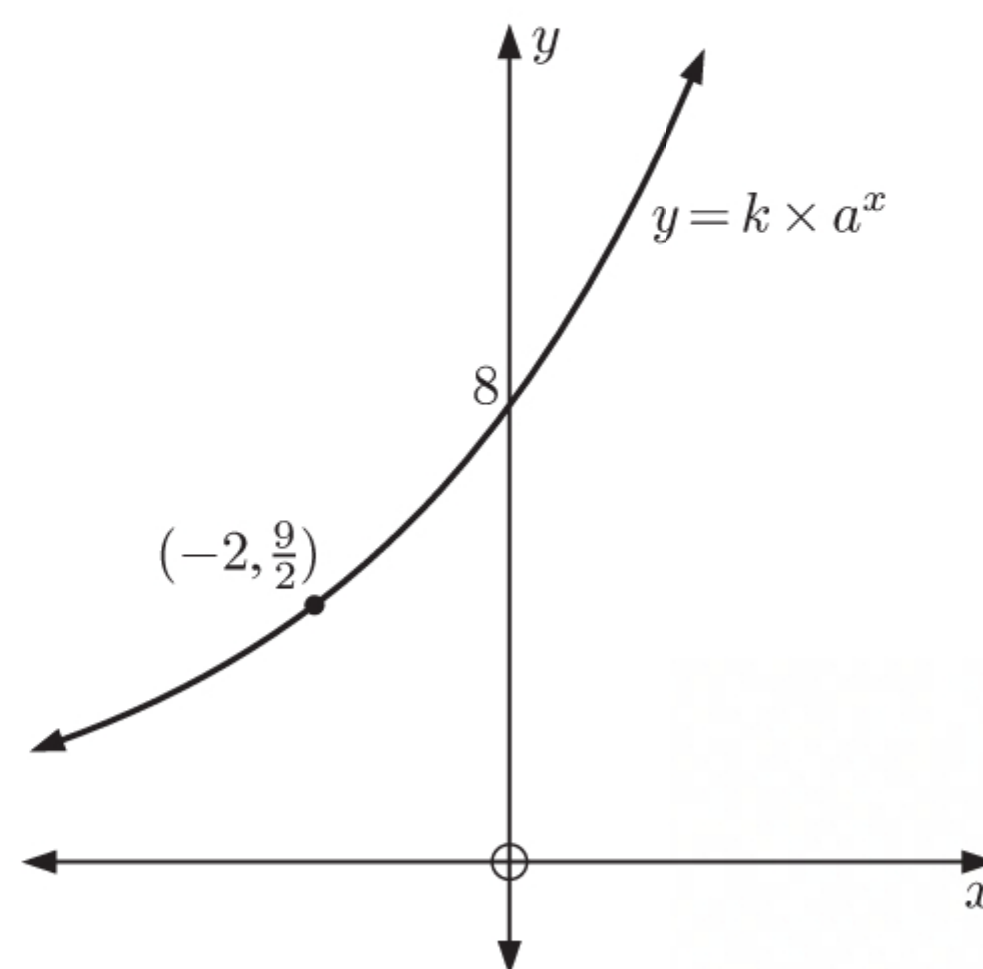
$$\therefore 9a^2 = 16$$

$$\therefore a^2 = \frac{16}{9}$$

$$\therefore a = \pm \sqrt{\frac{16}{9}}$$

$$\therefore a = \frac{4}{3} \quad \{\text{since } a > 0\}$$

$$\therefore k = 8, a = \frac{4}{3}$$



b As $x \rightarrow -\infty$, $y \rightarrow 0^+$, so the horizontal asymptote is $y = 0$.

c $y = 8 \times (\frac{4}{3})^x$

When $x = 2$, $y = 8 \times (\frac{4}{3})^2$

$$\therefore y = 8 \times \frac{16}{9}$$

$$\therefore y = \frac{128}{9}$$

8 a i When $x = 0.7$, $y = 3^{0.7}$

From point A, $y \approx 2.2$

$$\therefore 3^{0.7} \approx 2.2$$

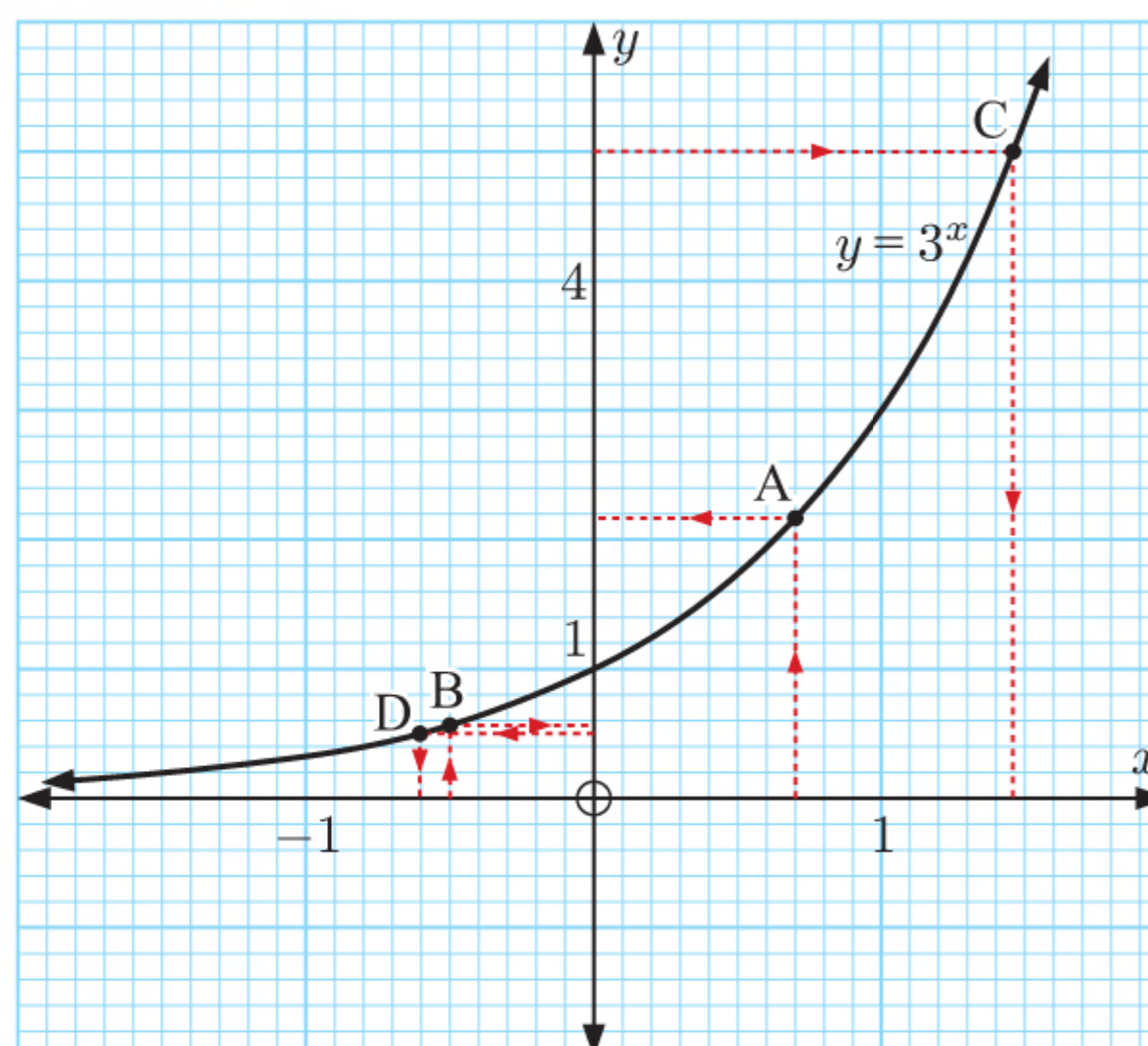
ii When $x = -0.5$, $y = 3^{-0.5}$

From point B, $y \approx 0.6$

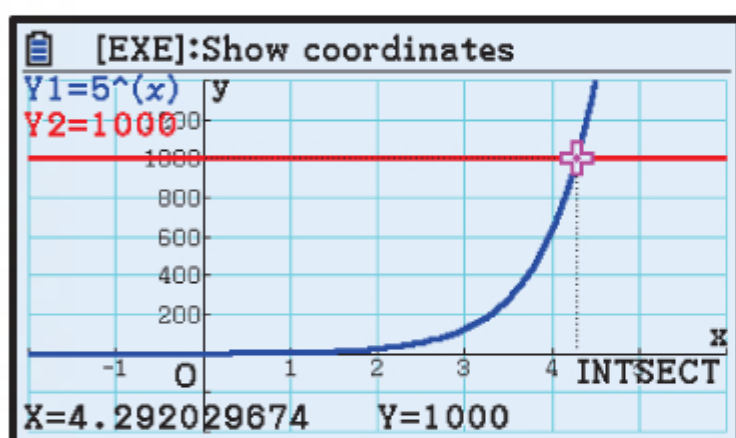
$$\therefore 3^{-0.5} \approx 0.6$$

b i When $3^x = 5$,
 $x \approx 1.5$ from point C.

ii When $3^x = \frac{1}{2}$,
 $x \approx -0.6$ from point D.

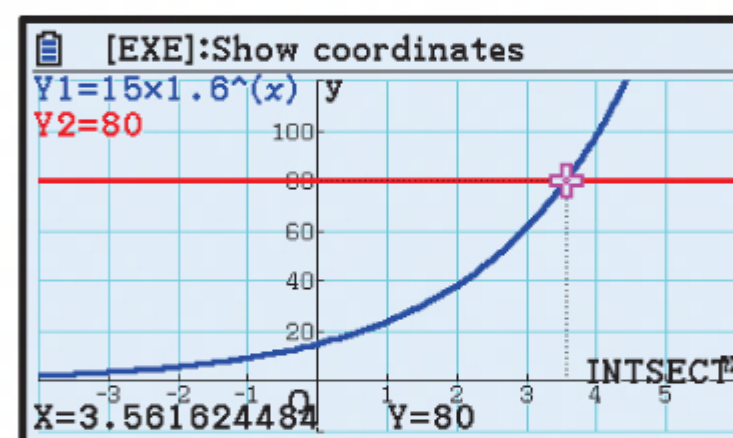


- 9 a We graph $Y_1 = 5^x$ and $Y_2 = 1000$ on the same set of axes, and find their point of intersection.



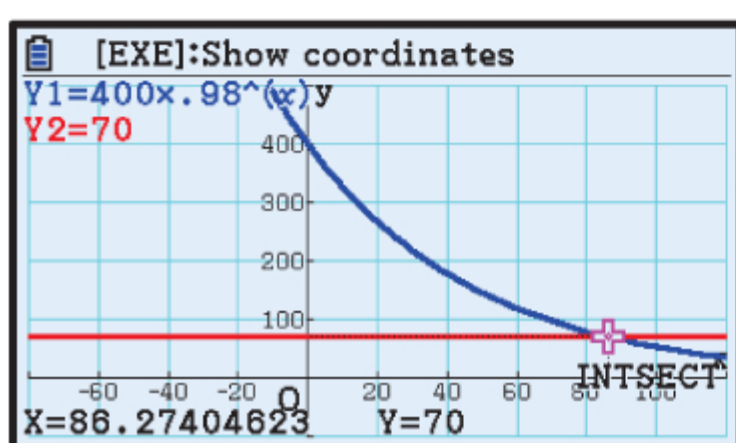
The solution is $x \approx 4.29$.

- b We graph $Y_1 = 15 \times (1.6)^x$ and $Y_2 = 80$ on the same set of axes, and find their point of intersection.



The solution is $x \approx 3.56$.

- c We graph $Y_1 = 400 \times (0.98)^x$ and $Y_2 = 70$ on the same set of axes, and find their point of intersection.



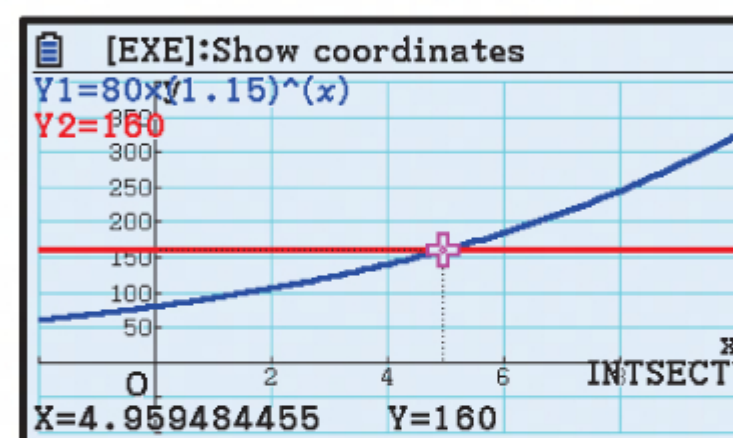
The solution is $x \approx 86.3$.

- 10 $P(t) = 80 \times (1.15)^t$ seals

a
$$\begin{aligned} P(0) &= 80 \times (1.15)^0 \\ &= 80 \times 1 \\ &= 80 \end{aligned}$$

So the initial population was 80 seals.

- b When $P(t) = 2 \times 80 = 160$, $160 = 80 \times (1.15)^t$.
Using technology, $t \approx 4.96$.



\therefore it took about 4.96 years or about 4 years and $11\frac{1}{2}$ months for the population to double in size.

c Percentage increase in first 4 years $= \left(\frac{P(4) - P(0)}{P(0)} \right) \times 100\%$

$$= \left(\frac{80 \times (1.15)^4 - 80}{80} \right) \times 100\%$$

$$\approx 74.9\%$$

11 a $V = k \times 0.6^t + c$ pounds

Substituting $(0, 800)$ into the equation gives

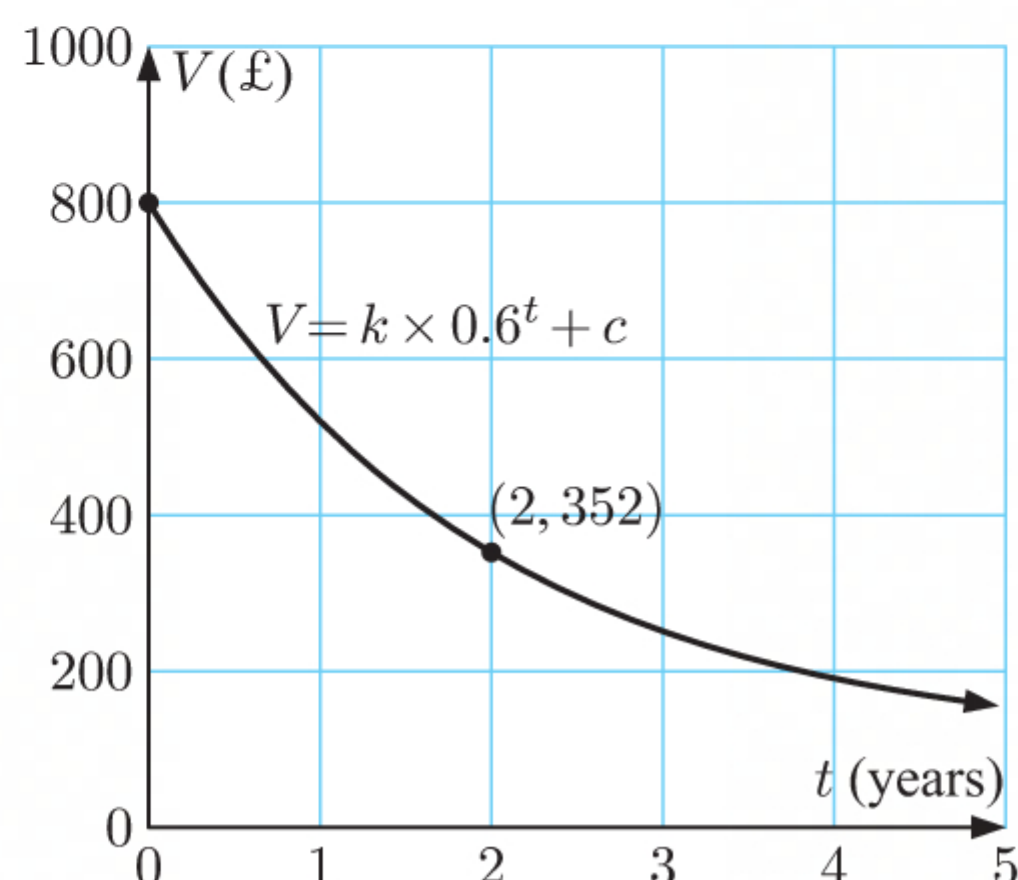
$$800 = k \times 0.6^0 + c$$

$$\therefore k + c = 800$$

Substituting $(2, 352)$ into the equation gives

$$352 = k \times 0.6^2 + c$$

$$\therefore 0.36k + c = 352$$



Solving the system of equations

$$\begin{cases} k + c = 800 \\ 0.36k + c = 352 \end{cases}$$

simultaneously gives $k = 700$,
 $c = 100$.

	a	b	c
1	1	1	800
2	0.36	1	352

352

	X	Y
1	700	100

700

b No, the value of the computer does not decrease by 40% each year. The computer depreciates by 40% each year above the computer's *minimum* price, but not overall.

c $V = 700 \times 0.6^t + 100$

When $t = 3$, $V = 700 \times 0.6^3 + 100$
 $= 251.2$

The value of the computer after 3 years is £251.20.

d The horizontal asymptote is $V = 100$. This means the computer will never be worth less than £100.

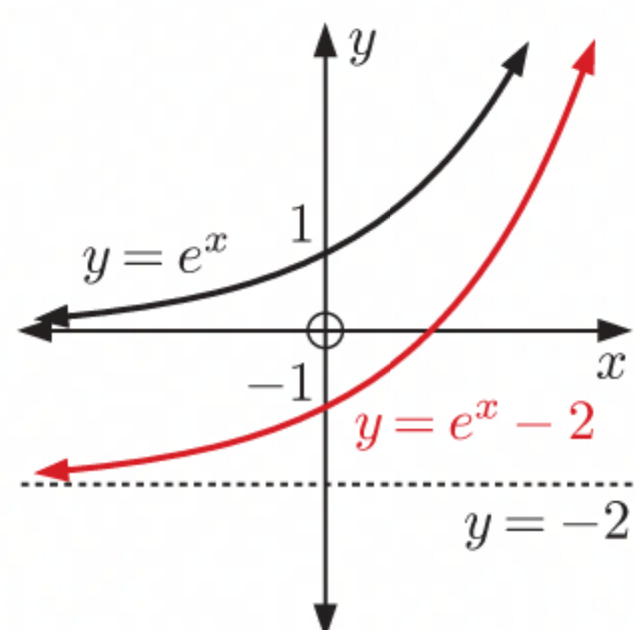
12 a $e^4 \approx 54.6$

b $e^{-2} \approx 0.135$

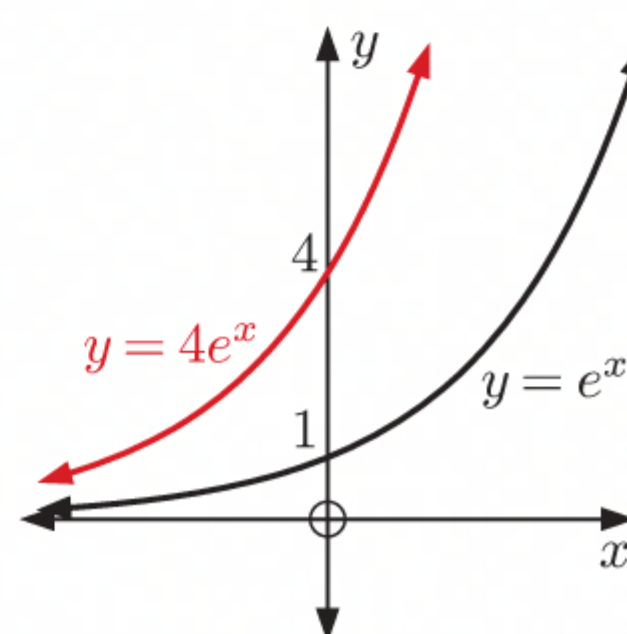
c $10e^{3.5} \approx 331$

d $40e^{-2.53} \approx 3.19$

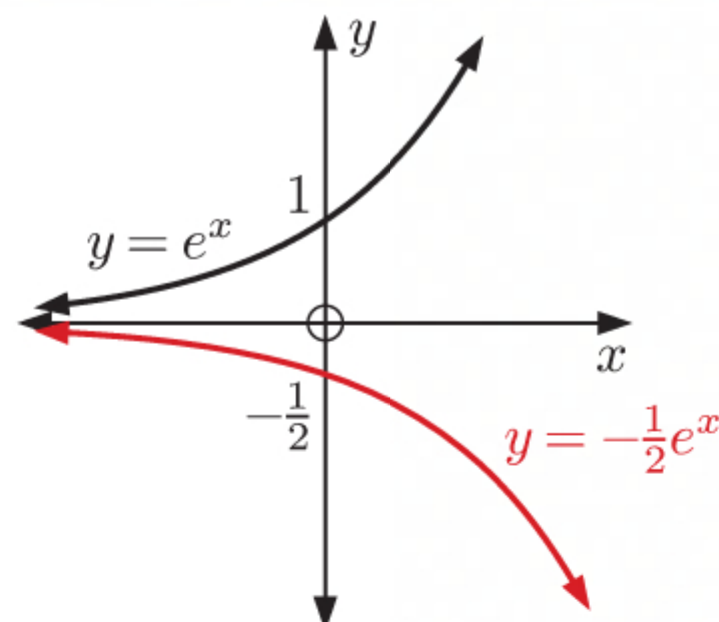
13 a



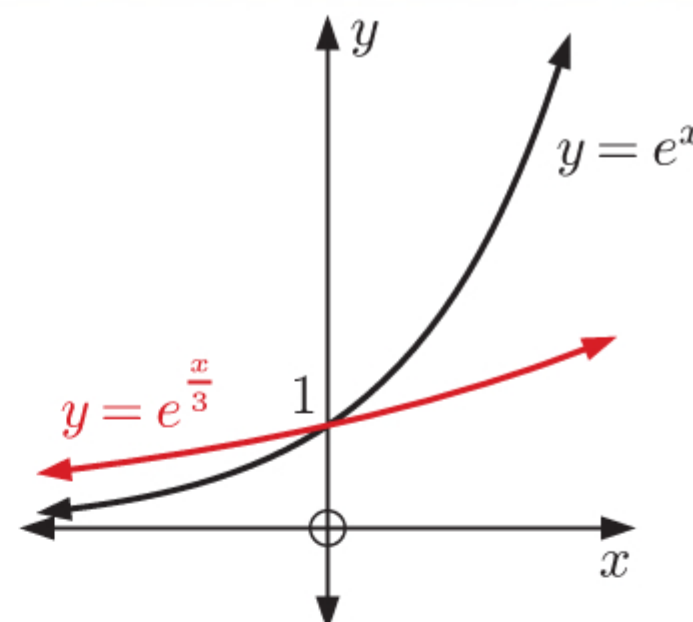
b



c



d



14 a $T = k \times e^{-0.1t} + c$

The horizontal asymptote is $T = 20$,
so $c = 20$

$$\therefore T = k \times e^{-0.1t} + 20$$

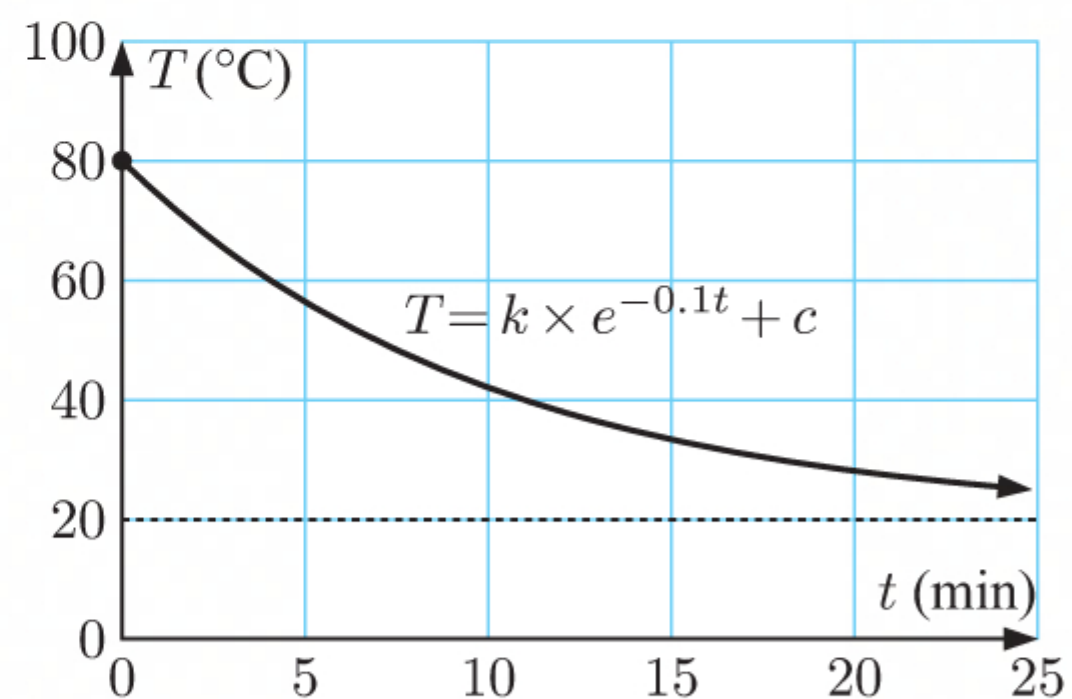
Substituting $(0, 80)$ into the equation gives

$$80 = k \times e^{-0.1(0)} + 20$$

$$\therefore k + 20 = 80$$

$$\therefore k = 60$$

\therefore the exponential model is $T = 60e^{-0.1t} + 20$.

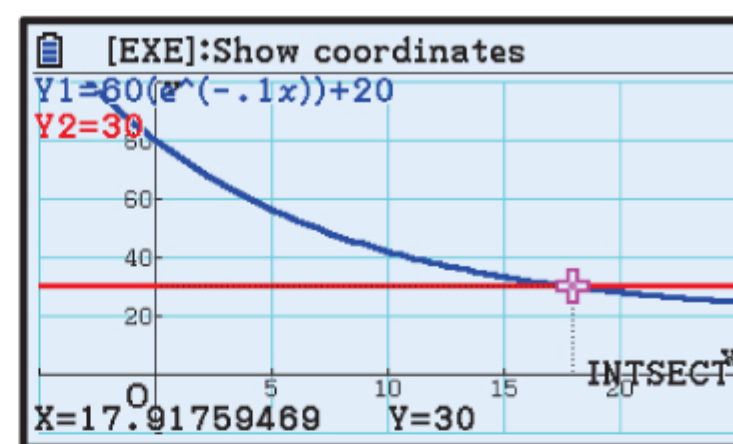


b When $t = 10$, $T = 60e^{-0.1(10)} + 20$
 $= 60e^{-1} + 20$
 ≈ 42.1

The temperature of the water after 10 minutes is about 42.1°C .

c When $T = 30$, $30 = 60e^{-0.1t} + 20$.

Using technology, $t \approx 17.9$.



It will take about 17.9 minutes for the temperature of the water to fall to 30°C .

15 a $\log 27$
 ≈ 1.431

b $\log(0.58)$
 ≈ -0.237

c $\log 400$
 ≈ 2.602

d $\ln 40$
 ≈ 3.689

16 a $32 = 10^{\log 32}$
 $\approx 10^{1.5051}$

b 0.0013
 $= 10^{\log(0.0013)}$
 $\approx 10^{-2.8861}$

c 8.963×10^{-5}
 $= 10^{\log(8.963)} \times 10^{-5}$
 $= 10^{\log(8.963) - 5}$
 $\approx 10^{-4.0475}$

17 $\ln k > 1$ Also, $\log k < 1$
 $\therefore e^{\ln k} > e^1$ $\therefore 10^{\log k} < 10^1$
 $\therefore k > e$ $\therefore k < 10$
 $\therefore k$ lies between e and 10 .

18 a $f(x) = 10^x$
 $\therefore f(2) = 10^2$
 $= 100$

\therefore the y -coordinate of P is 100.

b $f(x) = 10^x$ and $f^{-1}(x) = \log x$ are inverse functions.

\therefore the point corresponding to P(2, 100) on the inverse function is (100, 2).

$$19 \quad H = \frac{\ln 0.5}{\ln(1 - \frac{k}{100})} \text{ years}$$

$$\text{a} \quad \text{i} \quad \text{When } k = 5, \quad H = \frac{\ln 0.5}{\ln(1 - \frac{5}{100})} \approx 13.5$$

The half-life of the substance is about 13.5 years.

$$\text{b} \quad \text{When } k = 1.5, \quad H = \frac{\ln 0.5}{\ln(1 - \frac{1.5}{100})} \approx 45.9$$

Substance A's half-life is about 45.9 years.

So, substance B's half-life is $98.7 - 45.9 \approx 52.8$ years longer than substance A's half-life.

$$\text{c} \quad \text{When } H = 30, \quad 30 = \frac{\ln 0.5}{\ln(1 - \frac{k}{100})}.$$

Using technology, $k \approx 2.28$.

The substance decays by about 2.28% each year.

$$\text{ii} \quad \text{When } k = 2.2, \quad H = \frac{\ln 0.5}{\ln(1 - \frac{2.2}{100})} \approx 31.2$$

The half-life of the substance is about 31.2 years.

$$\text{When } k = 0.7, \quad H = \frac{\ln 0.5}{\ln(1 - \frac{0.7}{100})} \approx 98.7$$

Substance B's half-life is about 98.7 years.

Math Deg Norm1 d/c Real
Eq: 30 = $\frac{\ln 0.5}{\ln(1 - \frac{x}{100})}$
x = 2.284003157
Lft = 30
Rgt = 30
[REPEAT]

REVIEW SET 8B

$$1 \quad g(x) = 3^{-x} - 2$$

$$\text{a} \quad g(0) = 3^0 - 2 = 1 - 2 = -1$$

$$\text{c} \quad g(-1) = 3^{-(-1)} - 2 = 3^1 - 2 = 1$$

$$\text{b} \quad g(1) = 3^{-1} - 2 = \frac{1}{3} - 2 = -1\frac{2}{3}$$

$$\text{d} \quad g(3) = 3^{-3} - 2 = \frac{1}{3^3} - 2 = \frac{1}{27} - 2 = -1\frac{26}{27}$$

$$2 \quad f(x) = 3 \times 2^x - 12$$

$$\text{a} \quad f(0) = 3 \times 2^0 - 12 = 3 - 12 = -9$$

\therefore the y -intercept is -9 .

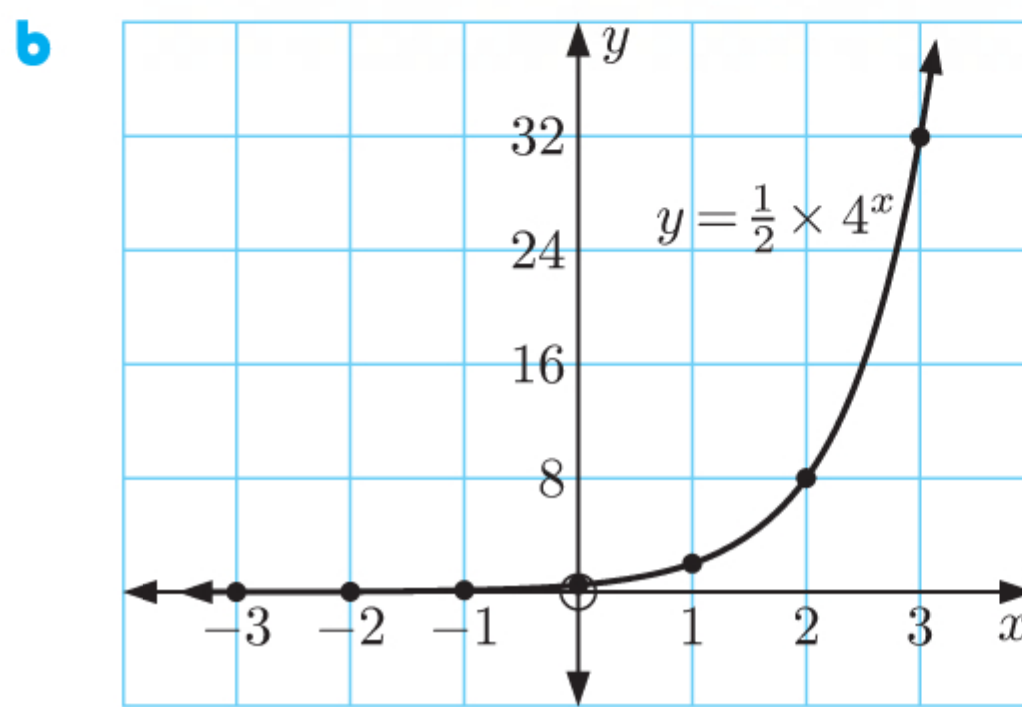
$$\text{b} \quad f(2) = 3 \times 2^2 - 12 = 3 \times 4 - 12 = 12 - 12 = 0$$

\therefore the x -intercept is 2.

3 $y = \frac{1}{2} \times 4^x$

a

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{128}$	$\frac{1}{32}$	$\frac{1}{8}$	$\frac{1}{2}$	2	8	32



4 a The horizontal asymptote of $y = -\frac{1}{2} \times 5^x + 3$ is $y = 3$.

b The horizontal asymptote of $y = 2^{-x} - 4$ is $y = -4$.

c The horizontal asymptote of $y = 6 - 8 \times \left(\frac{1}{3}\right)^x$ is $y = 6$.

5 $f(x) = -2 \times 3^x - 4$

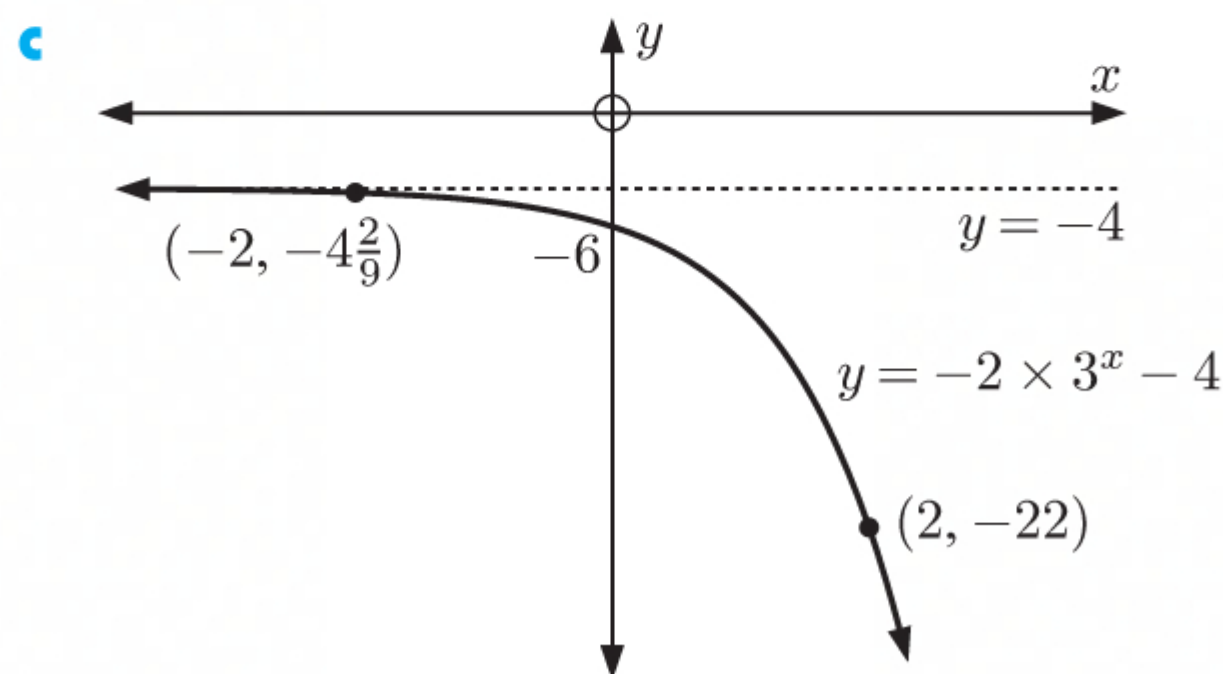
a $f(-2) = -2 \times 3^{-2} - 4$
 $= -2 \times \frac{1}{3^2} - 4$
 $= -\frac{2}{9} - 4$
 $= -4\frac{2}{9}$

$f(2) = -2 \times 3^2 - 4$
 $= -2 \times 9 - 4$
 $= -18 - 4$
 $= -22$

b i $f(0) = -2 \times 3^0 - 4$
 $= -2 - 4$
 $= -6$

\therefore the y -intercept is -6 .

ii The horizontal asymptote is $y = -4$.



d The domain is $\{x \mid x \in \mathbb{R}\}$. The range is $\{y \mid y < -4\}$.

6 a $y = k \times 2^{-x} + c$

Substituting $(0, 10)$ into the equation gives

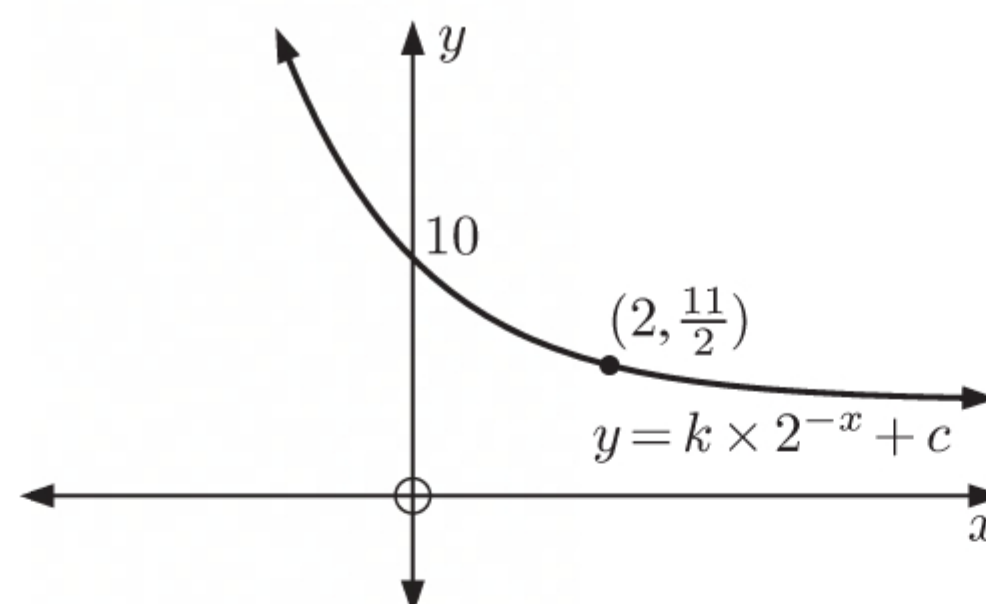
$$10 = k \times 2^0 + c$$

$$\therefore k + c = 10$$

Substituting $(2, \frac{11}{2})$ into the equation gives

$$\frac{11}{2} = k \times 2^{-2} + c$$

$$\therefore \frac{1}{4}k + c = \frac{11}{2}$$



Solving the system of equations

$$\begin{cases} k + c = 10 \\ \frac{1}{4}k + c = \frac{11}{2} \end{cases}$$

simultaneously gives $k = 6$,
 $c = 4$.

	a	b	c
1	1	1	10
2	0.25	1	5.5

5.5

SOLVE DELETE CLEAR EDIT

	a	b	c
X			6
Y			4

6

REPEAT

So, the exponential model is $y = 6 \times 2^{-x} + 4$.

b $y = k \times a^x + c$

The horizontal asymptote is $y = 5$, so $c = 5$.

$$\therefore y = k \times a^x + 5$$

Substituting $(0, 2)$ into the equation gives

$$2 = k \times a^0 + 5$$

$$\therefore k + 5 = 2$$

$$\therefore k = -3$$

$$\therefore y = -3 \times a^x + 5$$

Substituting $(-1, -1)$ into the equation gives

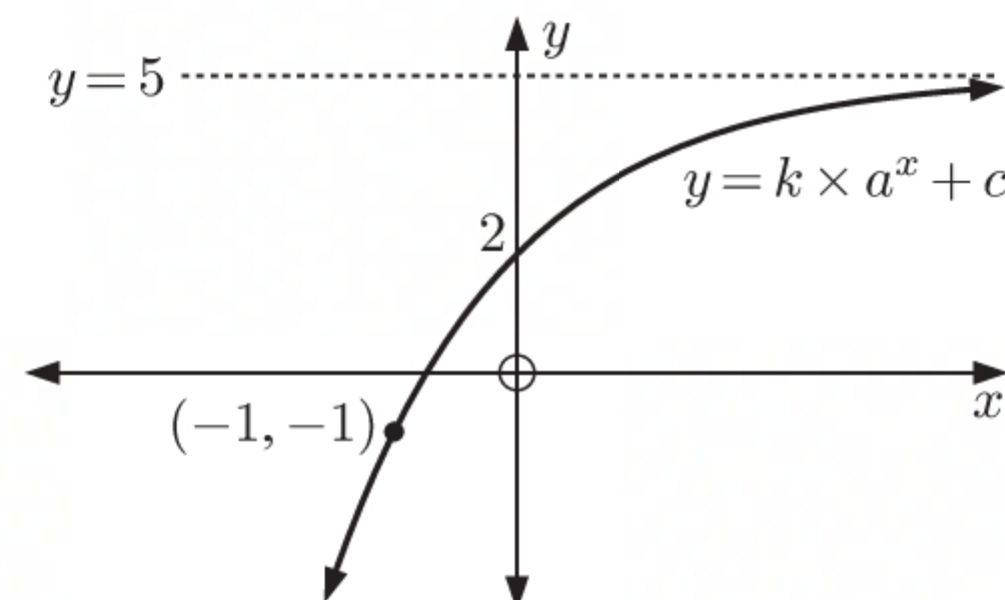
$$-1 = -3 \times a^{-1} + 5$$

$$\therefore -6 = \frac{-3}{a}$$

$$\therefore -6a = -3$$

$$\therefore a = \frac{1}{2}$$

So, the exponential model is $y = -3 \times \left(\frac{1}{2}\right)^x + 5$.



7 $y = k \times 5^{-x} + c$

a The x -intercept is 1.

When $x = 1$, $y = 0$

$$\therefore k \times 5^{-1} + c = 0$$

$$\therefore \frac{k}{5} + c = 0$$

The y -intercept is 8.

When $x = 0$, $y = 8$

$$\therefore k \times 5^0 + c = 8$$

$$\therefore k + c = 8$$

Solving the system of equations

$$\begin{cases} \frac{k}{5} + c = 0 \\ k + c = 8 \end{cases}$$

simultaneously gives $k = 10$,
 $c = -2$.

	a	b	c
1	0.2	1	0
2	1	1	8

8

SOLVE DELETE CLEAR EDIT

	a	b	c
X			10
Y			-2

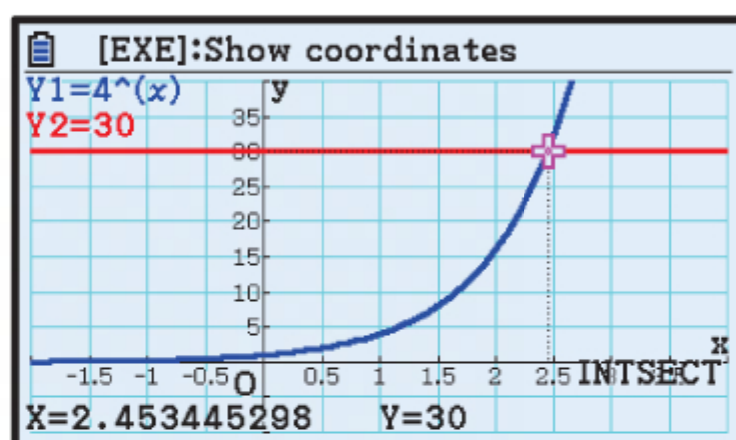
10

REPEAT

So, the exponential model is $y = 10 \times 5^{-x} - 2$.

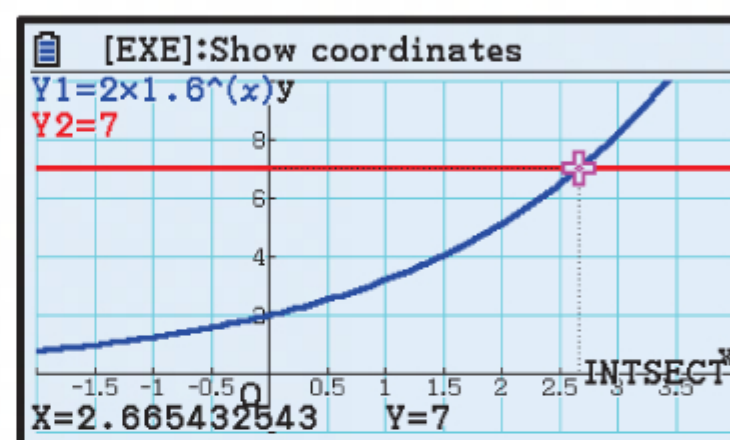
b When $x = -1$, $y = 10 \times 5^{-(-1)} - 2$
 $= 10 \times 5^1 - 2$
 $= 48$

- 8 a** We graph $Y_1 = 4^x$ and $Y_2 = 30$ on the same set of axes, and find their point of intersection.



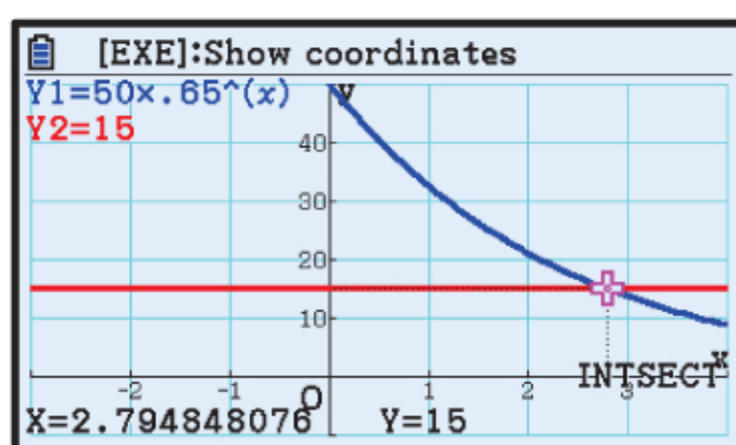
The solution is $x \approx 2.45$.

- b** We graph $Y_1 = 2 \times (1.6)^x$ and $Y_2 = 7$ on the same set of axes, and find their point of intersection.



The solution is $x \approx 2.67$.

- c** We graph $Y_1 = 50 \times (0.65)^x$ and $Y_2 = 15$ on the same set of axes, and find their point of intersection.



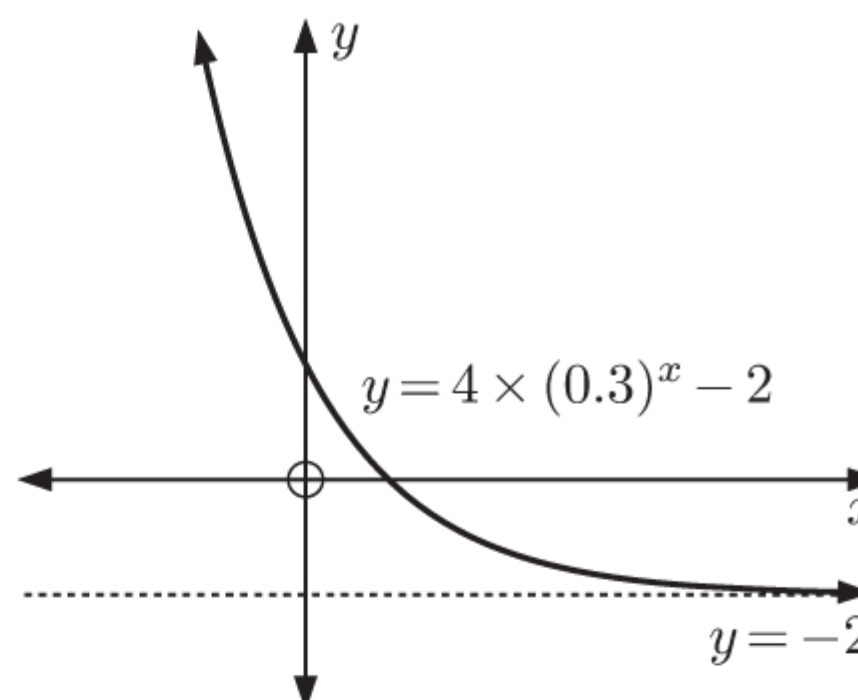
The solution is $x \approx 2.79$.

- 9** The graph of $y = 4 \times (0.3)^x - 2$ lies above its horizontal asymptote $y = -2$.

$\therefore y > -2$ for all x .

So, the equation $4 \times (0.3)^x - 2 = k$ has:

- a** 1 solution for $k > -2$
- b** no solutions for $k \leq -2$.



- 10** $W = 1500 \times (0.993)^t$ grams

- a** When $t = 0$, $W = 1500 \times (0.993)^0$
 $= 1500 \times 1$
 $= 1500$

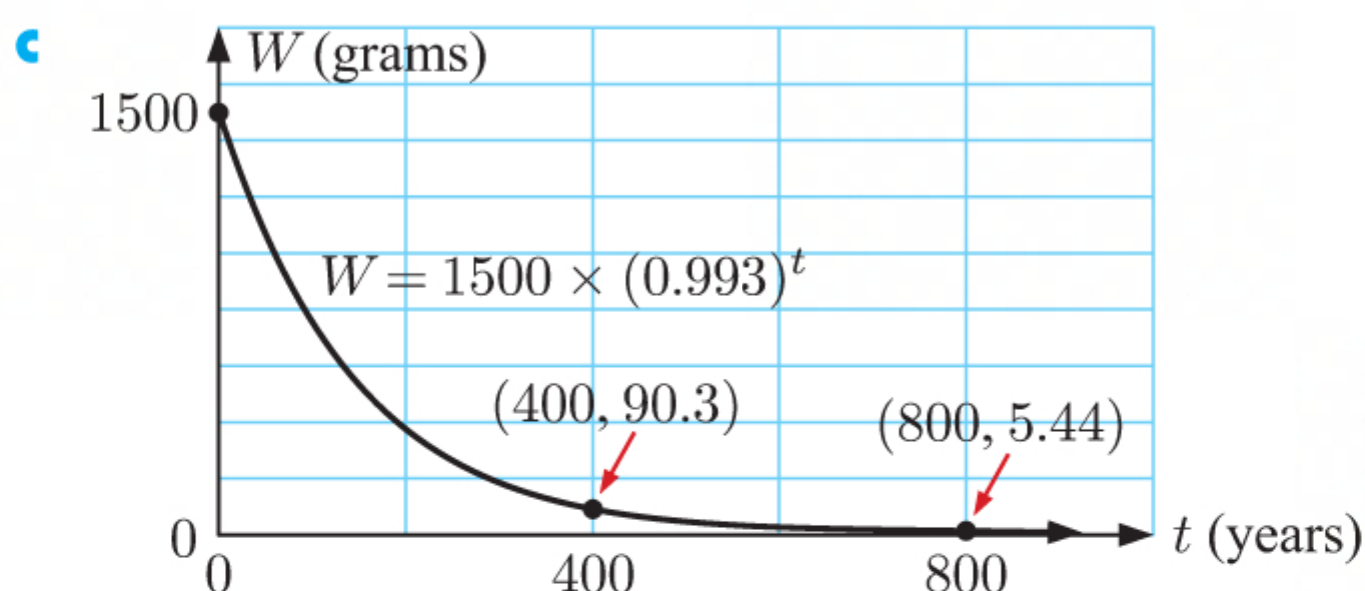
The initial weight of the radioactive substance was 1500 grams.

- b i** When $t = 400$,
 $W = 1500 \times (0.993)^{400}$
 ≈ 90.3

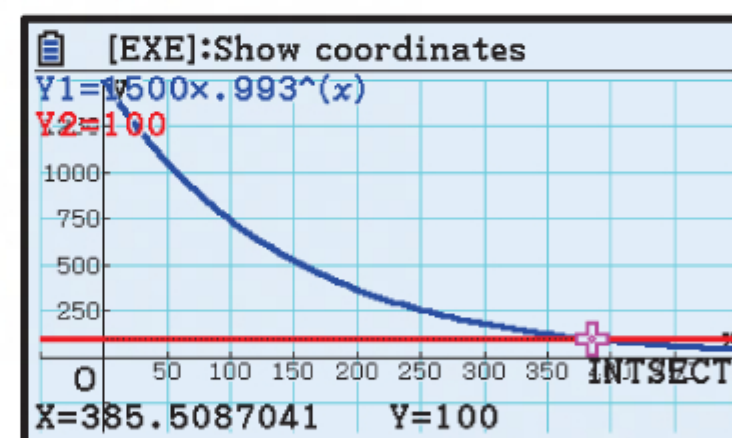
The weight remaining after 400 years was about 90.3 grams.

- ii** When $t = 800$,
 $W = 1500 \times (0.993)^{800}$
 ≈ 5.44

The weight remaining after 800 years was about 5.44 grams.



- d** When $W = 100$, $100 = 1500 \times (0.993)^t$.
Using technology, $t \approx 386$.



It takes about 386 years for the weight to reduce to 100 grams.

- 11** $P = k \times a^t$ birds, $t \geq 0$

- a** The population increases by 15% each year, so the multiplier $a = 1.15$.

b $P = k \times (1.15)^t$

When $t = 2$, $P = 1058$

$\therefore k \times (1.15)^2 = 1058$

$\therefore k = \frac{1058}{(1.15)^2}$

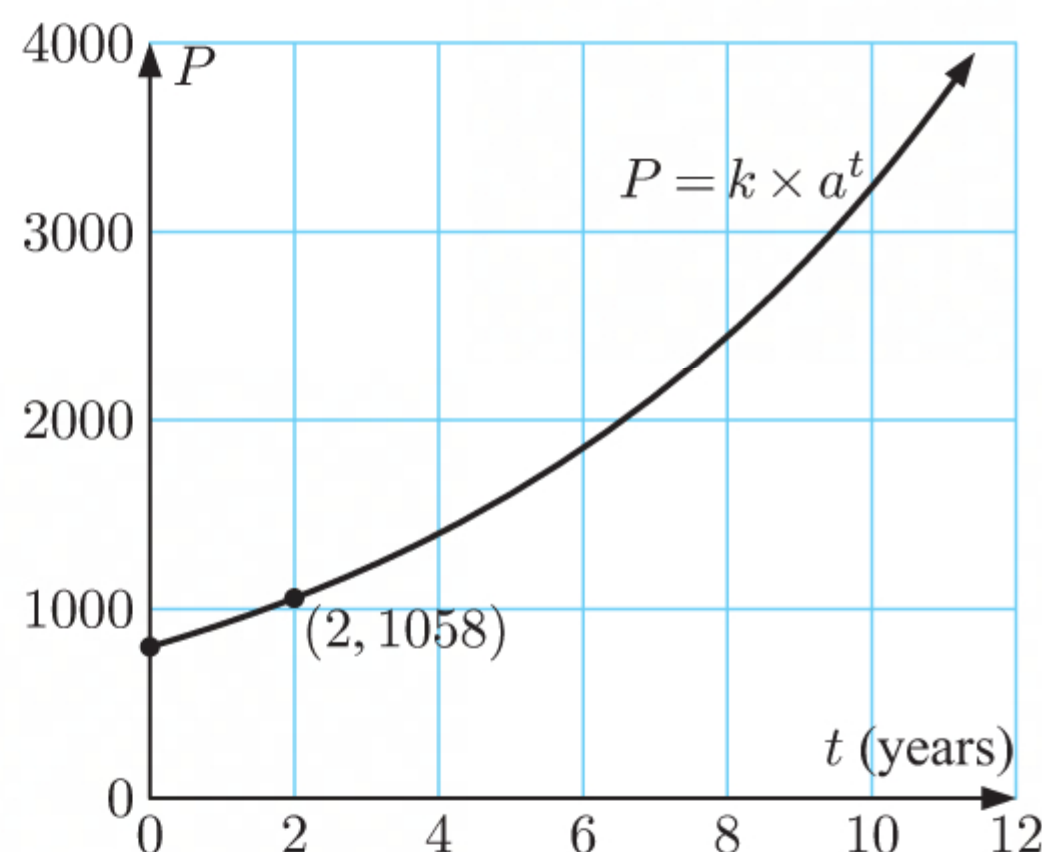
$\therefore k = 800$

The initial population was 800 birds.

- c** When $t = 5$, $P = 800 \times (1.15)^5$
 ≈ 1610

The population after 5 years was about 1610 birds.

- d** No, this model is unlikely to be accurate for very large values of t as the population of birds would be unrealistically large. The birds would eventually reach a limiting population.



- 12** Use the exponential function $y = ka^x + c$.

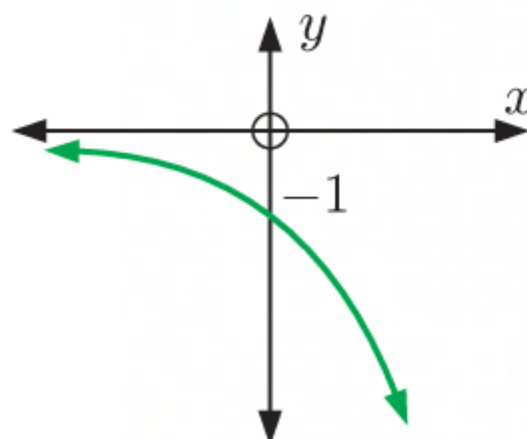
a $y = -e^x$

$k = -1 \therefore k < 0$
 $a = e \therefore a > 1$ } function is below horizontal asymptote and is decreasing

When $x = 0$, $y = -e^0 = -1$

\therefore the y -intercept is -1 .

\therefore the graph is **C**.



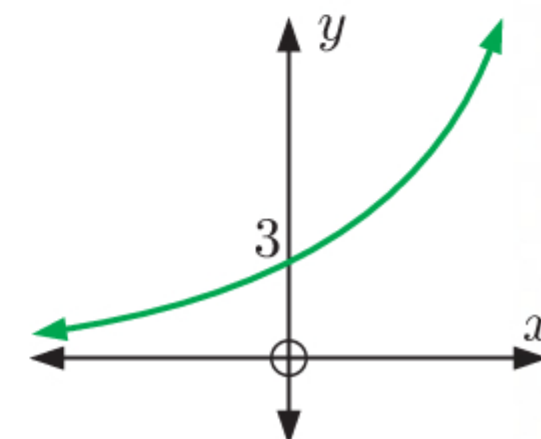
b $y = 3 \times 2^x$

$k = 3 \therefore k > 0$
 $a = 2 \therefore a > 1$ } function is above horizontal asymptote and is increasing

When $x = 0$, $y = 3 \times 2^0 = 3$

\therefore the y -intercept is 3 .

\therefore the graph is **E**.



c $y = e^x + 1$

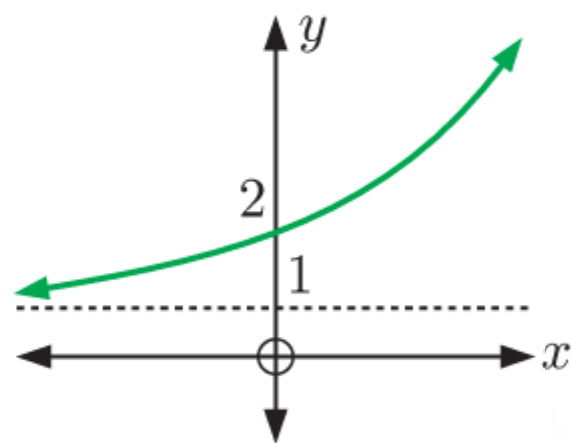
$k = 1 \quad \therefore k > 0$
 $a = e \quad \therefore a > 1$ } function is above
 horizontal asymptote
 and is increasing

When $x = 0$, $y = e^0 + 1 = 2$

\therefore the y -intercept is 2.

$c = 1$, so $y = 1$ is a horizontal asymptote.

\therefore the graph is **A**.



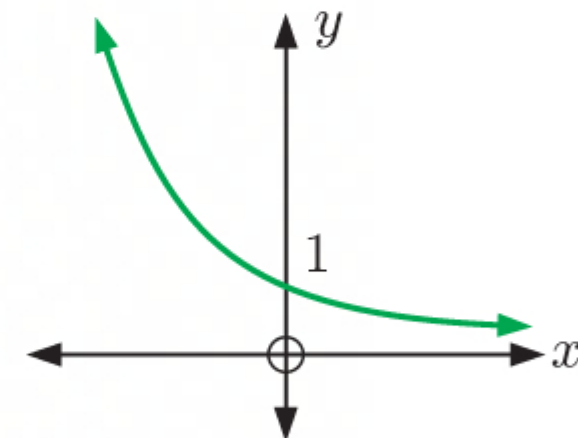
d $y = 3^{-x} = \frac{1}{3^x} = \left(\frac{1}{3}\right)^x$

$k = 1 \quad \therefore k > 0$
 $a = \frac{1}{3} \quad \therefore 0 < a < 1$ } function is above
 horizontal
 asymptote and
 is decreasing

When $x = 0$, $y = 3^0 = 1$

\therefore the y -intercept is 1.

\therefore the graph is **B**.



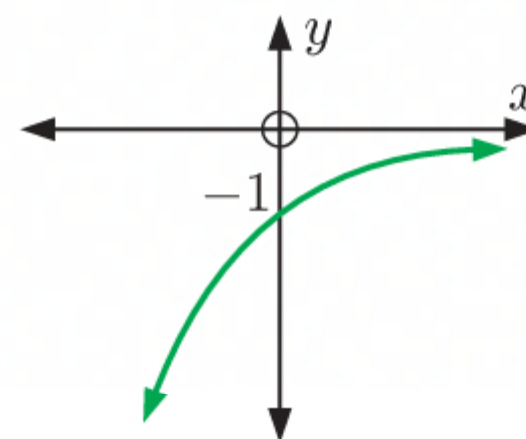
e $y = -e^{-x} = -\frac{1}{e^x} = -\left(\frac{1}{e}\right)^x$

$k = -1 \quad \therefore k < 0$
 $a = \frac{1}{e} \quad \therefore 0 < a < 1$ } function is below
 horizontal asymptote
 and is increasing

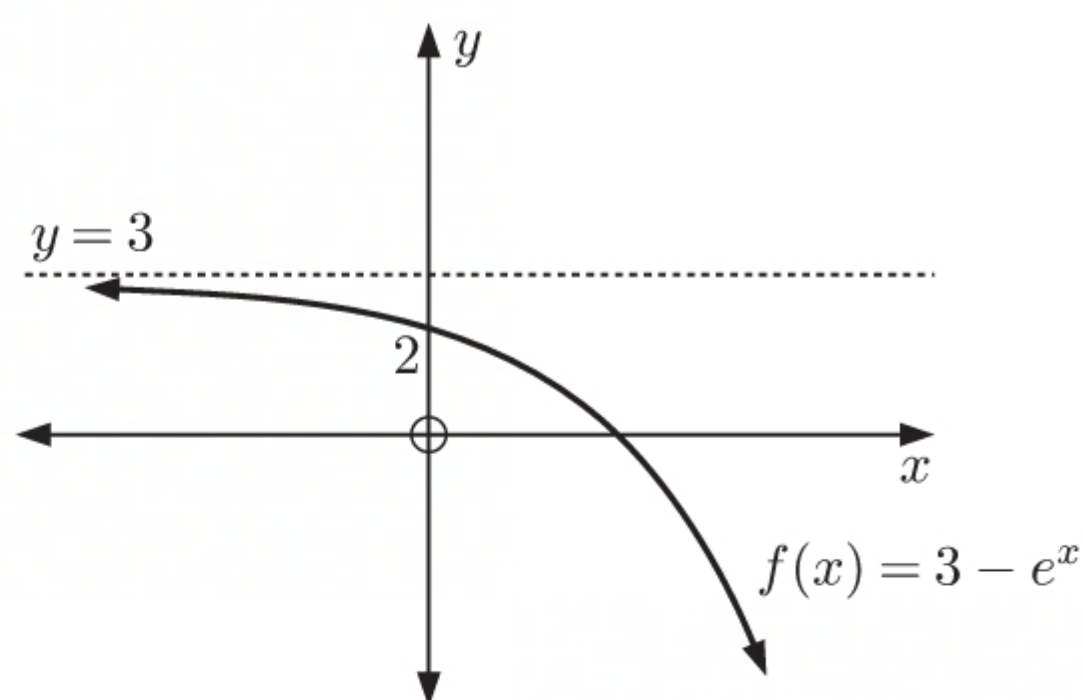
When $x = 0$, $y = -e^0 = -1$

\therefore the y -intercept is -1 .

\therefore the graph is **D**.



13 a



b The domain is $\{x \mid x \in \mathbb{R}\}$.

The range is $\{y \mid y < 3\}$.

c As $x \rightarrow \infty$, $f(x) \rightarrow -\infty$.

As $x \rightarrow -\infty$, $f(x) \rightarrow 3^-$.

14 a $V = k \times e^{-0.2t} + c$

The horizontal asymptote is $V = 50$, so $c = 50$.

$\therefore V = k \times e^{-0.2t} + 50$

Substituting $(0, 0)$ into the equation gives

$0 = k \times e^0 + 50$

$\therefore k + 50 = 0$

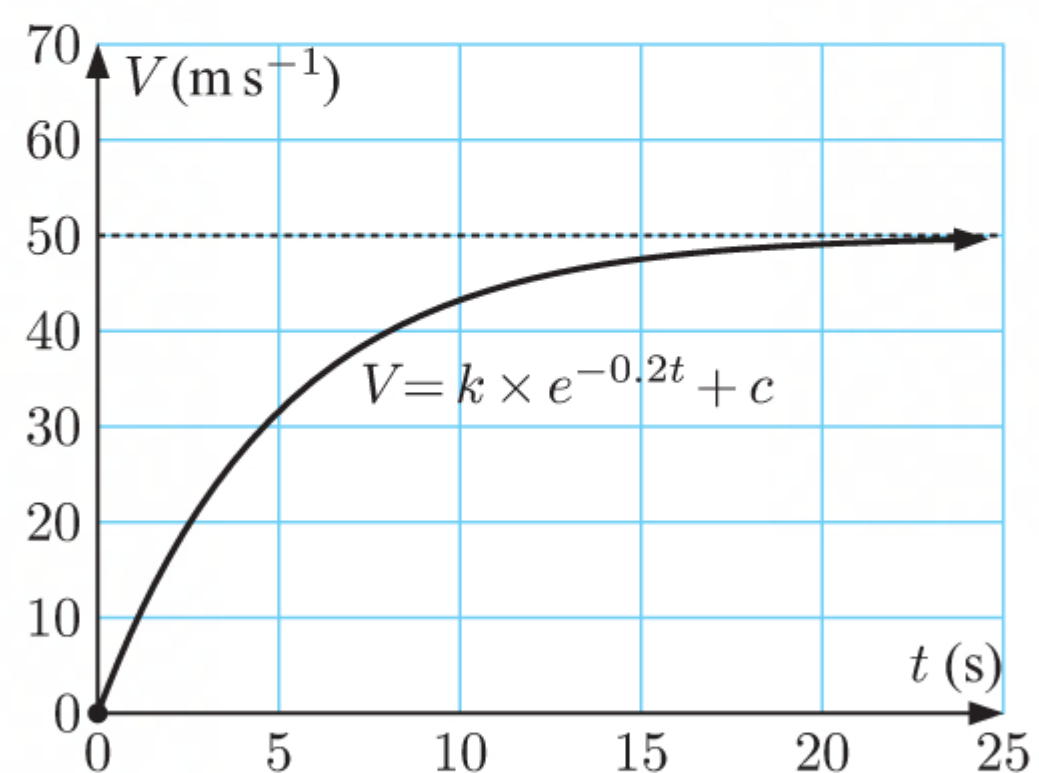
$\therefore k = -50$

b $V = -50e^{-0.2t} + 50$

When $t = 4$, $V = -50e^{-0.2(4)} + 50$

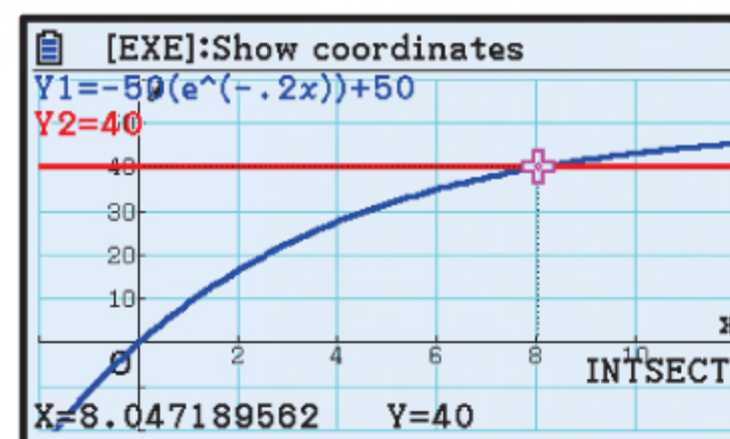
$= -50e^{-0.8} + 50$

≈ 27.5



After 4 seconds, the speed of the skydiver was about 27.5 m s^{-1} .

- c** When $V = 40$, $40 = -50e^{-0.2t} + 50$.
Using technology, $t \approx 8.05$.



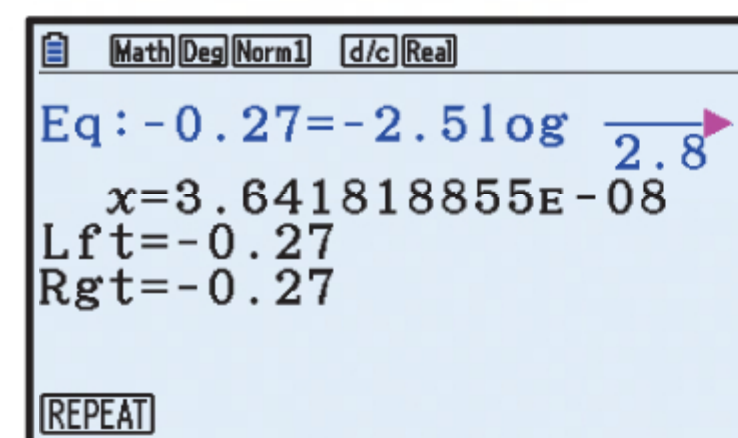
It will take about 8.05 seconds for the skydiver's speed to reach 40 m s^{-1} .

- 15** **a** $\log 10\,000 = \log(10^4)$
 $= 4$
- b** $\ln\left(\frac{1}{e^5}\right) = \ln(e^{-5})$
 $= -5$
- c** $\log\left(\frac{10^c}{1000}\right) = \log\left(\frac{10^c}{10^3}\right)$
 $= \log(10^{c-3})$
 $= c - 3$
- d** $e^{-\ln 4} = e^{\ln(4^{-1})}$
 $= 4^{-1}$
 $= \frac{1}{4}$
- 16** **a** $\log 125 \approx 2.097$
- b** $\log(0.03) \approx -1.523$
- c** $\ln 19 \approx 2.944$
- d** $\ln\left(\frac{2}{3}\right) \approx -0.4055$
- 17** $M = -2.5 \log\left(\frac{b}{2.84 \times 10^{-8}}\right)$

- a** When $b = 1.4 \times 10^3$, $M = -2.5 \log\left(\frac{1.4 \times 10^3}{2.84 \times 10^{-8}}\right)$
 ≈ -26.7

The apparent magnitude of the Sun is about -26.7 .

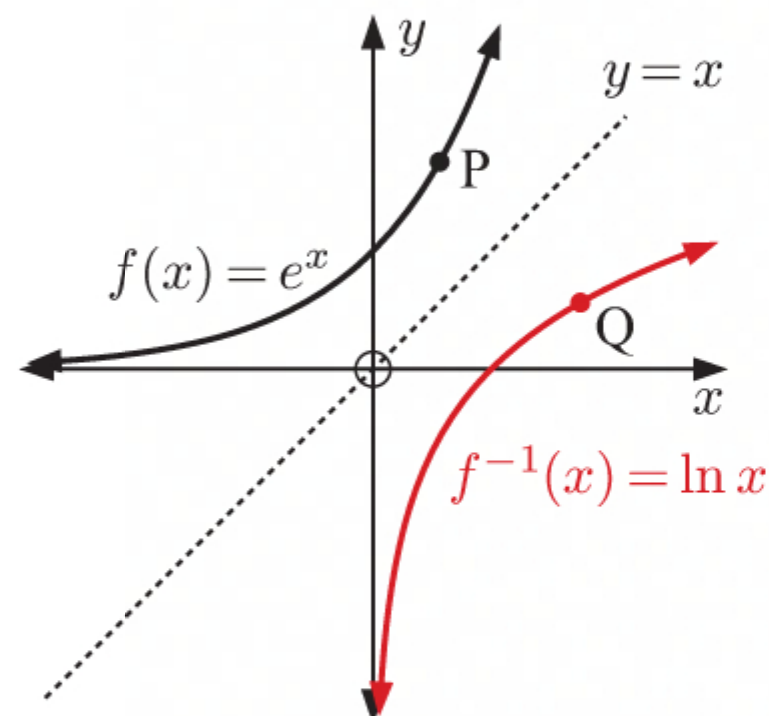
- b** When $M = -0.27$, $-0.27 = -2.5 \log\left(\frac{b}{2.84 \times 10^{-8}}\right)$
Using technology, $b \approx 3.64 \times 10^{-8}$.



The brightness of Alpha Centauri is about 3.64×10^{-8} Watts per m^2 .

- 18** **a** $20 = e^{\ln 20}$
 $\approx e^{2.9957}$
- b** $3000 = e^{\ln 3000}$
 $\approx e^{8.0064}$
- c** $0.075 = e^{\ln(0.075)}$
 $\approx e^{-2.5903}$

- 19** **a** $f(x) = e^x$
 $\therefore f(\ln 2) = e^{\ln 2}$
 $= 2$
 \therefore the y -coordinate of P is 2.
- b** $f(x) = e^x$ and $f^{-1}(x) = \ln x$ are inverse functions.
 \therefore the point corresponding to $P(\ln 2, 2)$ on the inverse function is $Q(2, \ln 2)$.



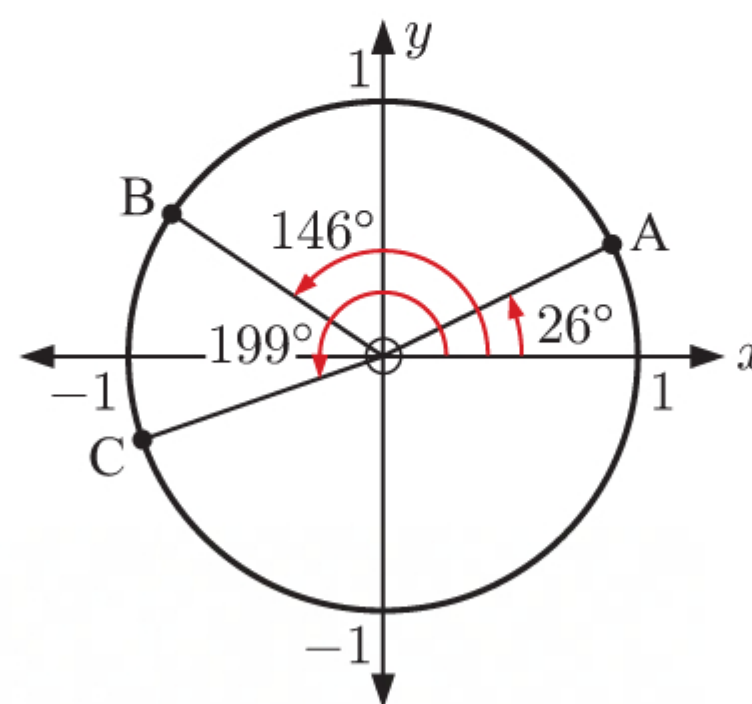
Chapter 9

TRIGONOMETRIC FUNCTIONS

EXERCISE 9A

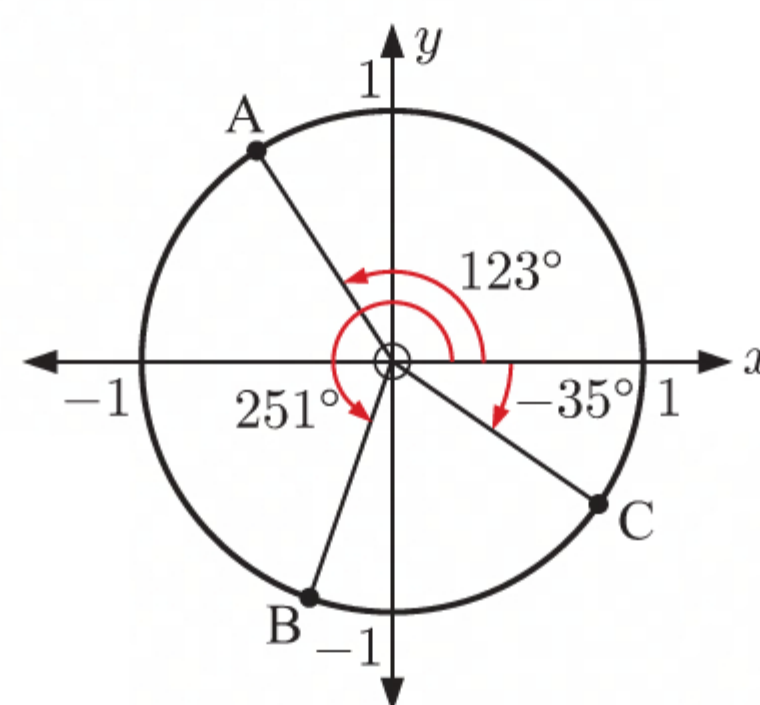
1 a i $A(\cos 26^\circ, \sin 26^\circ)$, $B(\cos 146^\circ, \sin 146^\circ)$,
 $C(\cos 199^\circ, \sin 199^\circ)$

ii $A(0.899, 0.438)$, $B(-0.829, 0.559)$,
 $C(-0.946, -0.326)$



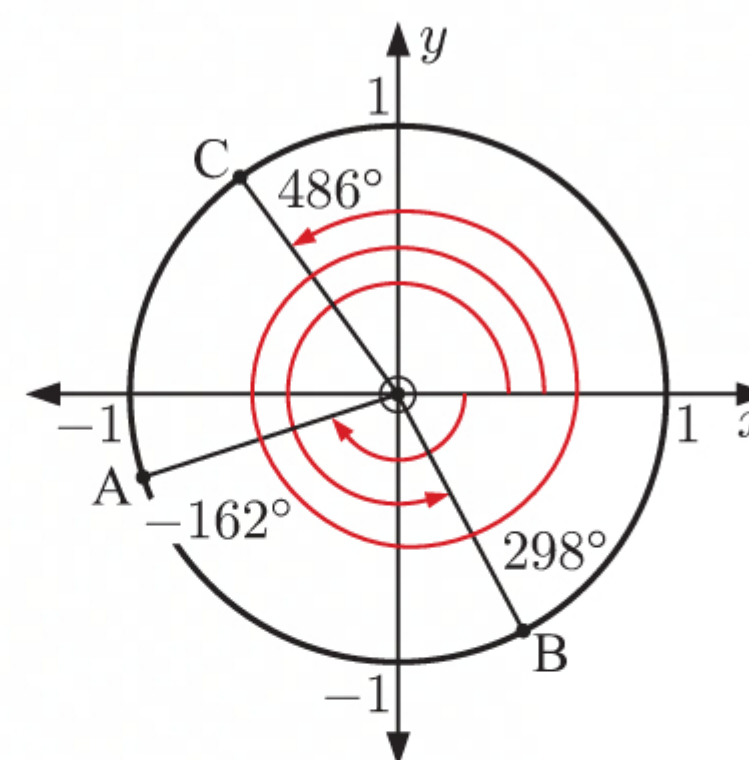
b i $A(\cos 123^\circ, \sin 123^\circ)$, $B(\cos 251^\circ, \sin 251^\circ)$,
 $C(\cos(-35^\circ), \sin(-35^\circ))$

ii $A(-0.545, 0.839)$, $B(-0.326, -0.946)$,
 $C(0.819, -0.574)$



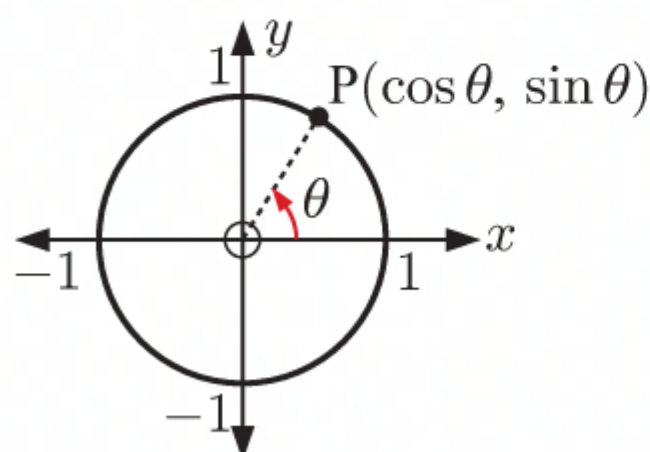
c i $A(\cos(-162^\circ), \sin(-162^\circ))$, $B(\cos 298^\circ, \sin 298^\circ)$,
 $C(\cos 486^\circ, \sin 486^\circ)$

ii $A(-0.951, -0.309)$, $B(0.469, -0.883)$,
 $C(-0.588, 0.809)$



2 Since any point P on the unit circle has coordinates $(\cos \theta, \sin \theta)$, then:

θ	0°	90°	180°	270°	360°
$\sin \theta$	0	1	0	-1	0
$\cos \theta$	1	0	-1	0	1



3 a i $\frac{1}{\sqrt{2}} \approx 0.707$ ii $\frac{\sqrt{3}}{2} \approx 0.866$

b

θ (degrees)	30°	45°	60°	135°	150°	240°	315°
sine	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$
cosine	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{1}{\sqrt{2}}$

4 a

Quadrant	Degree measure	$\cos \theta$	$\sin \theta$
1	$0^\circ < \theta < 90^\circ$	positive	positive
2	$90^\circ < \theta < 180^\circ$	negative	positive
3	$180^\circ < \theta < 270^\circ$	negative	negative
4	$270^\circ < \theta < 360^\circ$	positive	negative

- b
- i $\cos \theta$ is positive in quadrants 1 and 4.
 - ii $\cos \theta$ is negative in quadrants 2 and 3.
 - iii $\cos \theta$ and $\sin \theta$ are both negative in quadrant 3.
 - iv $\cos \theta$ is negative and $\sin \theta$ is positive in quadrant 2.

5 a $\sin \theta$ can take any value in the range $-1 \leq \sin \theta \leq 1$.

b $0^\circ \leq \theta \leq 360^\circ$

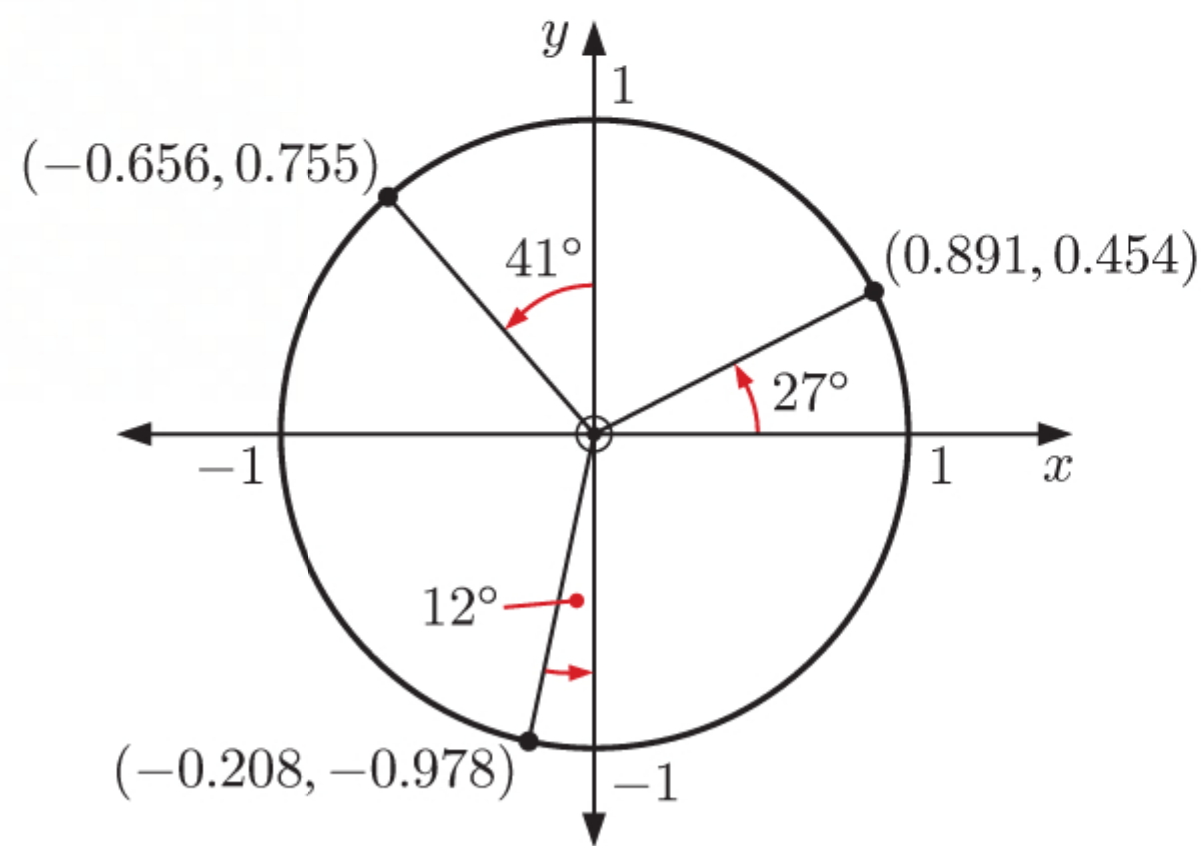
- i The maximum value of $\sin \theta$ is 1. This occurs when $\theta = 90^\circ$.
 $\therefore \sin \theta$ is maximised when $\theta = 90^\circ$.
- ii The minimum value of $\sin \theta$ is -1 . This occurs when $\theta = 270^\circ$.
 $\therefore \sin \theta$ is minimised when $\theta = 270^\circ$.
- iii $\sin \theta$ is zero when $\theta = 0^\circ, 180^\circ$, or 360° .
- iv $\sin \theta$ is positive when $0^\circ < \theta < 180^\circ$.
- v $\sin \theta$ is negative when $180^\circ < \theta < 360^\circ$.

6 a $\cos 400^\circ$
 $= \cos(360^\circ + 40^\circ)$
 $= \cos 40^\circ$

b $\sin 130^\circ$
 $= \sin(360^\circ - 230^\circ)$
 $= \sin(-230^\circ)$

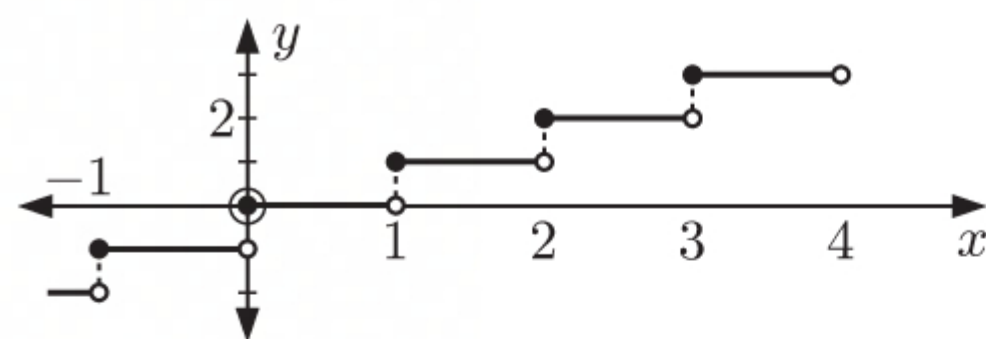
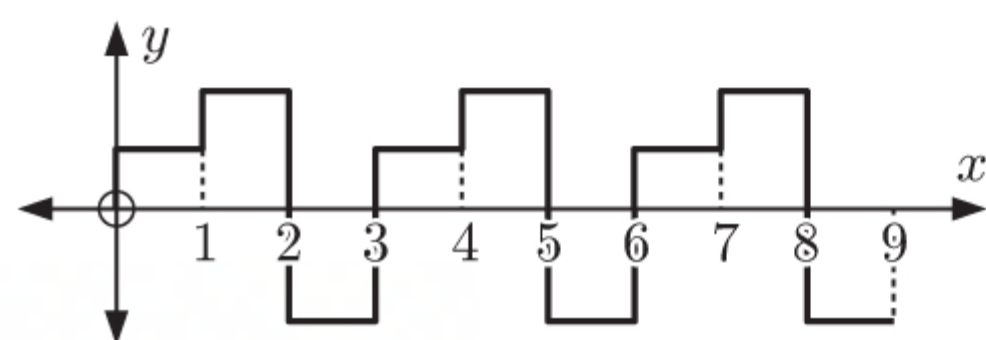
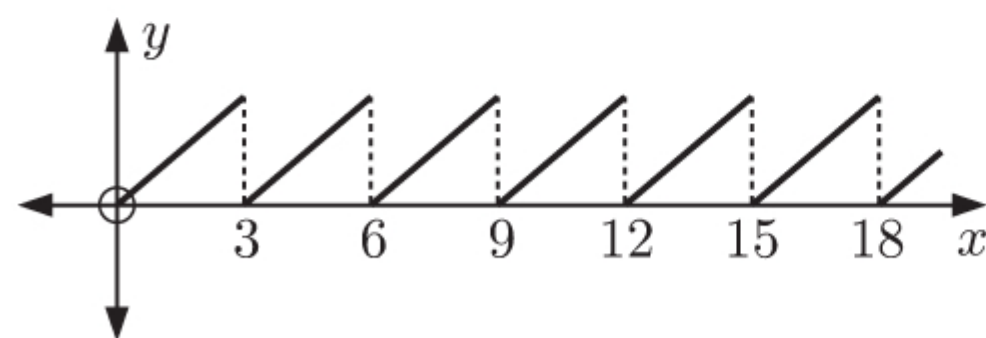
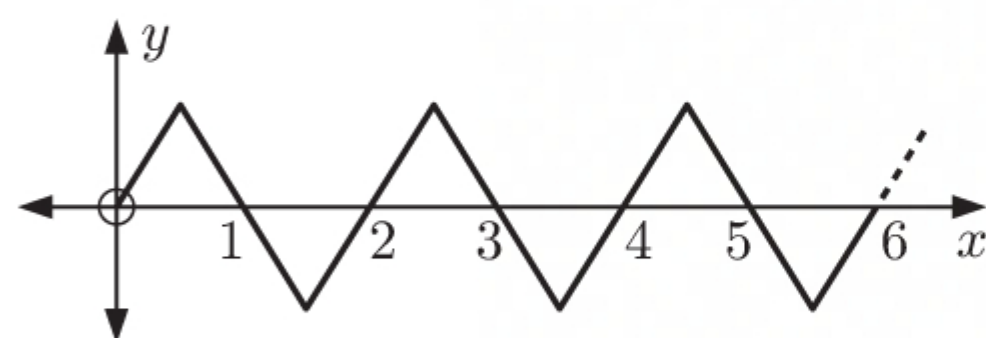
c $\cos 790^\circ$
 $= \cos(720^\circ + 70^\circ)$
 $= \cos 70^\circ$

- 7**
- a** $\cos 27^\circ \approx 0.891$
 - b** $\sin 27^\circ \approx 0.454$
 - c** $\sin 131^\circ = \sin(90^\circ + 41^\circ)$
 ≈ 0.755
 - d** $\cos 258^\circ = \cos(270^\circ - 12^\circ)$
 ≈ -0.208
 - e** $\cos 387^\circ = \cos(360^\circ + 27^\circ)$
 $= \cos 27^\circ$
 ≈ 0.891
 - f** $\cos 491^\circ = \cos(360^\circ + 131^\circ)$
 $= \cos 131^\circ$
 $= \cos(90^\circ + 41^\circ)$
 ≈ -0.656
 - g** $\sin(-102^\circ) = \sin(-90^\circ - 12^\circ)$
 ≈ -0.978
 - h** $\sin 747^\circ = \sin(720^\circ + 27^\circ)$
 $= \sin 27^\circ$
 ≈ 0.454

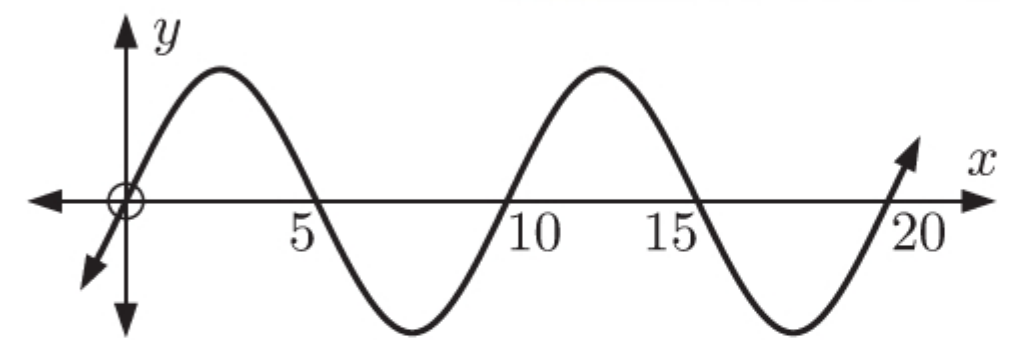


EXERCISE 9B

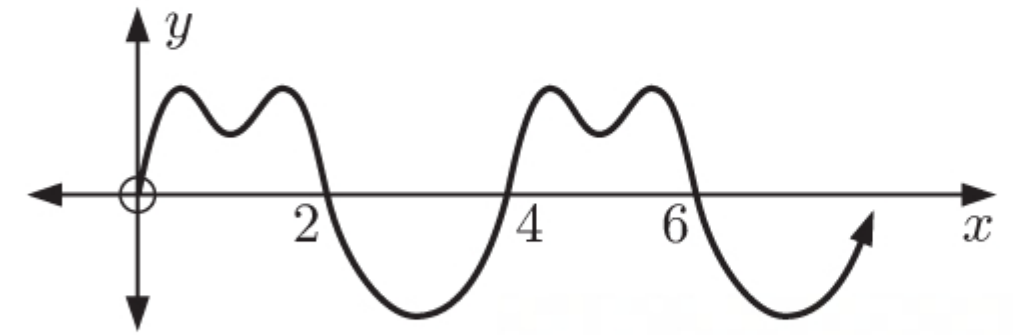
- 1**
- a** This graph repeats itself over and over in a horizontal direction, in intervals of the same length.
 \therefore this graph shows periodic behaviour.
 - b** This graph repeats itself over and over in a horizontal direction, in intervals of the same length.
 \therefore this graph shows periodic behaviour.
 - c** This graph repeats itself over and over in a horizontal direction, in intervals of the same length.
 \therefore this graph shows periodic behaviour.
 - d** This graph repeats itself over and over in intervals of the same length, but not in a horizontal direction.
 \therefore this graph does not show periodic behaviour.



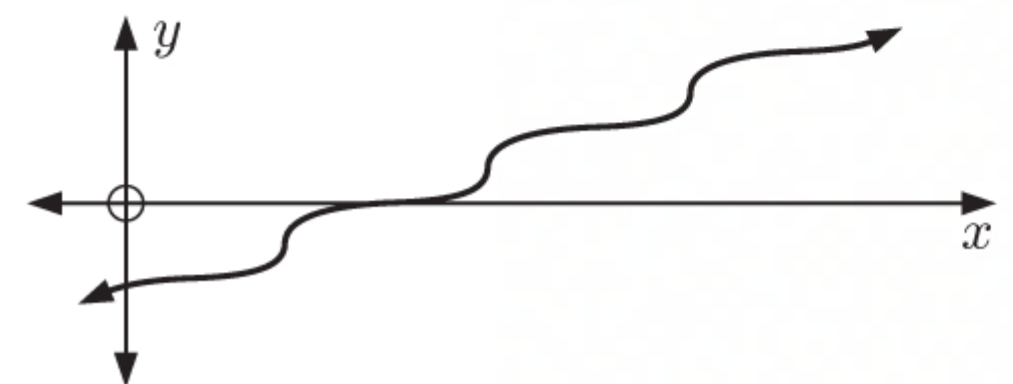
- e** This graph repeats itself over and over in a horizontal direction, in intervals of the same length.
 \therefore this graph shows periodic behaviour.



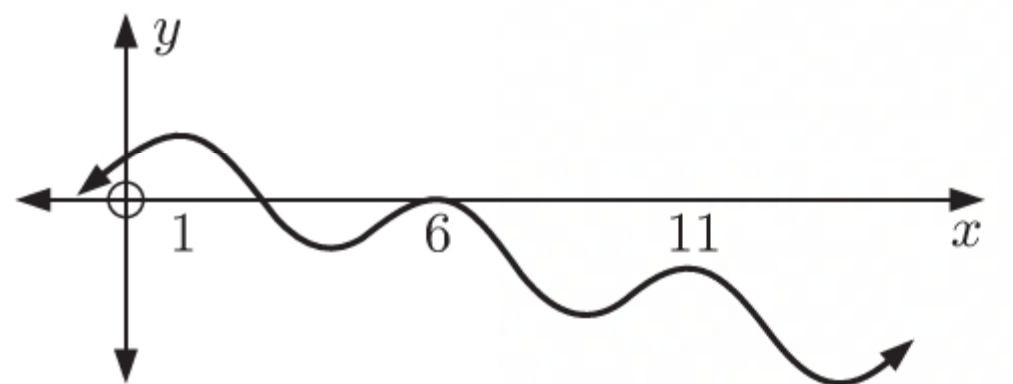
- f** This graph repeats itself over and over in a horizontal direction, in intervals of the same length.
 \therefore this graph shows periodic behaviour.



- g** This graph repeats itself over and over in intervals of the same length, but not in a horizontal direction.
 \therefore this graph does not show periodic behaviour.

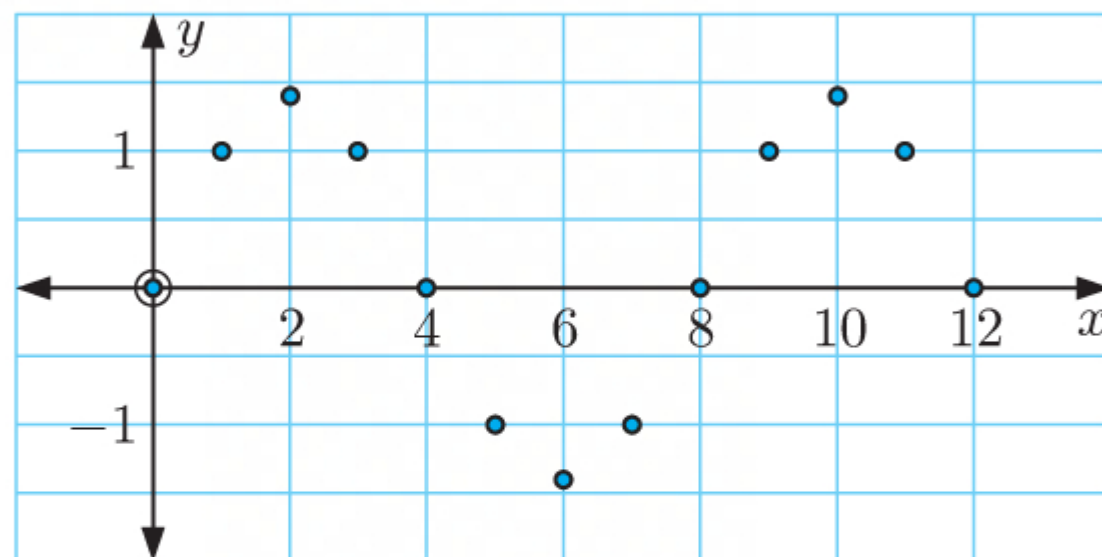


- h** This graph repeats itself over and over in intervals of the same length, but not in a horizontal direction.
 \therefore this graph does not show periodic behaviour.



2 a

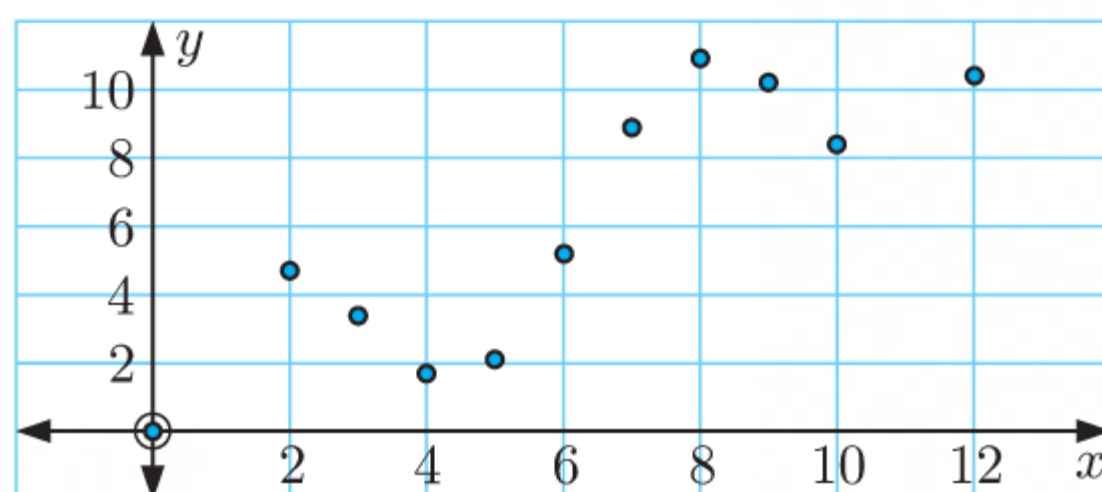
x	0	1	2	3	4	5	6	7	8	9	10	11	12
y	0	1	1.4	1	0	-1	-1.4	-1	0	1	1.4	1	0



This data exhibits periodic behaviour, as the graph repeats itself in intervals of the same length in a horizontal direction.

b

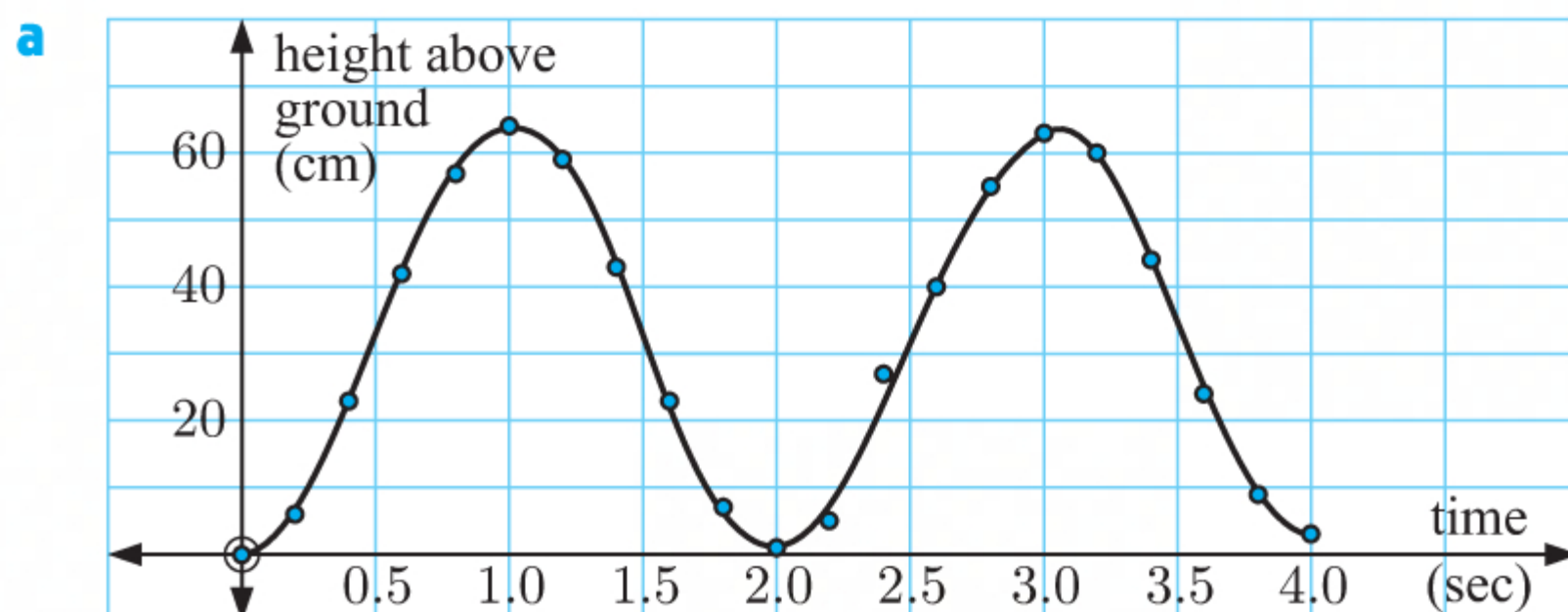
x	0	2	3	4	5	6	7	8	9	10	12
y	0	4.7	3.4	1.7	2.1	5.2	8.9	10.9	10.2	8.4	10.4



There is not enough information to say this data is periodic.

3	Time (seconds)	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
	Height above ground (cm)	0	6	23	42	57	64	59	43	23	7	1

Time (seconds)	2.2	2.4	2.6	2.8	3	3.2	3.4	3.6	3.8	4
Height above ground (cm)	5	27	40	55	63	60	44	24	9	3



- b** A curve can be fitted to the data, as time is continuous.
c The graph repeats itself in a horizontal direction, in intervals of the same length.
 \therefore this data shows periodic behaviour.

- i** The minimum data value is 0 cm and the maximum data value is 64 cm.

The principal axis is $y = \frac{\text{max} + \text{min}}{2}$

$$\approx \frac{64 + 0}{2}$$

$$\therefore y \approx 32$$

- ii** The maximum value is ≈ 64 cm.

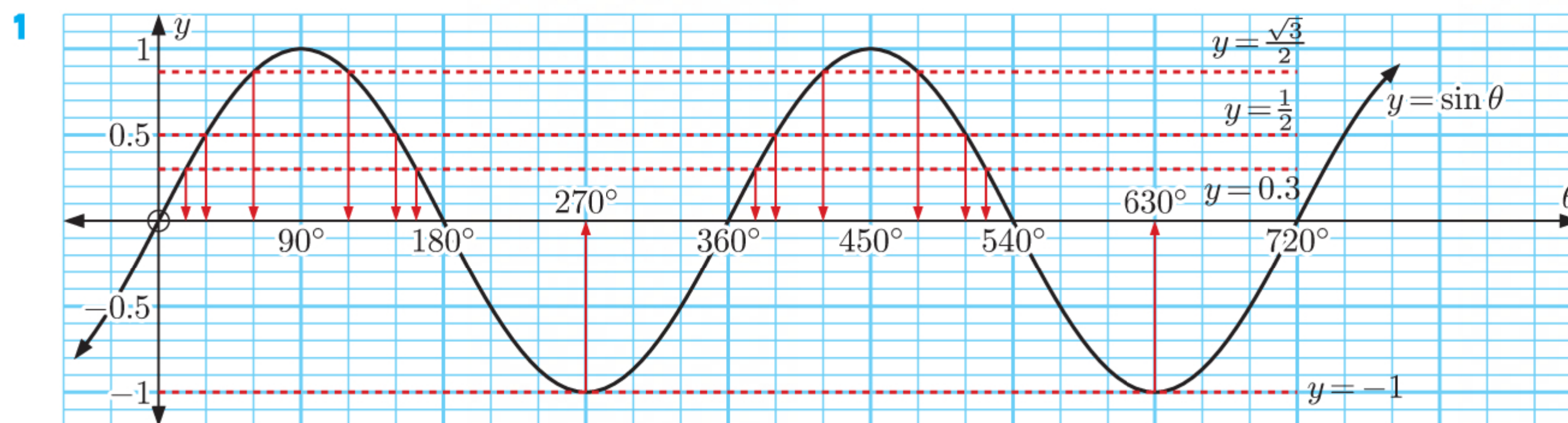
- iii** The period $\approx 3.0 - 1.0$
 ≈ 2 seconds

iv The amplitude $= \frac{\text{max} - \text{min}}{2}$

$$\approx \frac{64 - 0}{2}$$

$$\approx 32 \text{ cm}$$

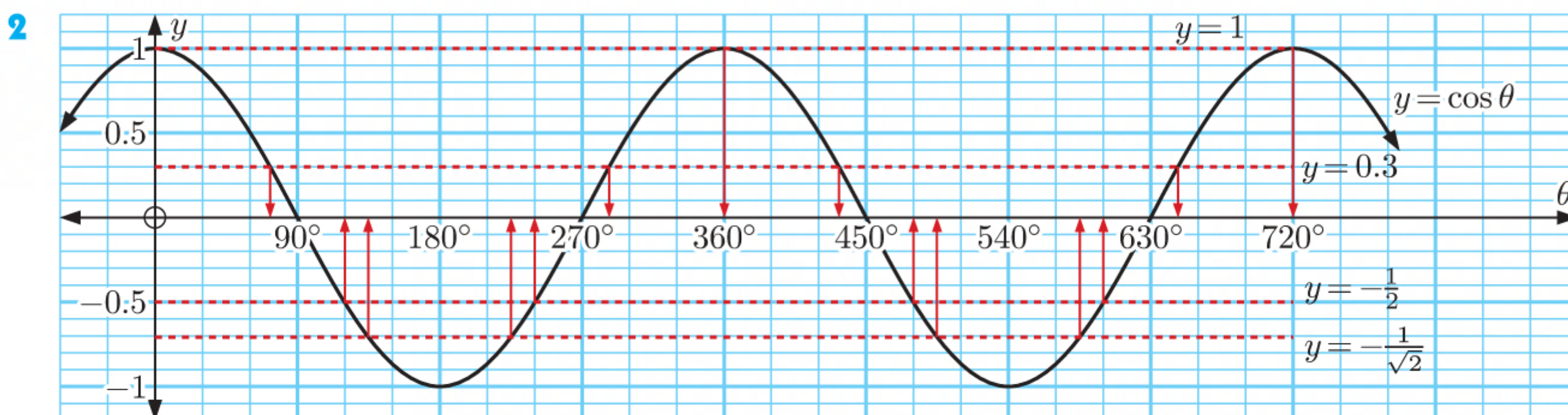
EXERCISE 9C



- a** The y -intercept is 0.

- b**
- i** When $\sin \theta = 0$, $0^\circ \leq \theta \leq 720^\circ$, $\theta = 0^\circ, 180^\circ, 360^\circ, 540^\circ$, or 720° .
 - ii** When $\sin \theta = -1$, $0^\circ \leq \theta \leq 720^\circ$, $\theta = 270^\circ$ or 630° .
 - iii** When $\sin \theta = \frac{1}{2}$, $0^\circ \leq \theta \leq 720^\circ$, $\theta = 30^\circ, 150^\circ, 390^\circ$, or 510° .

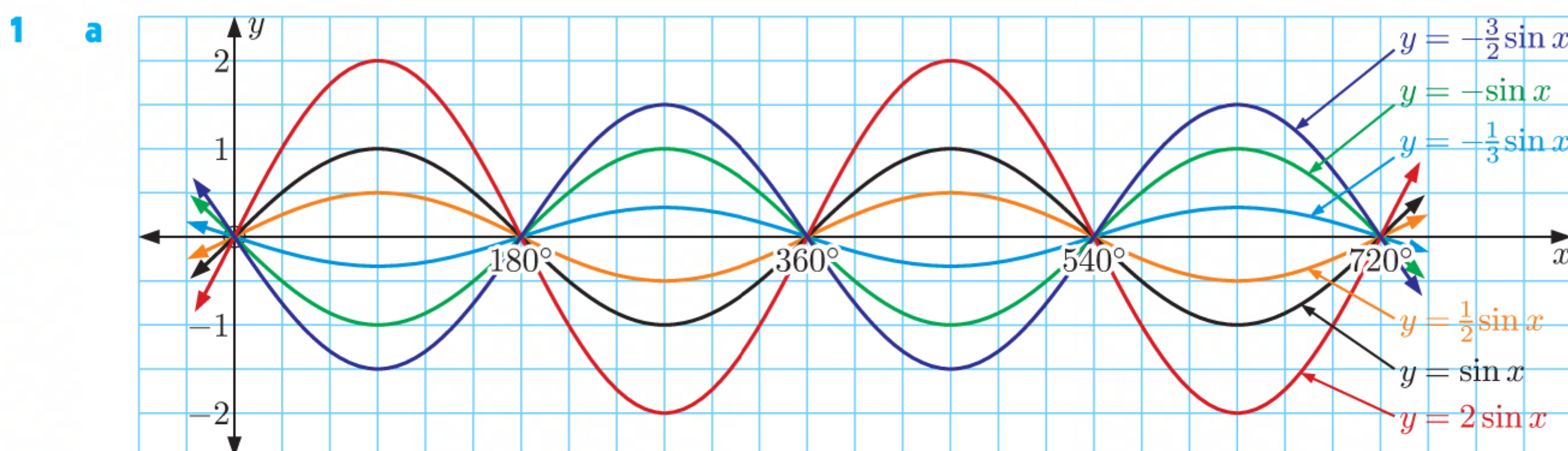
- iv When $\sin \theta = \frac{\sqrt{3}}{2}$, $0^\circ \leq \theta \leq 720^\circ$, $\theta = 60^\circ, 120^\circ, 420^\circ$, or 480° .
- c When $\sin \theta = 0.3$, $0^\circ \leq \theta \leq 720^\circ$, $\theta \approx 15^\circ, 165^\circ, 375^\circ$, or 525° .
- d i On $0^\circ \leq \theta \leq 720^\circ$, $\sin \theta$ is positive for $0^\circ < \theta < 180^\circ$, $360^\circ < \theta < 540^\circ$.
 ii On $0^\circ \leq \theta \leq 720^\circ$, $\sin \theta$ is negative for $180^\circ < \theta < 360^\circ$, $540^\circ < \theta < 720^\circ$.
- e The range is $\{y \mid -1 \leq y \leq 1\}$.



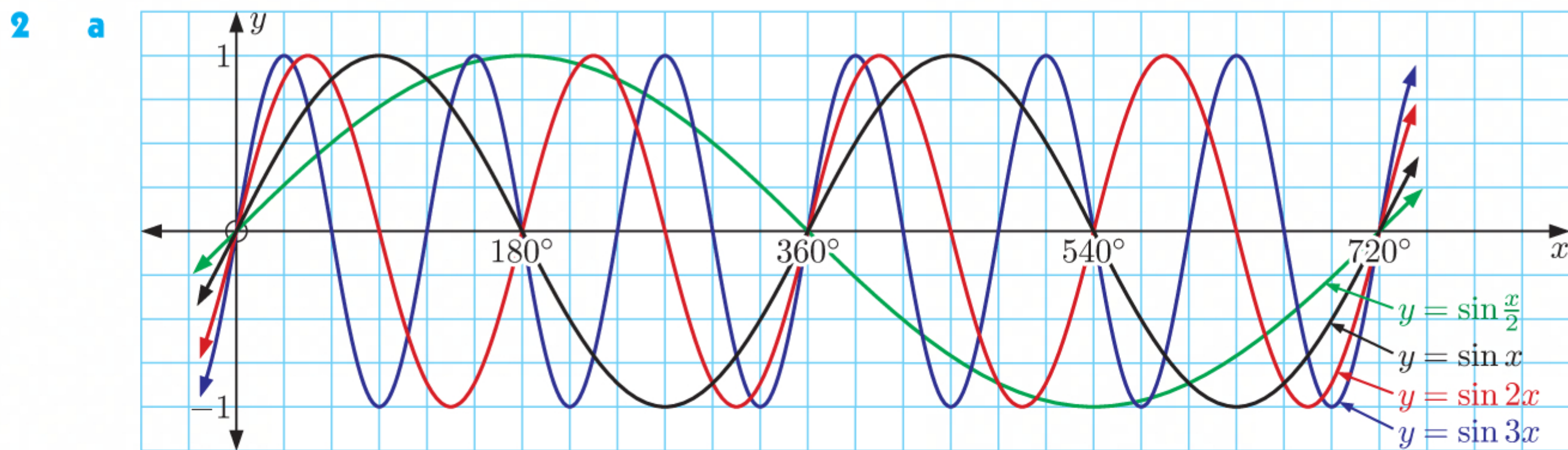
- a The y -intercept is 1.
- b i When $\cos \theta = 0$, $0^\circ \leq \theta \leq 720^\circ$, $\theta = 90^\circ, 270^\circ, 450^\circ$, or 630° .
 ii When $\cos \theta = 1$, $0^\circ \leq \theta \leq 720^\circ$, $\theta = 0^\circ, 360^\circ$, or 720° .
 iii When $\cos \theta = -\frac{1}{2}$, $0^\circ \leq \theta \leq 720^\circ$, $\theta = 120^\circ, 240^\circ, 480^\circ$, or 600° .
 iv When $\cos \theta = -\frac{1}{\sqrt{2}}$, $0^\circ \leq \theta \leq 720^\circ$, $\theta = 135^\circ, 225^\circ, 495^\circ$, or 585° .
- c When $\cos \theta = 0.3$, $0^\circ \leq \theta \leq 720^\circ$, $\theta \approx 75^\circ, 285^\circ, 435^\circ$, or 645° .
- d i On $0^\circ \leq \theta \leq 720^\circ$, $\cos \theta$ is positive for $0^\circ \leq \theta < 90^\circ$, $270^\circ < \theta < 450^\circ$, $630^\circ < \theta \leq 720^\circ$.
 ii On $0^\circ \leq \theta \leq 720^\circ$, $\cos \theta$ is negative for $90^\circ < \theta < 270^\circ$, $450^\circ < \theta < 630^\circ$.
- e The range is $\{y \mid -1 \leq y \leq 1\}$.

INVESTIGATION 1

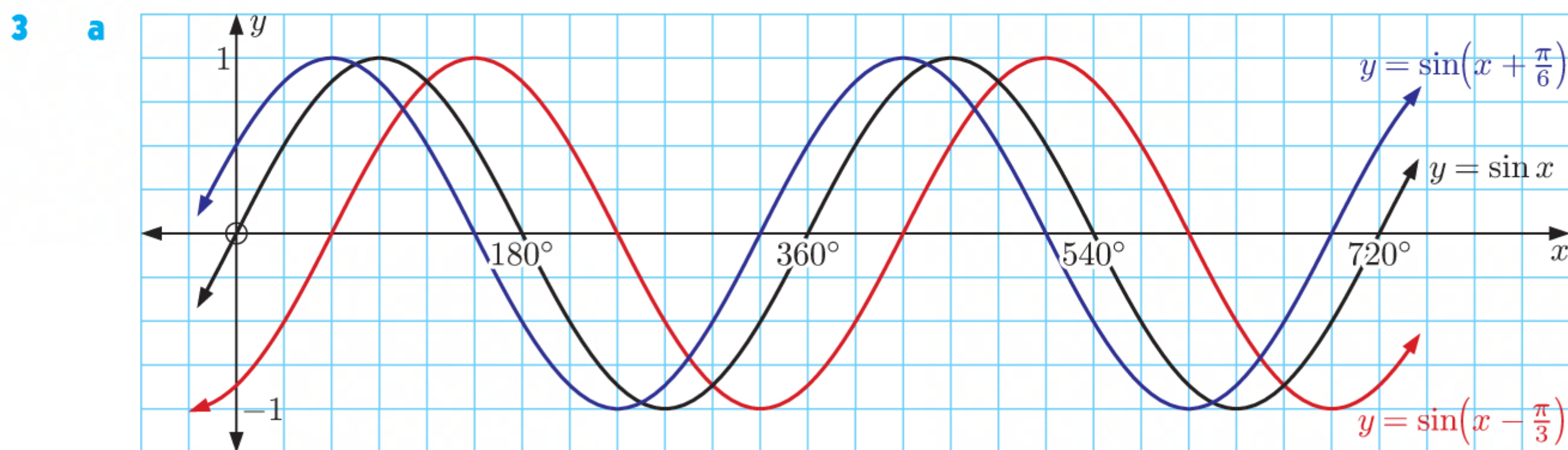
FAMILIES OF TRIGONOMETRIC FUNCTIONS



- b For graphs of the form $y = a \sin x$:
 - i the sign of a affects where the graph is positive or negative
 - ii the size of a affects the **amplitude** of the graph.



b For graphs of the form $y = \sin bx$, $b > 0$, the period is $\frac{360^\circ}{b}$.



b A vertical translation of d units moves $y = \sin x$ to $y = \sin x + d$.

c The principal axis of $y = \sin x + d$ is $y = d$.

4 $y = a \sin(bx) + d$ is obtained from $y = \sin x$ by a vertical stretch with scale factor $|a|$ and a horizontal stretch with scale factor $\frac{1}{b}$, a reflection in the x -axis if $a < 0$, and a vertical translation of d units.

EXERCISE 9D

- 1 a** The amplitude of $y = 4 \sin x$ is $|4| = 4$.
- b** The amplitude of $y = -2 \cos x + 1$ is $|-2| = 2$.
- c** The amplitude of $y = -\frac{1}{3} \sin 2x$ is $|\frac{1}{3}| = \frac{1}{3}$.
- 2 a** The period of $y = \cos 3x$ is $\frac{360^\circ}{3} = 120^\circ$.
- b** The period of $y = 5 \sin 4x + 2$ is $\frac{360^\circ}{4} = 90^\circ$.
- c** The period of $y = -\cos \frac{x}{2}$ is $\frac{360^\circ}{\frac{1}{2}} = 720^\circ$.
- 3 a** The principal axis of $y = \sin x - 3$ is $y = -3$.
- b** The principal axis of $y = -2 \cos x + 5$ is $y = 5$.
- c** The principal axis of $y = \frac{1}{4} \sin 15x$ is $y = 0$.

4 $y = \sin bx, \quad b > 0$

a period $= \frac{360^\circ}{b}$

$$\therefore 90^\circ = \frac{360^\circ}{b}$$

$$\therefore b = \frac{360^\circ}{90^\circ}$$

$$\therefore b = 4$$

b period $= \frac{360^\circ}{b}$

$$\therefore 24^\circ = \frac{360^\circ}{b}$$

$$\therefore b = \frac{360^\circ}{24^\circ}$$

$$\therefore b = 15$$

c period $= \frac{360^\circ}{b}$

$$\therefore 1440^\circ = \frac{360^\circ}{b}$$

$$\therefore b = \frac{360^\circ}{1440^\circ}$$

$$\therefore b = \frac{1}{4}$$

5 a $y = 4 \cos 2x$ has maximum value $4(1) = 4$ {when $\cos 2x = 1$ }
and minimum value $4(-1) = -4$ {when $\cos 2x = -1$ }

b $y = 3 \cos x + 5$ has maximum value $3(1) + 5 = 8$ {when $\cos x = 1$ }
and minimum value $3(-1) + 5 = 2$ {when $\cos x = -1$ }

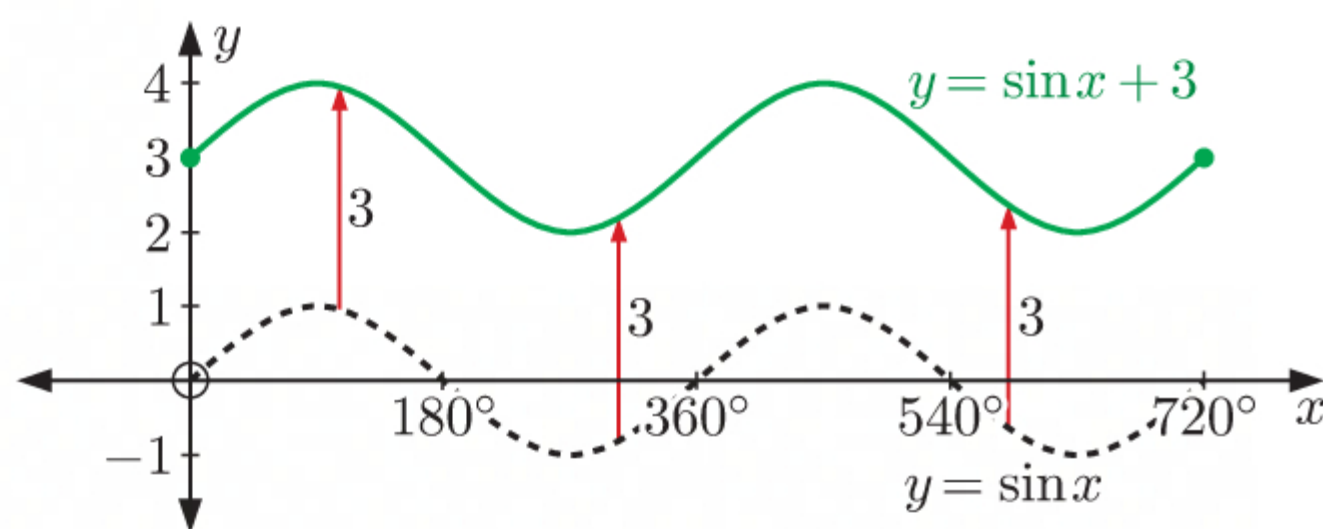
c $y = -2 \cos x - 4$ has maximum value $-2(-1) - 4 = -2$ {when $\cos x = -1$ }
and minimum value $-2(1) - 4 = -6$ {when $\cos x = 1$ }

6 a $y = \sin x - 1$ is a vertical translation of $y = \sin x$ downwards by 1 unit.
So, a vertical translation 1 unit downwards will map $y = \sin x$ onto $y = \sin x - 1$.

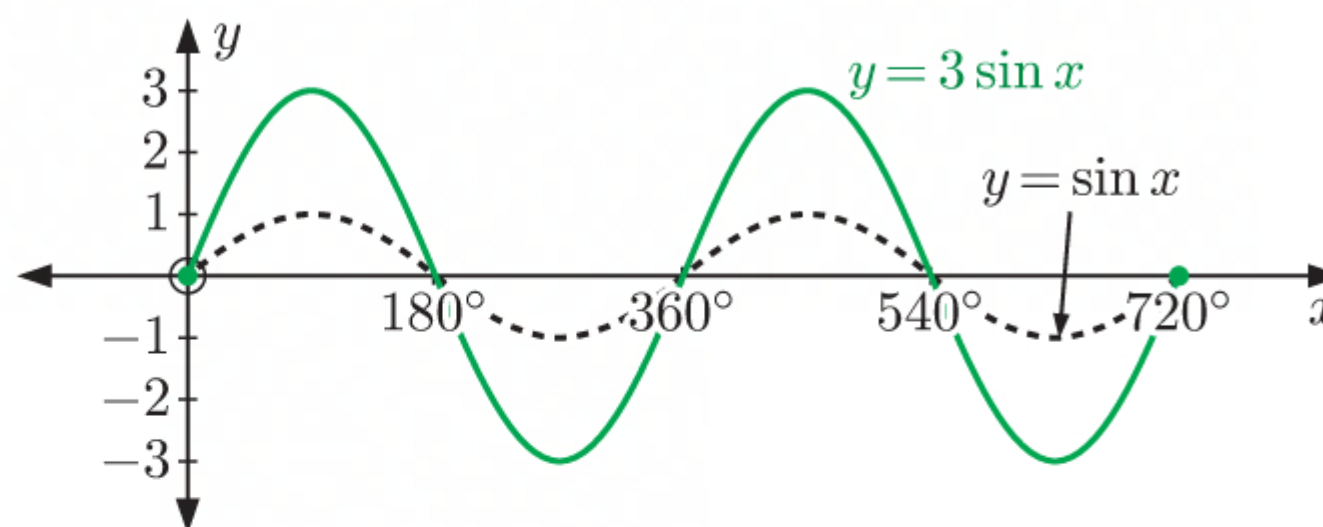
b $y = 2 \sin x$ is a vertical stretch of $y = \sin x$ with scale factor 2.
So, a vertical stretch with scale factor 2 will map $y = \sin x$ onto $y = 2 \sin x$.

c $y = \sin 4x$ is a horizontal stretch of $y = \sin x$ with scale factor $\frac{1}{4}$.
So, a horizontal stretch with scale factor $\frac{1}{4}$ will map $y = \sin x$ onto $y = \sin 4x$.

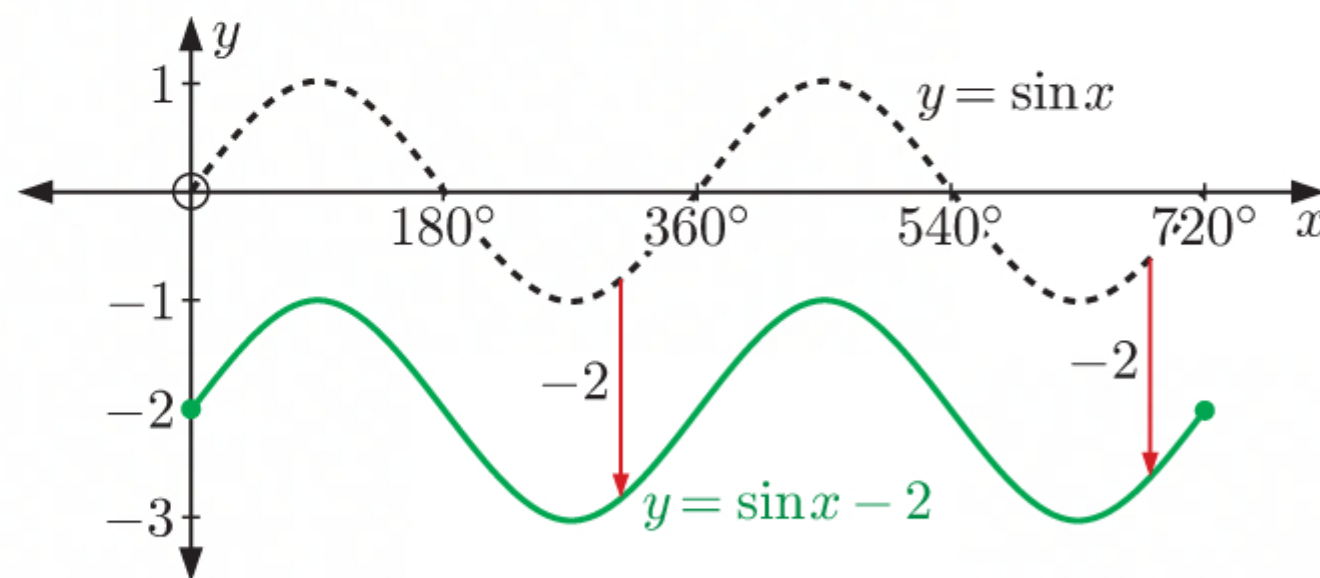
7 a $y = \sin x + 3$ is a vertical translation of $y = \sin x$ upwards by 3 units.



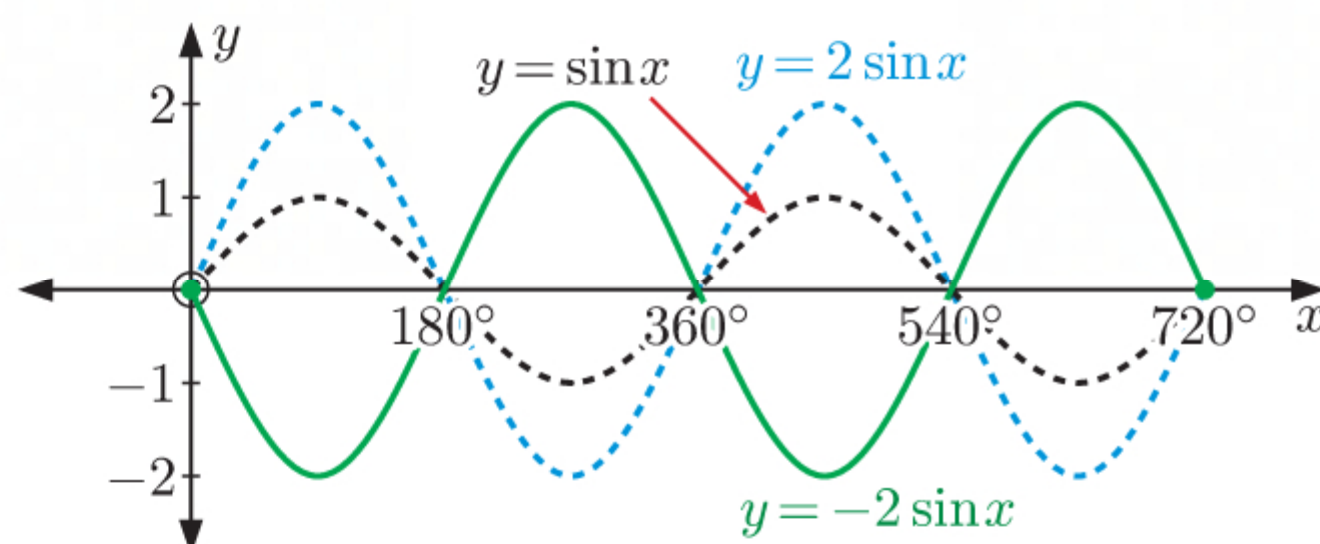
b $y = 3 \sin x$ is a vertical stretch of $y = \sin x$ with scale factor 3. The amplitude is 3.



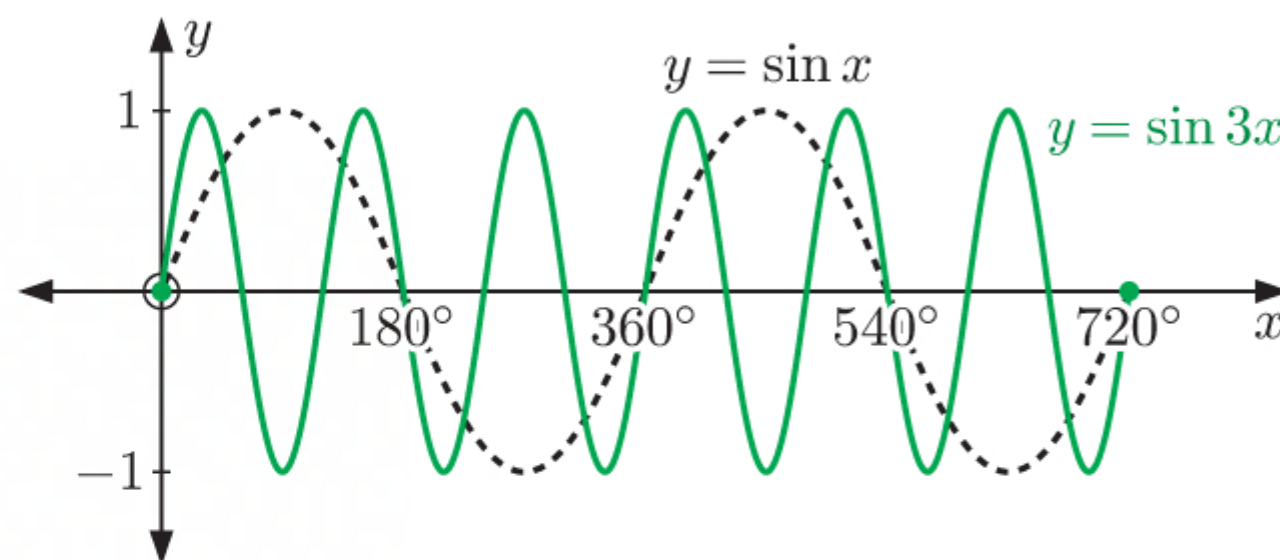
- c** $y = \sin x - 2$ is a vertical translation of $y = \sin x$ downwards by 2 units.



- d** $y = -2 \sin x$ is a vertical stretch of $y = \sin x$ with scale factor 2, followed by a reflection in the x -axis. The amplitude is 2.

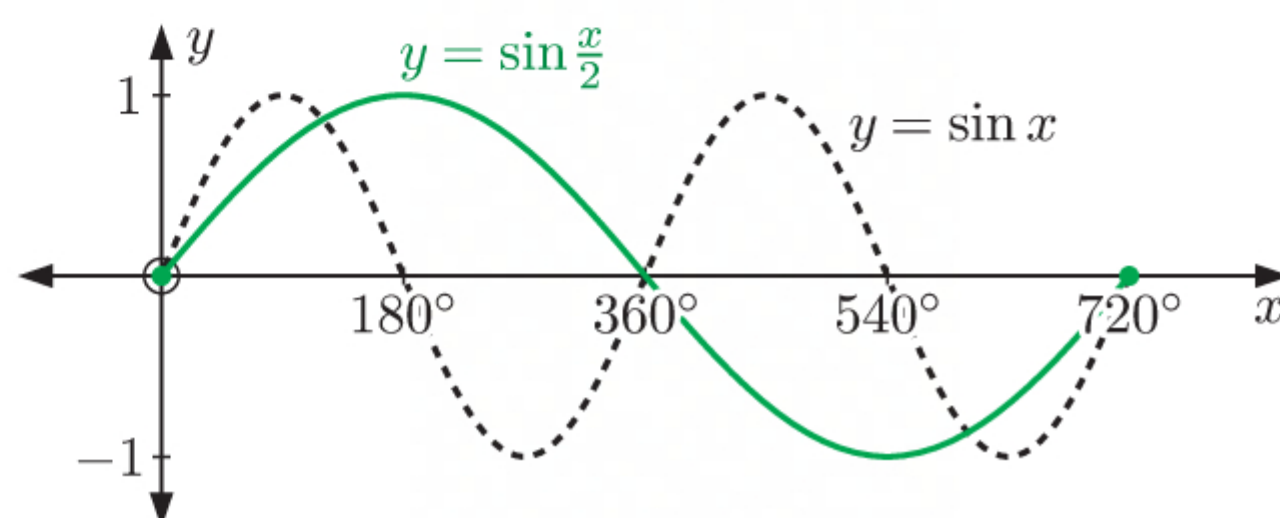


- e** $y = \sin 3x$ is a horizontal stretch of $y = \sin x$ with scale factor $\frac{1}{3}$.
The period is $\frac{360^\circ}{3} = 120^\circ$. \therefore the maximum values are 120° units apart.



- f** $y = \sin \frac{x}{2}$ is a horizontal stretch of $y = \sin x$ with scale factor $\frac{1}{\frac{1}{2}} = 2$.

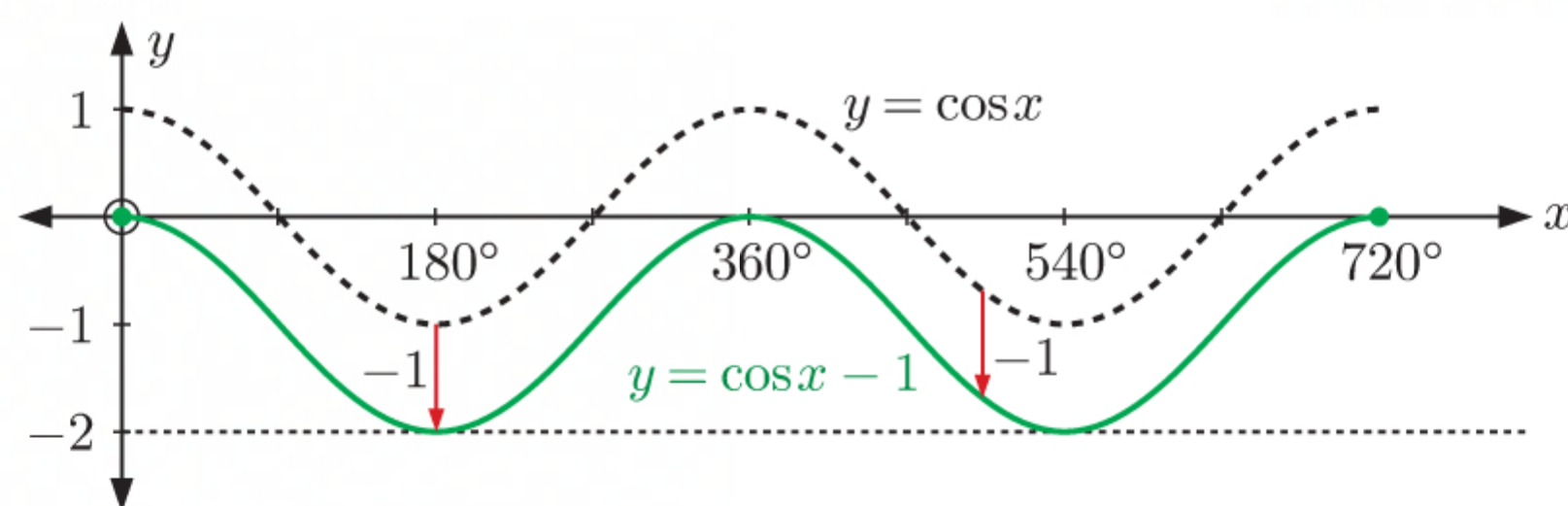
The period is $\frac{360^\circ}{\frac{1}{2}} = 720^\circ$. \therefore the maximum values are 720° units apart.



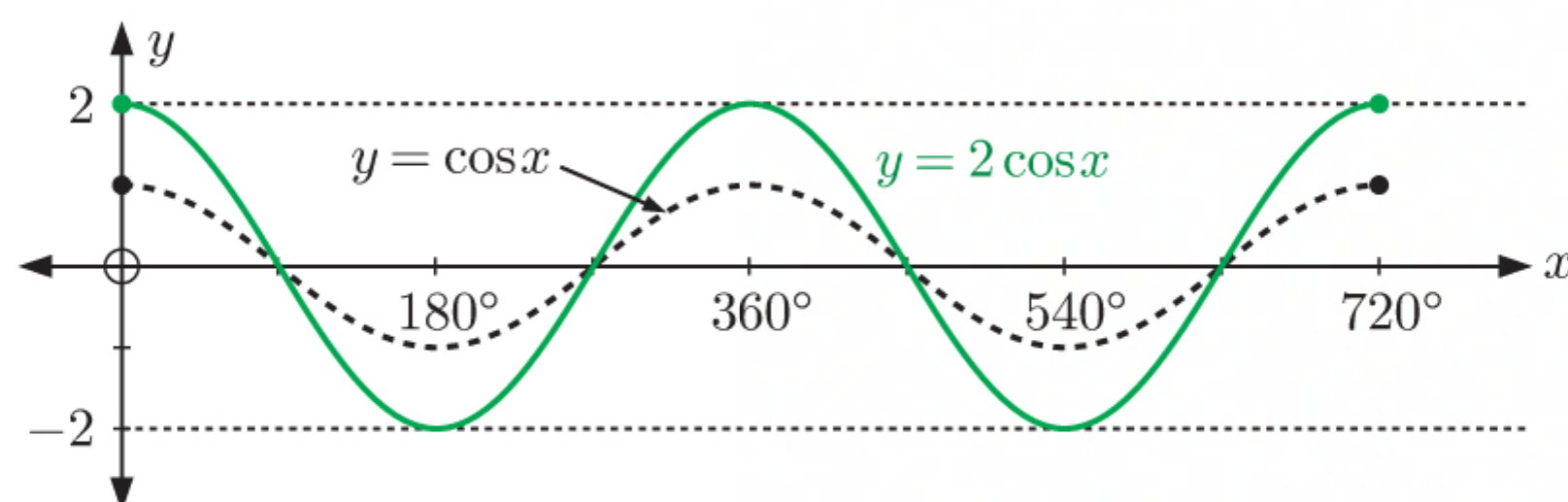
- 8 a** $y = \frac{1}{2} \cos x$ is a vertical stretch of $y = \cos x$ with scale factor $\frac{1}{2}$.
So, a vertical stretch with scale factor $\frac{1}{2}$ will map $y = \cos x$ onto $y = \frac{1}{2} \cos x$.
- b** $y = -\cos x$ is a reflection of $y = \cos x$ in the x -axis.
So, a reflection in the x -axis will map $y = \cos x$ onto $y = -\cos x$.

- c** $y = \cos x + 3$ is a vertical translation of $y = \cos x$ upwards by 3 units.
So, a vertical translation of 3 units upwards will map $y = \cos x$ onto $y = \cos x + 3$.

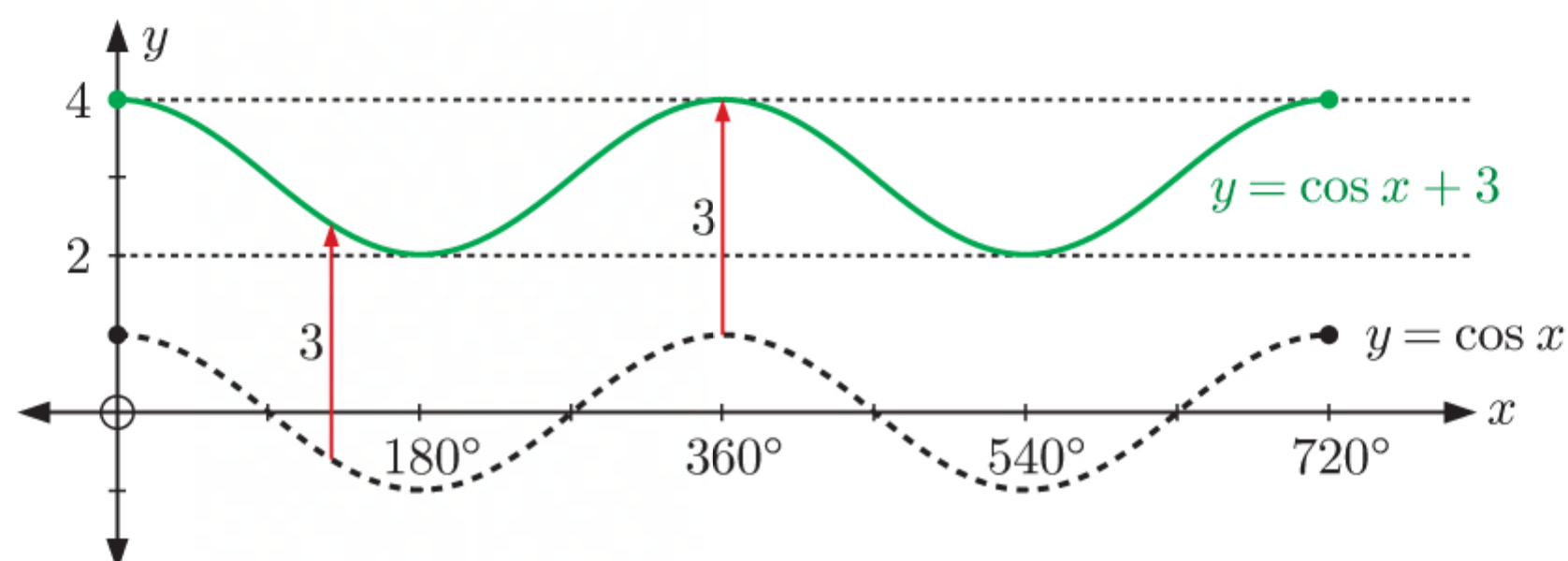
- 9 a** $y = \cos x - 1$ is a vertical translation of $y = \cos x$ downwards by 1 unit.



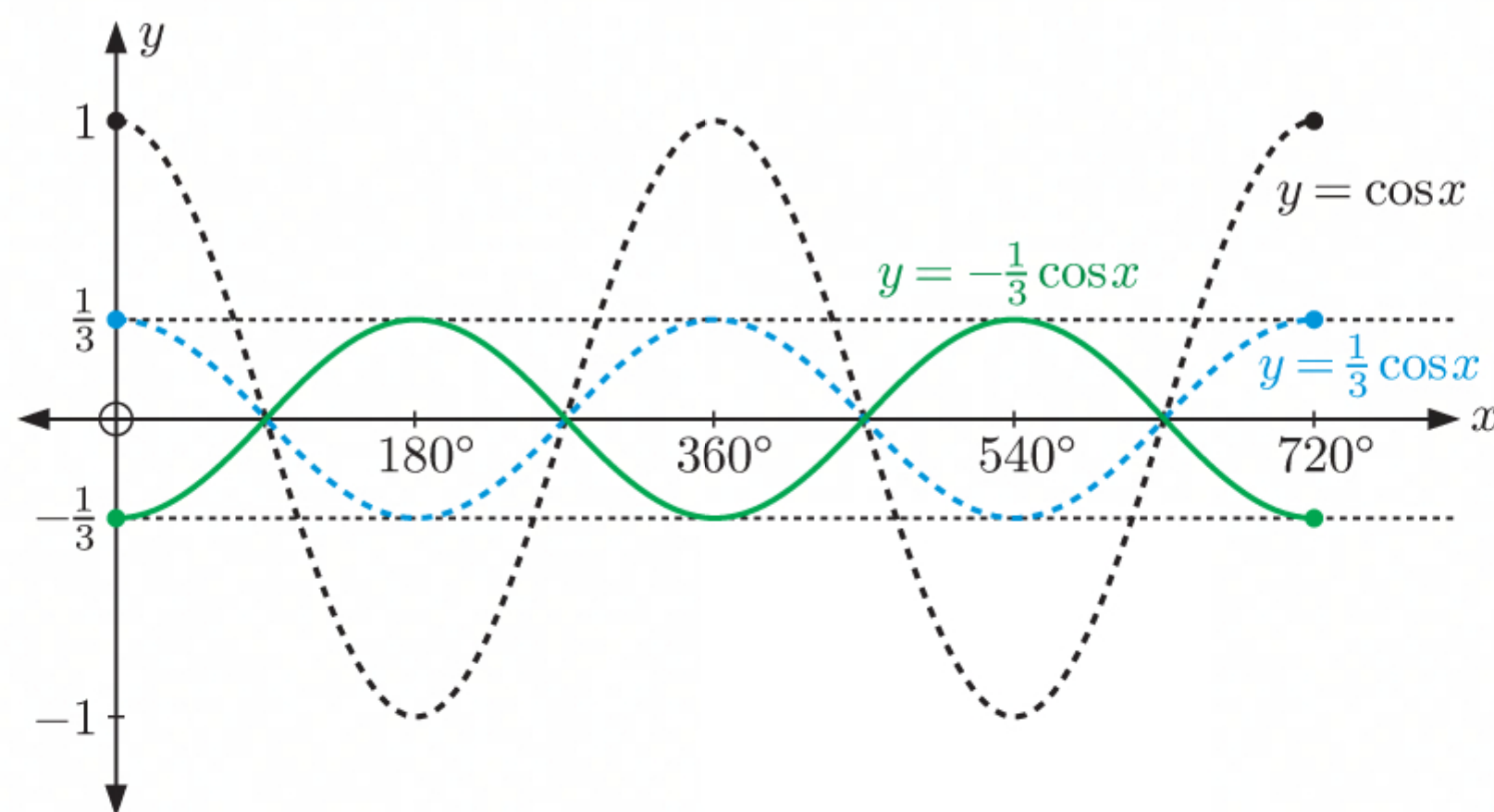
- b** $y = 2 \cos x$ is a vertical stretch of $y = \cos x$ with scale factor 2. The amplitude is 2.



- c** $y = \cos x + 3$ is a vertical translation of $y = \cos x$ upwards by 3 units.

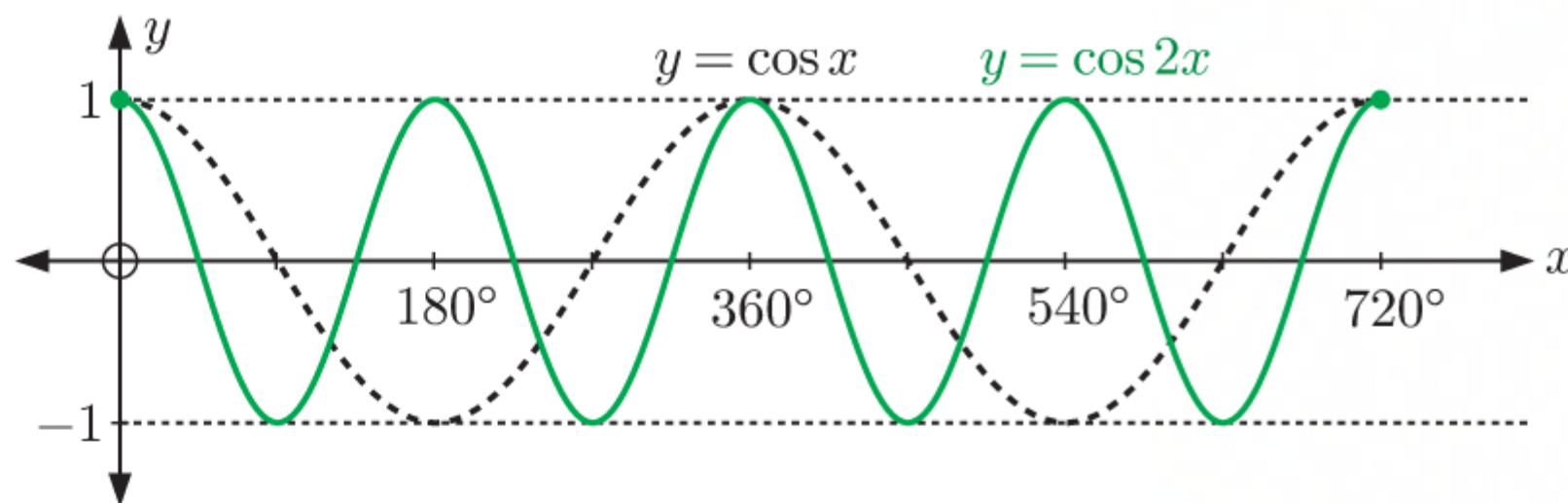


- d** $y = -\frac{1}{3} \cos x$ is a vertical stretch of $y = \cos x$ with scale factor $\frac{1}{3}$, followed by a reflection in the x -axis. The amplitude is $\frac{1}{3}$.



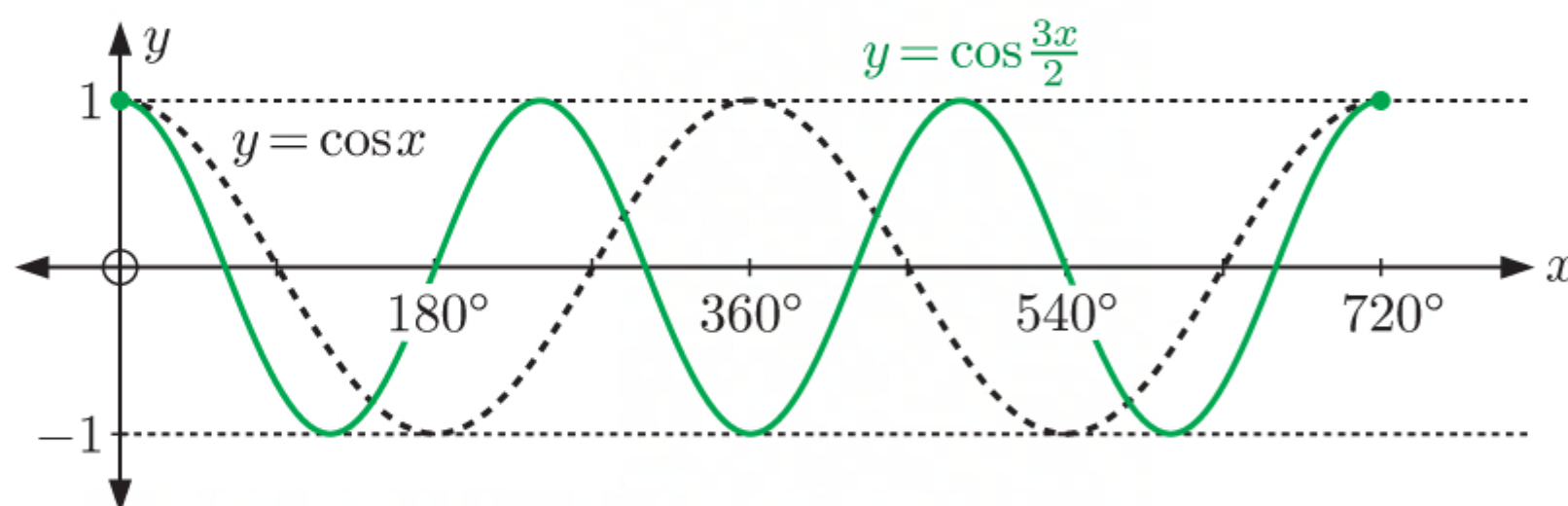
- e** $y = \cos 2x$ is a horizontal stretch of $y = \cos x$ with scale factor $\frac{1}{2}$.

The period is $\frac{360^\circ}{2} = 180^\circ$. \therefore the maximum values are 180° units apart.

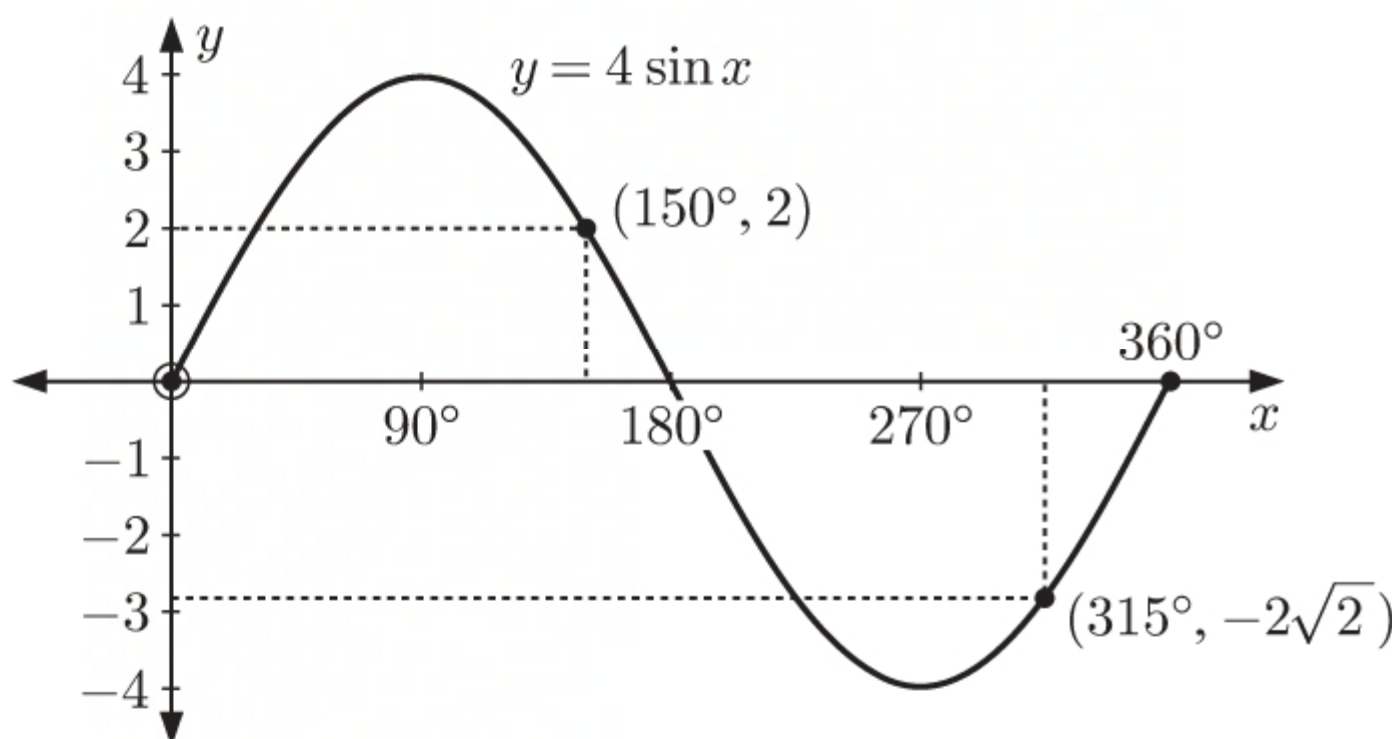


- f** $y = \cos \frac{3x}{2}$ is a horizontal stretch of $y = \cos x$ with scale factor $\frac{2}{3}$.

The period is $\frac{360^\circ}{\frac{3}{2}} = 240^\circ$. \therefore the maximum values are 240° units apart.



- 10 a** $y = 4 \sin x$ is a vertical stretch of $y = \sin x$ with scale factor 4. The amplitude is 4 and the period is 360° .



- b i** When $x = 150^\circ$, $y = 4 \sin 150^\circ$
 $= 4 \times \frac{1}{2}$
 $= 2$

- ii** When $x = 315^\circ$, $y = 4 \sin 315^\circ$
 $= 4 \times \left(-\frac{1}{\sqrt{2}}\right)$
 $= -2\sqrt{2}$
 ≈ -2.83

- 11** $y = 3 \cos x$ has minimum value -3 and maximum value 3 .

Now, $y = 3 \cos x + d$ is a vertical translation of $y = 3 \cos x$ by d units.

- a** $y = 3 \cos x + d$ will lie entirely above the x -axis when $y = 3 \cos x$ has been translated more than 3 units upwards.

$\therefore d > 3$

- b** $y = 3 \cos x + d$ will lie entirely below the x -axis when $y = 3 \cos x$ has been translated more than 3 units downwards.
 $\therefore d < -3$
- c** $y = 3 \cos x + d$ will lie partially above and partially below the x -axis when $y = 3 \cos x$ has been translated between 0 and 3 units upwards or downwards.
 $\therefore -3 < d < 3$

12 $y = 4 \sin 3x + 2$

- a** The amplitude is $|4| = 4$. **b** The period is $\frac{360^\circ}{3} = 120^\circ$.
- c** $y = 4 \sin 3x + 2$ has maximum value $4(1) + 2 = 6$ {when $\sin 3x = 1$ }
 and minimum value $4(-1) + 2 = -2$ {when $\sin 3x = -1$ }
 \therefore the range is $\{y \mid -2 \leq y \leq 6\}$.

13 a $\sin x \xrightarrow[\text{horizontal stretch}]{\text{scale factor } \frac{1}{3}} \sin 3x \xrightarrow[\text{vertical stretch}]{\text{scale factor } 2} 2 \sin 3x$

A horizontal stretch with scale factor $\frac{1}{3}$, then a vertical stretch with scale factor 2 maps $y = \sin x$ onto $y = 2 \sin 3x$.

b $\sin x \xrightarrow[\text{vertical stretch}]{\text{scale factor } 3} 3 \sin x \xrightarrow[\text{vertical translation}]{5 \text{ units downwards}} 3 \sin x - 5$

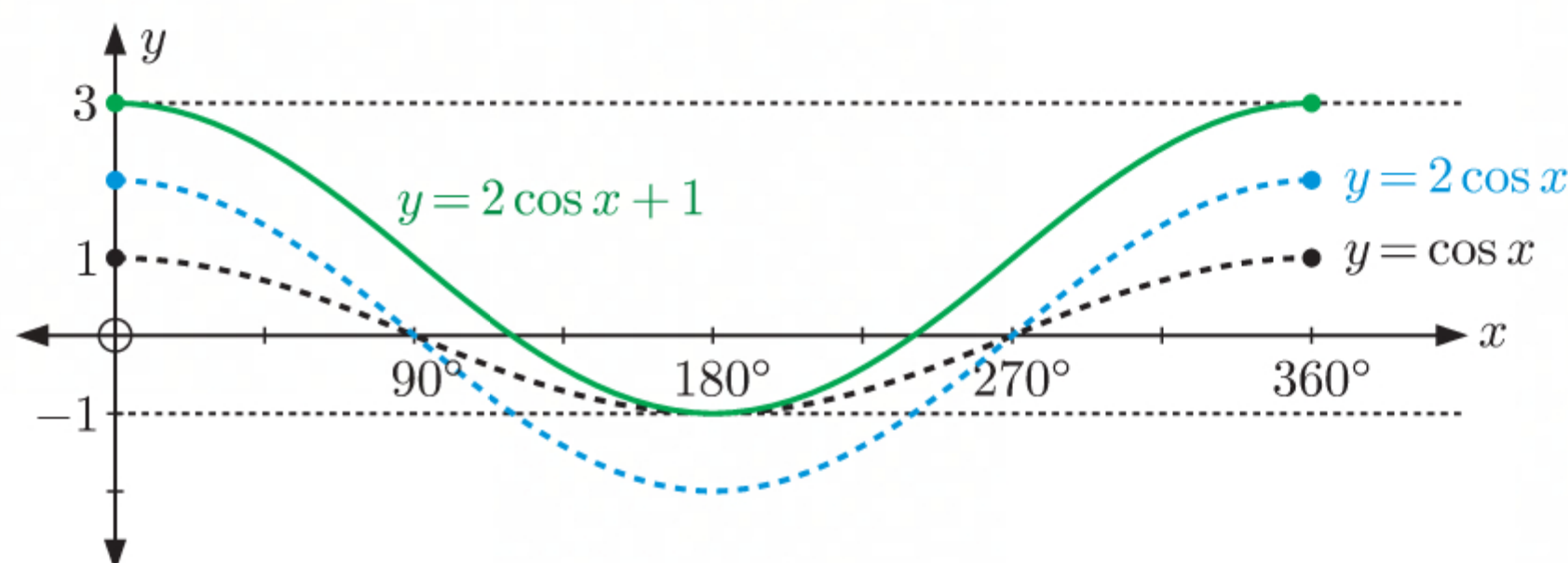
A vertical stretch with scale factor 3, then a translation 5 units downwards maps $y = \sin x$ onto $y = 3 \sin x - 5$.

c $\sin x \xrightarrow[\text{reflection in the } x\text{-axis}]{\text{vertical stretch}} -\sin x \xrightarrow[\text{scale factor } 2]{\text{vertical stretch}} -2 \sin x$

A reflection in the x -axis, then a vertical stretch with scale factor 2 maps $y = \sin x$ onto $y = -2 \sin x$.

14 a $a = 2$, so the amplitude is $|2| = 2$.
 $d = 1$, so the principal axis is $y = 1$.

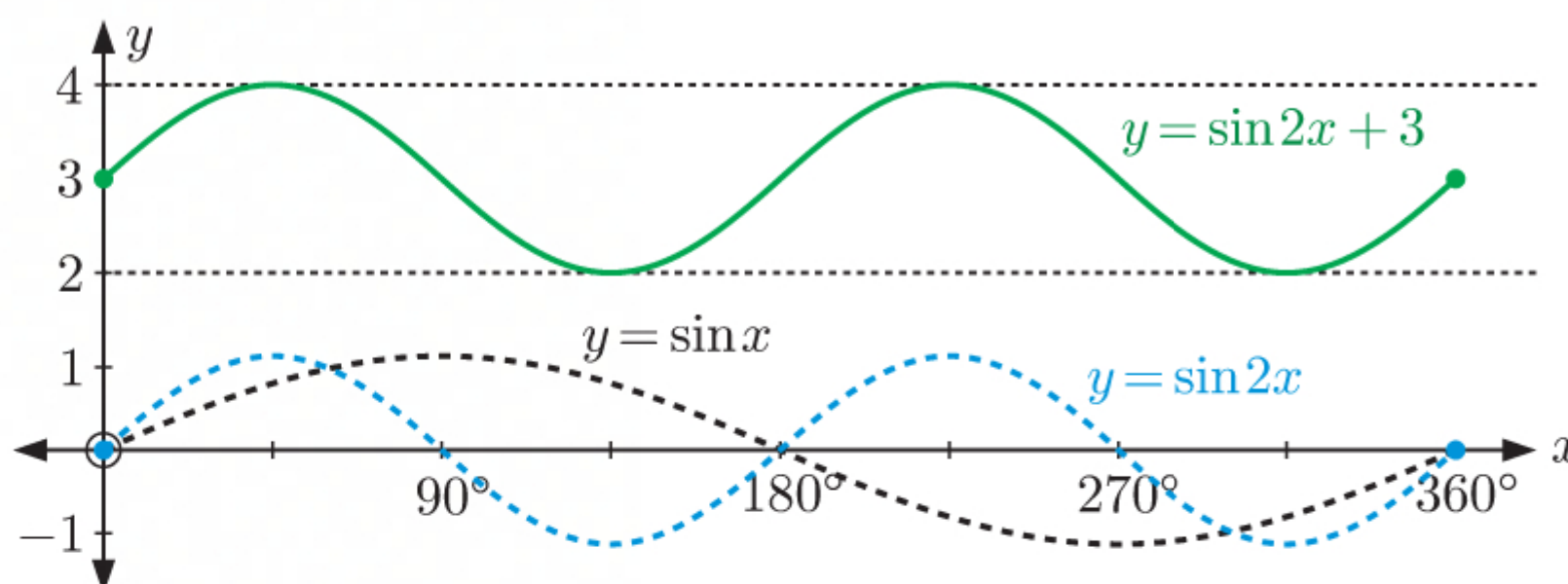
We stretch $y = \cos x$ vertically with scale factor 2 to give $y = 2 \cos x$, then translate $y = 2 \cos x$ upwards by 1 unit to give $y = 2 \cos x + 1$.



b $b = 2$, so the period is $\frac{360^\circ}{b} = \frac{360^\circ}{2} = 180^\circ$.

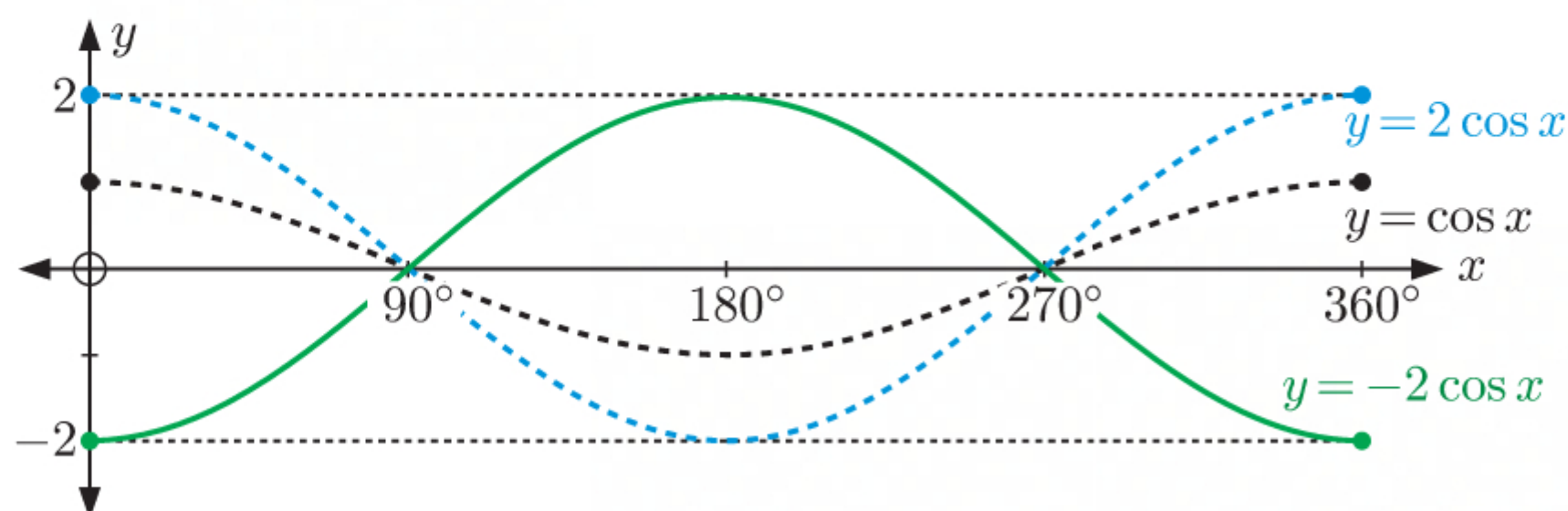
$d = 3$, so the principal axis is $y = 3$.

We stretch $y = \sin x$ horizontally with scale factor $\frac{1}{2}$ to give $y = \sin 2x$, then translate $y = \sin 2x$ upwards by 3 units to give $y = \sin 2x + 3$.



c $a = -2$, so the amplitude is $|-2| = 2$.

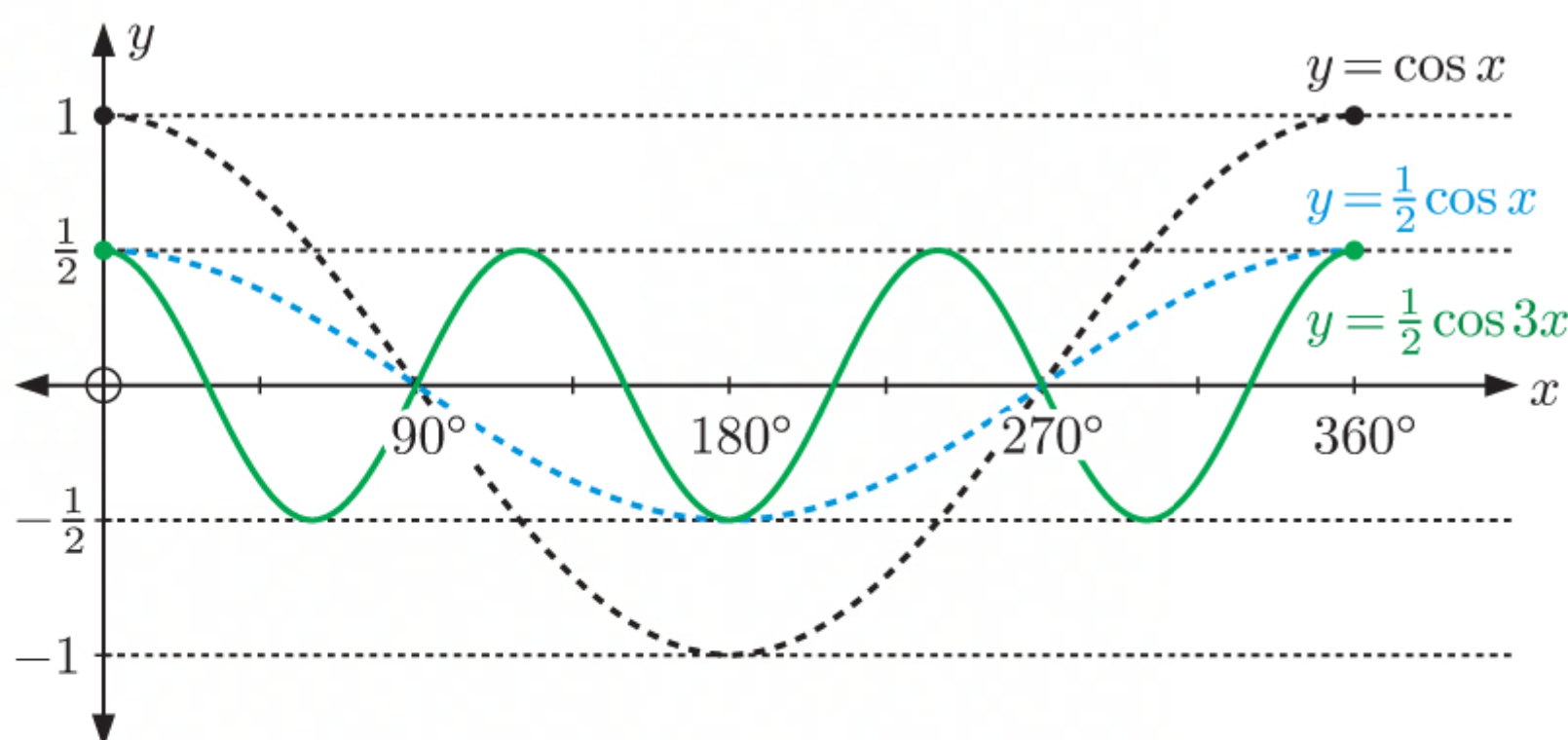
We stretch $y = \cos x$ vertically with scale factor 2 to give $y = 2 \cos x$, then reflect $y = 2 \cos x$ in the x -axis to give $y = -2 \cos x$.



d $a = \frac{1}{2}$, so the amplitude is $|\frac{1}{2}| = \frac{1}{2}$.

$b = 3$, so the period is $\frac{360^\circ}{b} = \frac{360^\circ}{3} = 120^\circ$.

We stretch $y = \cos x$ vertically with scale factor $\frac{1}{2}$ to give $y = \frac{1}{2} \cos x$, then stretch $y = \frac{1}{2} \cos x$ horizontally with scale factor $\frac{1}{3}$ to give $y = \frac{1}{2} \cos 3x$.

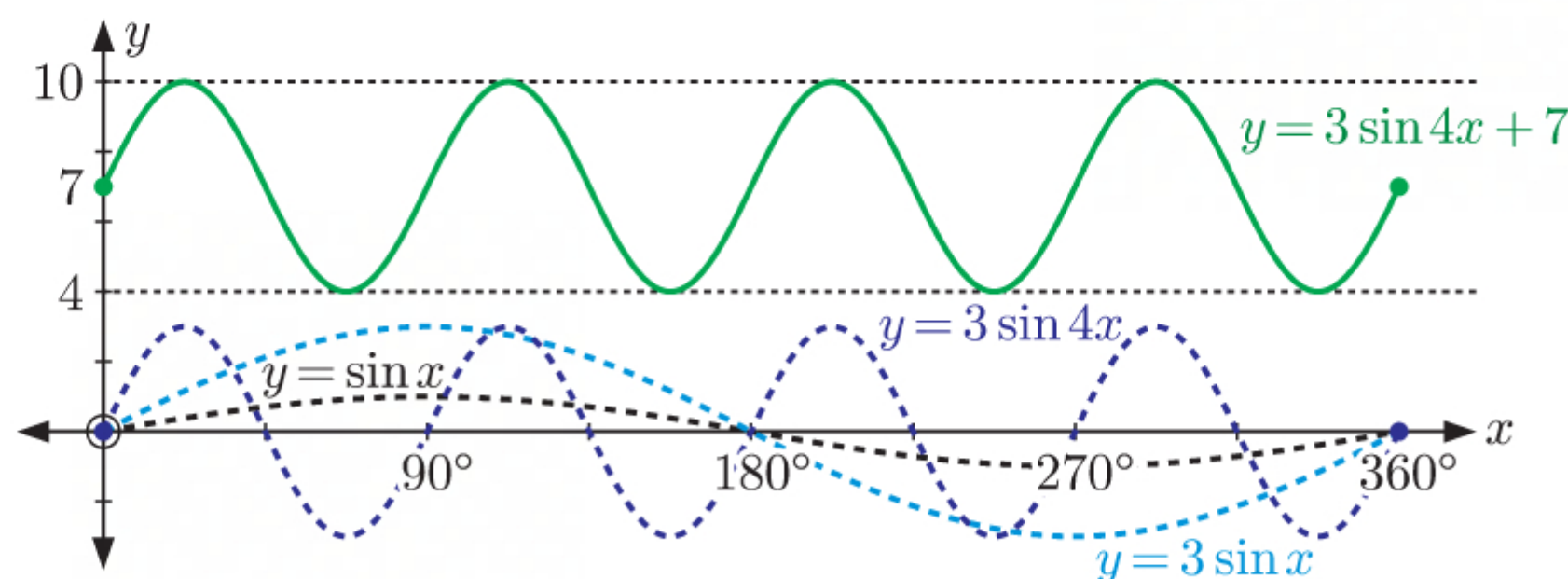


e $a = 3$, so the amplitude is $|3| = 3$.

$b = 4$, so the period is $\frac{360^\circ}{b} = \frac{360^\circ}{4} = 90^\circ$.

$d = 7$, so the principal axis is $y = 7$.

We stretch $y = \sin x$ vertically with scale factor 3 to give $y = 3 \sin x$, then stretch $y = 3 \sin x$ horizontally with scale factor $\frac{1}{4}$ to give $y = 3 \sin 4x$, then translate $y = 3 \sin 4x$ upwards by 7 units to give $y = 3 \sin 4x + 7$.

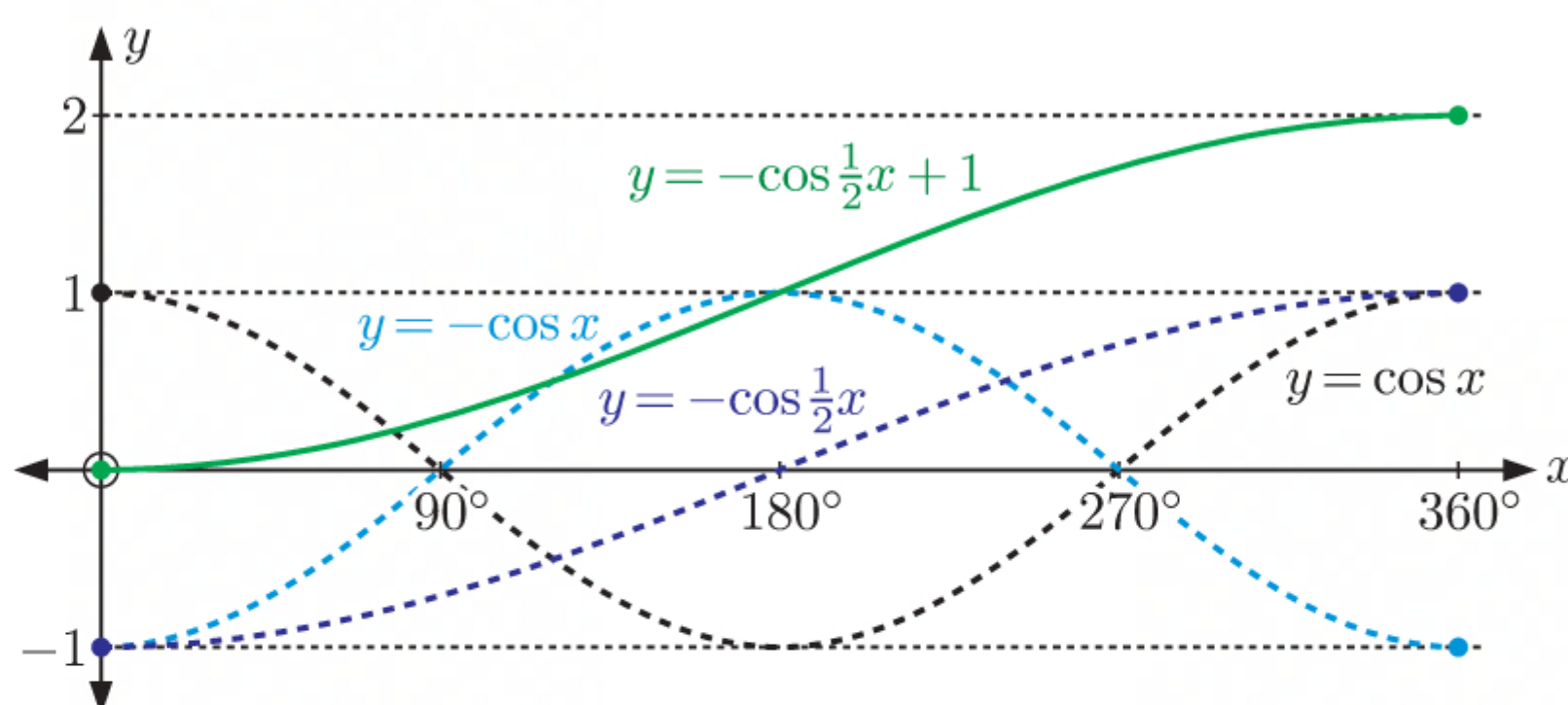


f $a = -1$, so the amplitude is $|-1| = 1$.

$b = \frac{1}{2}$, so the period is $\frac{360^\circ}{b} = \frac{360^\circ}{\frac{1}{2}} = 720^\circ$.

$d = 1$, so the principal axis is $y = 1$.

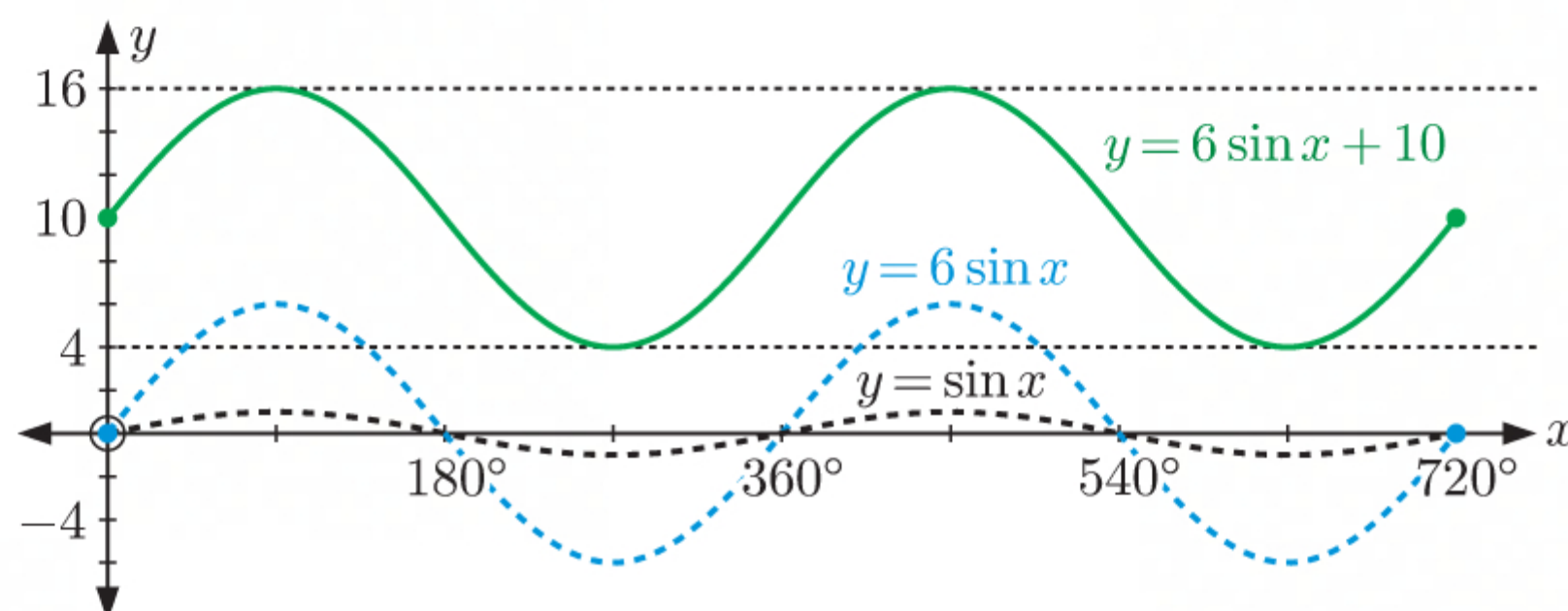
We reflect $y = \cos x$ in the x -axis to give $y = -\cos x$, then stretch $y = -\cos x$ horizontally with scale factor 2 to give $y = -\cos \frac{1}{2}x$, then translate $y = -\cos \frac{1}{2}x$ upwards by 1 unit to give $y = -\cos \frac{1}{2}x + 1$.



15 a $a = 6$, so the amplitude is $|6| = 6$.

$d = 10$, so the principal axis is $y = 10$.

We stretch $y = \sin x$ vertically with scale factor 6 to give $y = 6 \sin x$, then translate $y = 6 \sin x$ upwards by 10 units to give $y = 6 \sin x + 10$.

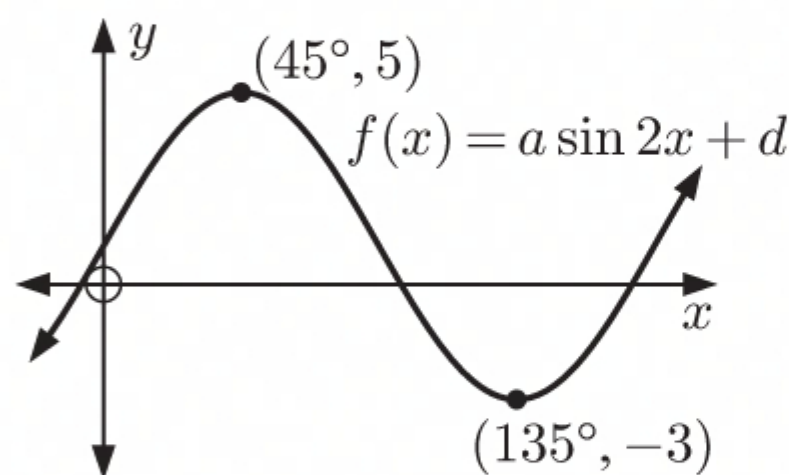


b When $x = 30^\circ$, $y = 6 \sin 30^\circ + 10$
 $= 6 \times \frac{1}{2} + 10$
 $= 13$

c From the graph, the maximum value of y is 16, which occurs when $\sin x = 1$.
 $\therefore x = 90^\circ$ or 450°

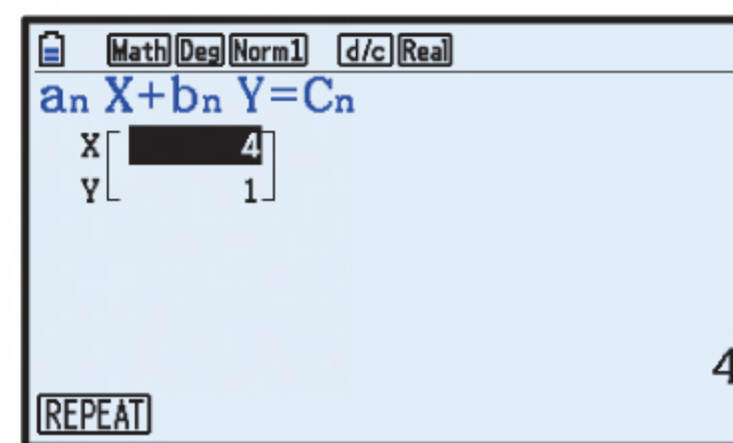
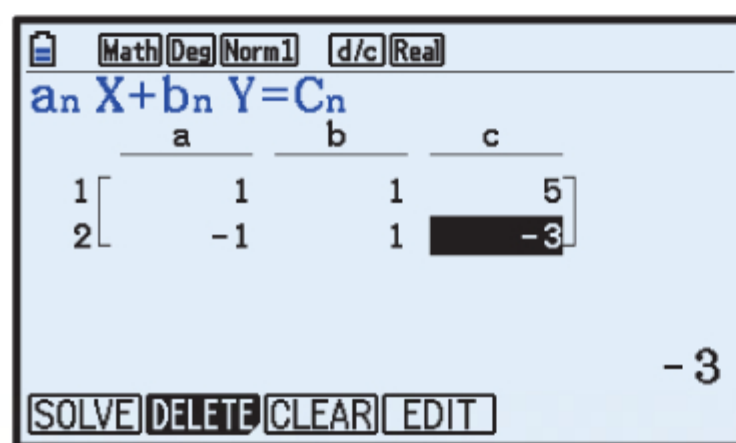
d From the graph, the minimum value of y is 4, which occurs when $\sin x = -1$.
 $\therefore x = 270^\circ$ or 630°

16 a



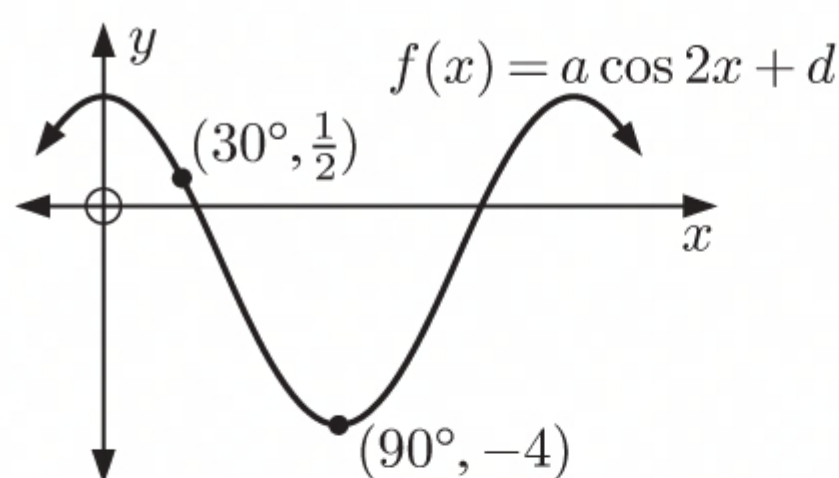
$$\begin{aligned} f(45^\circ) &= 5 \\ \therefore a \sin(2 \times 45^\circ) + d &= 5 \\ \therefore a \sin 90^\circ + d &= 5 \\ \therefore a + d &= 5 \quad \dots (1) \end{aligned}$$

$$\begin{aligned} f(135^\circ) &= -3 \\ \therefore a \sin(2 \times 135^\circ) + d &= -3 \\ \therefore a \sin 270^\circ + d &= -3 \\ \therefore -a + d &= -3 \quad \dots (2) \end{aligned}$$



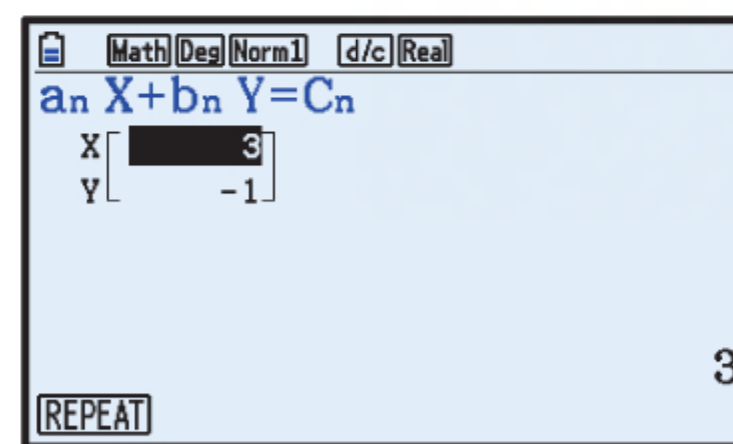
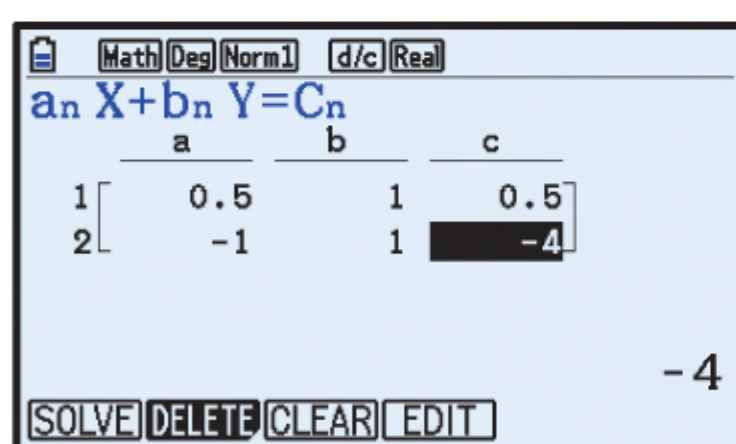
Solving (1) and (2) simultaneously using technology gives $a = 4$ and $d = 1$.

b

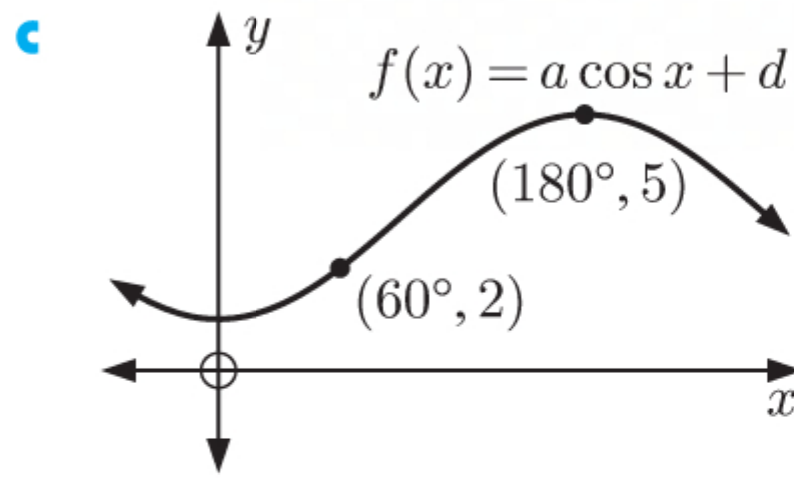


$$\begin{aligned} f(30^\circ) &= \frac{1}{2} \\ \therefore a \cos(2 \times 30^\circ) + d &= \frac{1}{2} \\ \therefore a \cos 60^\circ + d &= \frac{1}{2} \\ \therefore \frac{1}{2}a + d &= \frac{1}{2} \quad \dots (1) \end{aligned}$$

$$\begin{aligned} f(90^\circ) &= -4 \\ \therefore a \cos(2 \times 90^\circ) + d &= -4 \\ \therefore a \cos 180^\circ + d &= -4 \\ \therefore -a + d &= -4 \quad \dots (2) \end{aligned}$$



Solving (1) and (2) simultaneously using technology gives $a = 3$ and $d = -1$.



$$f(60^\circ) = 2$$

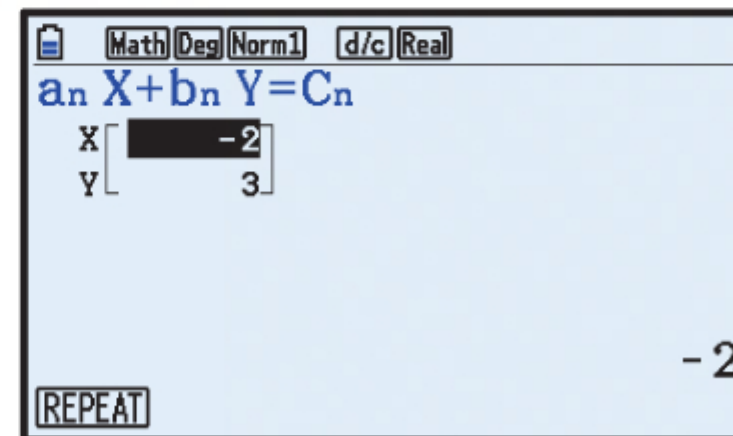
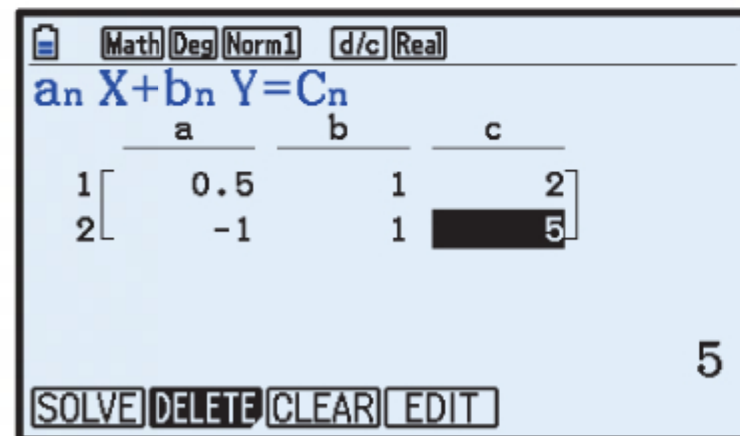
$$\therefore a \cos 60^\circ + d = 2$$

$$\therefore \frac{1}{2}a + d = 2 \quad \dots (1)$$

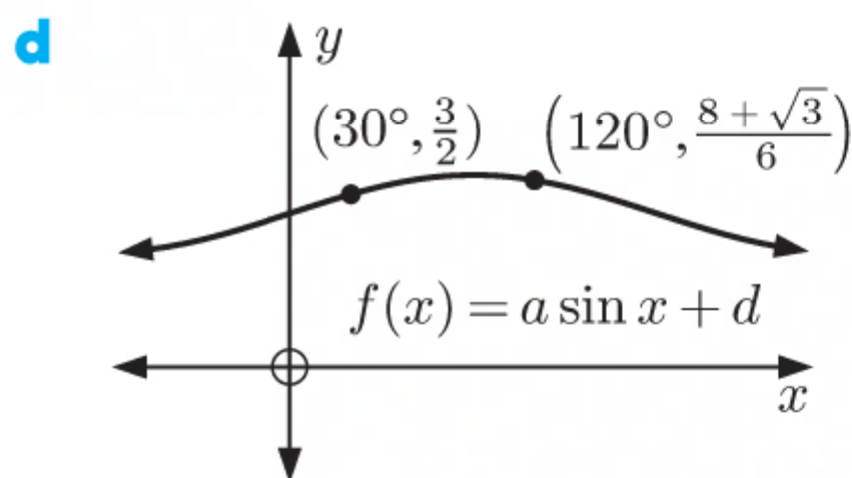
$$f(180^\circ) = 5$$

$$\therefore a \cos 180^\circ + d = 5$$

$$\therefore -a + d = 5 \quad \dots (2)$$



Solving (1) and (2) simultaneously using technology gives $a = -2$ and $d = 3$.



$$f(30^\circ) = \frac{3}{2}$$

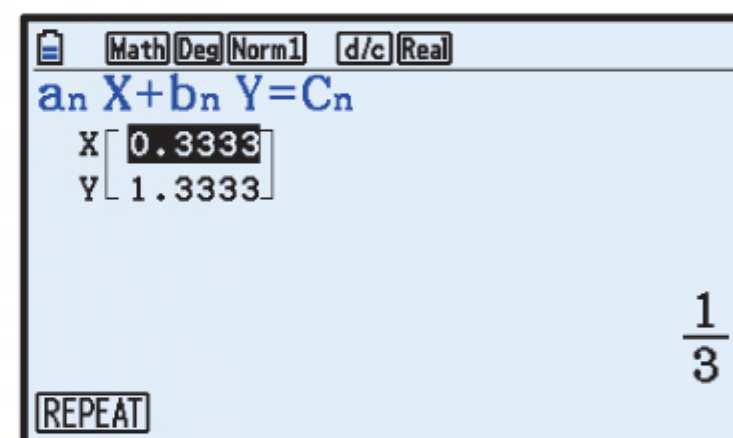
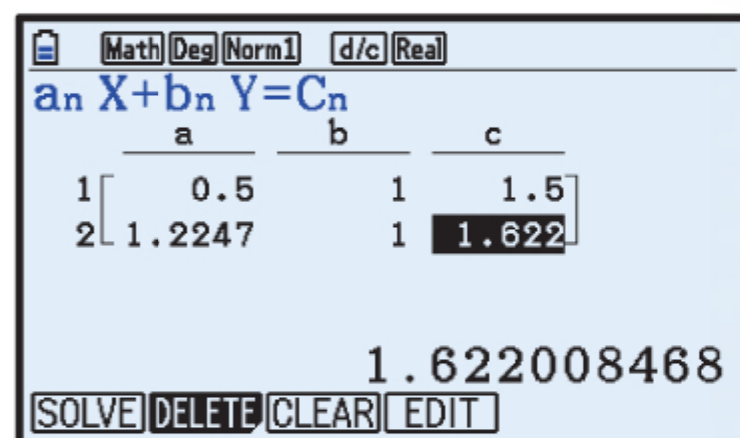
$$\therefore a \sin 30^\circ + d = \frac{3}{2}$$

$$\therefore \frac{1}{2}a + d = \frac{3}{2} \quad \dots (1)$$

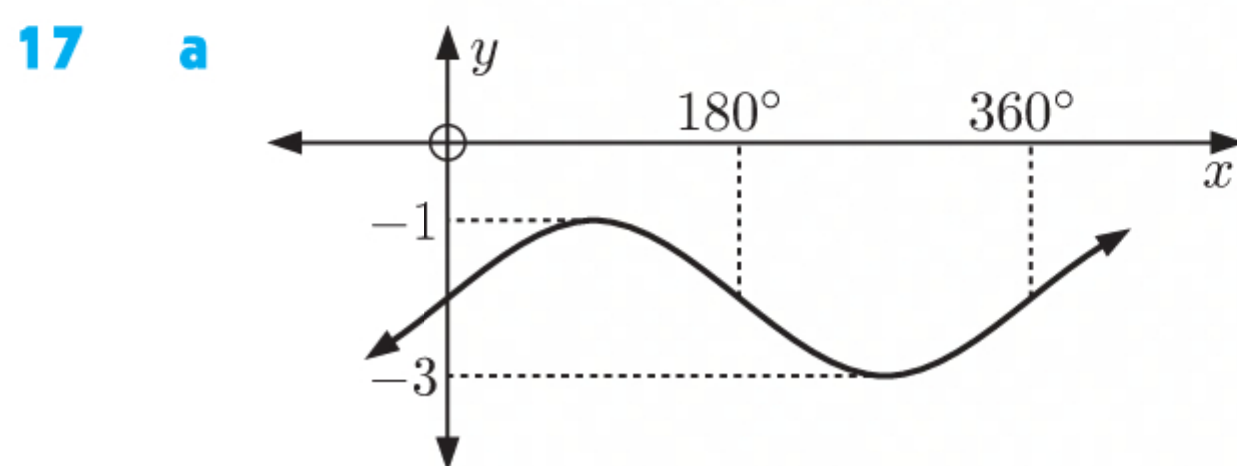
$$f(120^\circ) = \frac{8+\sqrt{3}}{6}$$

$$\therefore a \sin 120^\circ + d = \frac{8+\sqrt{3}}{6}$$

$$\therefore \frac{\sqrt{3}}{2}a + d = \frac{8+\sqrt{3}}{6} \quad \dots (2)$$



Solving (1) and (2) simultaneously using technology gives $a = \frac{1}{3}$ and $d = \frac{4}{3}$.

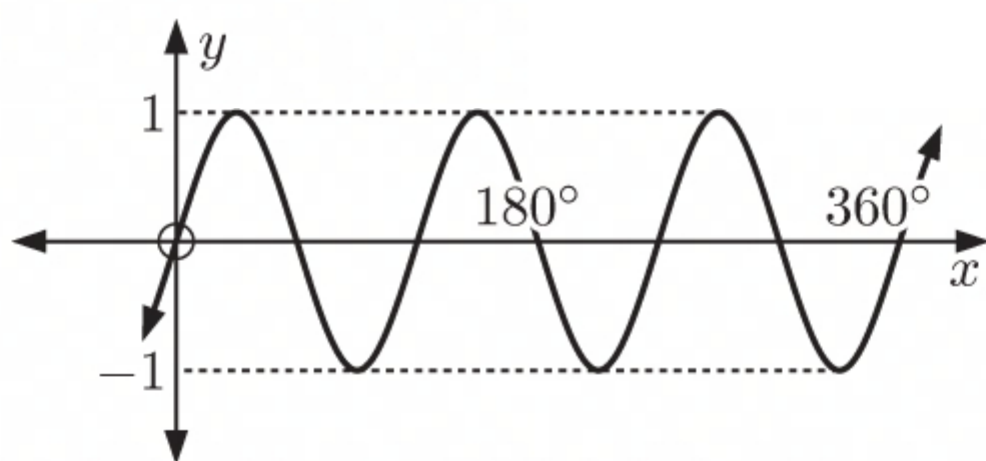


The amplitude is 1, so $a = 1$.

The period is 360° , so $\frac{360^\circ}{b} = 360^\circ$
 $\therefore b = 1$

The principal axis is $y = -2$, so $d = -2$.

The equation of the function is $y = \sin x - 2$.

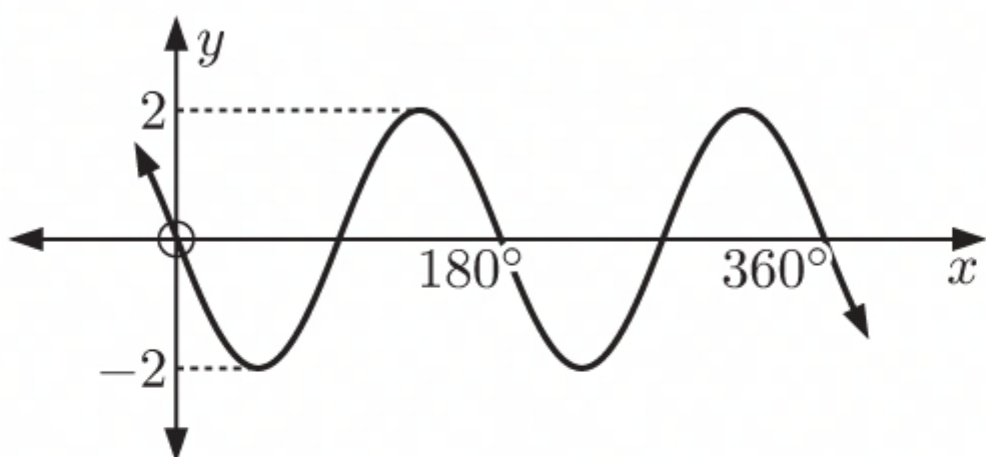
b

The amplitude is 1, so $a = 1$.

The period is 120° , so $\frac{360^\circ}{b} = 120^\circ$
 $\therefore b = 3$

The principal axis is $y = 0$, so $d = 0$.

The equation of the function is $y = \sin 3x$.

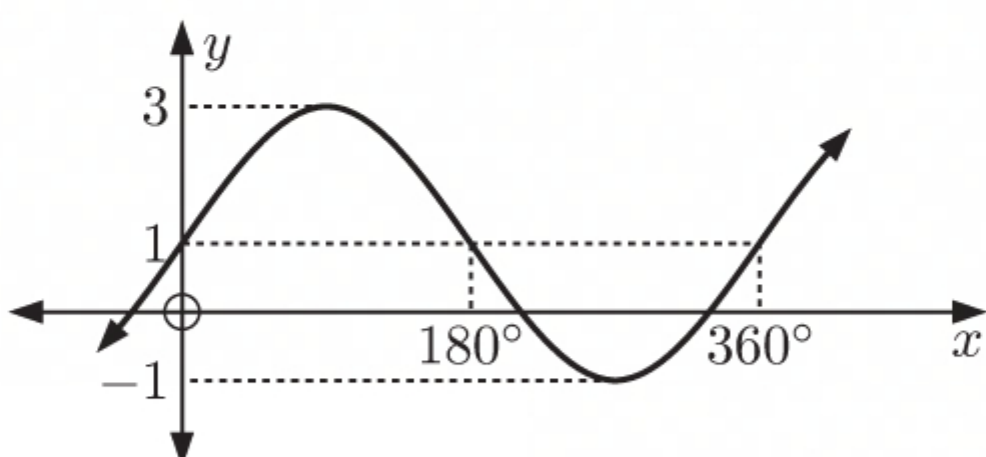
c

The amplitude is 2, and $a < 0$, so $a = -2$.

The period is 180° , so $\frac{360^\circ}{b} = 180^\circ$
 $\therefore b = 2$

The principal axis is $y = 0$, so $d = 0$.

The equation of the function is $y = -2 \sin 2x$.

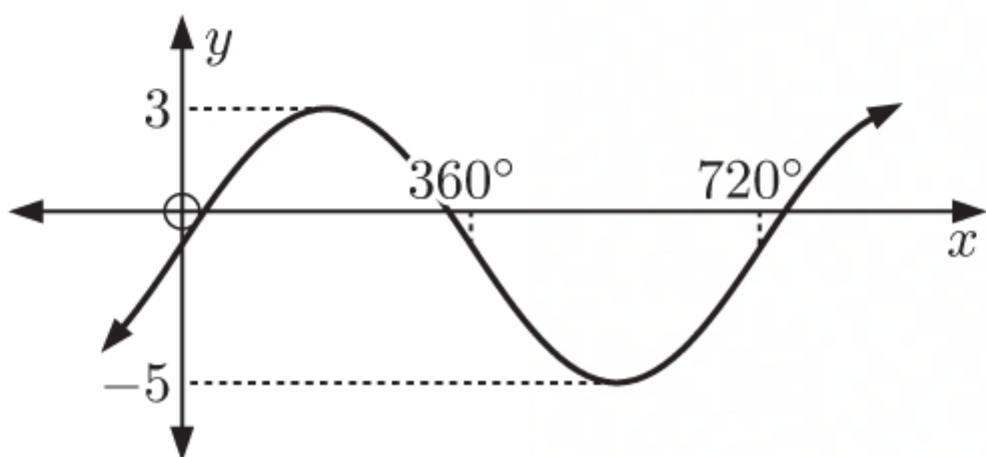
d

The amplitude is 2, so $a = 2$.

The period is 360° , so $\frac{360^\circ}{b} = 360^\circ$
 $\therefore b = 1$

The principal axis is $y = 1$, so $d = 1$.

The equation of the function is $y = 2 \sin x + 1$.

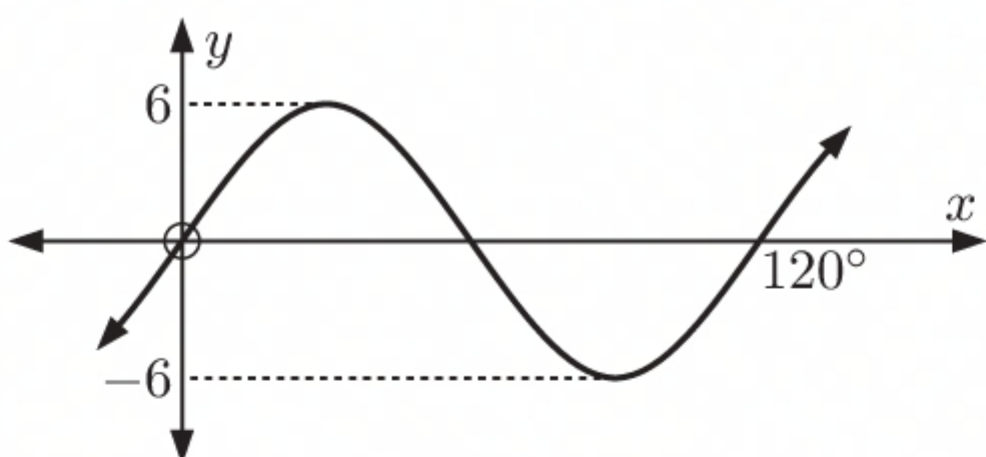
e

The amplitude is 4, so $a = 4$.

The period is 720° , so $\frac{360^\circ}{b} = 720^\circ$
 $\therefore b = \frac{1}{2}$

The principal axis is $y = -1$, so $d = -1$.

The equation of the function is $y = 4 \sin \frac{x}{2} - 1$.

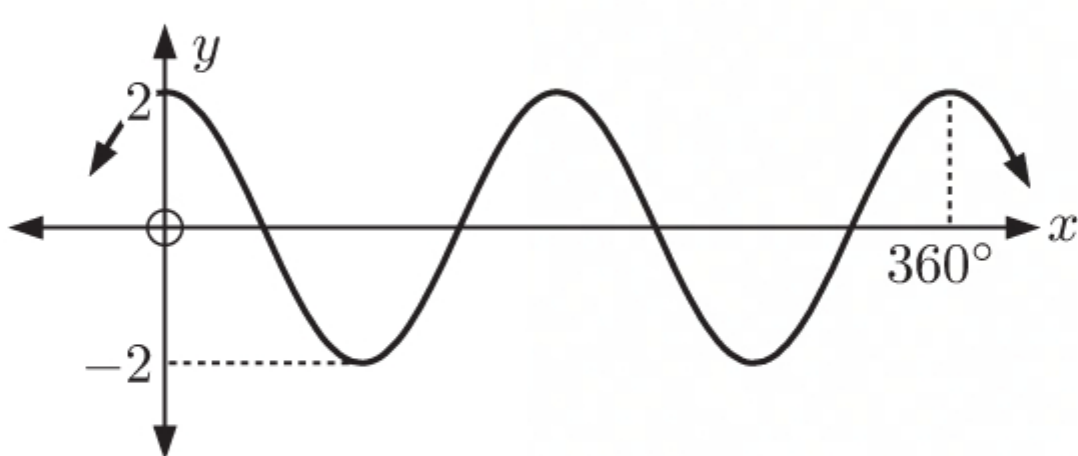
f

The amplitude is 6, so $a = 6$.

The period is 120° , so $\frac{360^\circ}{b} = 120^\circ$
 $\therefore b = 3$

The principal axis is $y = 0$, so $d = 0$.

The equation of the function is $y = 6 \sin 3x$.

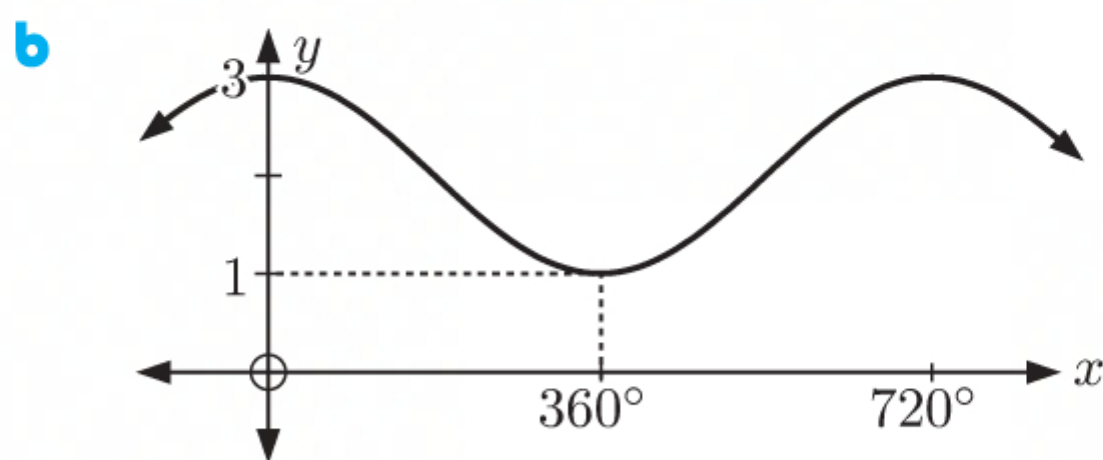
18 a

The amplitude is 2, so $a = 2$.

The period is 180° , so $\frac{360^\circ}{b} = 180^\circ$
 $\therefore b = 2$

The principal axis is $y = 0$, so $d = 0$.

The equation of the function is
 $y = 2 \cos 2x$.

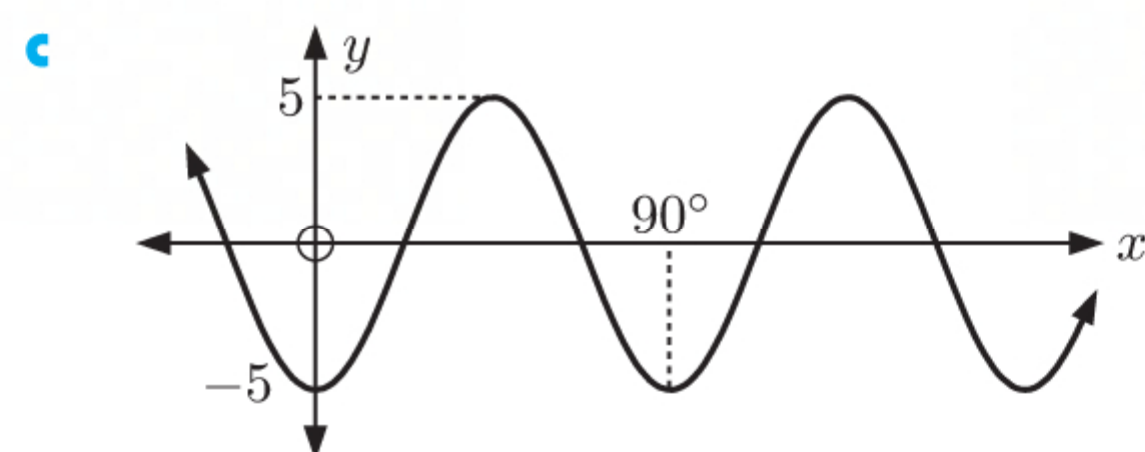


The amplitude is 1, so $a = 1$.

The period is 720° , so $\frac{360^\circ}{b} = 720^\circ$
 $\therefore b = \frac{1}{2}$

The principal axis is $y = 2$, so $d = 2$.

The equation of the function is
 $y = \cos \frac{x}{2} + 2$.

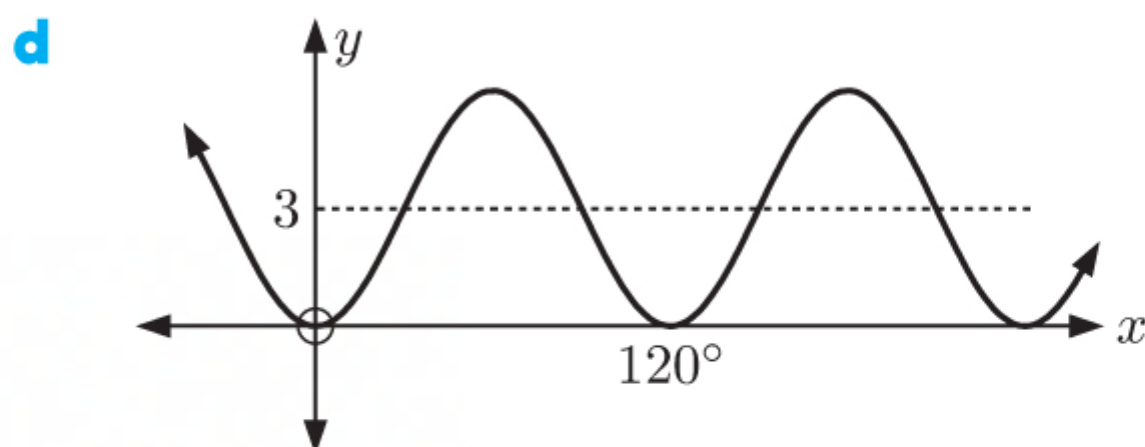


The amplitude is 5, and $a < 0$, so
 $a = -5$.

The period is 90° , so $\frac{360^\circ}{b} = 90^\circ$
 $\therefore b = 4$

The principal axis is $y = 0$, so $d = 0$.

The equation of the function is
 $y = -5 \cos 4x$.



The amplitude is 3, and $a < 0$, so
 $a = -3$.

The period is 120° , so $\frac{360^\circ}{b} = 120^\circ$
 $\therefore b = 3$

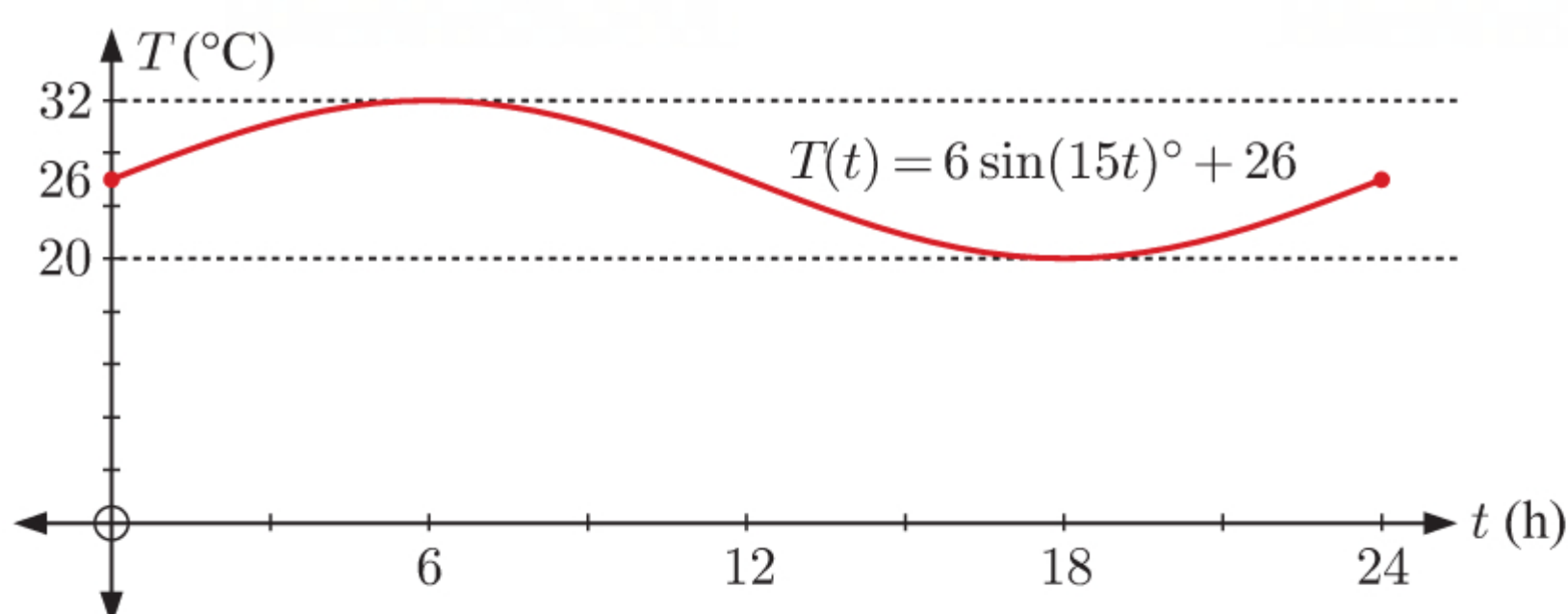
The principal axis is $y = 3$, so $d = 3$.

The equation of the function is
 $y = -3 \cos 3x + 3$.

EXERCISE 9E

1 a For $T(t) = 6 \sin(15t)^\circ + 26$:

- the amplitude is 6
- the period is $\frac{360}{15} = 24$ hours
- the principal axis is $T = 26$.



- b i** Midnight is 12 hours after midday.

When $t = 12$,

$$\begin{aligned} T &= 6 \sin(15 \times 12)^\circ + 26 \\ &= 6 \sin 180^\circ + 26 \\ &= 6 \times 0 + 26 \\ &= 26 \end{aligned}$$

\therefore at midnight the temperature inside Vanessa's house is 26°C .

- ii** 2 pm is 2 hours after midday.

When $t = 2$,

$$\begin{aligned} T &= 6 \sin(15 \times 2)^\circ + 26 \\ &= 6 \sin 30^\circ + 26 \\ &= 6 \times \frac{1}{2} + 26 \\ &= 29 \end{aligned}$$

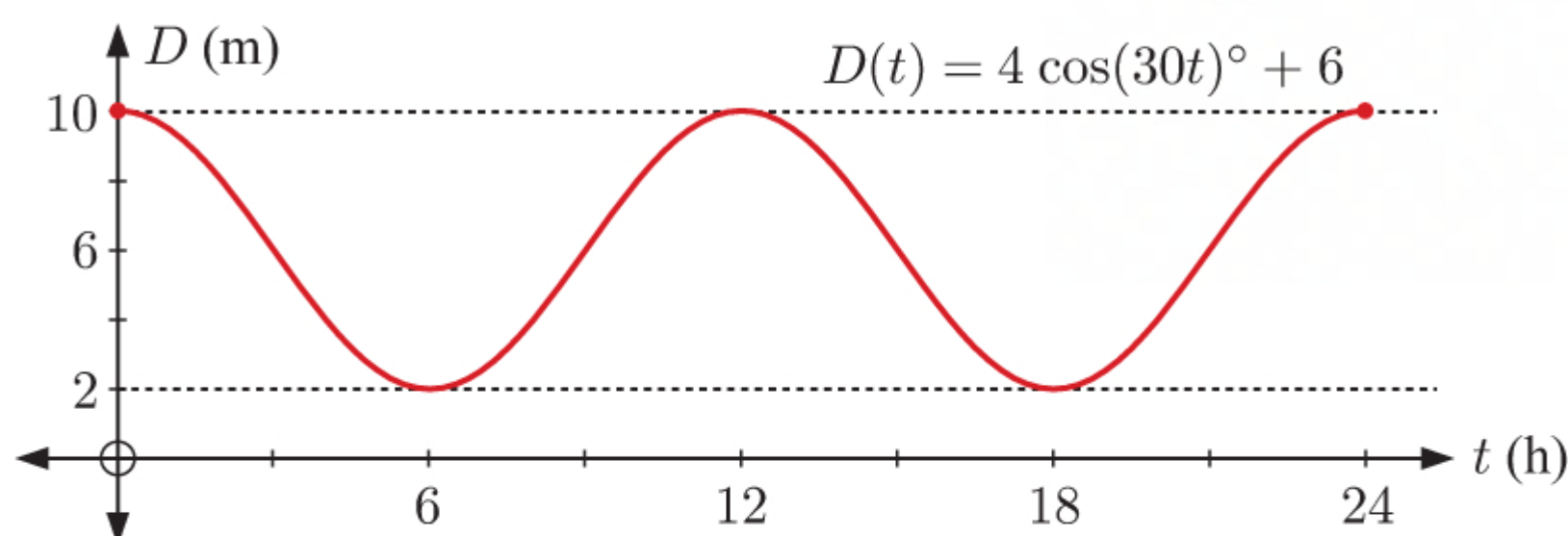
\therefore at 2 pm the temperature inside Vanessa's house is 29°C .

- c** The maximum temperature inside Vanessa's house is $26 + 6 = 32^\circ\text{C}$, which occurs when $t = 6$.

So, the maximum temperature inside Vanessa's house occurs at 6 pm.

- 2 a** For $D(t) = 4 \cos(30t)^\circ + 6$:

- the amplitude is 4
- the period is $\frac{360}{30} = 12$ hours
- the principal axis is $D = 6$.



- b** The highest water depth is $6 + 4 = 10$ metres, which occurs when $t = 0, 12$, or 24 . So, the highest water depth occurs at midnight, midday, and midnight the next day. The lowest water depth is $6 - 4 = 2$ metres, which occurs when $t = 6$ or 18 . So, the lowest water depth occurs at 6 am or 6 pm.

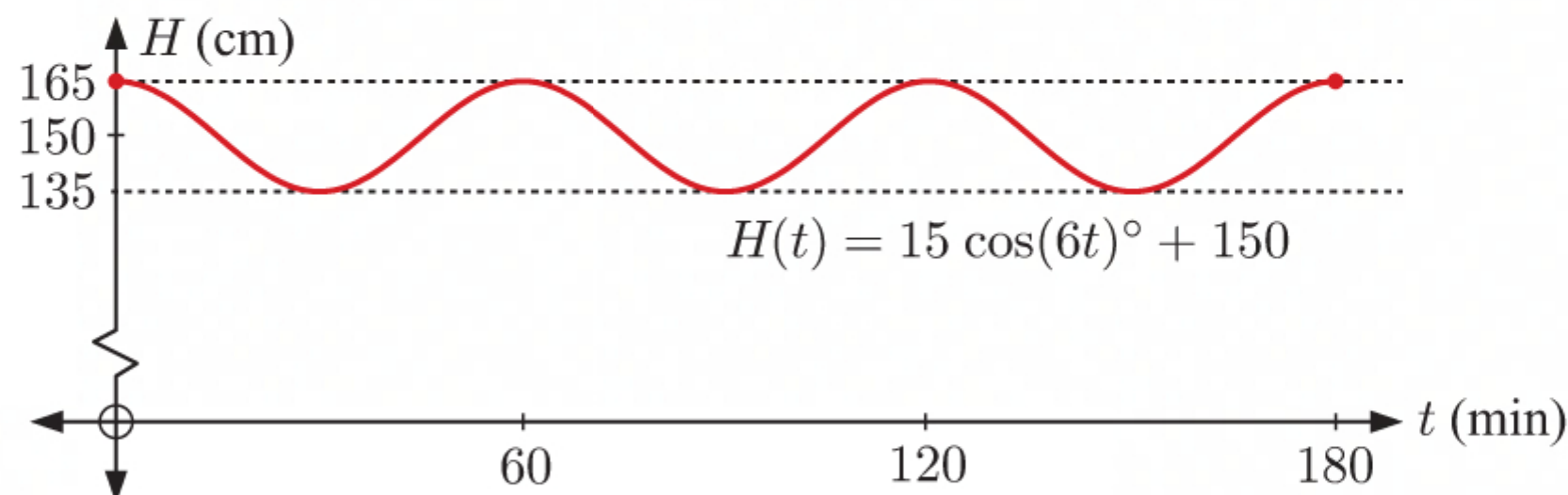
- c** 8 pm is 20 hours after midnight.

$$\begin{aligned} \text{When } t = 20, \quad D &= 4 \cos(30 \times 20)^\circ + 6 \\ &= 4 \cos 600^\circ + 6 \\ &= 4 \times \left(-\frac{1}{2}\right) + 6 \\ &= 4 \end{aligned}$$

\therefore at 8 pm the water depth is 4 metres, but the boat requires 5 metres, so it cannot enter the harbour at that time.

3 a For $H(t) = 15 \cos(6t)^\circ + 150$:

- the amplitude is 15
- the period is $\frac{360}{6} = 60$ minutes
- the principal axis is $H = 150$.



b The minute hand's length is represented by the amplitude in the function.
 \therefore the minute hand's length is 15 cm.

c i 5:08 pm is 8 minutes after 5 pm.

When $t = 8$,

$$\begin{aligned} H &= 15 \cos(6 \times 8)^\circ + 150 \\ &= 15 \cos 48^\circ + 150 \\ &\approx 160.037 \end{aligned}$$

\therefore at 5:08 pm, the minute hand's tip is approximately 160.0 cm above ground level.

ii 5:37 pm is 37 minutes after 5 pm.

When $t = 37$,

$$\begin{aligned} H &= 15 \cos(6 \times 37)^\circ + 150 \\ &= 15 \cos 222^\circ + 150 \\ &\approx 138.853 \end{aligned}$$

\therefore at 5:37 pm, the minute hand's tip is approximately 138.9 cm above ground level.

iii 5:51 pm is 51 minutes after 5 pm.

When $t = 51$,

$$\begin{aligned} H &= 15 \cos(6 \times 51)^\circ + 150 \\ &= 15 \cos 306^\circ + 150 \\ &\approx 158.817 \end{aligned}$$

\therefore at 5:51 pm, the minute hand's tip is approximately 158.8 cm above ground level.

iv 6:23 pm is 83 minutes after 5 pm.

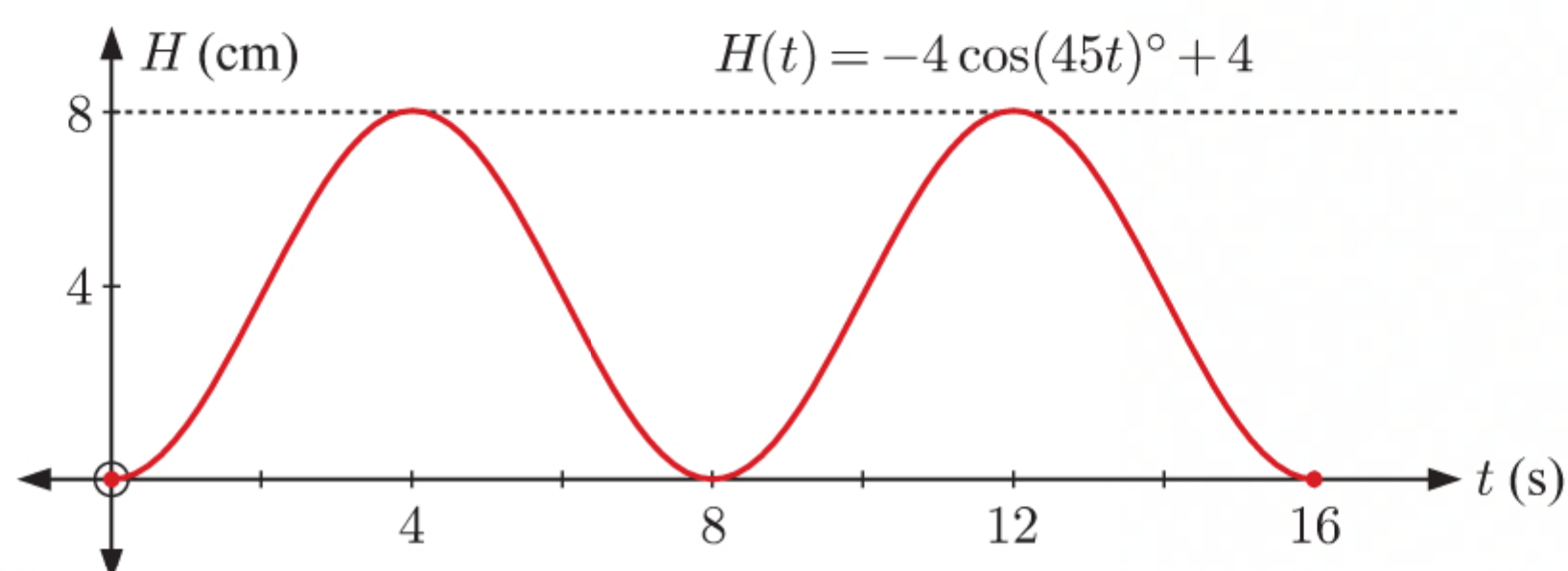
When $t = 83$,

$$\begin{aligned} H &= 15 \cos(6 \times 83)^\circ + 150 \\ &= 15 \cos 498^\circ + 150 \\ &\approx 138.853 \end{aligned}$$

\therefore at 6:23 pm, the minute hand's tip is approximately 138.9 cm above ground level.

4 a For $H(t) = -4 \cos(45t)^\circ + 4$:

- the amplitude is $|-4| = 4$
- the period is $\frac{360}{45} = 8$ seconds
- the principal axis is $H = 4$.



b When $t = 2$, $H = -4 \cos(45 \times 2)^\circ + 4$
 $= -4 \cos 90^\circ + 4$
 $= -4 \times 0 + 4$
 $= 4$

\therefore 2 seconds after the gate touches the ground it is 4 cm above ground level.

c When $t = 5.3 + 1 = 6.3$,
 $H = -4 \cos(45 \times 6.3)^\circ + 4$
 $= -4 \cos 283.5^\circ + 4$
 ≈ 3.066

\therefore the ball will not pass through the entrance as its diameter is $2 \times 2.14 = 4.28$ cm but the gate height is only approximately 3.07 cm above ground level.

5 $y = a \cos(bt)^\circ + d$

a The period is 12, so $\frac{360}{b} = 12$
 $\therefore b = \frac{360}{12}$
 $\therefore b = 30$

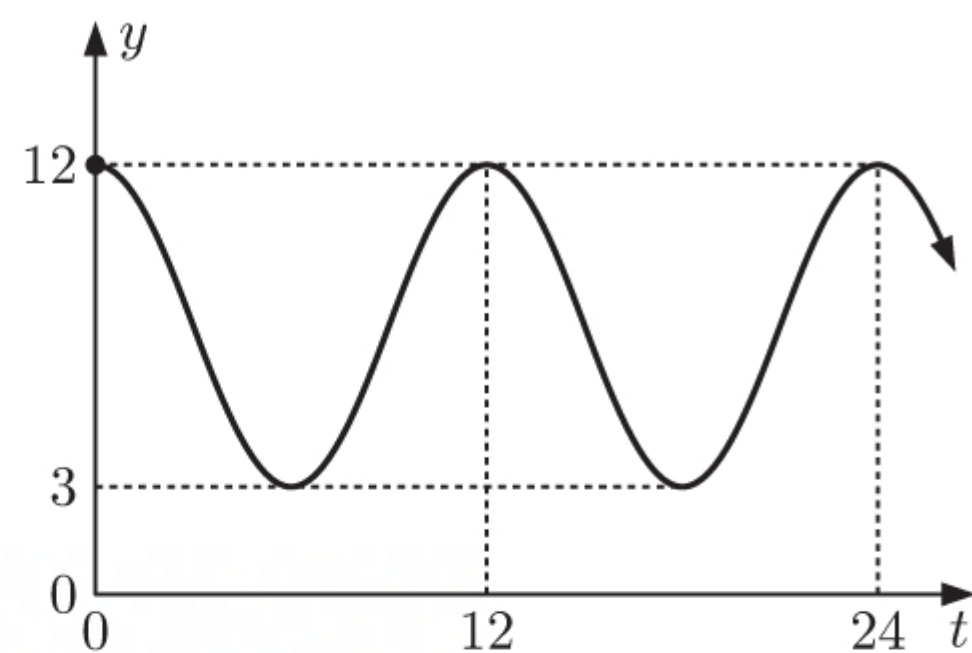
b The amplitude is $\frac{12-3}{2} = 4.5$, so $a = 4.5$.
 The principal axis is $\frac{12+3}{2} = 7.5$, so $d = 7.5$.

c $y = 4.5 \cos(30t)^\circ + 7.5$ {from **a** and **b**}

May is 4 months after January.

When $t = 4$, $y = 4.5 \cos(30 \times 4)^\circ + 7.5$
 $= 4.5 \cos 120^\circ + 7.5$
 $= 5.25$

The average UV index for Santiago during May is about 5 units.



6 $C(t) = a \sin(15t)^\circ + d$

a The principal axis is 120, so $d = 120$.

b $b = 15$, so the period is $\frac{360}{b} = \frac{360}{15} = 24$.

So, the cycle of petrol prices repeats itself every 24 days.

c $C(10) = 129$
 $\therefore a \sin(15 \times 10)^\circ + 120 = 129$
 $\therefore a \sin 150^\circ = 9$
 $\therefore \frac{1}{2}a = 9$
 $\therefore a = 18$

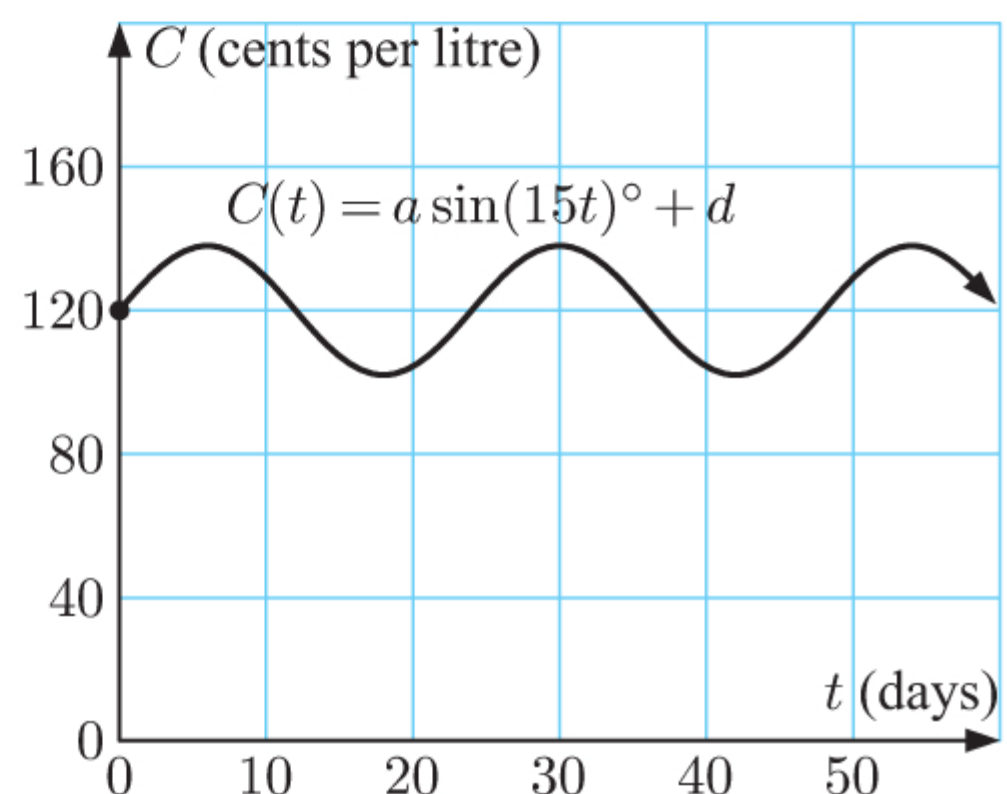
d $C(t) = 18 \sin(15t)^\circ + 120$ {from **a**, **b**, and **c**}

The minimum value is $18(-1) + 120 = 102$ {when $\sin(15t)^\circ = -1$ }

\therefore the minimum price of petrol at this service station is 102 cents per litre.

e $C(17) = 18 \sin(15 \times 17)^\circ + 120$
 $= 18 \sin 255^\circ + 120$
 ≈ 102.6

The petrol price on day 17 was about 103 cents per litre.



7 $F(t) = a \sin(bt)^\circ + d$

- a** The amount of feed fluctuates in a yearly cycle, so the period is 365 days.

$$\therefore \frac{360}{b} = 365$$

$$\therefore b = \frac{360}{365}$$

$$\therefore b \approx 0.986$$

b

$$F(40) = 10.63$$

$$\therefore a \sin\left(\frac{360}{365} \times 40\right)^\circ + d = 10.63 \quad \dots (1)$$

$$F(132) = 11.32$$

$$\therefore a \sin\left(\frac{360}{365} \times 132\right)^\circ + d = 11.32 \quad \dots (2)$$

	a	b	c
1	0.6354	1	10.63
2	0.7638	1	11.32

$\sin(132 \times 360 \div 365)$

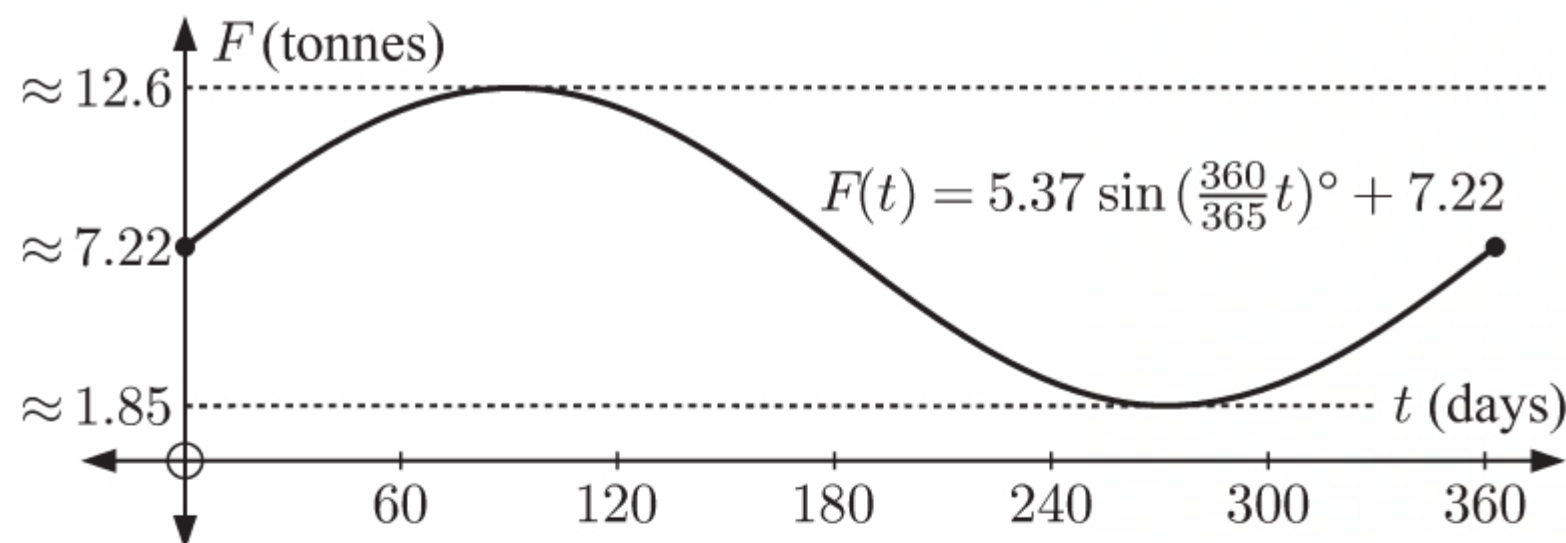
	X	Y
	5.3714	7.2168

5.371444696

REPEAT

Solving (1) and (2) simultaneously using technology gives $a \approx 5.37$ and $d \approx 7.22$.

c



d

$$F(t) \approx 5.37 \sin\left(\frac{360}{365}t\right)^\circ + 7.22 \quad \{\text{from a and b}\}$$

$$\therefore F(252) \approx 5.37 \sin\left(\frac{360}{365} \times 252\right)^\circ + 7.22$$

$$\approx 2.22$$

There are about 2.22 tonnes of feed available on day 252.

e

$F(t)$ has minimum value of about $5.37(-1) + 7.22 \approx 1.85$ {when $\sin\left(\frac{360}{365}t\right)^\circ = -1$ }

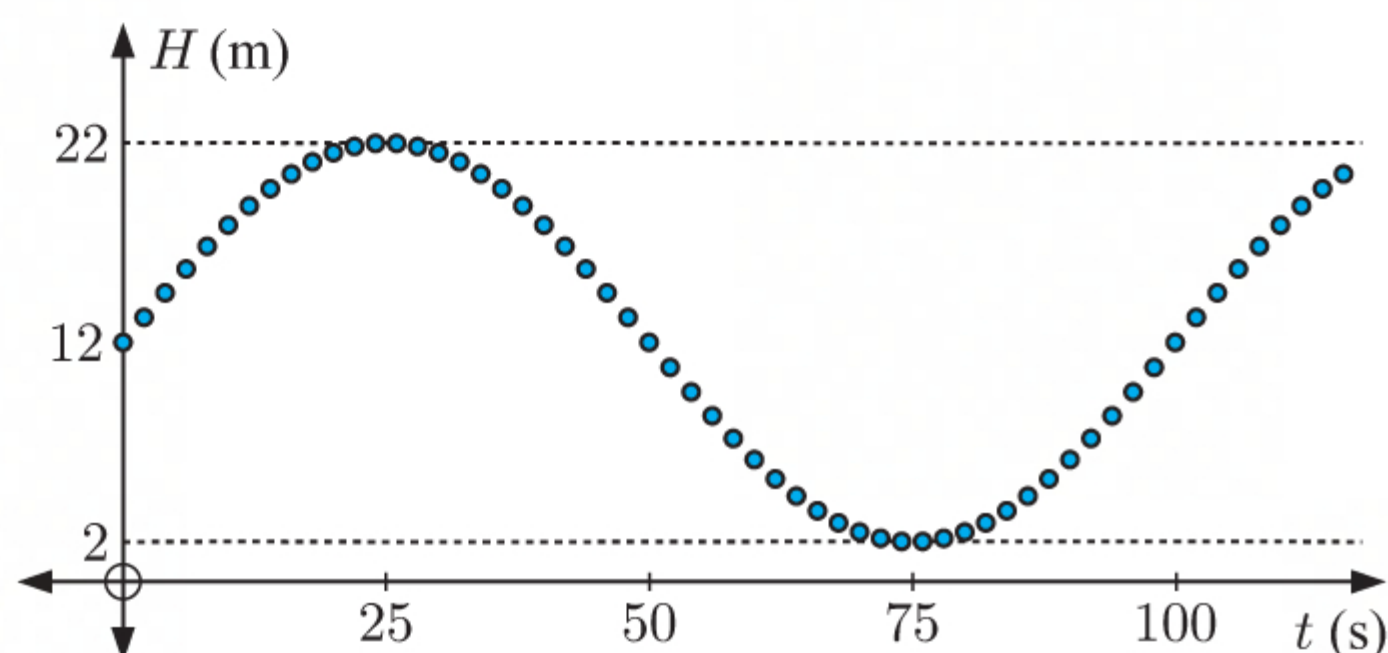
\therefore the minimum amount of feed available is about 1.85 tonnes.

$F(t)$ has maximum value of about $5.37(1) + 7.22 \approx 12.6$ {when $\sin\left(\frac{360}{365}t\right)^\circ = 1$ }

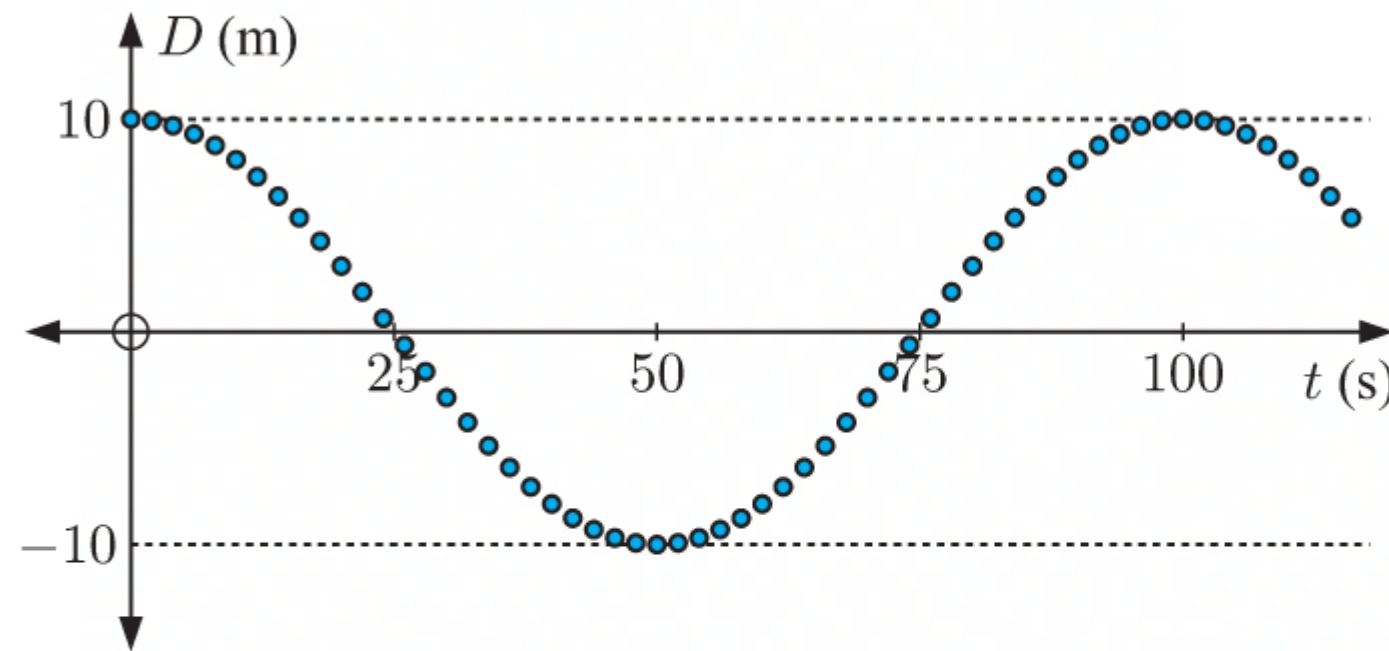
\therefore the maximum amount of feed available is about 12.6 tonnes.

8

- a** The light is initially $10 + 2 = 12$ m above the ground (its mean height) and oscillates with amplitude 10 m and period 100 seconds.



- b** Assume that when the light is directly above or below the centre of the Ferris wheel, its horizontal position is 0 m. The horizontal position of the light oscillates with amplitude 10 m and period 100 seconds. The light initially has horizontal position 10 m.



- c** Both graphs are periodic with an amplitude of 10 m and a period of 100 s. The graph of H is a sine graph with principal axis $H = 12$. The graph of D is a cosine graph with principal axis $D = 0$.

- d i** For the height, H :

The amplitude is 10, so $a = 10$.

The period is $\frac{360}{b} = 100$

$$\therefore 100b = 360$$

$$\therefore b = \frac{18}{5}$$

The principal axis is $H = 12$, so $d = 12$.

$$\therefore H = 10 \sin\left(\frac{18}{5}t\right)^\circ + 12 \text{ m}$$

- ii** The graph of D has the same amplitude and period as the graph of H .

The graph of D is a cosine graph with principal axis $D = 0$.

$$\therefore D = 10 \cos\left(\frac{18}{5}t\right)^\circ \text{ m}$$

Note: The function of horizontal displacement of the light will be different depending on how the coordinate system is defined.

- 9 a** For $H(t) = a \sin(bt)^\circ + d$:

The wheel has radius 3 metres, so the amplitude $a = 3$.

The period is 4 seconds, so $\frac{360}{b} = 4$
 $\therefore b = 90$

When $t = 0$, X is $3 + 1 = 4$ m above the bottom of the boat.

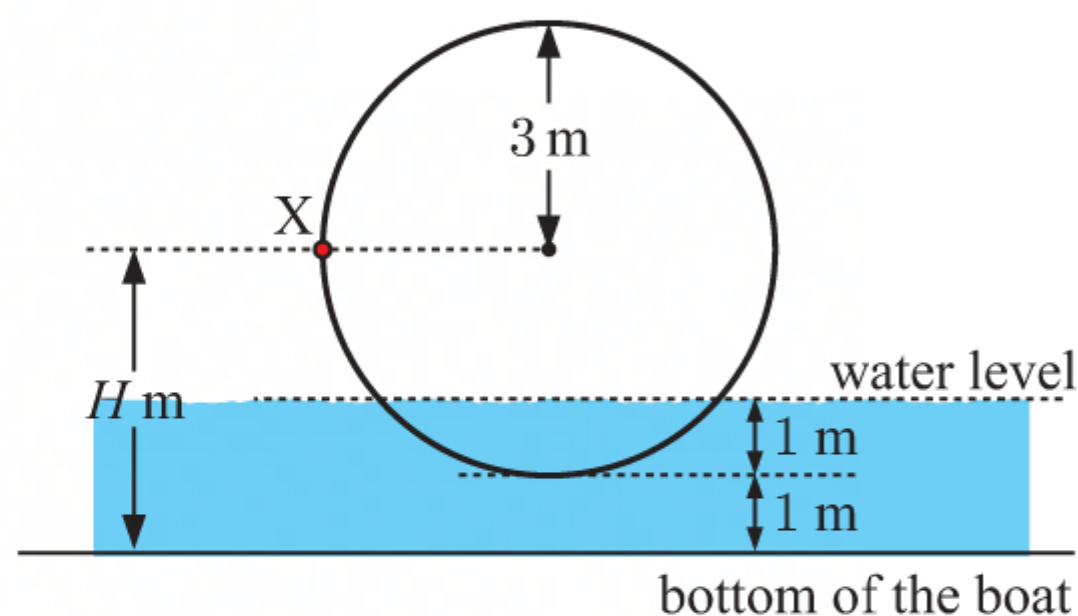
$$\therefore H(0) = 4, \text{ so } d = 4$$

$$\therefore H(t) = 3 \sin(90t)^\circ + 4 \text{ m}$$

- b** $H(6.5) = 3 \sin(90 \times 6.5)^\circ + 4$
 ≈ 1.88

After 6.5 seconds, X is about 1.88 m above the bottom of the boat.

Since the water level is $1 + 1 = 2$ m above the bottom of the boat, X must be below the water after 6.5 seconds.



- 10 a** Let $h = a \cos(bt)^\circ + d$ cm be the height of the yo-yo above ground level after t seconds. The string is 60 cm long, so the amplitude $a = 60$. The yo-yo completes two revolutions every second, so the period is $\frac{1}{2}$ second.

$$\therefore \frac{360}{b} = \frac{1}{2}$$

$$\therefore b = 720$$

$$\text{So, } h = 60 \cos(720t)^\circ + d$$

The yo-yo is initially at its highest point, 130 cm above ground level.

$$\text{So, when } t = 0, \quad h = 130$$

$$\therefore 130 = 60 \cos(720 \times 0)^\circ + d$$

$$\therefore 130 = 60 \cos 0^\circ + d$$

$$\therefore 130 = 60 \times 1 + d$$

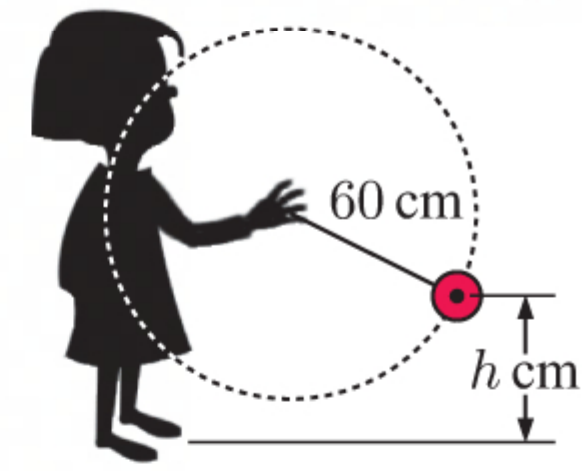
$$\therefore 130 = 60 + d$$

$$\therefore d = 70$$

$$\therefore h = 60 \cos(720t)^\circ + 70 \text{ cm}$$

- b** When $t = 3.9$, $h = 60 \cos(720 \times 3.9)^\circ + 70$
 ≈ 88.5

After 3.9 seconds, the height of the yo-yo above ground level is about 88.5 cm.

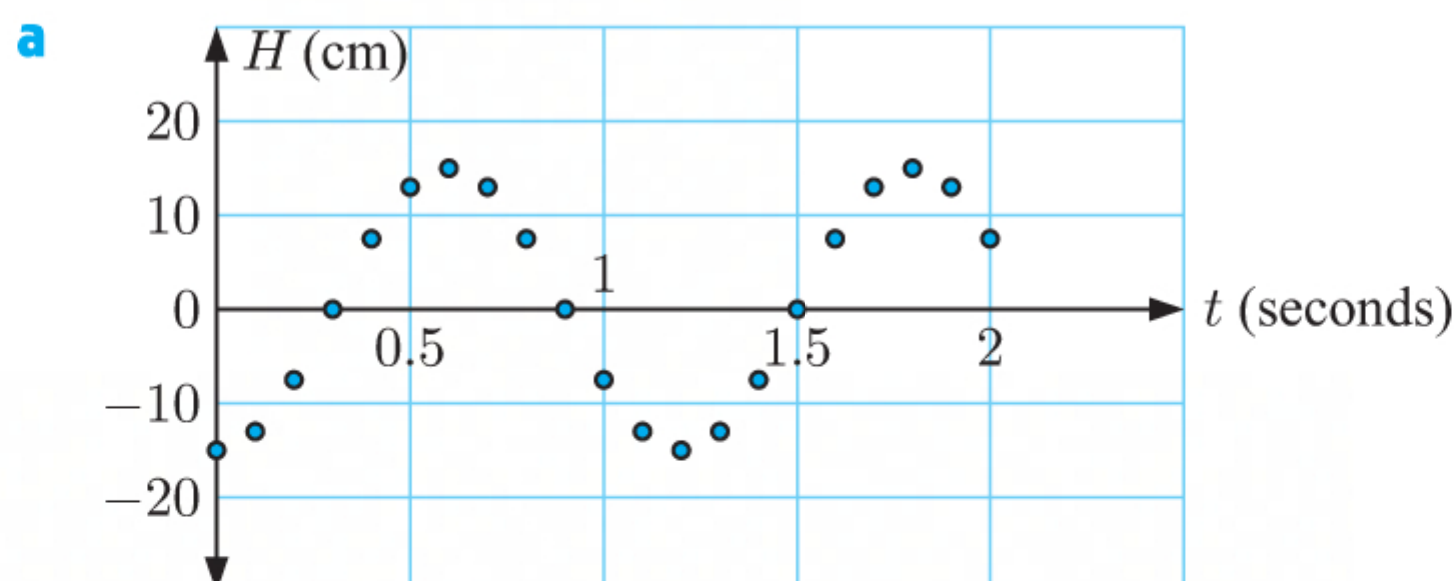


INVESTIGATION 2 MODELLING WITH TRIGONOMETRIC FUNCTIONS

Task 1: The undamped spring

1	Time (t seconds)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
	Height (H cm)	-15	-13	-7.5	0	7.5	13	15	13	7.5	0

	Time (t seconds)	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
	Height (H cm)	-7.5	-13	-15	-13	-7.5	0	7.5	13	15	13	7.5

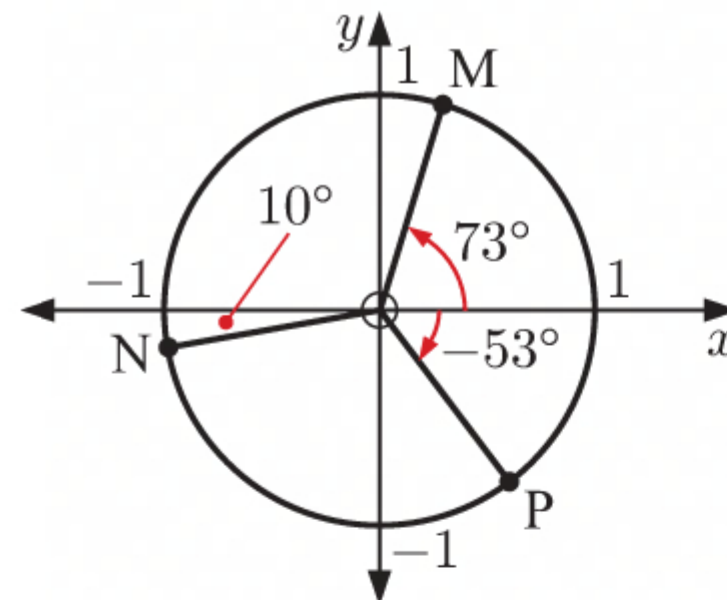


- b** The data appears to repeat itself in a horizontal direction, in intervals of the same length.
 \therefore the diagram shows periodic behaviour, so a trigonometric model is appropriate.
- 2 a** The principal axis of the oscillation is $H = 0$.
- b** The principal axis is $H = d$ $\therefore d = 0$.

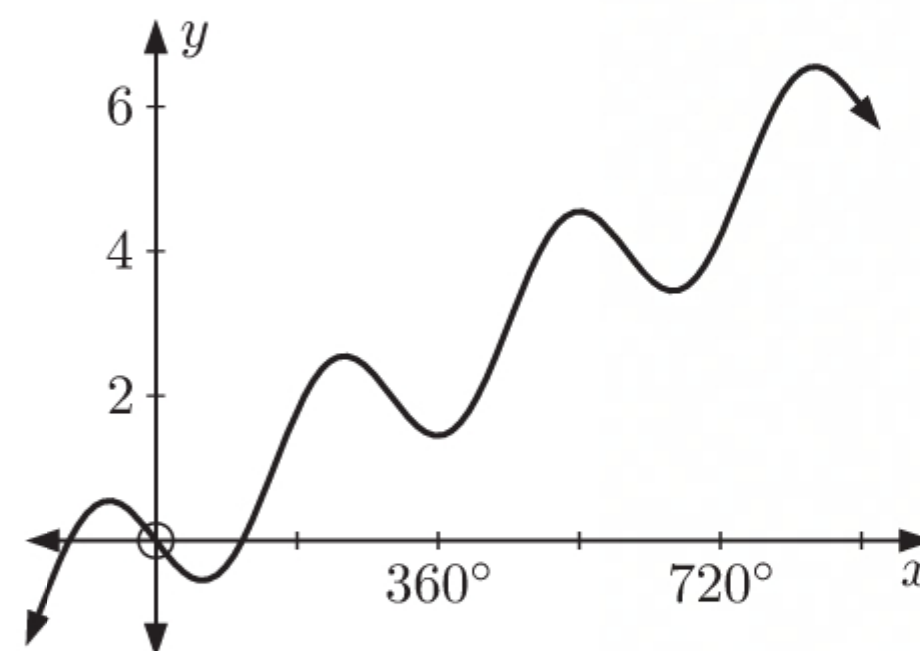
- 3 a** The amplitude of the oscillation is 15 cm.
b The spring starts below its resting position.
 $\therefore a < 0$, and so $a = -15$ {using **a**}
- 4 a** The period of the oscillation is 1.2 seconds.
b The period is 1.2 seconds, so $\frac{360}{b} = 1.2$
 $\therefore b = \frac{360}{1.2}$
 $\therefore b = 300$
- 5** The height of the object relative to its resting position after t seconds is $H = -15 \cos(300t)^\circ$ cm.
- 6** When $t = 4.25$, $H = -15 \cos(300 \times 4.25)^\circ$
 ≈ 14.5
 The height of the object after 4.25 seconds is about 14.5 cm above its resting position.
- 7** This model is unrealistic since the spring will not oscillate indefinitely. It will slow down and come to a stop.

REVIEW SET 9A

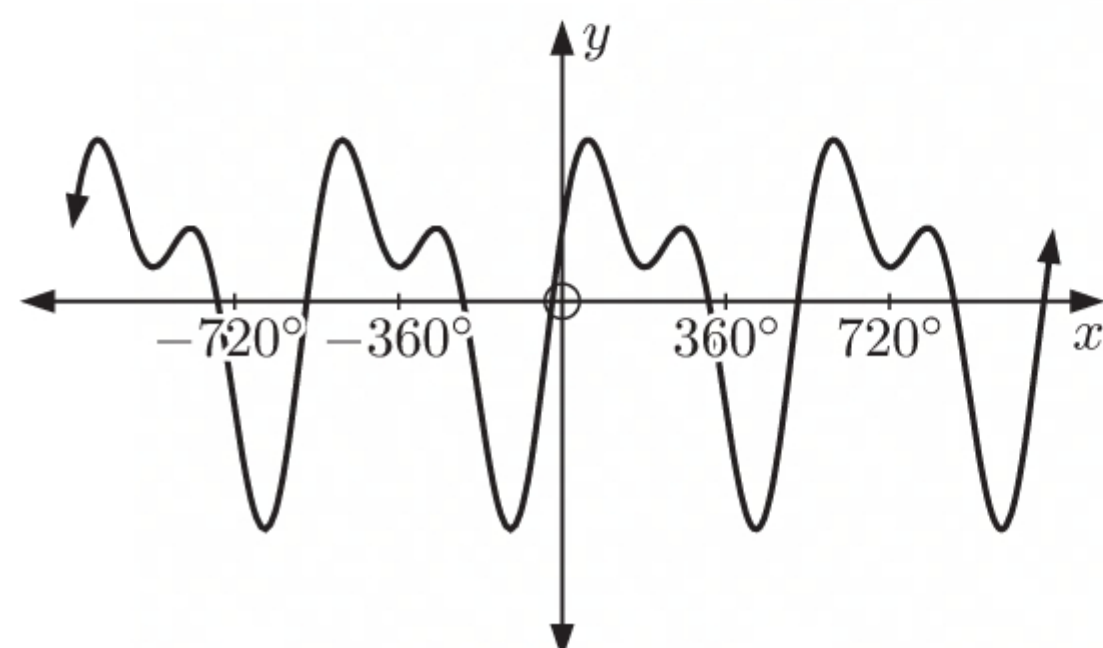
- 1** $M(\cos 73^\circ, \sin 73^\circ) \approx M(0.292, 0.956)$,
 $N(\cos 190^\circ, \sin 190^\circ) \approx N(-0.985, -0.174)$,
 $P(\cos(-53^\circ), \sin(-53^\circ)) = P(\cos 307^\circ, \sin 307^\circ)$
 $\approx P(0.602, -0.799)$



- 2 a** This graph repeats itself over and over in intervals of the same length, but not in a horizontal direction.
 \therefore this graph does not show periodic behaviour.

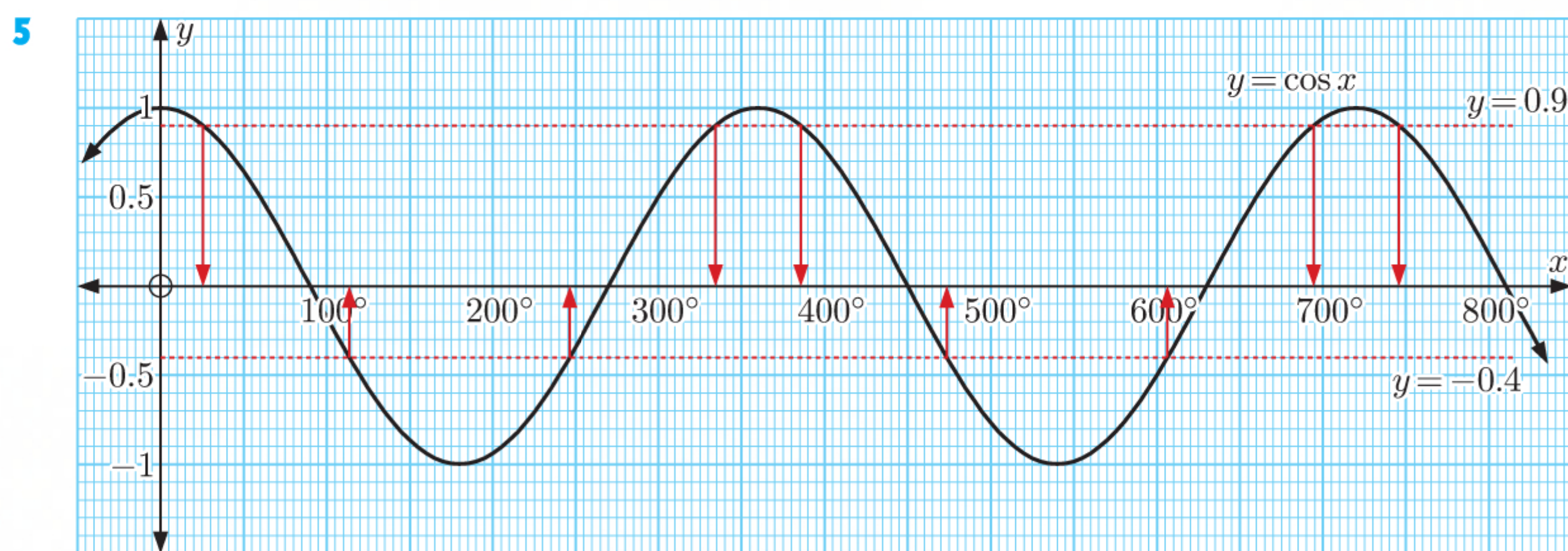


- b** This graph repeats itself over and over in a horizontal direction, in intervals of the same length.
 \therefore this graph shows periodic behaviour.



- 3 a** The amplitude of $y = 5 \cos 2x + 3$ is $|5| = 5$.
- b** The amplitude of $y = -\frac{1}{4} \sin x - 2$ is $|\frac{1}{4}| = \frac{1}{4}$.

- 4 a** The period of $y = 4 \sin \frac{x}{5}$ is $\frac{360^\circ}{\frac{1}{5}} = 1800^\circ$.
- b** The period of $y = -2 \cos 4x$ is $\frac{360^\circ}{4} = 90^\circ$.
- c** The period of $y = 4 \cos \frac{x}{2} + 4$ is $\frac{360^\circ}{\frac{1}{2}} = 720^\circ$.
- d** The period of $y = \frac{1}{2} \sin 3x$ is $\frac{360^\circ}{3} = 120^\circ$.



- a** When $\cos x = -0.4$, $0^\circ \leq x \leq 800^\circ$, $x \approx 115^\circ, 245^\circ, 475^\circ$, or 605° .
- b** When $\cos x = 0.9$, $0^\circ \leq x \leq 800^\circ$, $x \approx 25^\circ, 335^\circ, 385^\circ, 695^\circ$, or 745° .
- 6** $y = -3 \sin \frac{x}{4} + 1$ has period $= \frac{360^\circ}{\frac{1}{4}} = 1440^\circ$

$$\text{amplitude} = |-3| = 3$$

$$\text{maximum value} = -3(-1) + 1 = 4 \quad \{\text{when } \sin \frac{x}{4} = -1\}$$

$$\text{and minimum value} = -3(1) + 1 = -2 \quad \{\text{when } \sin \frac{x}{4} = 1\}$$

$$y = 2 \cos 5x - 7 \text{ has period} = \frac{360^\circ}{5} = 72^\circ$$

$$\text{amplitude} = |2| = 2$$

$$\text{maximum value} = 2(1) - 7 = -5 \quad \{\text{when } \cos 5x = 1\}$$

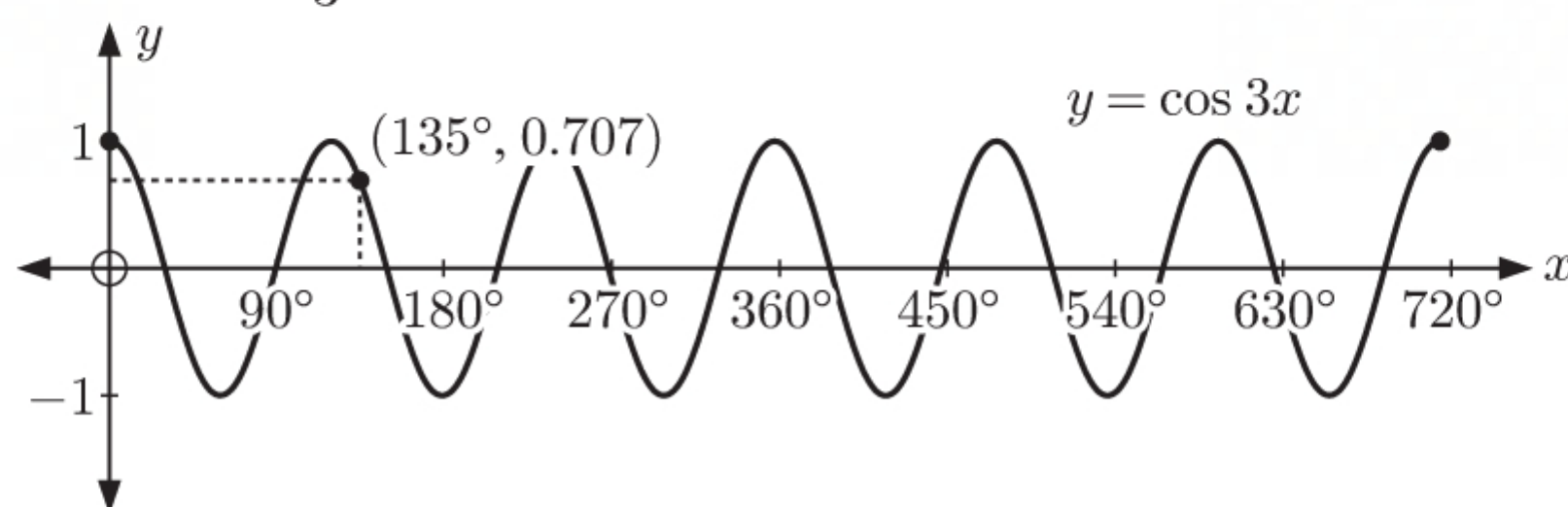
$$\text{and minimum value} = 2(-1) - 7 = -9 \quad \{\text{when } \cos 5x = -1\}$$

So,

Function	Period	Amplitude	Range
$y = -3 \sin \frac{x}{4} + 1$	1440°	3	$-2 \leq y \leq 4$
$y = 2 \cos 5x - 7$	72°	2	$-9 \leq y \leq -5$

- 7 a** $y = \cos 3x$ is a horizontal stretch of $y = \cos x$ with scale factor $\frac{1}{3}$.

The period is $\frac{360^\circ}{3} = 120^\circ$.



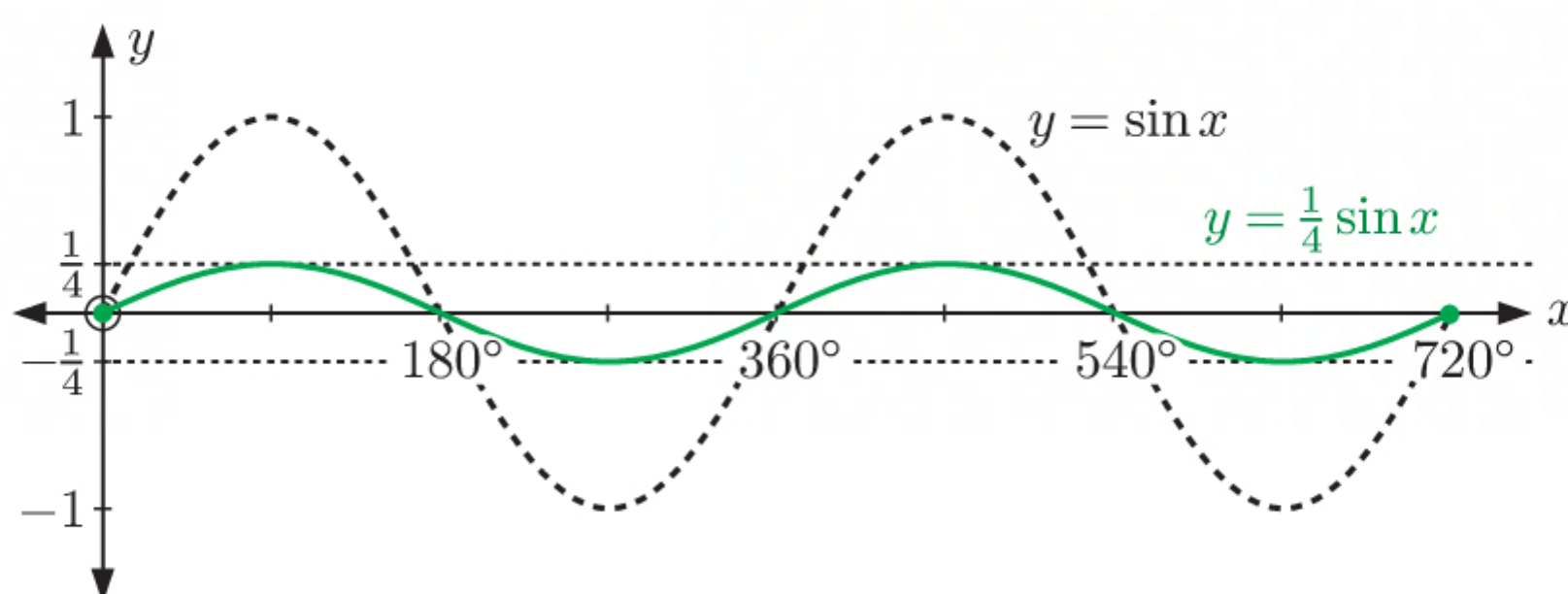
- b** When $x = 135^\circ$, $y = \cos(3 \times 135^\circ)$
 $= \cos 405^\circ$
 $= \cos(45^\circ + 360^\circ)$
 $= \cos 45^\circ \quad \{\cos(\theta + 360^\circ) = \cos \theta\}$
 $= \frac{1}{\sqrt{2}} \approx 0.707$

- 8** $\sin x \xrightarrow[\text{horizontal stretch}]{\text{scale factor } \frac{1}{2}} \sin 2x \xrightarrow[\text{vertical stretch}]{\text{scale factor } 3} 3 \sin 2x$

A horizontal stretch with scale factor $\frac{1}{2}$, then a vertical stretch with scale factor 3 maps $y = \sin x$ onto $y = 3 \sin 2x$.

- 9 a** $a = \frac{1}{4}$, so the amplitude is $|\frac{1}{4}| = \frac{1}{4}$.

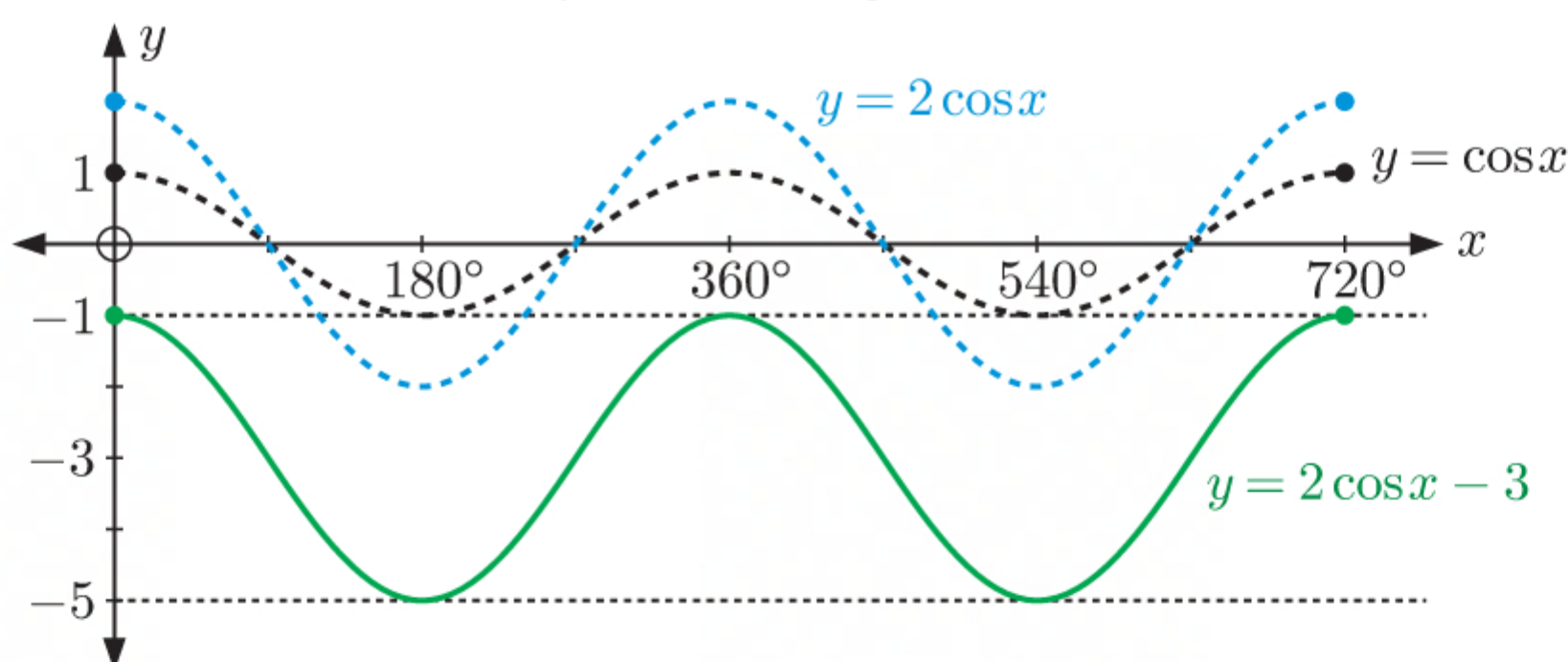
We stretch $y = \sin x$ vertically with scale factor $\frac{1}{4}$ to give $y = \frac{1}{4} \sin x$.



- b** $a = 2$, so the amplitude is $|2| = 2$.

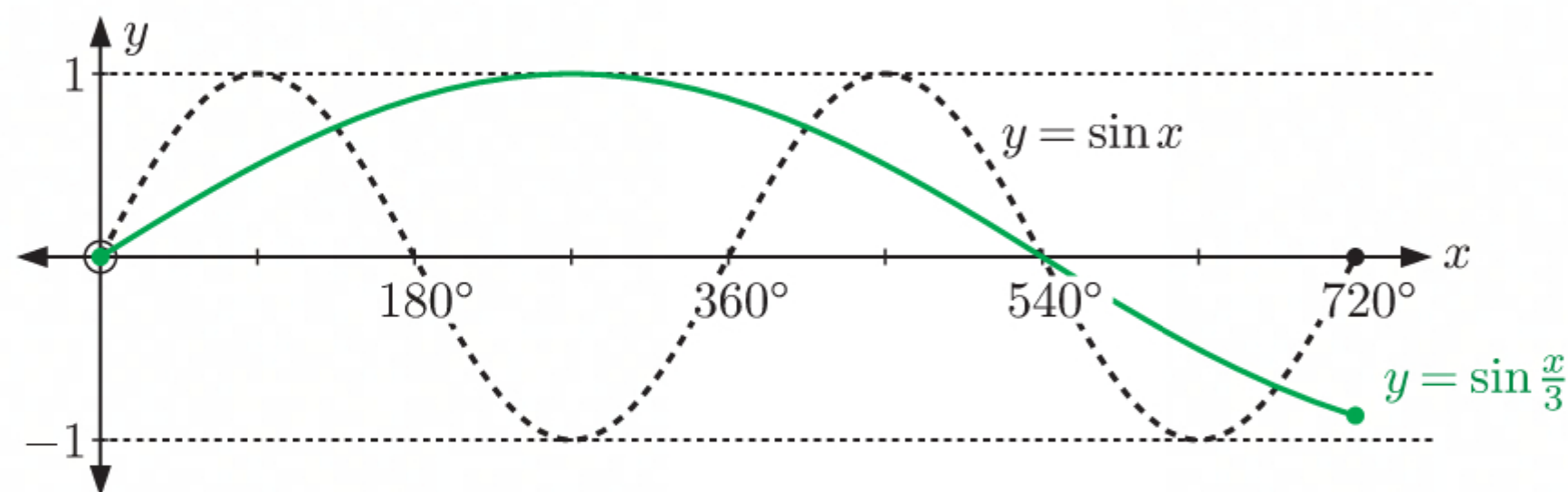
$d = -3$, so the principal axis is $y = -3$.

We stretch $y = \cos x$ vertically with scale factor 2 to give $y = 2 \cos x$, then translate $y = 2 \cos x$ downwards by 3 units to give $y = 2 \cos x - 3$.



- c $b = \frac{1}{3}$, so the period is $\frac{360^\circ}{b} = \frac{360^\circ}{\frac{1}{3}} = 1080^\circ$.

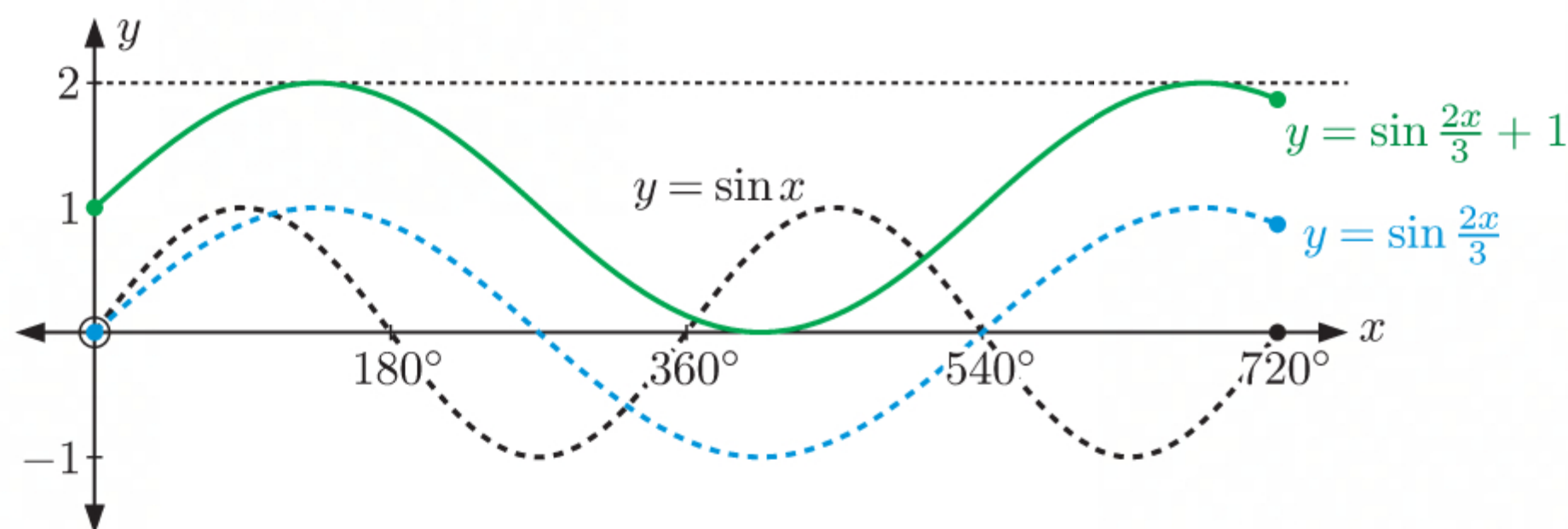
We stretch $y = \sin x$ horizontally with scale factor 3 to give $y = \sin \frac{x}{3}$.



- d $b = \frac{2}{3}$, so the period is $\frac{360^\circ}{b} = \frac{360^\circ}{\frac{2}{3}} = 540^\circ$.

$d = 1$, so the principal axis is $y = 1$.

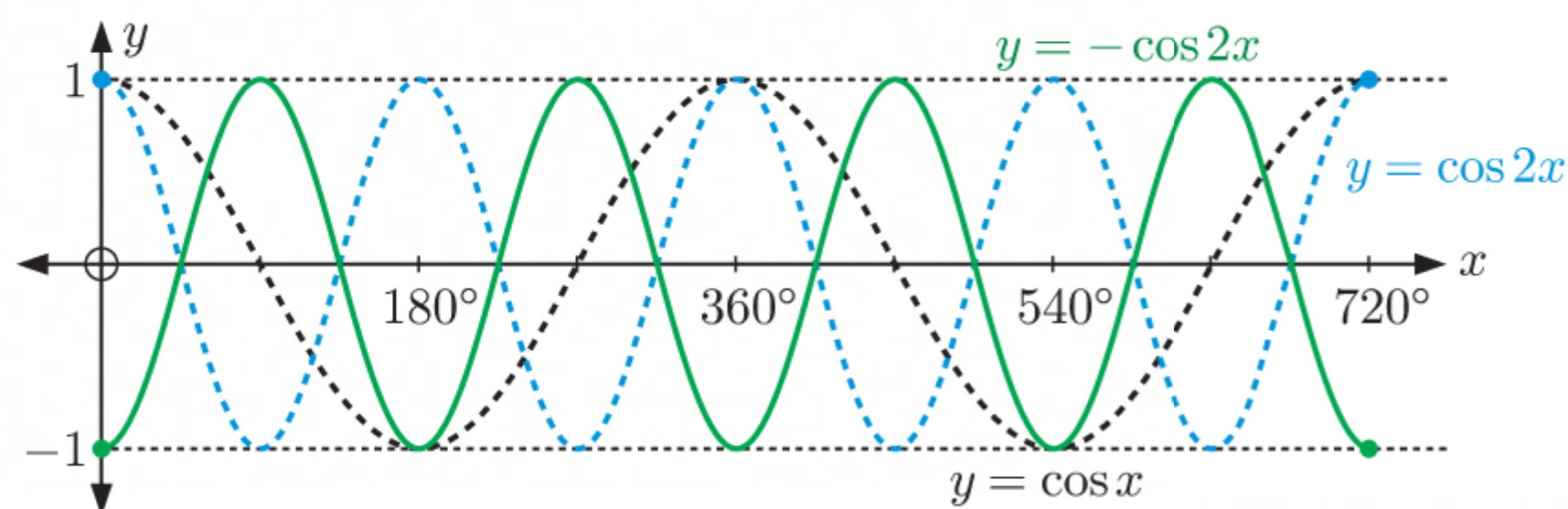
We stretch $y = \sin x$ horizontally with scale factor $\frac{3}{2}$ to give $y = \sin \frac{2x}{3}$, then translate $y = \sin \frac{2x}{3}$ upwards by 1 unit to give $y = \sin \frac{2x}{3} + 1$.



- e $a = -1$, so the amplitude is $|-1| = 1$.

$b = 2$, so the period is $\frac{360^\circ}{b} = \frac{360^\circ}{2} = 180^\circ$.

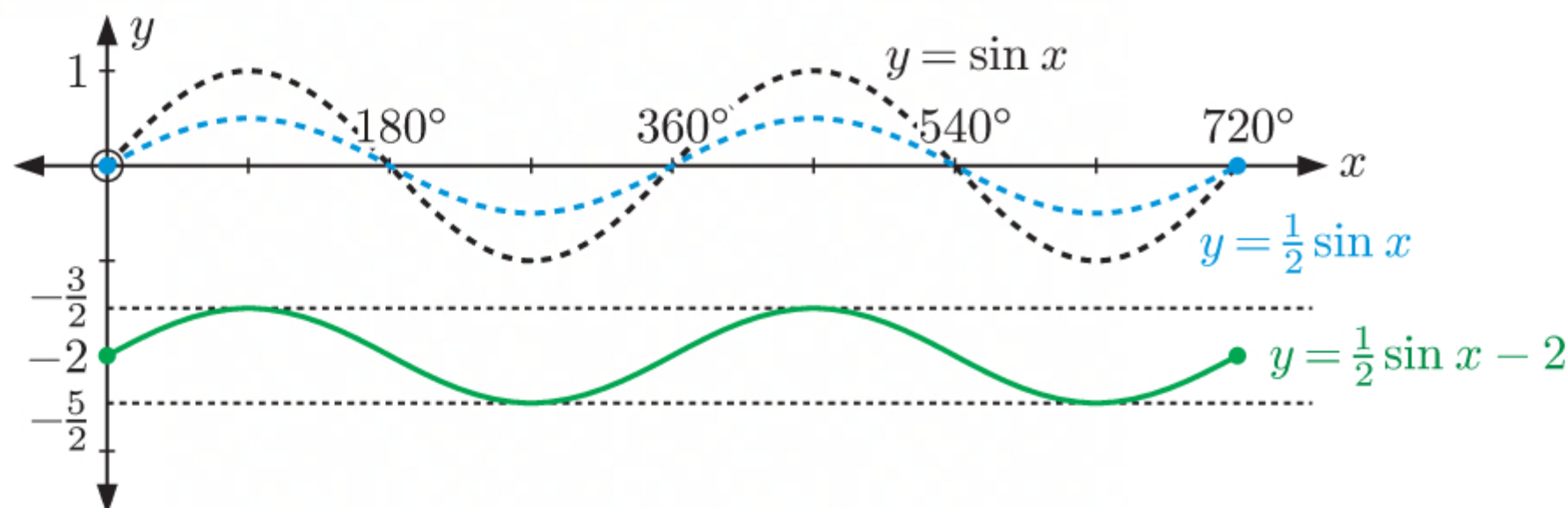
We stretch $y = \cos x$ horizontally with scale factor $\frac{1}{2}$ to give $y = \cos 2x$, then reflect $y = \cos 2x$ in the x -axis to give $y = -\cos 2x$.



f $a = \frac{1}{2}$, so the amplitude is $|\frac{1}{2}| = \frac{1}{2}$.

$d = -2$, so the principal axis is $y = -2$.

We stretch $y = \sin x$ vertically with scale factor $\frac{1}{2}$ to give $y = \frac{1}{2} \sin x$, then translate $y = \frac{1}{2} \sin x$ downwards by 2 units to give $y = \frac{1}{2} \sin x - 2$.

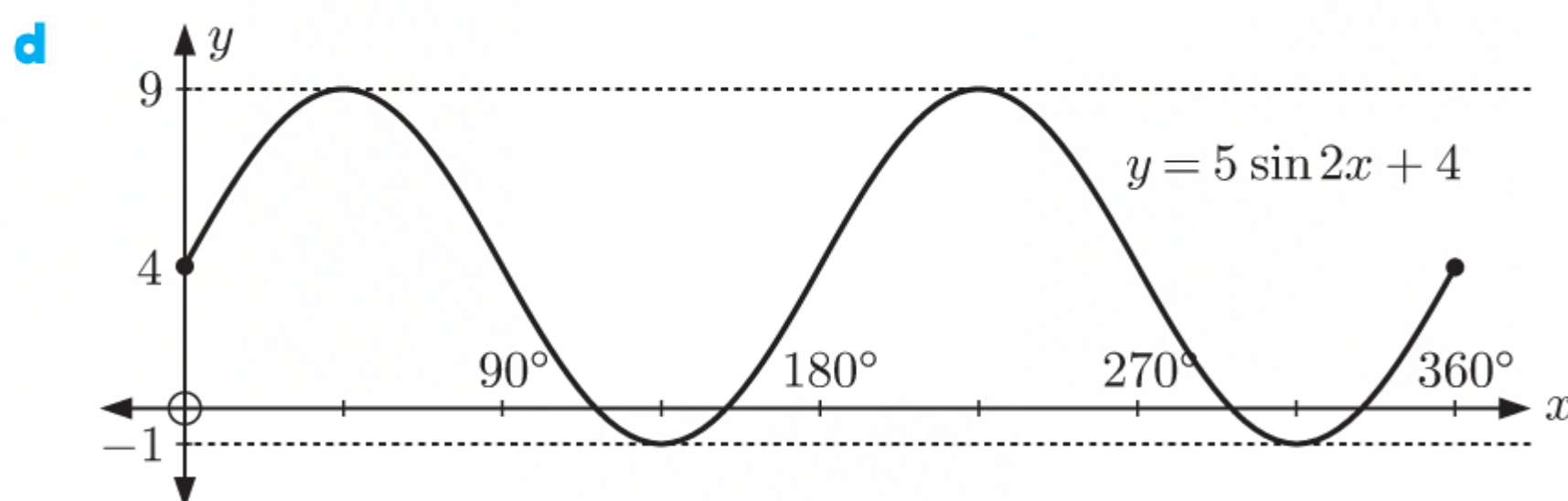


10 $y = 5 \sin 2x + 4$

a The amplitude is $|5| = 5$.

b The principal axis is $y = 4$.

c The period is $\frac{360^\circ}{2} = 180^\circ$.

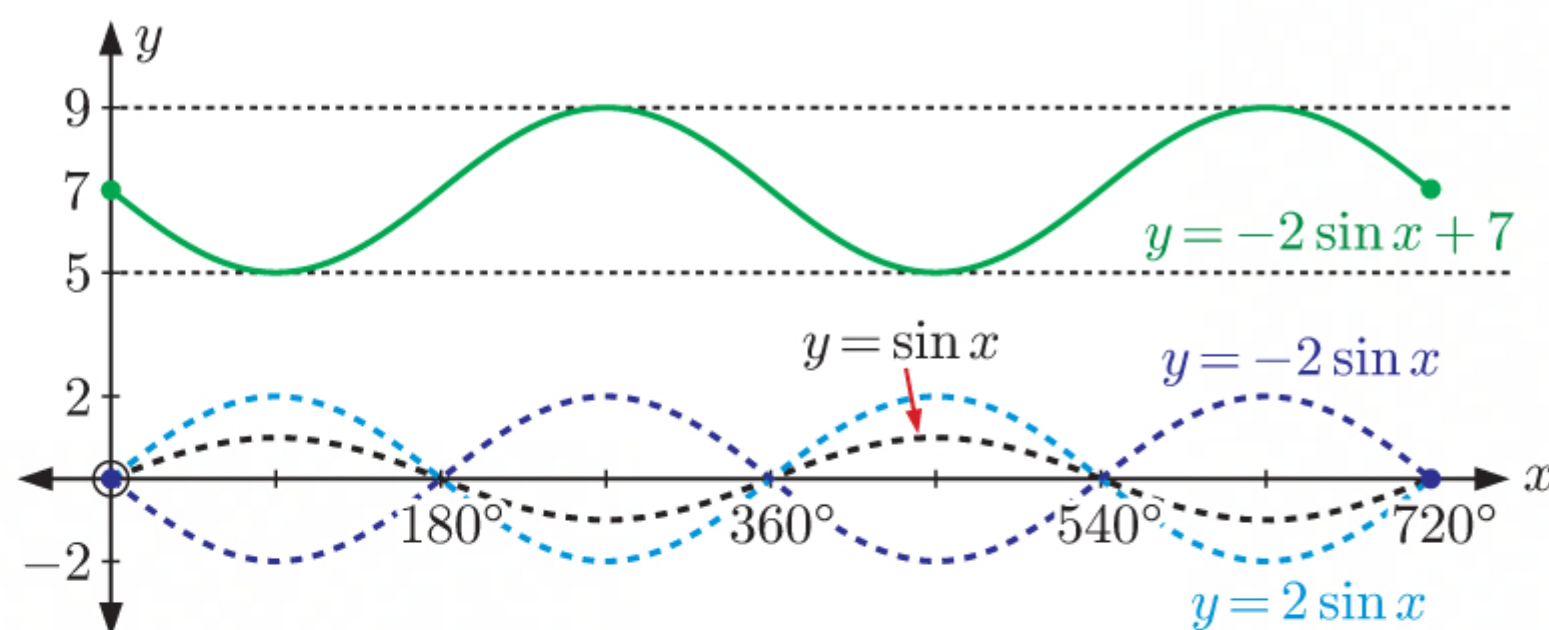


e When $x = 45^\circ$, $y = 5 \sin(2 \times 45^\circ) + 4$
 $= 5 \sin 90^\circ + 4$
 $= 5 \times 1 + 4$
 $= 9$

11 a $a = -2$, so the amplitude is $|-2| = 2$.

$d = 7$, so the principal axis is $y = 7$.

We stretch $y = \sin x$ vertically with scale factor 2 to give $y = 2 \sin x$, then reflect $y = 2 \sin x$ in the x -axis to give $y = -2 \sin x$, then translate $y = -2 \sin x$ upwards by 7 units to give $y = -2 \sin x + 7$.

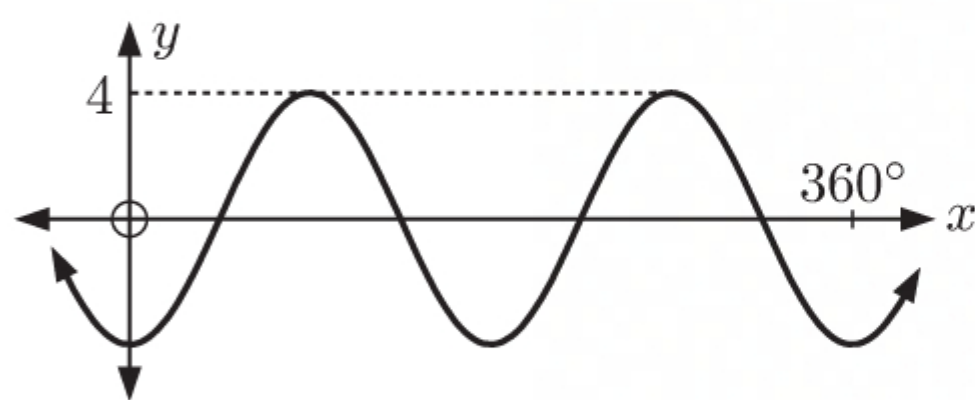


b When $x = 150^\circ$, $y = -2 \sin 150^\circ + 7$
 $= -2 \times \frac{1}{2} + 7$
 $= 6$

- c From the graph, the maximum value of y is 9, which occurs when $\sin x = -1$.
 $\therefore x = 270^\circ$ or 630°

- d From the graph, the minimum value of y is 5, which occurs when $\sin x = 1$.
 $\therefore x = 90^\circ$ or 450°

12 a



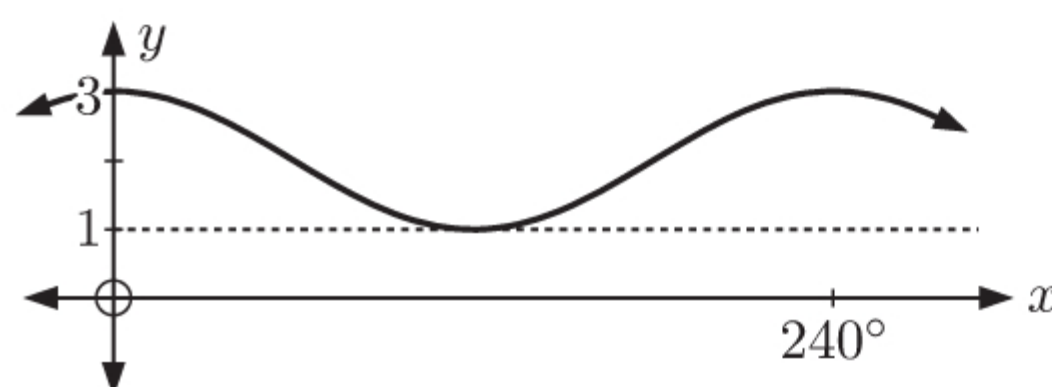
The amplitude is 4, and $a < 0$, so $a = -4$.

The period is 180° , so $\frac{360^\circ}{b} = 180^\circ$
 $\therefore b = 2$

The principal axis is $y = 0$, so $d = 0$.

The equation of the function is $y = -4 \cos 2x$.

b



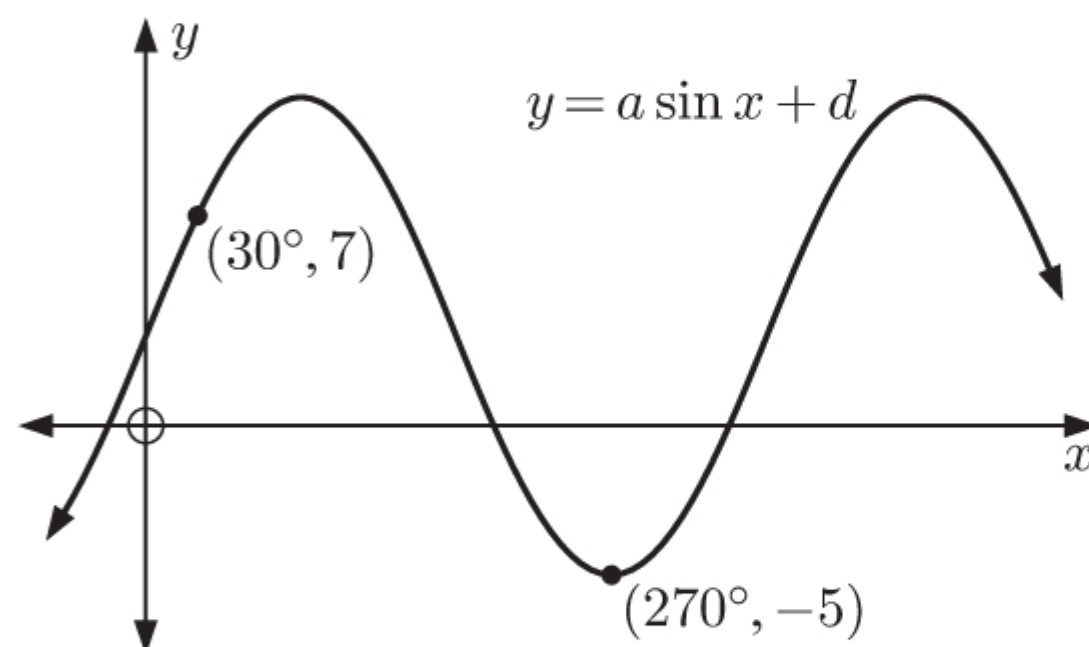
The amplitude is 1, so $a = 1$.

The period is 240° , so $\frac{360^\circ}{b} = 240^\circ$
 $\therefore b = \frac{3}{2}$

The principal axis is $y = 2$, so $d = 2$.

The equation of the function is $y = \cos \frac{3x}{2} + 2$.

13 a



When $x = 30^\circ$, $y = 7$

$$\therefore a \sin 30^\circ + d = 7$$

$$\therefore \frac{1}{2}a + d = 7 \quad \dots (1)$$

When $x = 270^\circ$, $y = -5$

$$\therefore a \sin 270^\circ + d = -5$$

$$\therefore -a + d = -5 \quad \dots (2)$$

	a	b	c
1	0.5	1	7
2	-1	1	-5

-5

SOLVE DELETE CLEAR EDIT

	X	Y
1	8	3

8

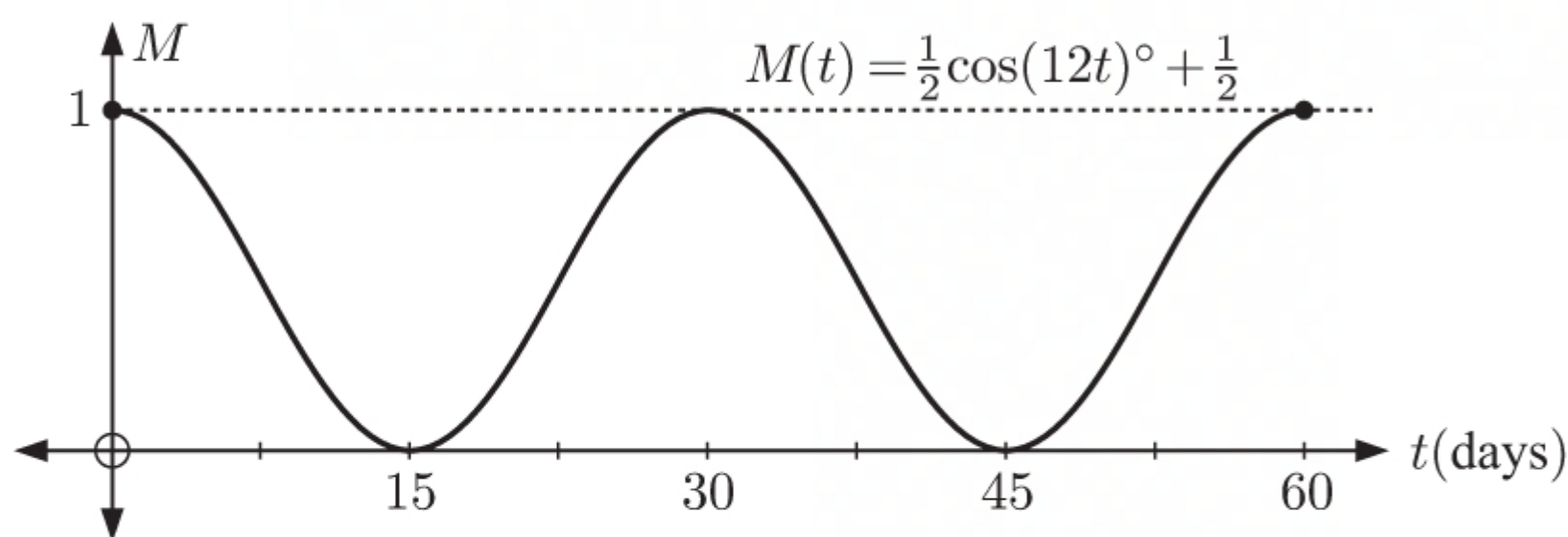
REPEAT

Solving (1) and (2) simultaneously using technology gives $a = 8$ and $d = 3$.

- b** $y = 8 \sin x + 3$ has maximum value $8(1) + 3 = 11$ {when $\sin x = 1$ }
 and minimum value $8(-1) + 3 = -5$ {when $\sin x = -1$ }
 \therefore the range is $\{y \mid -5 \leq y \leq 11\}$.

14 a For $M(t) = \frac{1}{2} \cos(12t)^\circ + \frac{1}{2}$:

- the amplitude is $\left| \frac{1}{2} \right| = \frac{1}{2}$
- the period is $\frac{360}{12} = 30$ days
- the principal axis is $M = \frac{1}{2}$.



b i January 6th is 5 days after January 1st.

$$\begin{aligned} M(5) &= \frac{1}{2} \cos(12 \times 5)^\circ + \frac{1}{2} \\ &= \frac{1}{2} \cos 60^\circ + \frac{1}{2} \\ &= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \\ &= \frac{1}{4} + \frac{1}{2} \\ &= \frac{3}{4} = 0.75 \end{aligned}$$

\therefore the fraction of the moon illuminated on the night of January 6th is 0.75.

ii January 21st is 20 days after January 1st.

$$\begin{aligned} M(20) &= \frac{1}{2} \cos(12 \times 20)^\circ + \frac{1}{2} \\ &= \frac{1}{2} \cos 240^\circ + \frac{1}{2} \\ &= \frac{1}{2} \times \left(-\frac{1}{2}\right) + \frac{1}{2} \\ &= -\frac{1}{4} + \frac{1}{2} \\ &= \frac{1}{4} = 0.25 \end{aligned}$$

\therefore the fraction of the moon illuminated on the night of January 21st is 0.25.

iii January 27th is 26 days after January 1st.

$$\begin{aligned} M(26) &= \frac{1}{2} \cos(12 \times 26)^\circ + \frac{1}{2} \\ &\approx 0.835 \end{aligned}$$

\therefore the fraction of the moon illuminated on the night of January 27th is about 0.835.

iv February 19th is 49 days after January 1st.

$$\begin{aligned} M(49) &= \frac{1}{2} \cos(12 \times 49)^\circ + \frac{1}{2} \\ &\approx 0.165 \end{aligned}$$

\therefore the fraction of the moon illuminated on the night of February 19th is about 0.165.

c The period is 30 days, so a full moon occurs once every 30 days.

- d** The moon is not illuminated at all when $M = 0$, which occurs when $t = 15$ or 45 .
 $t = 15$ corresponds to January 16, and $t = 45$ corresponds to February 15.
 \therefore the moon is not illuminated at all on January 16th and February 15th.

15 $H(t) = a \cos(bt)^\circ + d$

- a** The amplitude $|a| = \frac{14}{2}$
 $\therefore |a| = 7$

The tide is initially high, so $a > 0 \therefore a = 7$.

The period is about 12.4 hours, so $\frac{360}{b} = 12.4$
 $\therefore b = \frac{360}{12.4}$
 $\therefore b = \frac{900}{31}$

- b** High tide is 16.2 m.

\therefore low tide is $16.2 - 14 = 2.2$

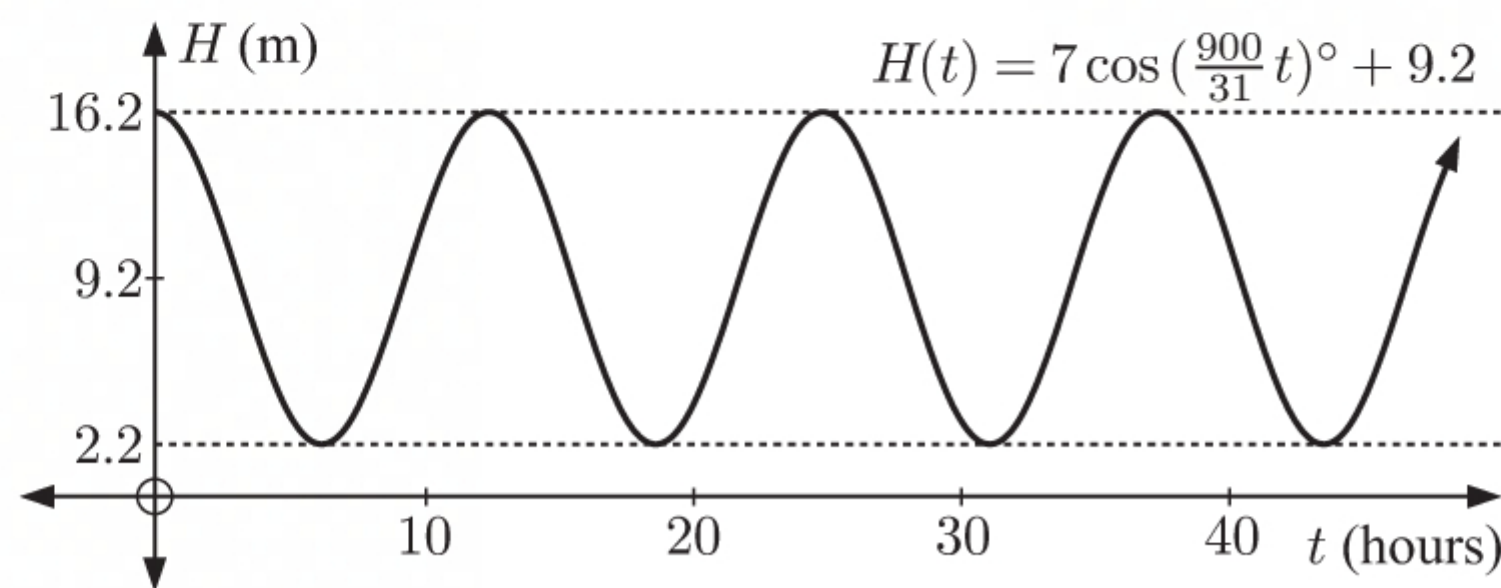
\therefore the principal axis is $y = \frac{16.2 + 2.2}{2}$

$$\therefore y = 9.2$$

So, $d = 9.2$

- c** For $H(t) = 7 \cos\left(\frac{900}{31}t\right)^\circ + 9.2$:

- the amplitude is 7
- the period is 12.4 hours
- the principal axis is $H = 9.2$.



- d** $H(8) = 7 \cos\left(\frac{900}{31} \times 8\right)^\circ + 9.2$
 ≈ 4.92

The height of the tide 8 hours after the first high tide is about 4.92 m.

16 a $H = a \sin(bt)^\circ + d$

The diameter of the wheel is 16 cm, so the amplitude $|a| = \frac{16}{2} = 8$.

P is initially moving upwards, so $a > 0$

$\therefore a = 8$.

The hamster rotates through one revolution each second, so the period is 1 second.

$\therefore \frac{360}{b} = 1$

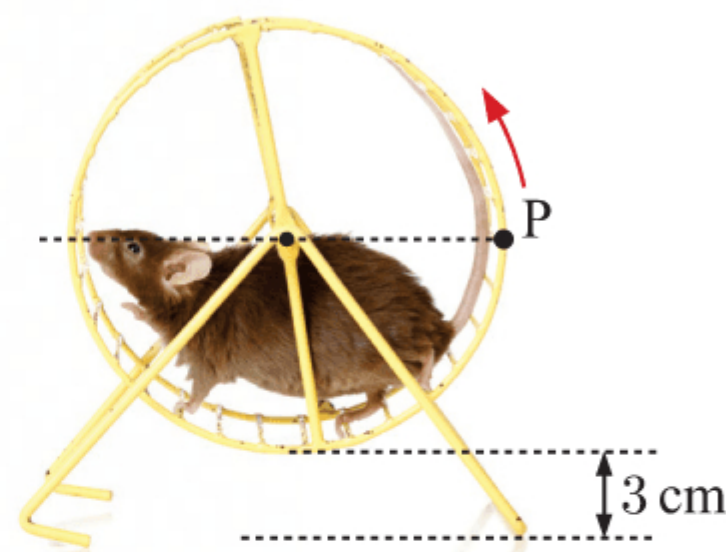
$\therefore b = 360$

The lowest position is 3 cm above ground level, so $d = \frac{3 + (3 + 16)}{2}$
 $\therefore d = 11$

So, $H = 8 \sin(360t)^\circ + 11$ cm.

b When $t = 4.7$, $H = 8 \sin(360 \times 4.7)^\circ + 11$
 ≈ 3.39

After 4.7 seconds, P is about 3.39 cm above ground level.

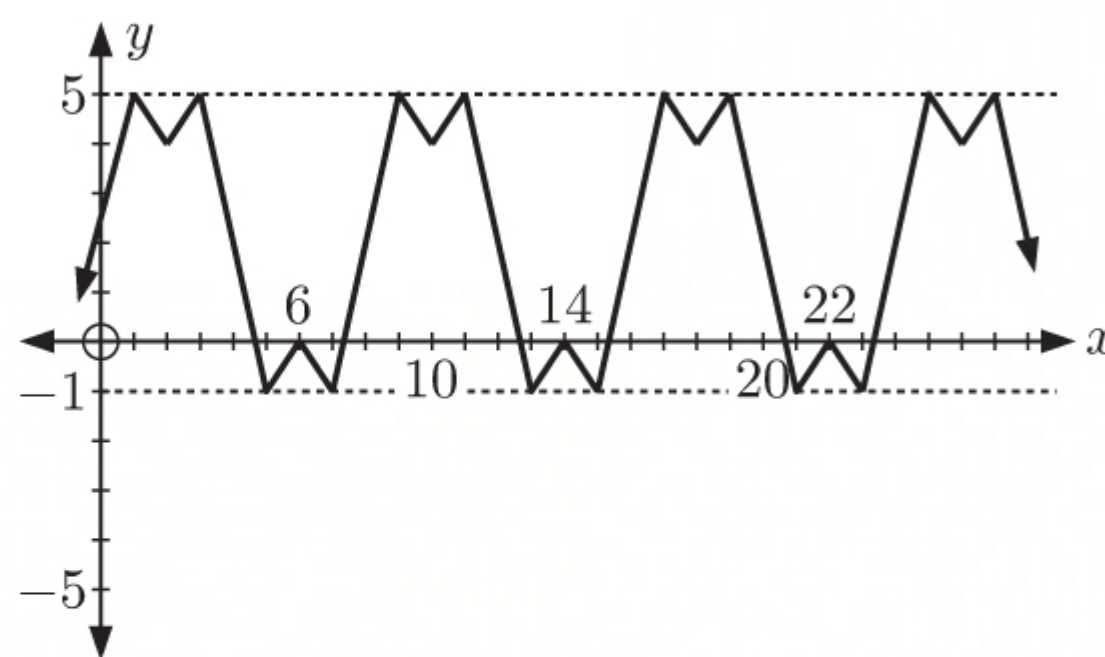


REVIEW SET 9B

1 a $\cos 500^\circ = \cos(360^\circ + 140^\circ)$
 $= \cos 140^\circ$

b $\sin(-100^\circ) = \sin(620^\circ - 720^\circ)$
 $= \sin(620^\circ)$

2 a The function repeats itself over and over in a horizontal direction, in intervals of length 8 units.



b i period = 8

ii maximum value = 5

iii minimum value = -1

3 a The period of $y = 2 \sin 3x$ is $\frac{360^\circ}{3} = 120^\circ$.

b The period of $y = -4 \cos \frac{x}{2} - 1$ is $\frac{360^\circ}{\frac{1}{2}} = 720^\circ$.

4 a The principal axis of $y = -\frac{1}{3} \sin x + 5$ is $y = 5$.

b The principal axis of $y = 2 \cos \frac{x}{3} - 4$ is $y = -4$.

5 $y = \sin bx, \quad b > 0$

a period = $\frac{360^\circ}{b}$

$$\therefore 1080^\circ = \frac{360^\circ}{b}$$

$$\therefore b = \frac{360^\circ}{1080^\circ}$$

$$\therefore b = \frac{1}{3}$$

b period = $\frac{360^\circ}{b}$

$$\therefore 15^\circ = \frac{360^\circ}{b}$$

$$\therefore b = \frac{360^\circ}{15^\circ}$$

$$\therefore b = 24$$

c period = $\frac{360^\circ}{b}$

$$\therefore 9^\circ = \frac{360^\circ}{b}$$

$$\therefore b = \frac{360^\circ}{9^\circ}$$

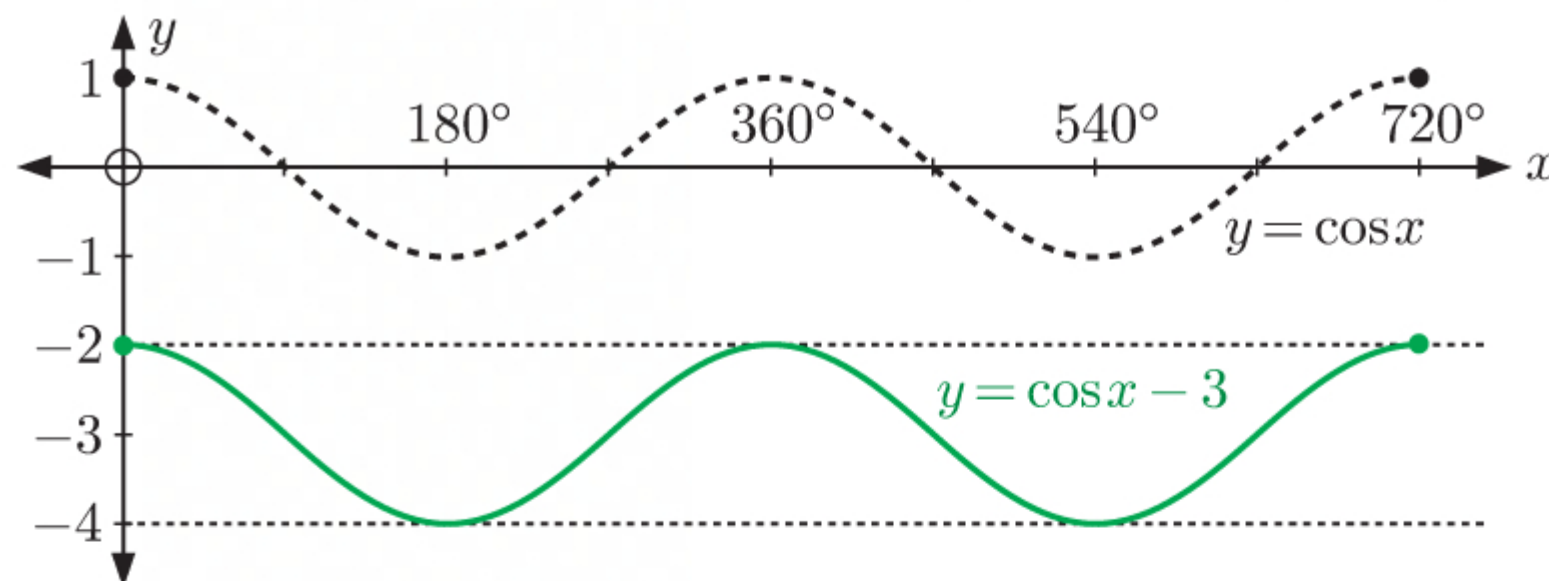
$$\therefore b = 40$$

6 a $y = 5 \sin x - 3$ has minimum value $5(-1) - 3 = -8$ {when $\sin x = -1$ }
and maximum value $5(1) - 3 = 2$ {when $\sin x = 1$ }

b $y = \frac{1}{3} \cos x + 1$ has minimum value $\frac{1}{3}(-1) + 1 = \frac{2}{3}$ {when $\cos x = -1$ }
and maximum value $\frac{1}{3}(1) + 1 = \frac{4}{3}$ {when $\cos x = 1$ }

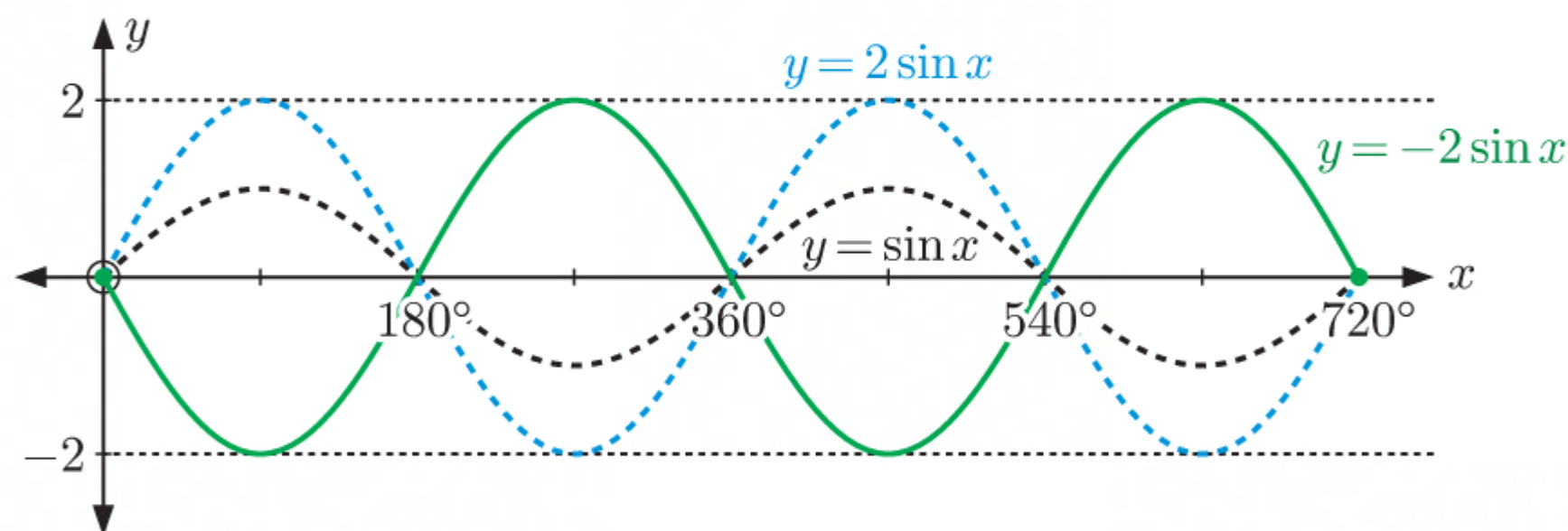
7 a $d = -3$, so the principal axis is $y = -3$.

We translate $y = \cos x$ downwards by 3 units to give $y = \cos x - 3$.



b $a = -2$, so the amplitude is $|-2| = 2$.

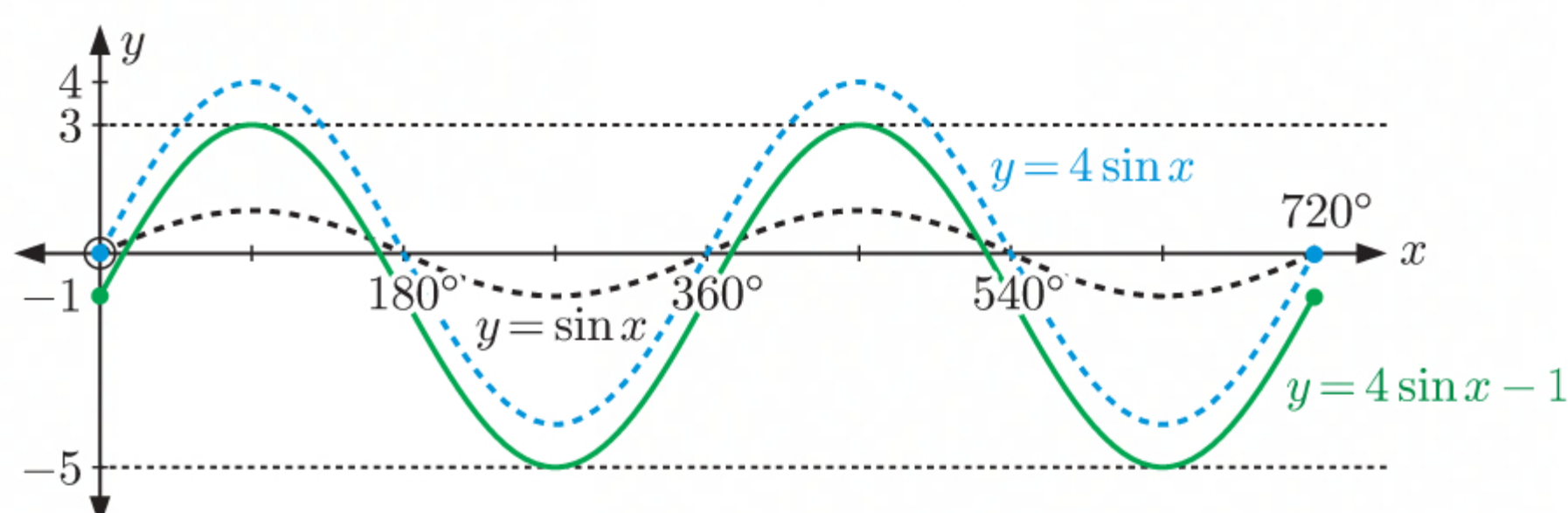
We stretch $y = \sin x$ vertically with scale factor 2 to give $y = 2 \sin x$, then reflect $y = 2 \sin x$ in the x -axis to give $y = -2 \sin x$.



c $a = 4$, so the amplitude is $|4| = 4$.

$d = -1$, so the principal axis is $y = -1$.

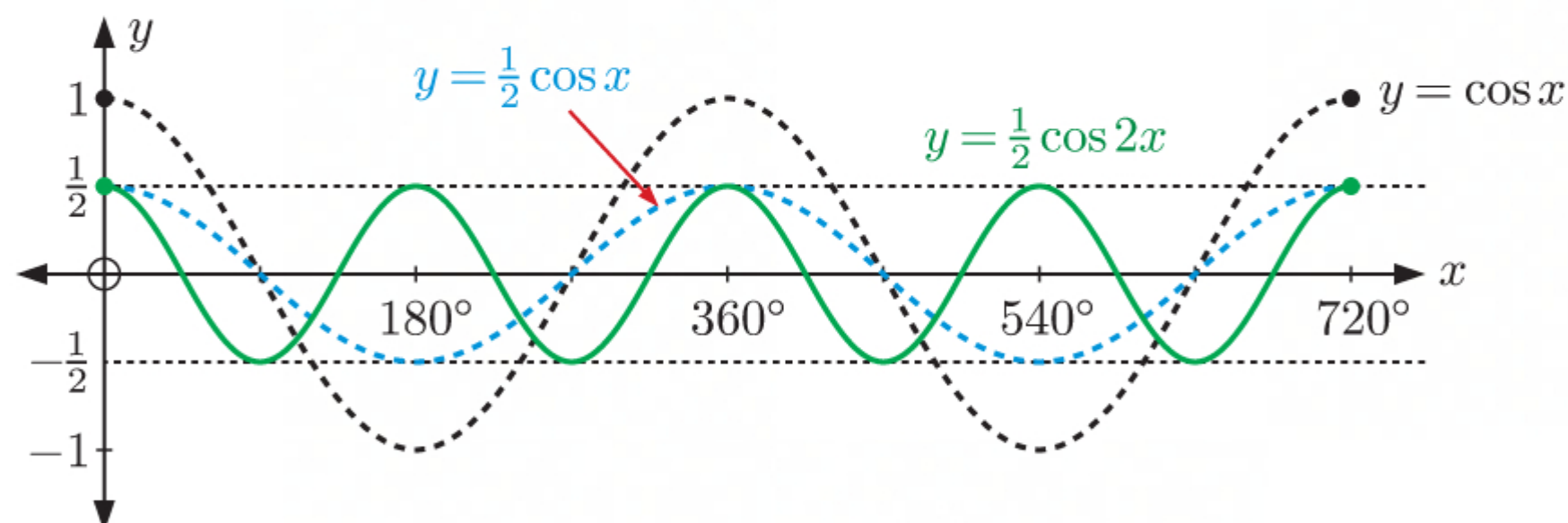
We stretch $y = \sin x$ vertically with scale factor 4 to give $y = 4 \sin x$, then translate $y = 4 \sin x$ downwards by 1 unit to give $y = 4 \sin x - 1$.



d $a = \frac{1}{2}$, so the amplitude is $\left| \frac{1}{2} \right| = \frac{1}{2}$.

$b = 2$, so the period is $\frac{360^\circ}{b} = \frac{360^\circ}{2} = 180^\circ$.

We stretch $y = \cos x$ vertically with scale factor $\frac{1}{2}$ to give $y = \frac{1}{2} \cos x$, then stretch $y = \frac{1}{2} \cos x$ horizontally with scale factor $\frac{1}{2}$ to give $y = \frac{1}{2} \cos 2x$.



8 a $\cos x \xrightarrow[\text{vertical stretch}]{\text{scale factor } \frac{1}{3}} \frac{1}{3} \cos x \xrightarrow[\text{vertical translation}]{1 \text{ unit upwards}} \frac{1}{3} \cos x + 1$

A vertical stretch with scale factor $\frac{1}{3}$, then a translation 1 unit upwards maps $y = \cos x$ onto $y = \frac{1}{3} \cos x + 1$.

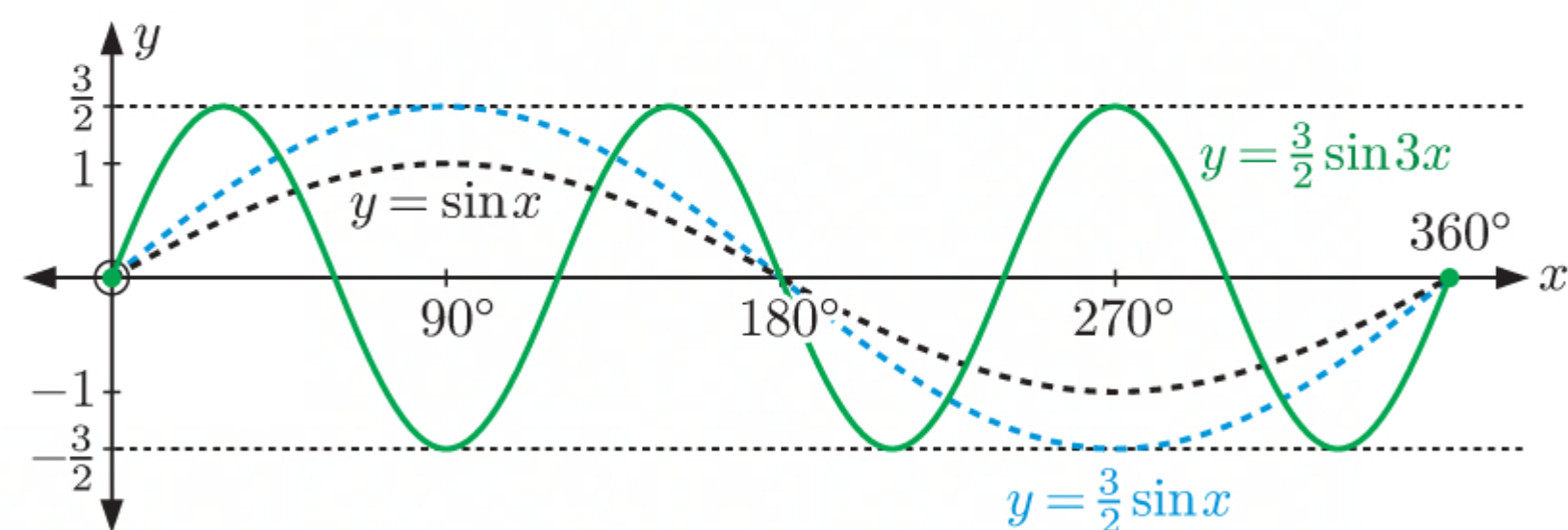
b $\sin x \xrightarrow[\text{reflection in}]{x\text{-axis}} -\sin x \xrightarrow[\text{horizontal stretch}]{\text{scale factor } \frac{2}{3}} -\sin \frac{3}{2}x$

A reflection in the x -axis, then a horizontal stretch with scale factor $\frac{2}{3}$ maps $y = \sin x$ onto $y = -\sin \frac{3}{2}x$.

9 a $a = \frac{3}{2}$, so the amplitude is $\left| \frac{3}{2} \right| = \frac{3}{2}$.

$b = 3$, so the period is $\frac{360^\circ}{b} = \frac{360^\circ}{3} = 120^\circ$.

We stretch $y = \sin x$ vertically with scale factor $\frac{3}{2}$ to give $y = \frac{3}{2} \sin x$, then stretch $y = \frac{3}{2} \sin x$ horizontally with scale factor $\frac{1}{3}$ to give $y = \frac{3}{2} \sin 3x$.

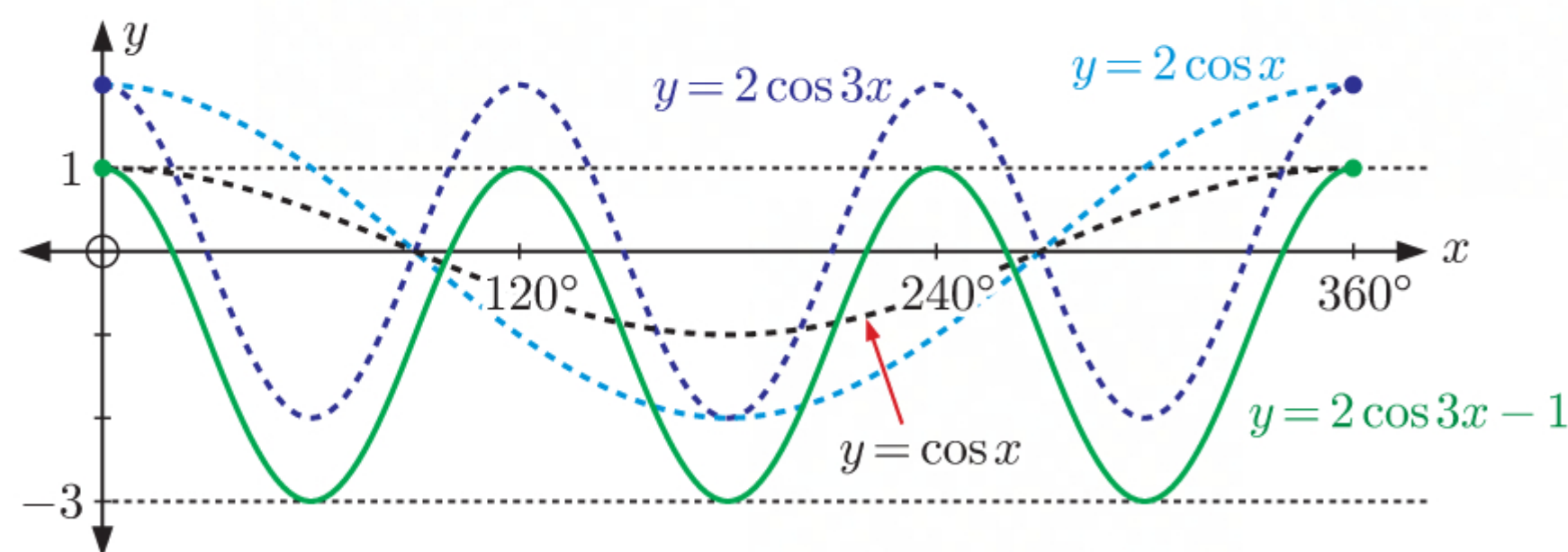


b $a = 2$, so the amplitude is $|2| = 2$.

$b = 3$, so the period is $\frac{360^\circ}{b} = \frac{360^\circ}{3} = 120^\circ$.

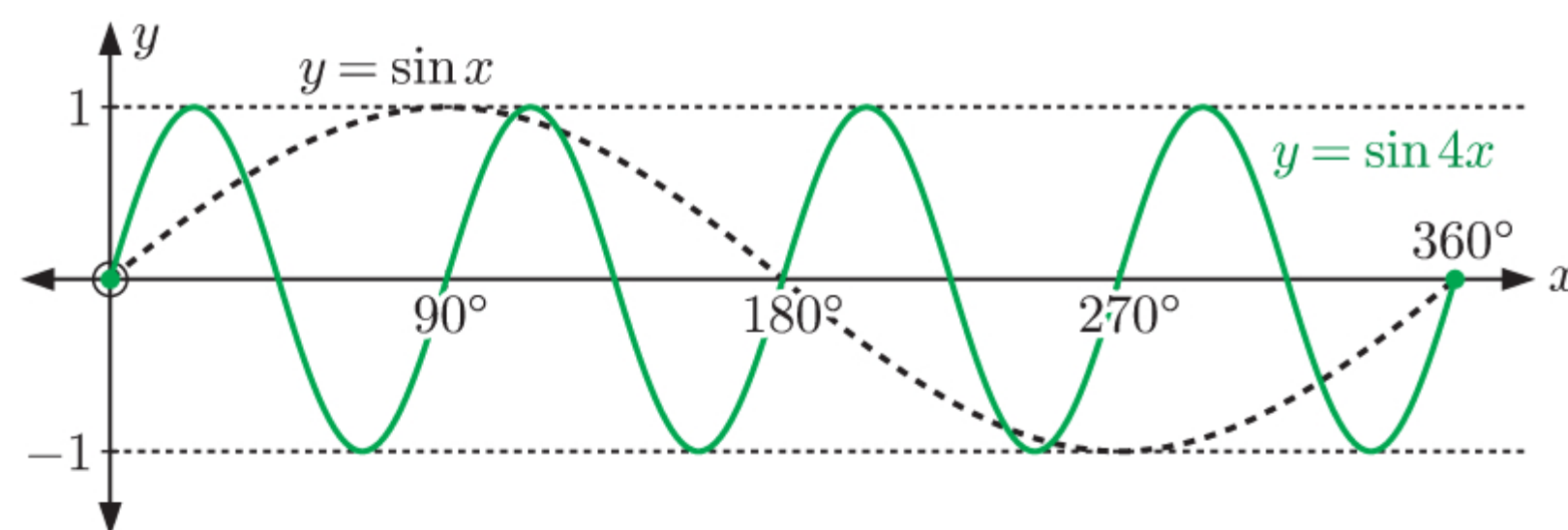
$d = -1$, so the principal axis is $y = -1$.

We stretch $y = \cos x$ vertically with scale factor 2 to give $y = 2 \cos x$, then stretch $y = 2 \cos x$ horizontally with scale factor $\frac{1}{3}$ to give $y = 2 \cos 3x$, then translate $y = 2 \cos 3x$ downwards by 1 unit to give $y = 2 \cos 3x - 1$.



c $b = 4$, so the period is $\frac{360^\circ}{b} = \frac{360^\circ}{4} = 90^\circ$.

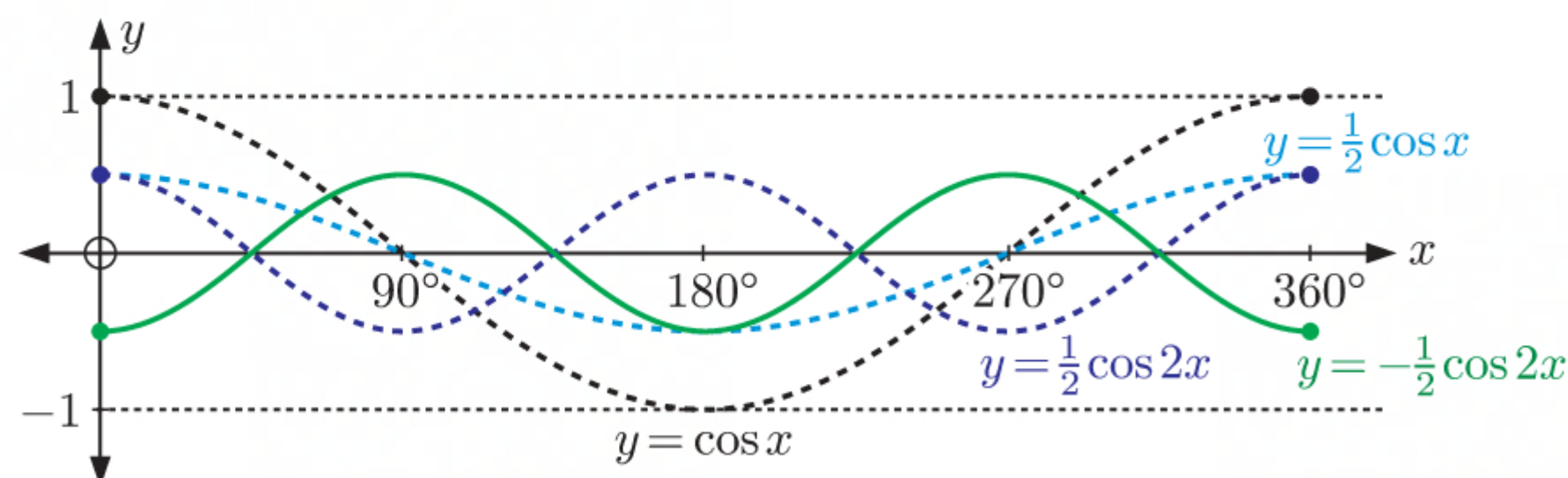
We stretch $y = \sin x$ horizontally with scale factor $\frac{1}{4}$ to give $y = \sin 4x$.



d $a = -\frac{1}{2}$, so the amplitude is $|\frac{1}{2}| = \frac{1}{2}$.

$b = 2$, so the period is $\frac{360^\circ}{b} = \frac{360^\circ}{2} = 180^\circ$.

We stretch $y = \cos x$ vertically with scale factor $\frac{1}{2}$ to give $y = \frac{1}{2} \cos x$, then stretch $y = \frac{1}{2} \cos x$ horizontally with scale factor $\frac{1}{2}$ to give $y = \frac{1}{2} \cos 2x$, then reflect $y = \frac{1}{2} \cos 2x$ in the x -axis to give $y = -\frac{1}{2} \cos 2x$.

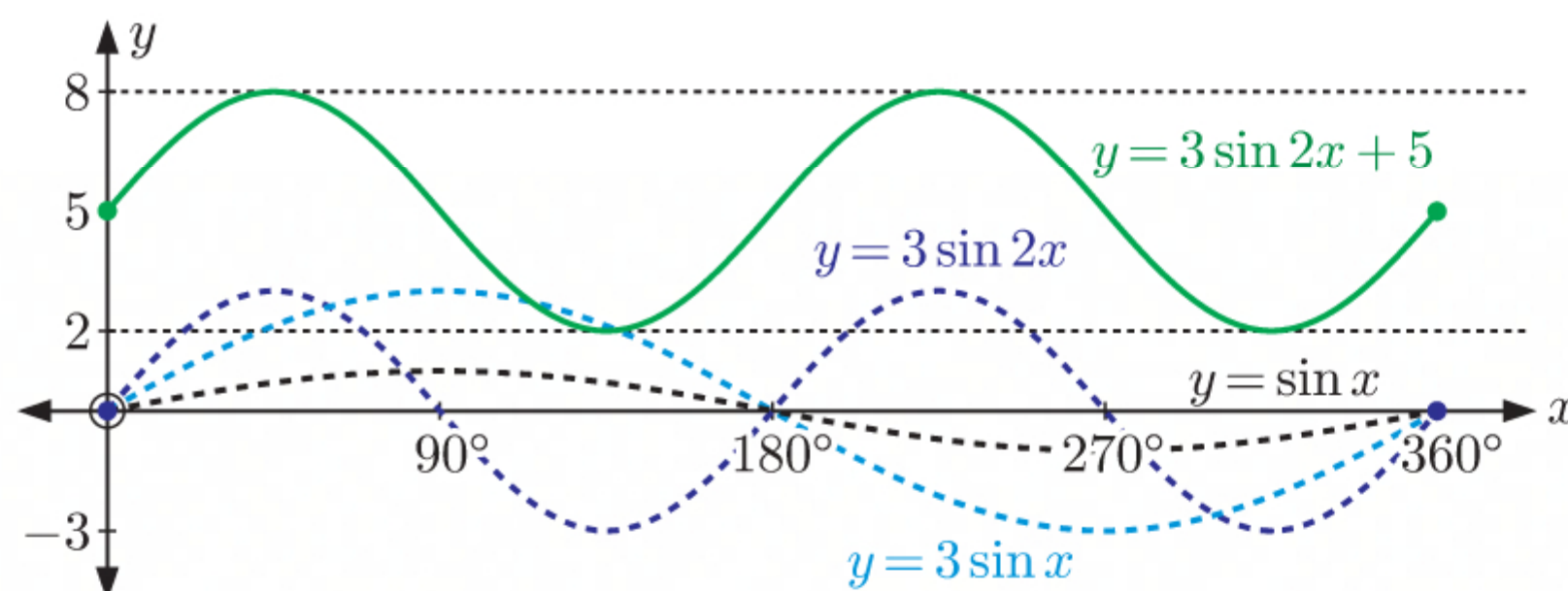


10 a $a = 3$, so the amplitude is $|3| = 3$.

$b = 2$, so the period is $\frac{360^\circ}{b} = \frac{360^\circ}{2} = 180^\circ$.

$d = 5$, so the principal axis is $y = 5$.

We stretch $y = \sin x$ vertically with scale factor 3 to give $y = 3 \sin x$, then stretch $y = 3 \sin x$ horizontally with scale factor $\frac{1}{2}$ to give $y = 3 \sin 2x$, then translate $y = 3 \sin 2x$ upwards by 5 units to give $y = 3 \sin 2x + 5$.



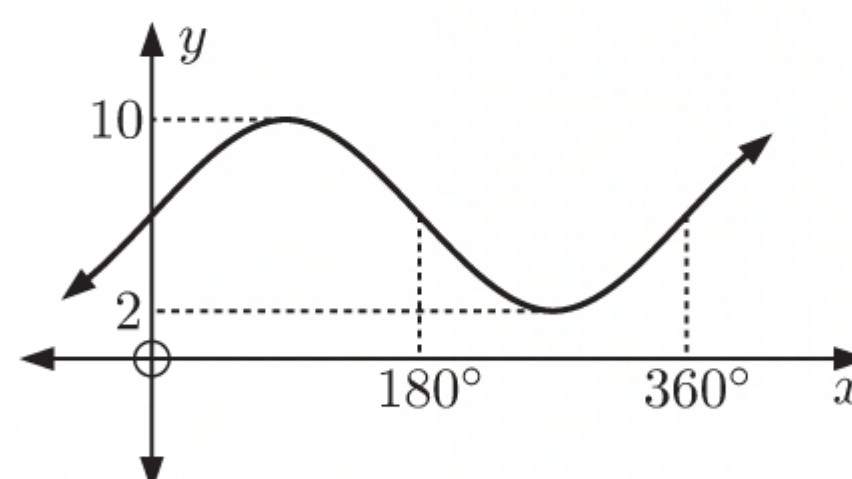
b When $x = 30^\circ$, $y = 3 \sin(2 \times 30^\circ) + 5$
 $= 3 \sin 60^\circ + 5$
 $= 3 \times \frac{\sqrt{3}}{2} + 5$
 $= \frac{3\sqrt{3}}{2} + 5 \approx 7.60$

11 The amplitude is 4, so $a = 4$.

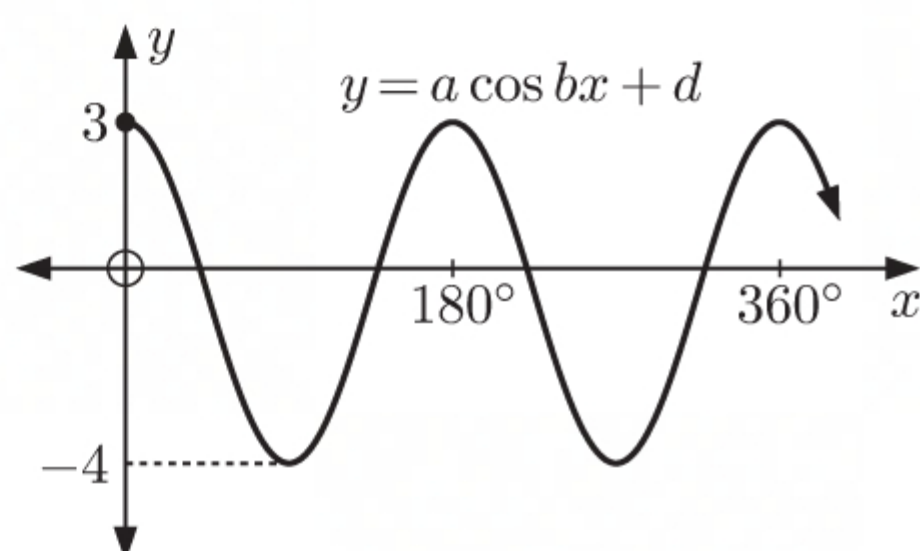
The period is 360° , so $\frac{360^\circ}{b} = 360^\circ$
 $\therefore b = 1$

The principal axis is $y = 6$, so $d = 6$.

The equation of the function is $y = 4 \sin x + 6$.



12 a

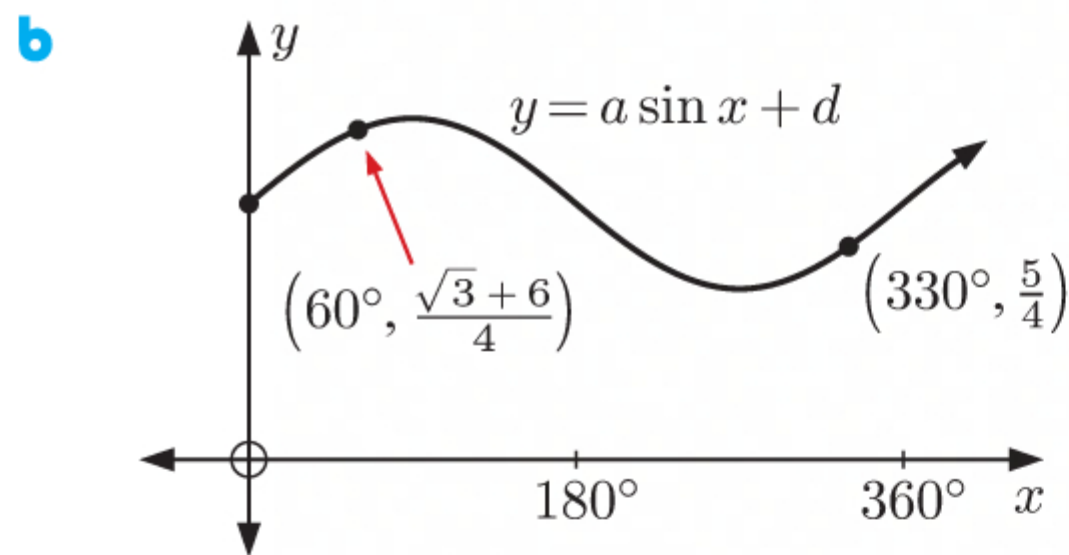


The amplitude is $\frac{3 - (-4)}{2} = \frac{7}{2}$, so
 $a = \frac{7}{2}$.

The period is 180° , so $\frac{360^\circ}{b} = 180^\circ$
 $\therefore b = 2$

The principal axis is $y = \frac{-4 + 3}{2} = -\frac{1}{2}$,
so $d = -\frac{1}{2}$.

So, $a = \frac{7}{2}$, $b = 2$, and $d = -\frac{1}{2}$.



When $x = 60^\circ$, $y = \frac{\sqrt{3}+6}{4}$

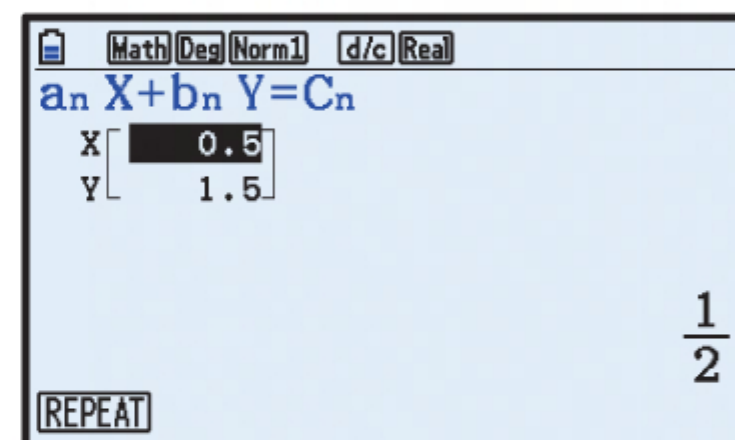
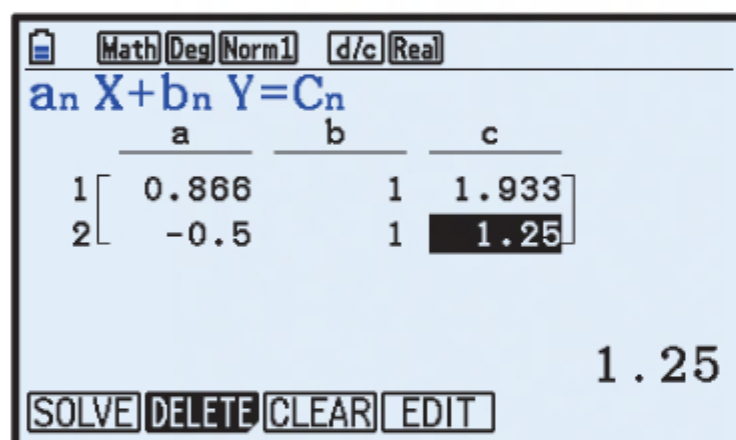
$$\therefore a \sin 60^\circ + d = \frac{\sqrt{3}+6}{4}$$

$$\therefore \frac{\sqrt{3}}{2}a + d = \frac{\sqrt{3}+6}{4} \quad \dots (1)$$

When $x = 330^\circ$, $y = \frac{5}{4}$

$$\therefore a \sin 330^\circ + d = \frac{5}{4}$$

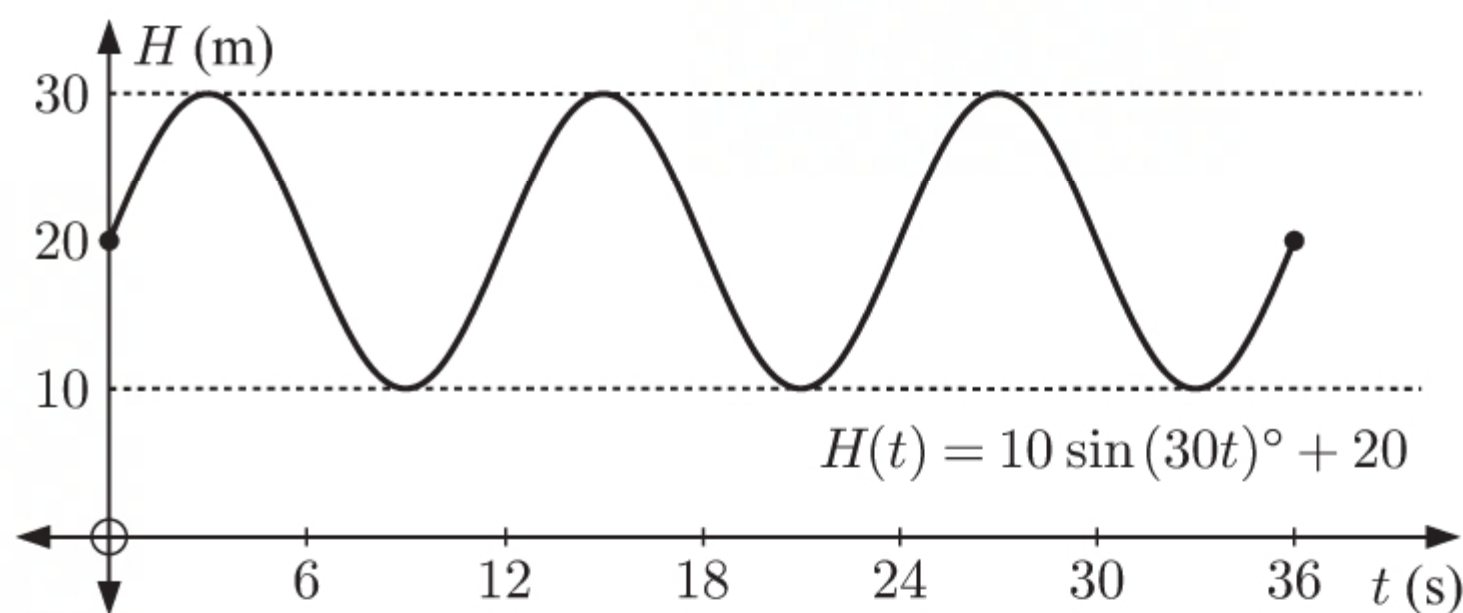
$$\therefore -\frac{1}{2}a + d = \frac{5}{4} \quad \dots (2)$$



Solving (1) and (2) simultaneously using technology gives $a = \frac{1}{2}$ and $d = \frac{3}{2}$.

13 a For $H(t) = 10 \sin(30t)^\circ + 20$:

- $a = 10$, so the amplitude is $|10| = 10$
- $b = 30$, so the period is $\frac{360}{b} = \frac{360}{30} = 12$ seconds
- the principal axis is $H = 20$.



b $H(9) = 10 \sin(30 \times 9)^\circ + 20$

$$= 10 \sin 270^\circ + 20$$

$$= 10 \times (-1) + 20$$

$$= 10$$

So, the height of the blade's tip after 9 seconds is 10 m above the ground.

c $H(t) = 10 \sin(30t)^\circ + 20$ has minimum value $10(-1) + 20 = 10$ {when $\sin(30t)^\circ = -1$ }
So, the minimum height of the blade's tip is 10 m.

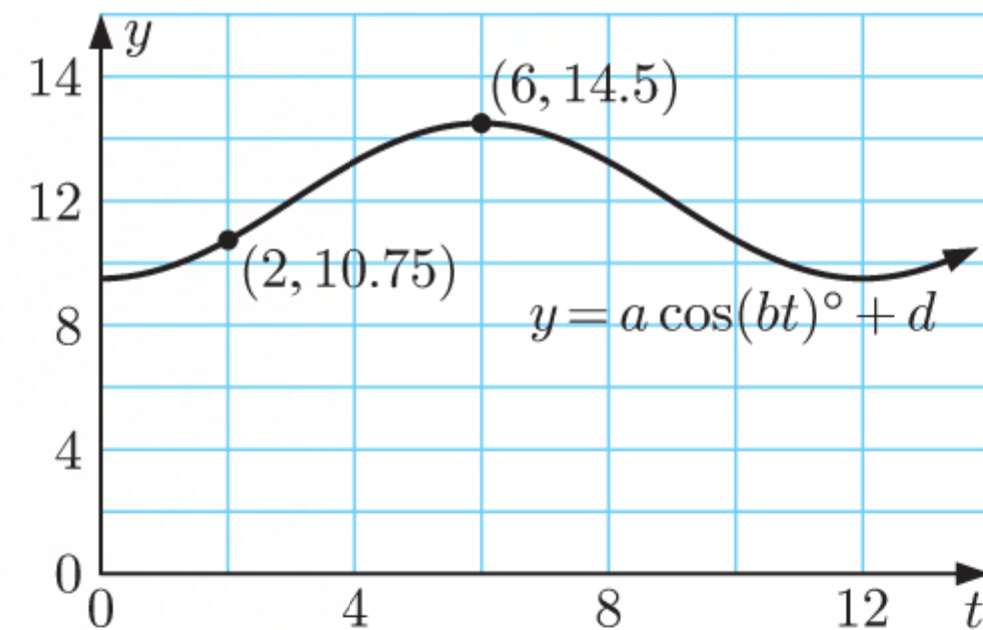
d From part **a**, the period is 12 seconds, so it takes 12 seconds for the blade to complete a full revolution.

14 $y = a \cos(bt)^\circ + d$

- a** The graph is at a minimum when $t = 0$ months, and again when $t = 12$ months, so the period is $12 - 0 = 12$ months.

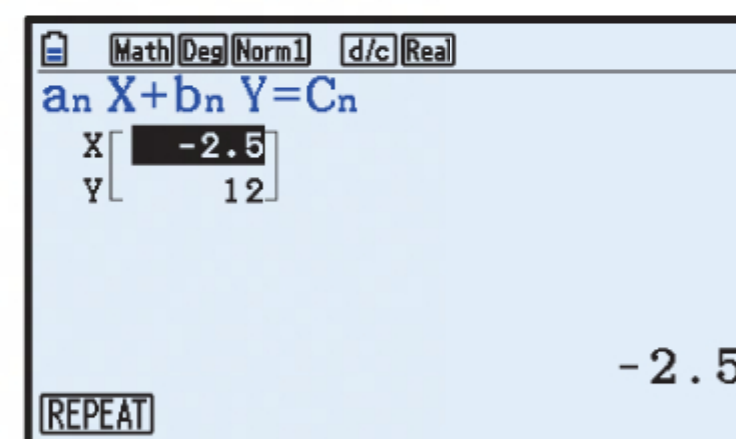
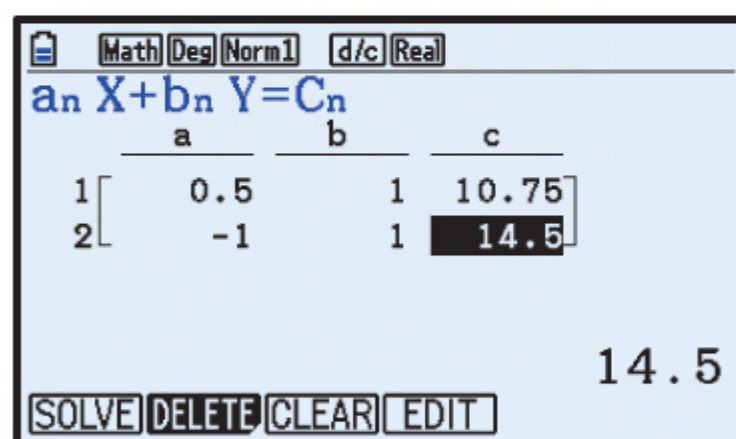
$$\text{So, } \frac{360}{b} = 12$$

$$\therefore b = 30$$



- b** When $t = 2$, $y = 10.75$
 $\therefore a \cos(30 \times 2)^\circ + d = 10.75$
 $\therefore a \cos 60^\circ + d = 10.75$
 $\therefore \frac{1}{2}a + d = 10.75 \quad \dots (1)$

- When $t = 6$, $y = 14.5$
 $\therefore a \cos(30 \times 6)^\circ + d = 14.5$
 $\therefore a \cos 180^\circ + d = 14.5$
 $\therefore -a + d = 14.5 \quad \dots (2)$



Solving (1) and (2) simultaneously using technology gives $a = -\frac{5}{2}$ and $d = 12$.

c $y = -\frac{5}{2} \cos(30t)^\circ + 12$

September is 8 months after January.

$$\begin{aligned} \text{When } t = 8, \quad y &= -\frac{5}{2} \cos(30 \times 8)^\circ + 12 \\ &= -\frac{5}{2} \cos 240^\circ + 12 \\ &= -\frac{5}{2} \times \left(-\frac{1}{2}\right) + 12 \\ &= 13.25 \end{aligned}$$

The average number of daylight hours in Los Angeles in September is 13.25 hours.

- d** The parameters b and d will stay the same since the number of daylight hours should be periodic over 12 months, with a mean of 12 daylight hours per day. The parameter a will change depending on the latitude of the city.

- 15 a** The paint spot is initially at its highest point which is 2 m above the ground.

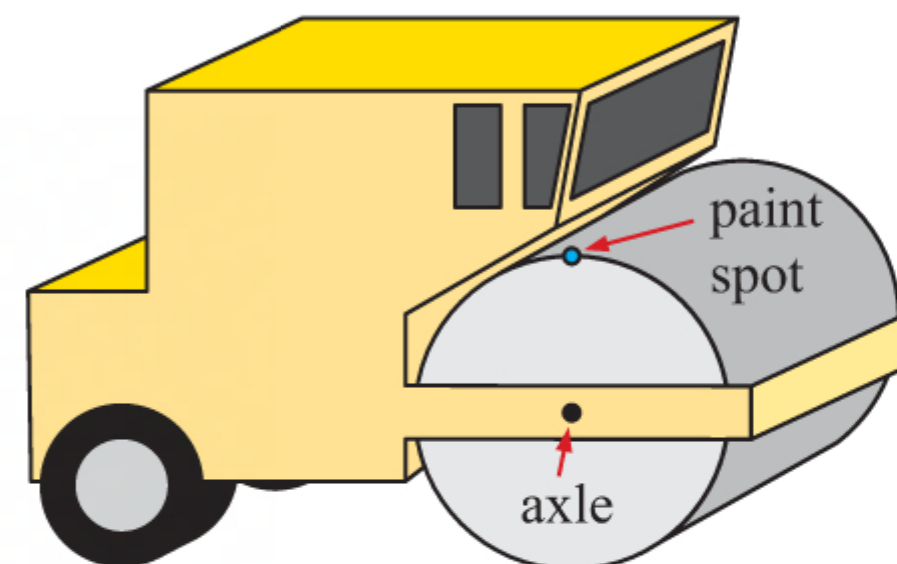
The amplitude $a = 1$ m.

$$\text{The period is 2 seconds, so } \frac{360}{b} = 2$$

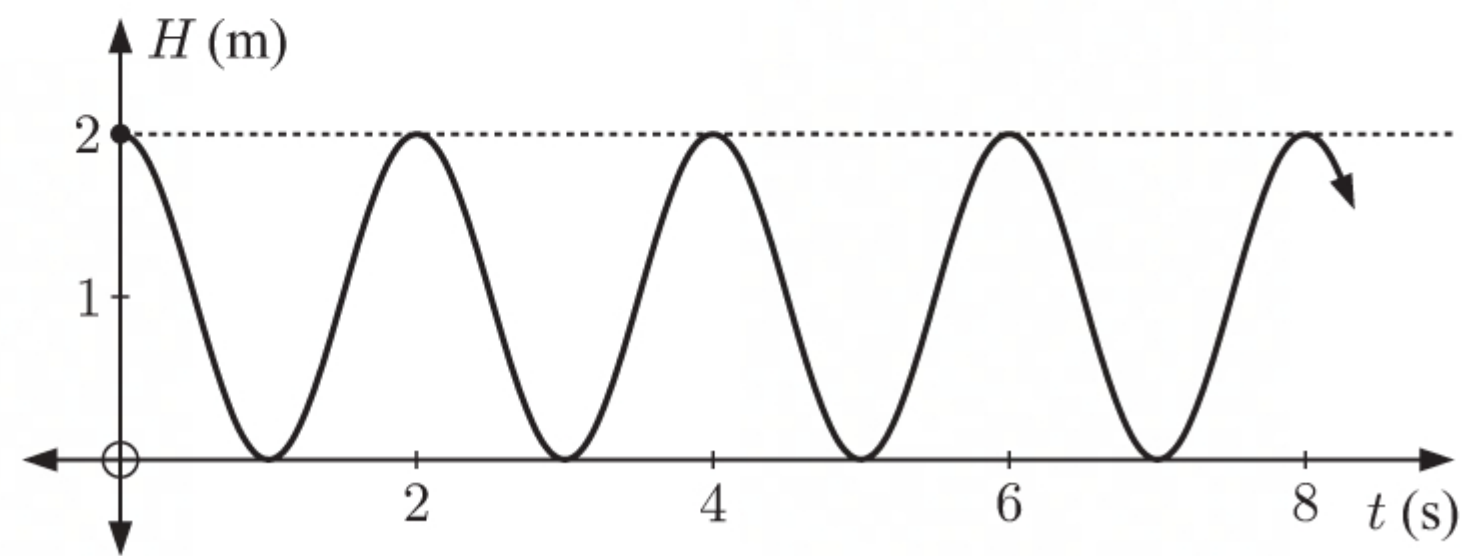
$$\therefore b = 180$$

$$\text{The principal axis is } y = \frac{\max + \min}{2} = \frac{2 + 0}{2} = 1$$

$$\therefore d = 1$$



The graph is:



b $H(t) = a \cos(bt)^\circ + d$
 $\therefore H(t) = \cos(180t)^\circ + 1 \text{ m}$ {using **a**}

c $H(3.6) = \cos(180 \times 3.6)^\circ + 1$
 ≈ 1.31

The height of the spot after 3.6 seconds is about 1.31 m.

Chapter 10

DIFFERENTIATION

EXERCISE 10A.1

1 a 62 beats per minute means that in one minute, Kirsten's heart is expected to beat 62 times.

$$\begin{aligned} \text{b Number of beats per hour} &= 62 \text{ beats per minute} \times 60 \text{ minutes} \\ &= 3720 \text{ beats} \end{aligned}$$

$$\begin{aligned} 2 \text{ a Number of words} &= 14 \times 380 \\ &= 5320 \text{ words} \end{aligned}$$

$$\begin{aligned} \therefore \text{error rate} &= \frac{8 \text{ errors}}{5320 \text{ words}} \\ &\approx 0.00150 \text{ errors per word} \end{aligned}$$

$$\text{b } 5320 \text{ words} = 53.2 \times 100 \text{ words}$$

$$\begin{aligned} \therefore \text{error rate} &= \frac{8 \text{ errors}}{53.2 (100 \text{ words})} \\ &\approx 0.150 \text{ errors per 100 words} \end{aligned}$$

$$\begin{aligned} 3 \text{ Niko's hourly rate} &= \frac{\pounds 148.20}{12 \text{ hours}} \\ &= \pounds 12.35 \text{ per hour} \end{aligned}$$

\therefore Niko worked for a better rate.

$$\begin{aligned} \text{Marita's hourly rate} &= \frac{\pounds 157.95}{13 \text{ hours}} \\ &= \pounds 12.15 \text{ per hour} \end{aligned}$$

$$\begin{aligned} 4 \text{ a Tyre wear} &= 8 - 2.3 = 5.7 \text{ mm} \\ \therefore \text{wearing rate} &= \frac{5.7 \text{ mm}}{32178 \text{ km}} \\ &\approx 0.000177 \text{ mm per km} \end{aligned}$$

$$\begin{aligned} \text{b } 32178 \text{ km} &= 3.2178 \times 10000 \text{ km} \\ \therefore \text{wearing rate} &= \frac{5.7 \text{ mm}}{3.2178(10000 \text{ km})} \\ &\approx 1.77 \text{ mm per } 10000 \text{ km} \end{aligned}$$

$$\begin{aligned} 5 \text{ a Time taken from 11:43 am to 3:39 pm} \\ &= 17 \text{ min} + 3 \text{ h } 39 \text{ min} \\ &= 3 \text{ h } 56 \text{ min or } 3\frac{56}{60} \text{ hours} \end{aligned}$$

$$\begin{aligned} \therefore \text{average speed} &= \frac{350 \text{ km}}{3\frac{56}{60} \text{ hours}} \\ &\approx 89.0 \text{ km h}^{-1} \end{aligned}$$

$$\begin{aligned} \text{b } 350 \text{ km} &= 350000 \text{ m} \\ 3\frac{56}{60} \text{ hours} &= 3\frac{56}{60} \times 60 \times 60 \\ &= 14160 \text{ seconds} \end{aligned}$$

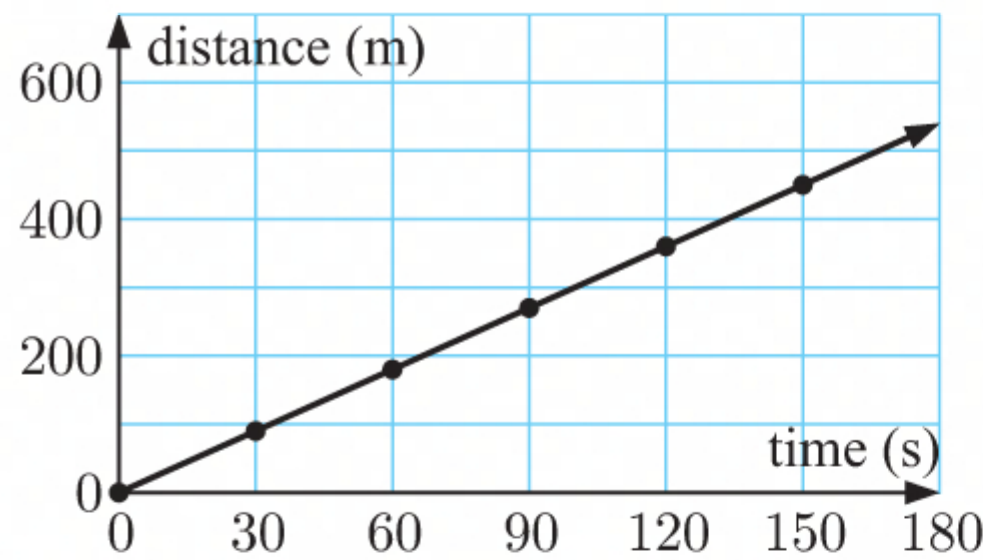
$$\begin{aligned} \therefore \text{average speed} &= \frac{350000 \text{ m}}{14160 \text{ seconds}} \\ &\approx 24.7 \text{ m s}^{-1} \end{aligned}$$

EXERCISE 10A.2**1 a**

Time (seconds)	0	30	60	90	120	150
Distance (metres)	0	90	180	270	360	450

$\xrightarrow{+90}$ $\xrightarrow{+90}$ $\xrightarrow{+90}$ $\xrightarrow{+90}$ $\xrightarrow{+90}$

The distance travelled increases by the same amount each time interval.
 \therefore the jogger is travelling at a constant speed.

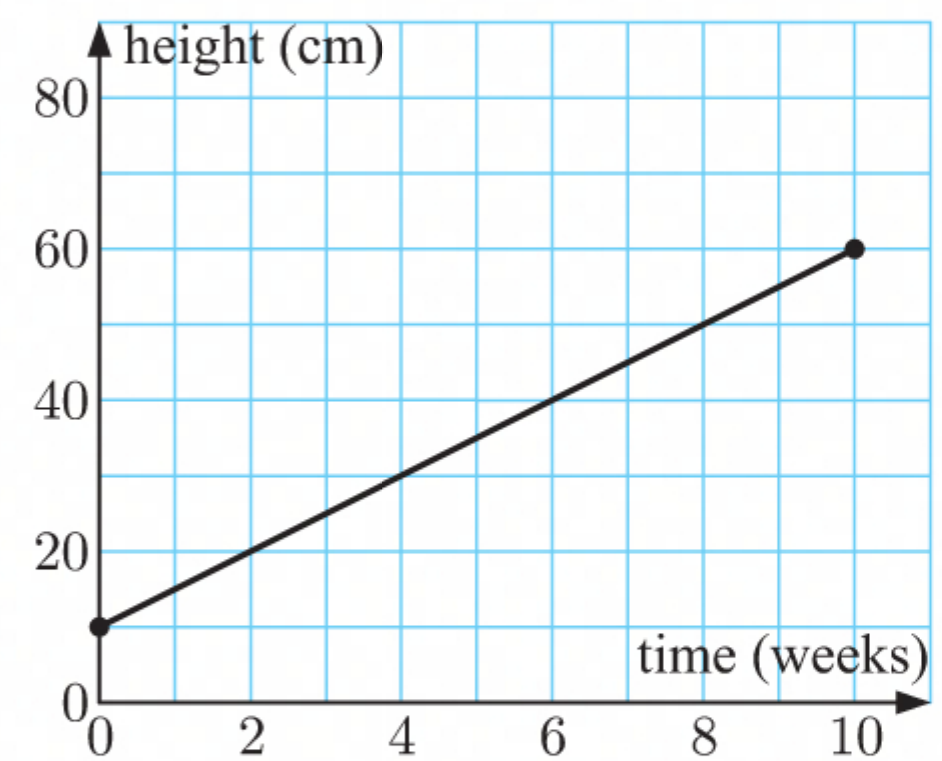
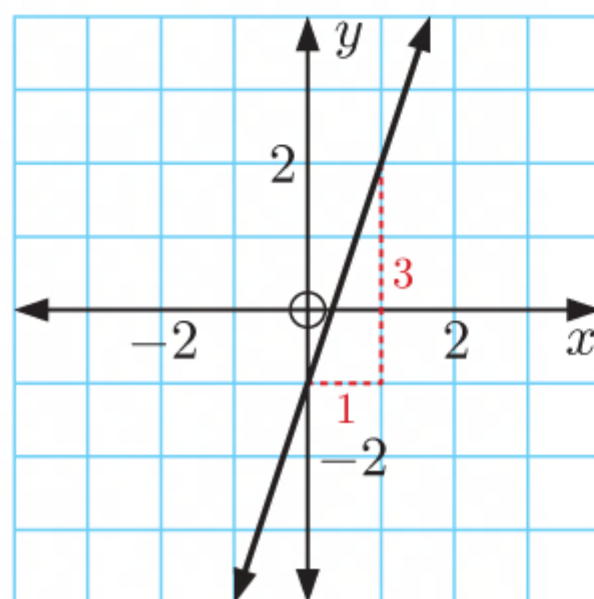
b

$$\begin{aligned}
 \text{speed} &= \frac{(90 - 0) \text{ m}}{(30 - 0) \text{ s}} \\
 &= 3 \text{ m s}^{-1}
 \end{aligned}$$

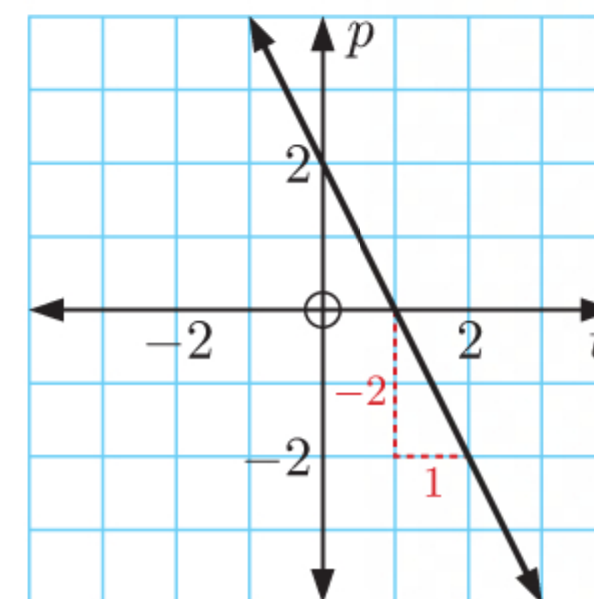
2 a

The graph of height against time is a straight line.
 \therefore the rate of change in height is constant.

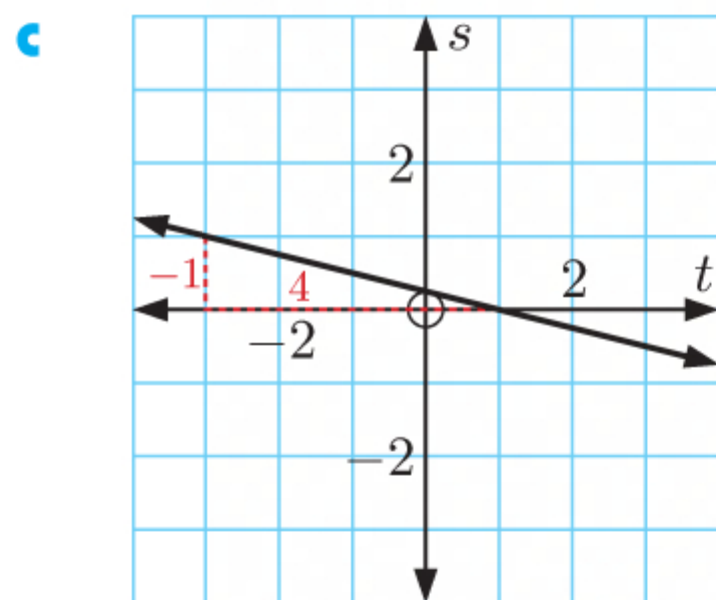
$$\begin{aligned}
 \text{rate of change} &= \frac{(60 - 10) \text{ cm}}{(10 - 0) \text{ weeks}} \\
 &= \frac{50 \text{ cm}}{10 \text{ weeks}} \\
 &= 5 \text{ cm per week}
 \end{aligned}$$

**3 a**

$$\begin{aligned}
 \text{rate of change} &= \text{gradient of line} \\
 &= \frac{3}{1} \\
 &= 3
 \end{aligned}$$

b

$$\begin{aligned}
 \text{rate of change} &= \text{gradient of line} \\
 &= \frac{-2}{1} \\
 &= -2
 \end{aligned}$$



rate of change = gradient of line

$$= \frac{-1}{4}$$

$$= -\frac{1}{4}$$

EXERCISE 10A.3

- 1 a** The graph of distance against time is not a straight line.

\therefore Aileen did not travel at a constant speed.

- b i** average speed from $t = 0$ to $t = 5$ h

$$= \frac{(300 - 0) \text{ km}}{(5 - 0) \text{ h}}$$

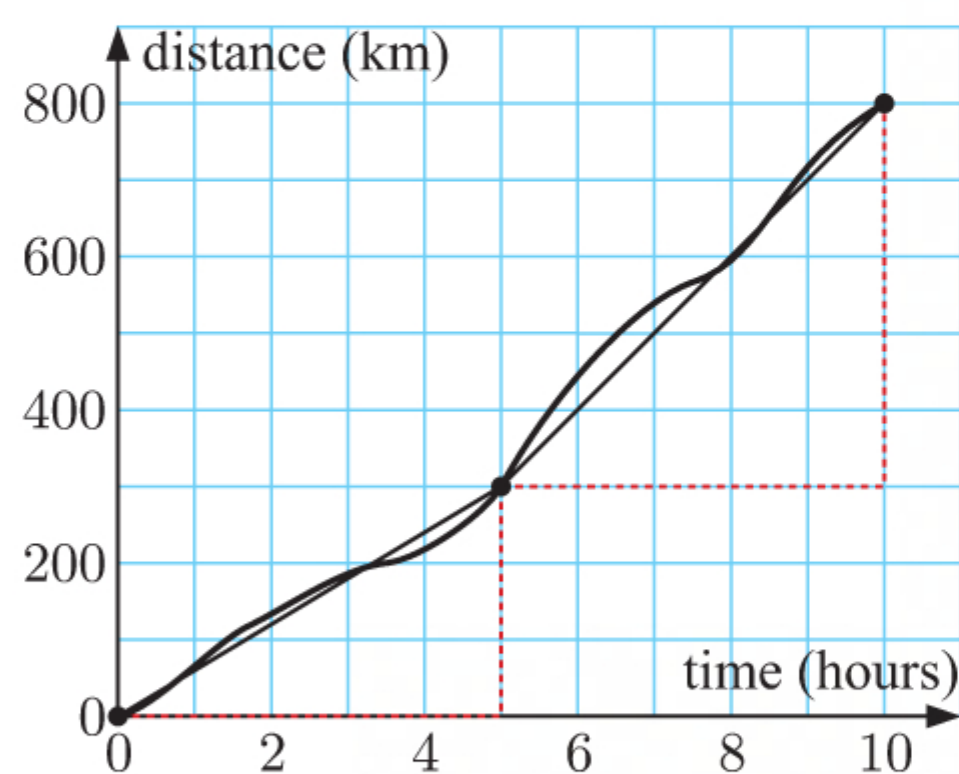
$$= 60 \text{ km h}^{-1}$$

- ii** average speed from $t = 5$ h to $t = 10$ h

$$= \frac{(800 - 300) \text{ km}}{(10 - 5) \text{ h}}$$

$$= \frac{500}{5} \text{ km h}^{-1}$$

$$= 100 \text{ km h}^{-1}$$



- 2 a** average rate of change from $t = 1$ h to $t = 2.5$ h

$$= \frac{(250 - 100) \text{ m}}{(2.5 - 1) \text{ h}}$$

$$= \frac{150}{1.5} \text{ m per hour}$$

$$= 100 \text{ m per hour}$$

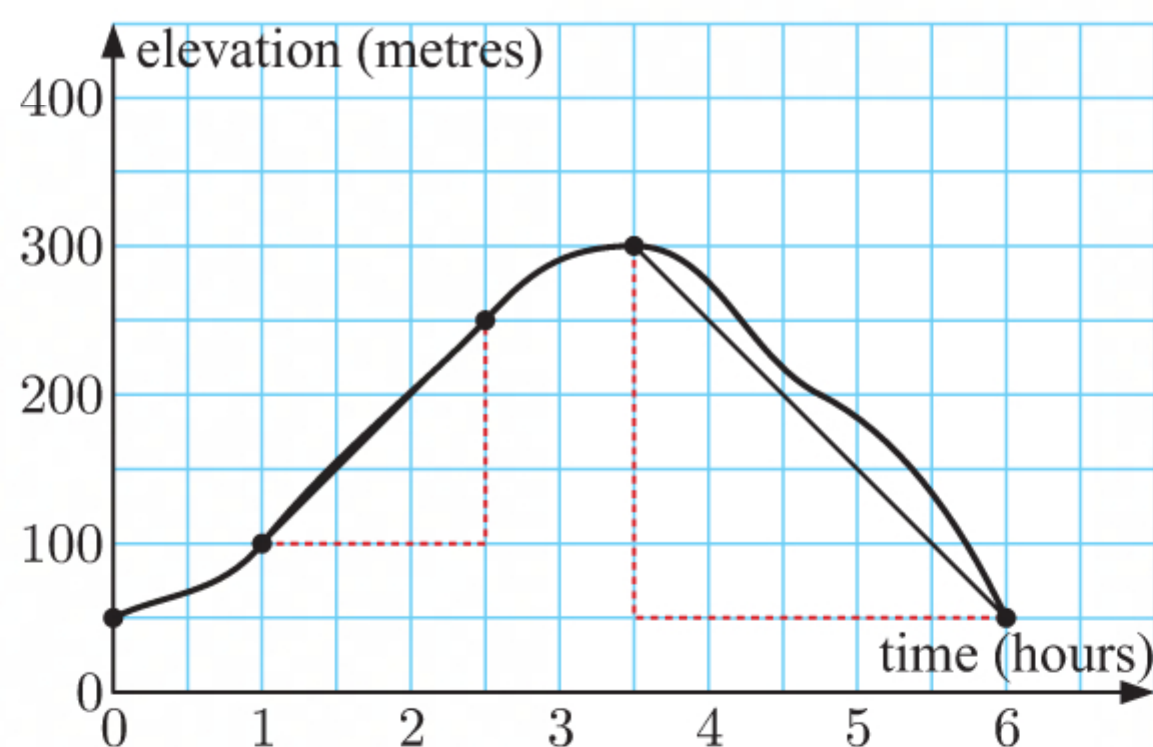
- b** average rate of change from $t = 3.5$ h to $t = 6$ h

$$= \frac{(50 - 300) \text{ m}}{(6 - 3.5) \text{ h}}$$

$$= \frac{-250}{2.5} \text{ m per hour}$$

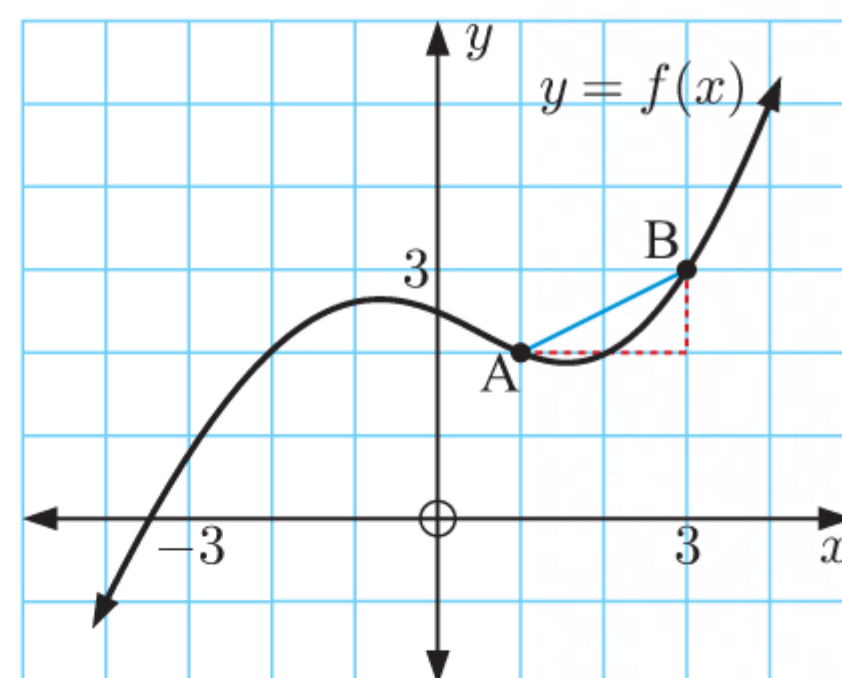
$$= -100 \text{ m per hour}$$

$$= 100 \text{ m per hour (downwards)}$$



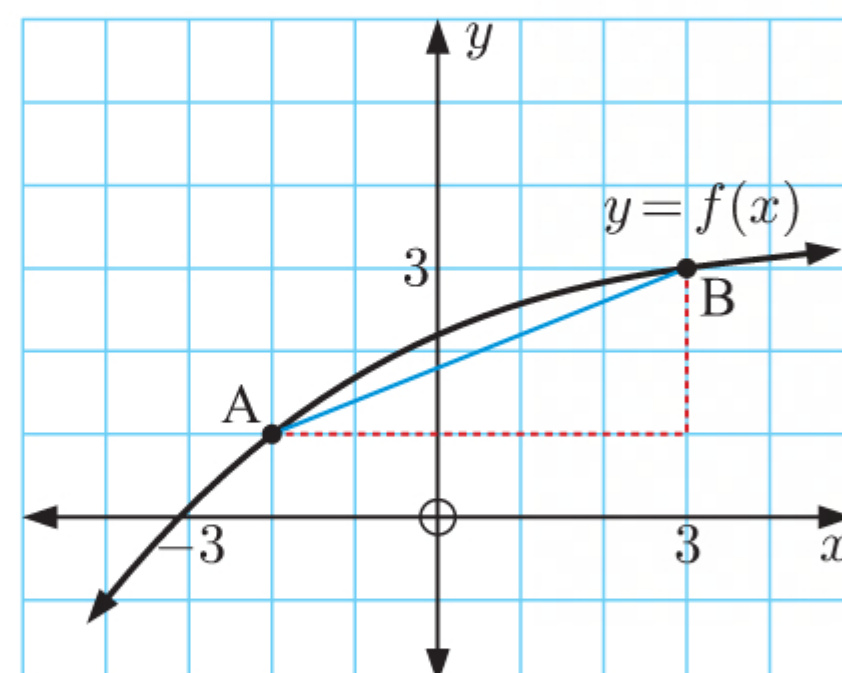
- 3 a** average rate of change in $f(x)$ from A to B

$$\begin{aligned}
 &= \frac{f(b) - f(a)}{b - a} \\
 &= \frac{3 - 2}{3 - 1} \\
 &= \frac{1}{2}
 \end{aligned}$$



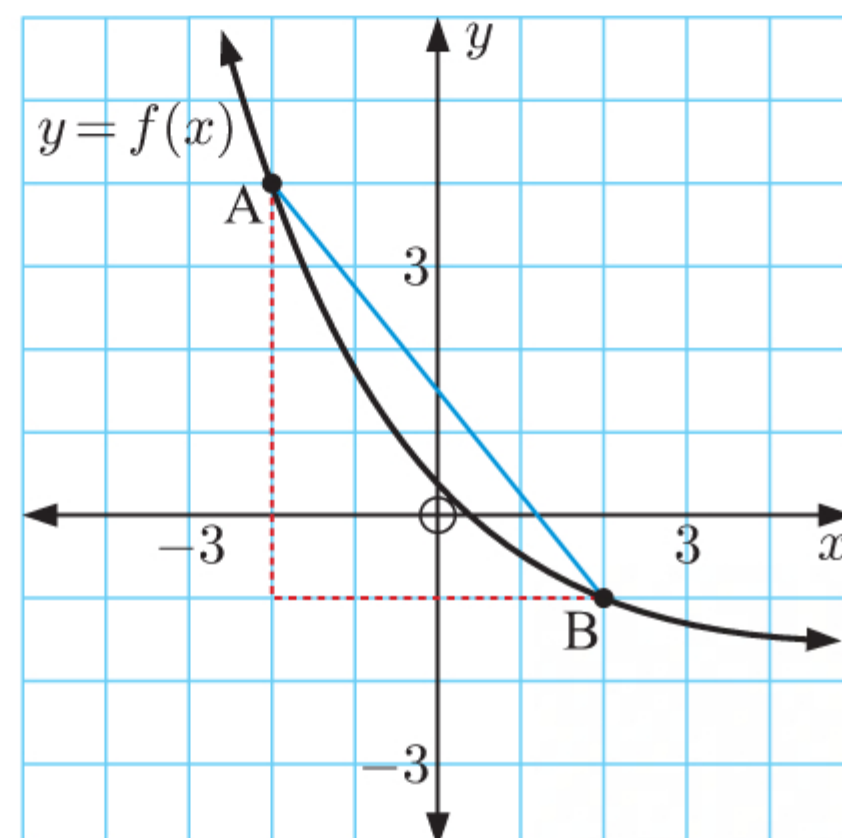
- b** average rate of change in $f(x)$ from A to B

$$\begin{aligned}
 &= \frac{f(b) - f(a)}{b - a} \\
 &= \frac{3 - 1}{3 - (-2)} \\
 &= \frac{2}{5}
 \end{aligned}$$



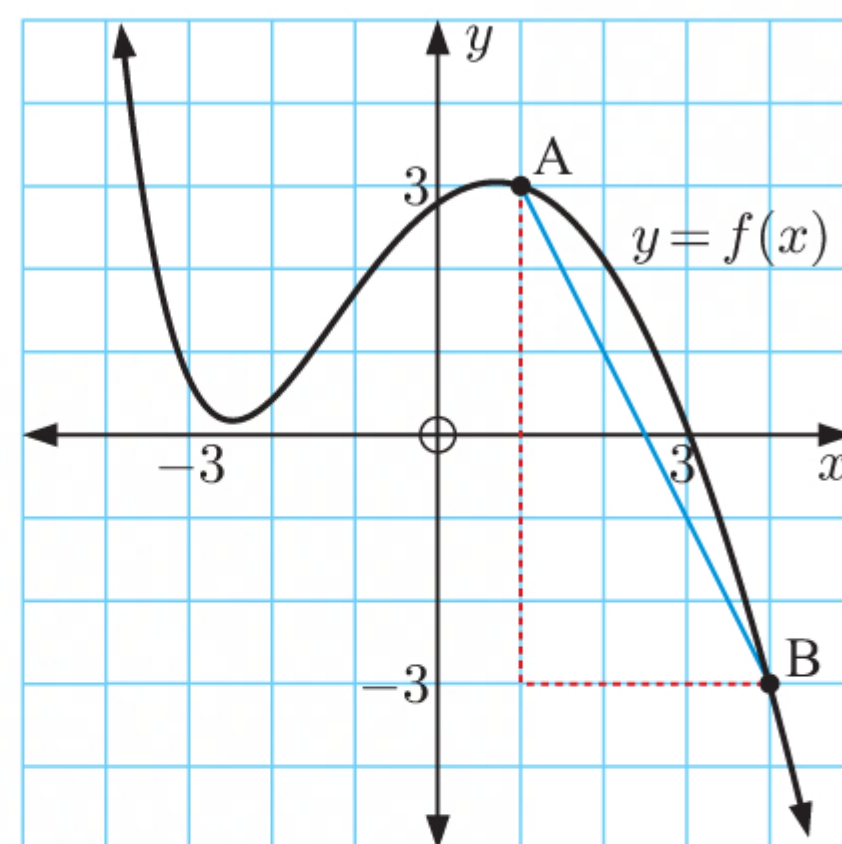
- c** average rate of change in $f(x)$ from A to B

$$\begin{aligned}
 &= \frac{f(b) - f(a)}{b - a} \\
 &= \frac{-1 - 4}{2 - (-2)} \\
 &= -\frac{5}{4}
 \end{aligned}$$



- d** average rate of change in $f(x)$ from A to B

$$\begin{aligned}
 &= \frac{f(b) - f(a)}{b - a} \\
 &= \frac{-3 - 3}{4 - 1} \\
 &= \frac{-6}{3} \\
 &= -2
 \end{aligned}$$



- 4 a i** average rate of change in $f(x)$ from $x = 1$ to $x = 2$

$$\begin{aligned} &= \frac{f(2) - f(1)}{2 - 1} \\ &= \frac{4 - 1}{1} \\ &= 3 \end{aligned}$$

- ii** average rate of change in $f(x)$ from $x = 1$ to $x = 1.5$

$$\begin{aligned} &= \frac{f(1.5) - f(1)}{1.5 - 1} \\ &= \frac{2.25 - 1}{0.5} \\ &= 2.5 \end{aligned}$$

- iii** average rate of change in $f(x)$ from $x = 1$ to $x = 1.1$

$$\begin{aligned} &= \frac{f(1.1) - f(1)}{1.1 - 1} \\ &= \frac{1.21 - 1}{0.1} \\ &= 2.1 \end{aligned}$$

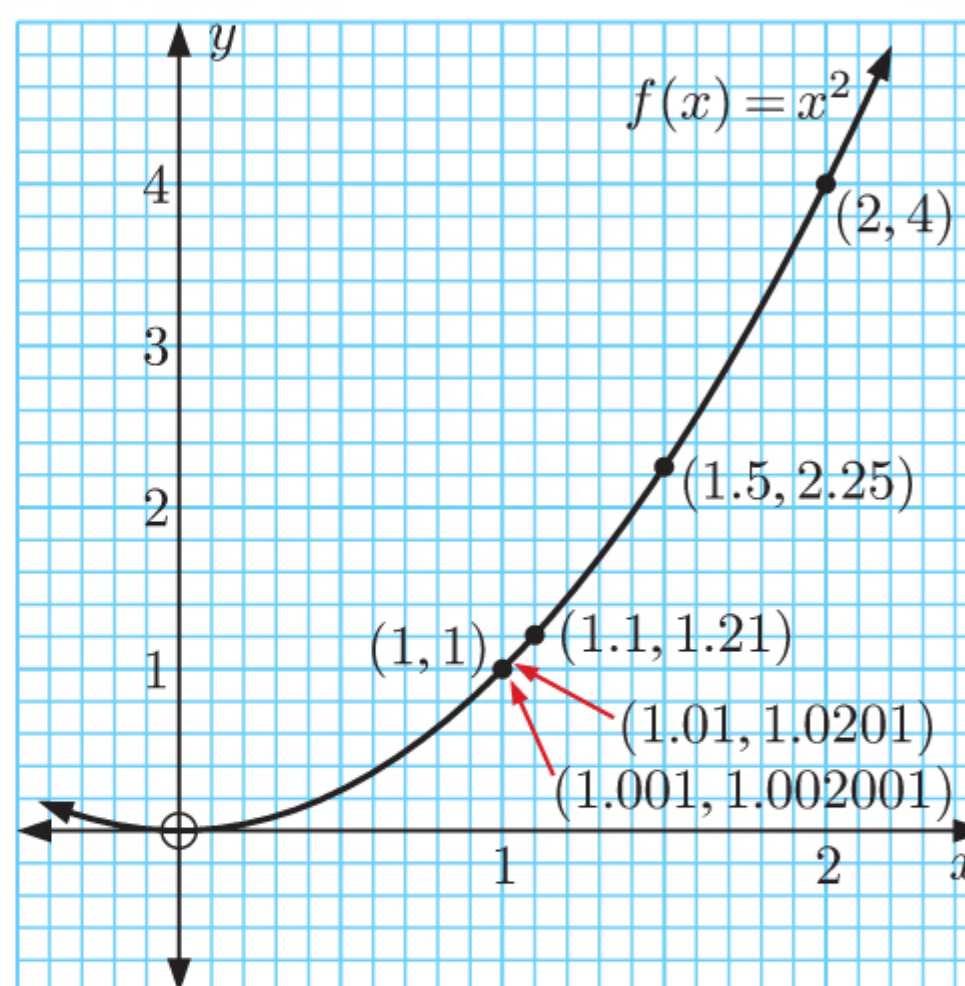
- iv** average rate of change in $f(x)$ from $x = 1$ to $x = 1.01$

$$\begin{aligned} &= \frac{f(1.01) - f(1)}{1.01 - 1} \\ &= \frac{1.0201 - 1}{0.01} \\ &= 2.01 \end{aligned}$$

- v** average rate of change in $f(x)$ from $x = 1$ to $x = 1.001$

$$\begin{aligned} &= \frac{f(1.001) - f(1)}{1.001 - 1} \\ &= \frac{1.002\,001 - 1}{0.001} \\ &= 2.001 \end{aligned}$$

- b** The average rate of change approaches 2.

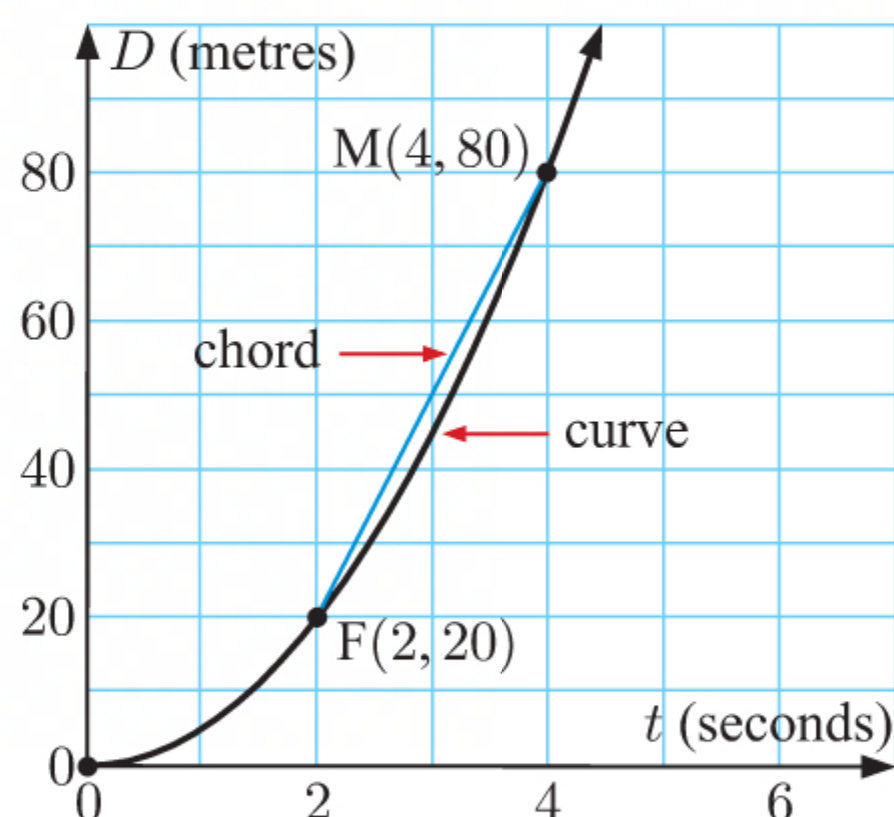


INVESTIGATION 1

INSTANTANEOUS SPEED

2

t	gradient of [FM]
4	30
3	25
2.5	22.5
2.1	20.5
2.01	20.05



3 As M approaches F , the gradient of [FM] approaches 20. However, when M reaches F , the gradient is undefined since we cannot divide by zero.

4 As t approaches 2 from the right, the gradient of [FM] approaches 20.

We suspect that the instantaneous speed of the ball bearing when $t = 2$ seconds is 20 m s^{-1} .

5

t	gradient of [FM]
0	10
1.5	17.5
1.9	19.5
1.99	19.95

6 As t approaches 2 from the left, the gradient of [FM] approaches 20.

The instantaneous speed of the ball bearing when $t = 2$ seconds appears to be 20 m s^{-1} , which agrees with our result in **4**.

EXERCISE 10B

1 a The tangent at A has gradient

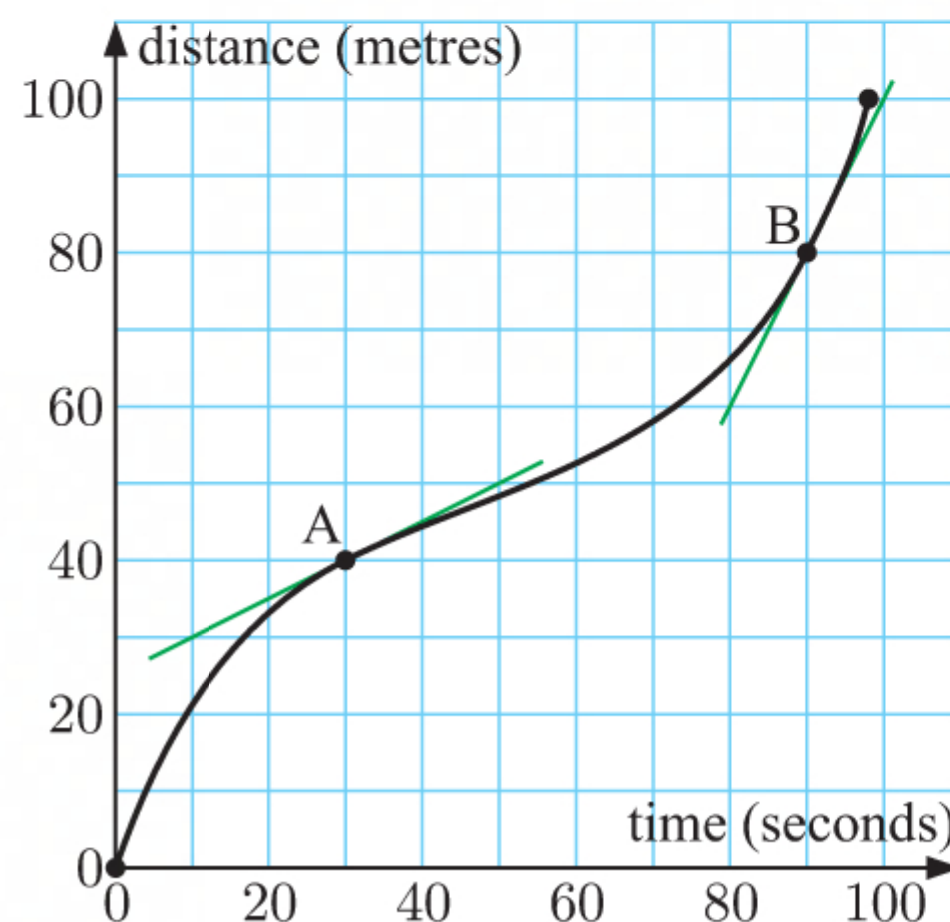
$$\frac{50 - 40}{50 - 30} = \frac{10}{20} = \frac{1}{2}.$$

\therefore the swimmer's instantaneous speed after 30 seconds is 0.5 m s^{-1} .

b The tangent at B has gradient

$$\frac{80 - 60}{90 - 80} = \frac{20}{10} = 2.$$

\therefore the swimmer's instantaneous speed after 90 seconds is 2 m s^{-1} .



- 2 a** The tangent at $x = -1$ has gradient

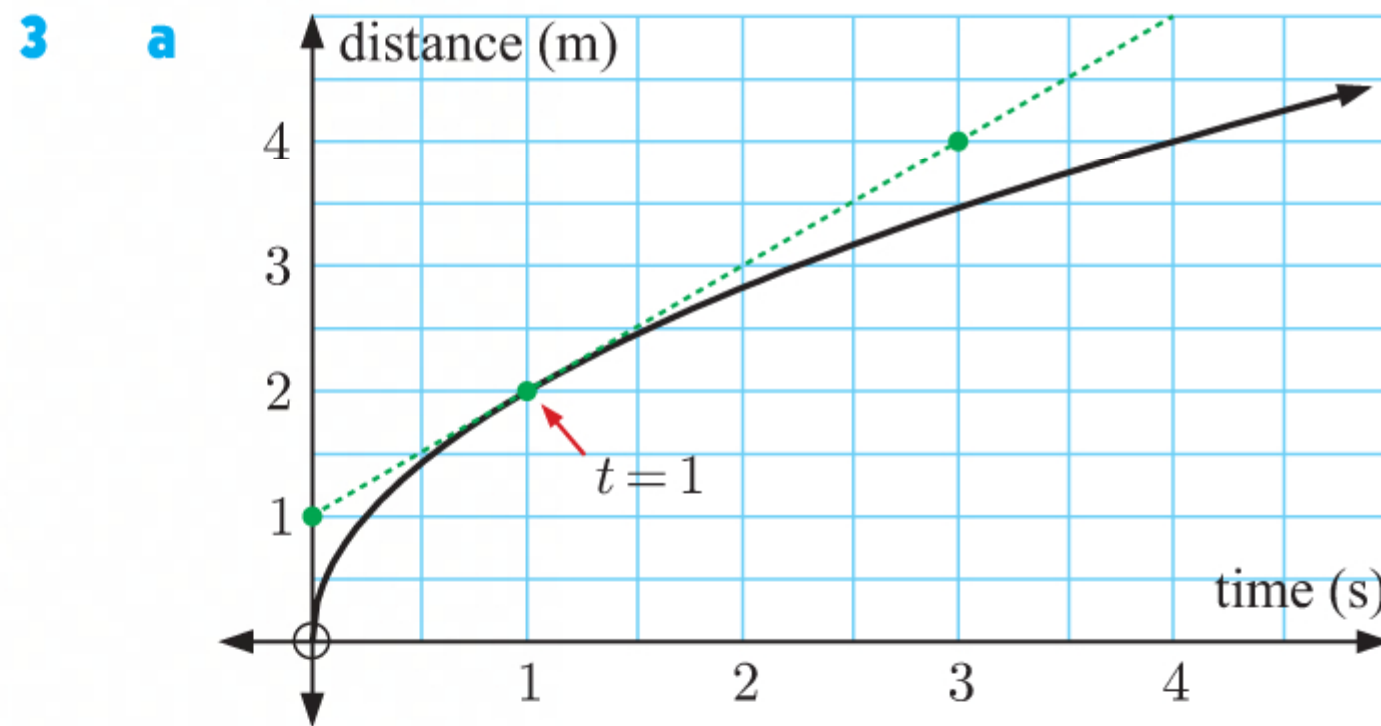
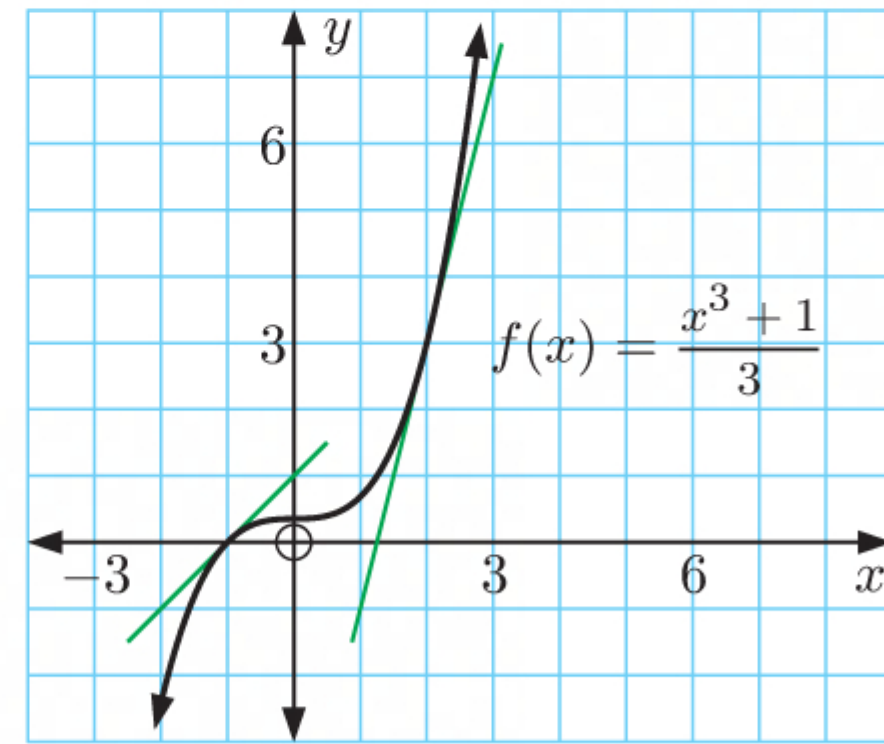
$$\frac{1 - (-1)}{0 - (-2)} = \frac{2}{2} = 1.$$

\therefore the instantaneous rate of change in $f(x)$ at $x = -1$ is 1.

- b** The tangent at $x = 2$ has gradient

$$\frac{3 - (-1)}{2 - 1} = \frac{4}{1} = 4.$$

\therefore the instantaneous rate of change in $f(x)$ at $x = 2$ is 4.



The tangent at $t = 1$ passes through $(0, 1)$ and $(3, 4)$.

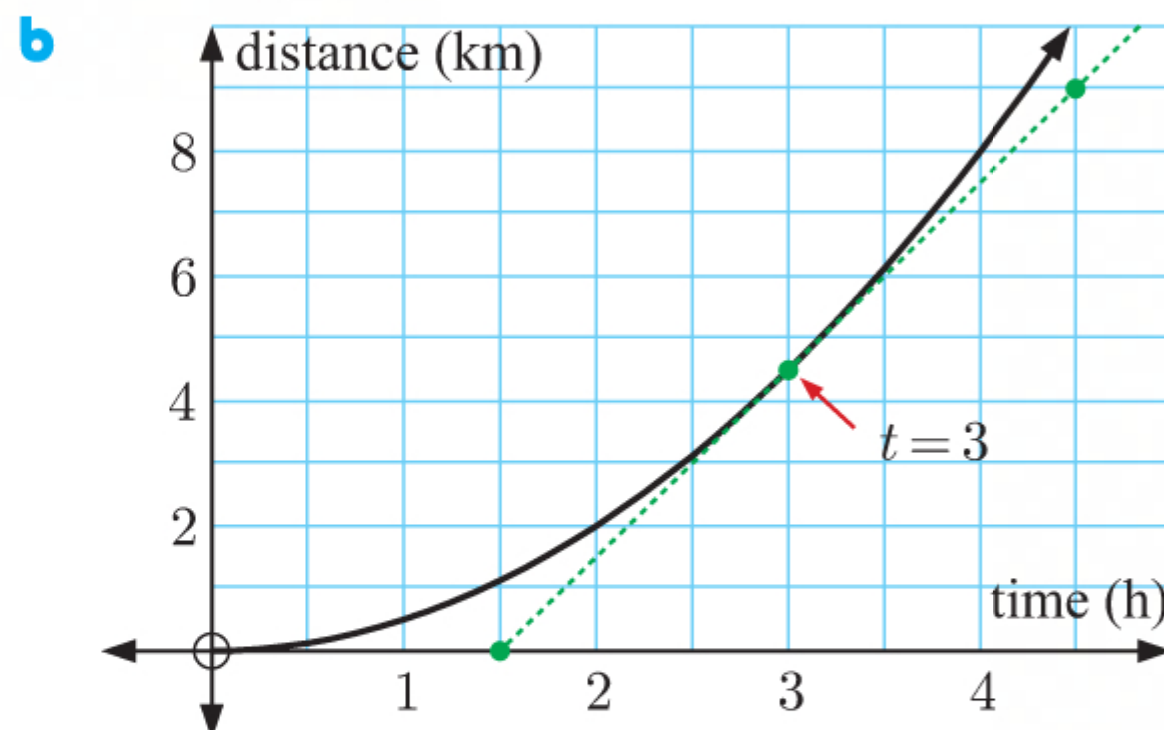
\therefore the rate of change at $t = 1$

= gradient of tangent

$$\approx \frac{(4 - 1) \text{ m}}{(3 - 0) \text{ s}}$$

$$\approx \frac{3}{3} \text{ m s}^{-1}$$

$$\approx 1 \text{ m s}^{-1}$$



The tangent at $t = 3$ passes through $(1.5, 0)$ and $(4.5, 9)$.

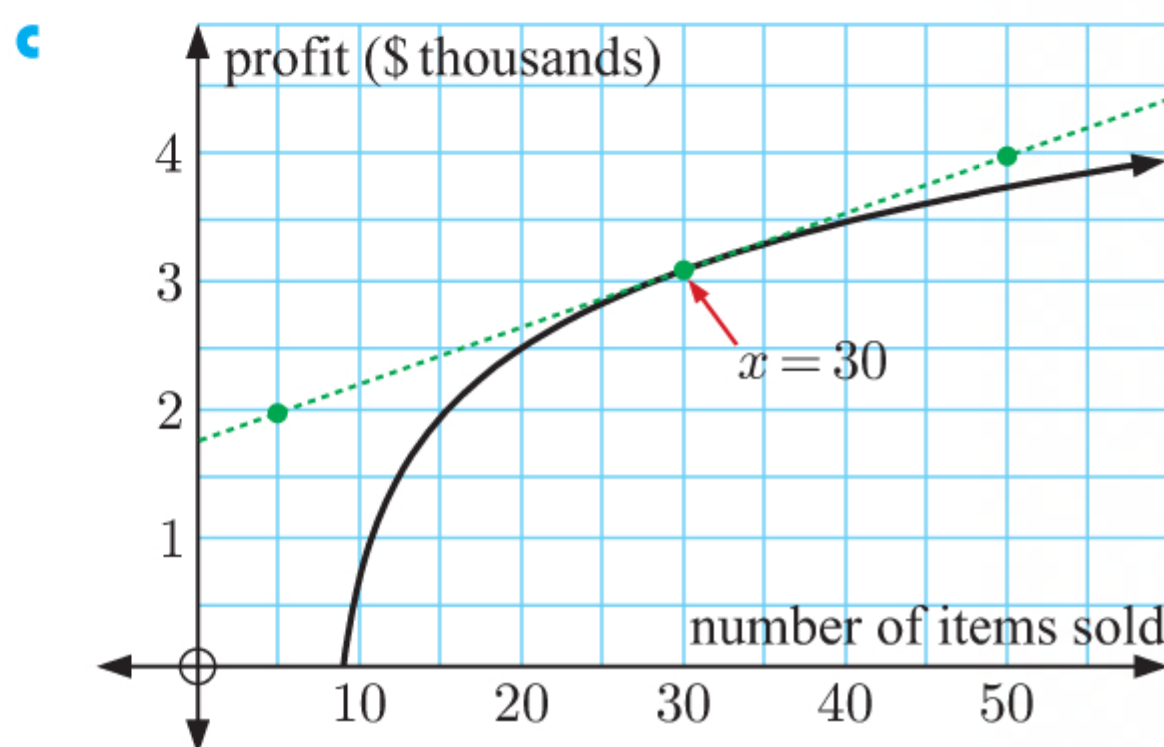
\therefore the rate of change at $t = 3$

= gradient of tangent

$$\approx \frac{(9 - 0) \text{ km}}{(4.5 - 1.5) \text{ h}}$$

$$\approx \frac{9}{3} \text{ km h}^{-1}$$

$$\approx 3 \text{ km h}^{-1}$$



The tangent at $x = 30$ passes through $(5, 2)$ and $(50, 4)$.

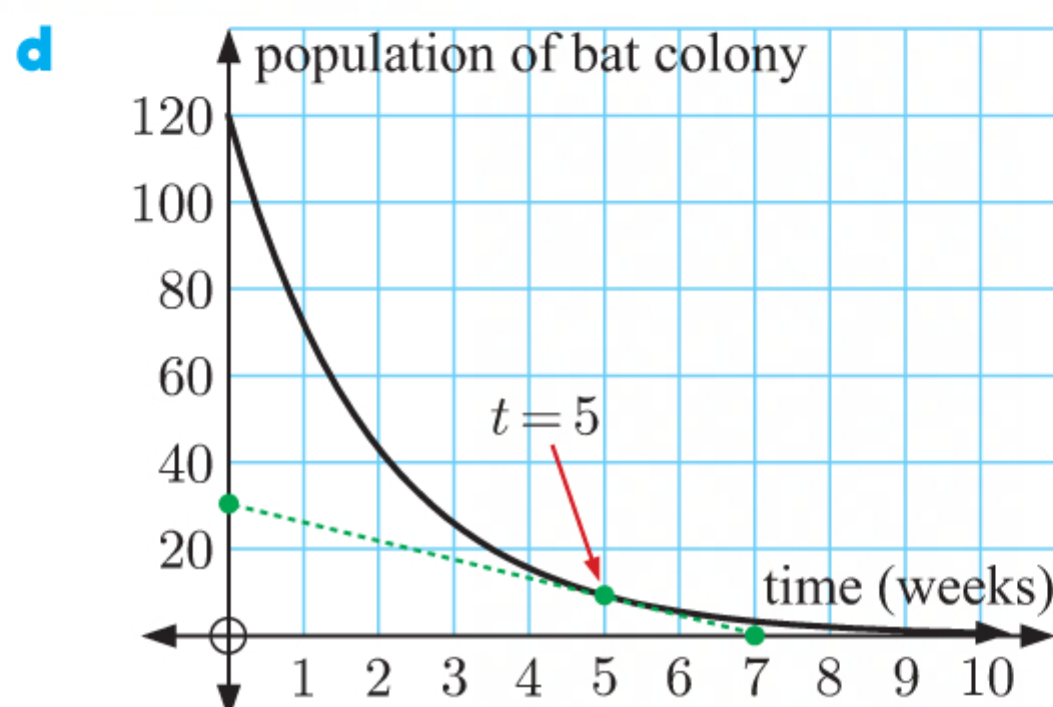
\therefore the rate of change at $x = 30$

= gradient of tangent

$$\approx \frac{(4 - 2) \text{ thousand \$}}{(50 - 5) \text{ items}}$$

$$\approx 0.0444 \text{ thousand \$ per item sold}$$

$$\approx \$44.40 \text{ per item sold}$$



The tangent at $t = 5$ passes through $(0, 30)$ and $(7, 0)$.

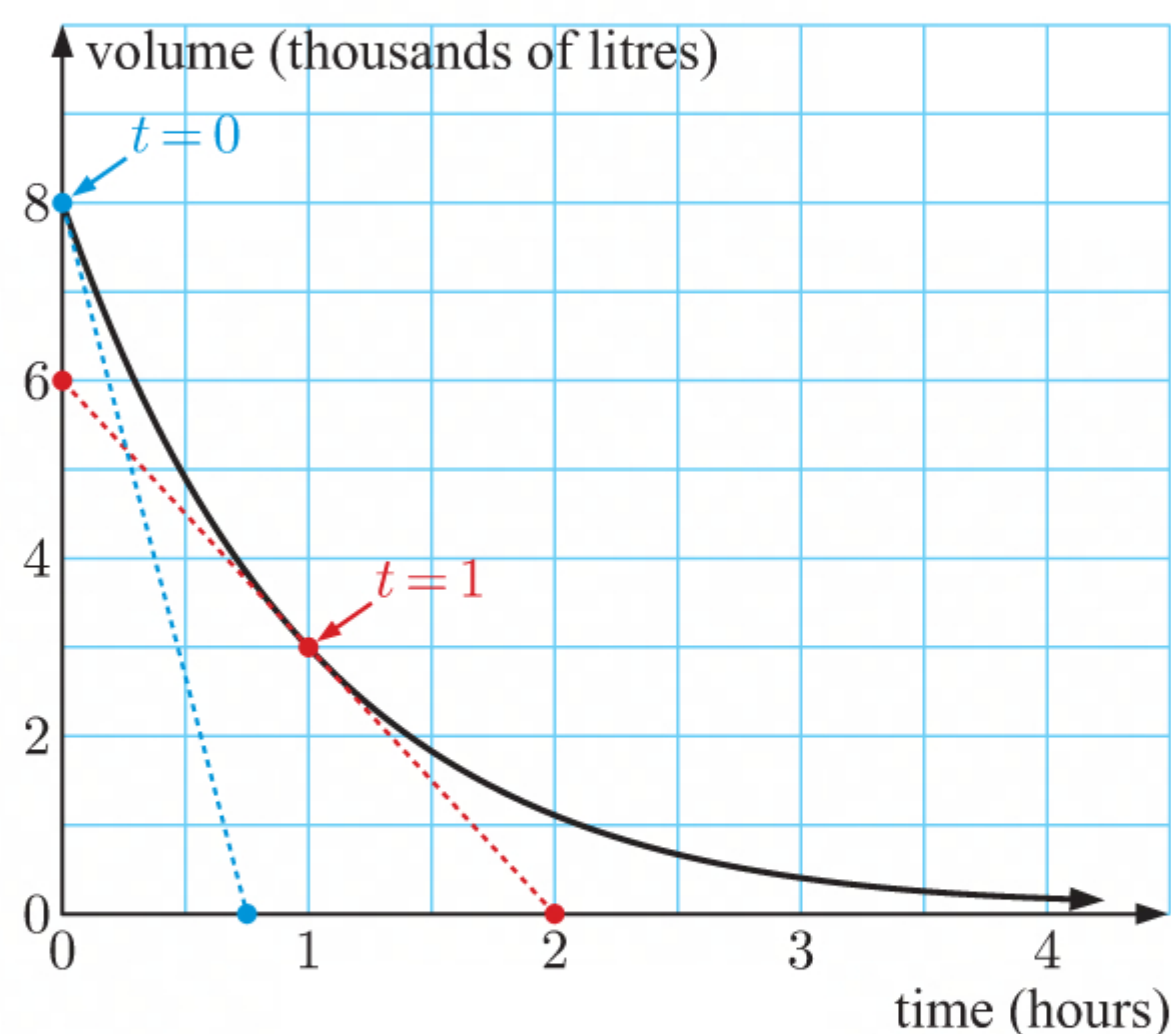
$$\begin{aligned} \therefore \text{the rate of change at } t = 5 &= \text{gradient of tangent} \\ &\approx \frac{(0 - 30) \text{ bats}}{(7 - 0) \text{ weeks}} \\ &\approx -\frac{30}{7} \text{ bats per week} \\ &\approx -4.29 \text{ bats per week} \end{aligned}$$

- 4 a** Initially at time 0 hours, the volume is 8 thousands of litres.
So, there were 8000 L in the tank originally.

- b** At time 1 hour, the volume is 3 thousands of litres.
So, after 1 hour there were 3000 L in the tank.

- c** The tangent at time $t = 0$ hours passes through $(0, 8)$ and $(0.75, 0)$.

$$\begin{aligned} \therefore \text{initial rate of water loss} &= \text{gradient of tangent at time 0 hours} \\ &\approx \frac{(0 - 8) \text{ thousand L}}{(0.75 - 0) \text{ hours}} \\ &\approx -10.667 \text{ thousand L per hour} \\ \therefore \text{initially the rate of water loss was} &\text{ about } 10\,667 \text{ L per hour.} \end{aligned}$$



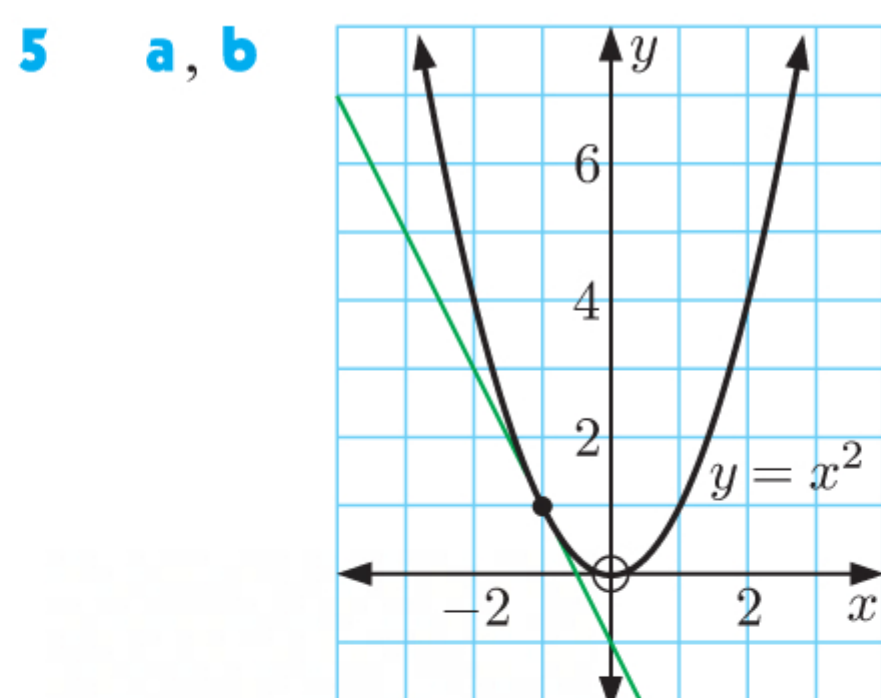
- d** The tangent at time $t = 1$ passes through $(0, 6)$ and $(2, 0)$.

\therefore rate of water loss after 1 hour = gradient of tangent at time 1 hour

$$\begin{aligned} &\approx \frac{(0 - 6) \text{ thousand L}}{(2 - 0) \text{ hours}} \\ &\approx -3 \text{ thousand L per hour} \end{aligned}$$

\therefore after 1 hour the rate of water loss was about 3000 L per hour.

- e** The rate at which the tank is leaking water is decreasing.



c The tangent at $x = -1$ has gradient $\frac{1 - (-1)}{-1 - 0} = -2$.

\therefore the instantaneous rate of change in $y = x^2$ when $x = -1$ is -2 .

EXERCISE 10C

1 a As $x \rightarrow 3$, $x + 4 \rightarrow 7$
 $\therefore \lim_{x \rightarrow 3} (x + 4) = 7$

c As $x \rightarrow 4$, $3x - 1 \rightarrow 11$
 $\therefore \lim_{x \rightarrow 4} (3x - 1) = 11$

d As $x \rightarrow 2$, $5x^2 - 3x + 2 \rightarrow 5(4) - 3(2) + 2 = 16$
 $\therefore \lim_{x \rightarrow 2} (5x^2 - 3x + 2) = 16$

e As $h \rightarrow 0$, $h^2 \rightarrow 0$ and $1 - h \rightarrow 1$
 $\therefore \lim_{h \rightarrow 0} h^2(1 - h) = 0 \times 1 = 0$

b As $x \rightarrow -1$, $5 - 2x \rightarrow 7$
 $\therefore \lim_{x \rightarrow -1} (5 - 2x) = 7$

f As $x \rightarrow 0$, $x^2 + 5 \rightarrow 5$
 $\therefore \lim_{x \rightarrow 0} (x^2 + 5) = 5$

2 a $\lim_{x \rightarrow 0} 5 = 5$

b $\lim_{h \rightarrow 2} 7 = 7$

c $\lim_{x \rightarrow 0} c = c$ (when c is a constant)

3 a $\frac{x^2 - 3x}{x}$ can be made as close as we like to -2 by making x sufficiently close to 1 .
 $\therefore \lim_{x \rightarrow 1} \frac{x^2 - 3x}{x} = -2$

b $\frac{h^2 + 5h}{h}$ can be made as close as we like to 7 by making h sufficiently close to 2 .
 $\therefore \lim_{h \rightarrow 2} \frac{h^2 + 5h}{h} = 7$

c $\frac{x - 1}{x + 1}$ can be made as close as we like to -1 by making x sufficiently close to 0 .
 $\therefore \lim_{x \rightarrow 0} \frac{x - 1}{x + 1} = -1$

4 a $\lim_{x \rightarrow 0} \frac{x}{x}$
 $= \lim_{x \rightarrow 0} 1$ {since $x \neq 0$ }
 $= 1$

b $\lim_{x \rightarrow 0} \frac{x^2 - 3x}{x}$
 $= \lim_{x \rightarrow 0} \frac{\cancel{x}(x - 3)}{\cancel{x}}$
 $= \lim_{x \rightarrow 0} (x - 3)$ {since $x \neq 0$ }
 $= -3$

c $\lim_{x \rightarrow 0} \frac{x^2 + 5x}{x}$
 $= \lim_{x \rightarrow 0} \frac{\cancel{x}(x + 5)}{\cancel{x}}$
 $= \lim_{x \rightarrow 0} (x + 5)$ {since $x \neq 0$ }
 $= 5$

d $\lim_{x \rightarrow 0} \frac{2x^2 - x}{x}$
 $= \lim_{x \rightarrow 0} \frac{\cancel{x}(2x - 1)}{\cancel{x}}$
 $= \lim_{x \rightarrow 0} (2x - 1)$ {since $x \neq 0$ }
 $= -1$

$$\begin{aligned}
 \text{e} \quad & \lim_{h \rightarrow 0} \frac{2h^2 + 6h}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{2}h(h+3)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} 2(h+3) \quad \{\text{since } h \neq 0\} \\
 &= 6
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & \lim_{h \rightarrow 0} \frac{h^3 - 8h}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(h^2 - 8)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (h^2 - 8) \quad \{\text{since } h \neq 0\} \\
 &= -8
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & \lim_{x \rightarrow 2} \frac{x^2 - 2x}{x - 2} \\
 &= \lim_{x \rightarrow 2} \frac{x(\cancel{x-2})}{\cancel{x-2}} \\
 &= \lim_{x \rightarrow 2} x \quad \{\text{since } x \neq 2\} \\
 &= 2
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \lim_{h \rightarrow 0} \frac{3h^2 - 4h}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(3h - 4)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (3h - 4) \quad \{\text{since } h \neq 0\} \\
 &= -4
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & \lim_{x \rightarrow 1} \frac{x^2 - x}{x - 1} \\
 &= \lim_{x \rightarrow 1} \frac{x(\cancel{x-1})}{\cancel{x-1}} \\
 &= \lim_{x \rightarrow 1} x \quad \{\text{since } x \neq 1\} \\
 &= 1
 \end{aligned}$$

EXERCISE 10D

- 1 a M has x -coordinate $3 + h$ and lies on the graph of $f(x) = x^2$.
 \therefore its y -coordinate is $(3 + h)^2$.

$$\begin{aligned}
 \text{b} \quad & \text{The gradient of [FM]} = \frac{y_M - y_F}{x_M - x_F} \\
 &= \frac{(3 + h)^2 - 9}{(3 + h) - 3} \\
 &= \frac{\cancel{9} + 6h + h^2 - \cancel{9}}{h} \\
 &= \frac{6h + h^2}{h} \\
 &= \frac{\cancel{h}(6 + h)}{\cancel{h}} \\
 &= 6 + h \quad \text{provided } h \neq 0
 \end{aligned}$$

- c i M has x -coordinate $3 + h$.
 \therefore at the point $(4, 16)$, $3 + h = 4$
 $\therefore h = 1$
 The gradient of [FM] is $6 + h$.
{from b}
 \therefore the gradient of [FM] at $(4, 16)$ is
 $6 + 1 = 7$.

- ii M has x -coordinate $3 + h$.
 \therefore at the point $(3.5, 12.25)$,
 $3 + h = 3.5$
 $\therefore h = 0.5$
 The gradient of [FM] is $6 + h$.
{from b}
 \therefore the gradient of [FM] at
 $(3.5, 12.25)$ is $6 + 0.5 = 6.5$.

iii M has x -coordinate $3 + h$.

\therefore at the point $(3.1, 9.61)$,

$$3 + h = 3.1$$

$$\therefore h = 0.1$$

The gradient of [FM] is $6 + h$.

{from b}

\therefore the gradient of [FM] at $(3.1, 9.61)$

$$\text{is } 6 + 0.1 = 6.1.$$

iv M has x -coordinate $3 + h$.

\therefore at the point $(3.01, 9.0601)$,

$$3 + h = 3.01$$

$$\therefore h = 0.01$$

The gradient of [FM] is $6 + h$.

{from b}

\therefore the gradient of [FM] at

$(3.01, 9.0601)$ is

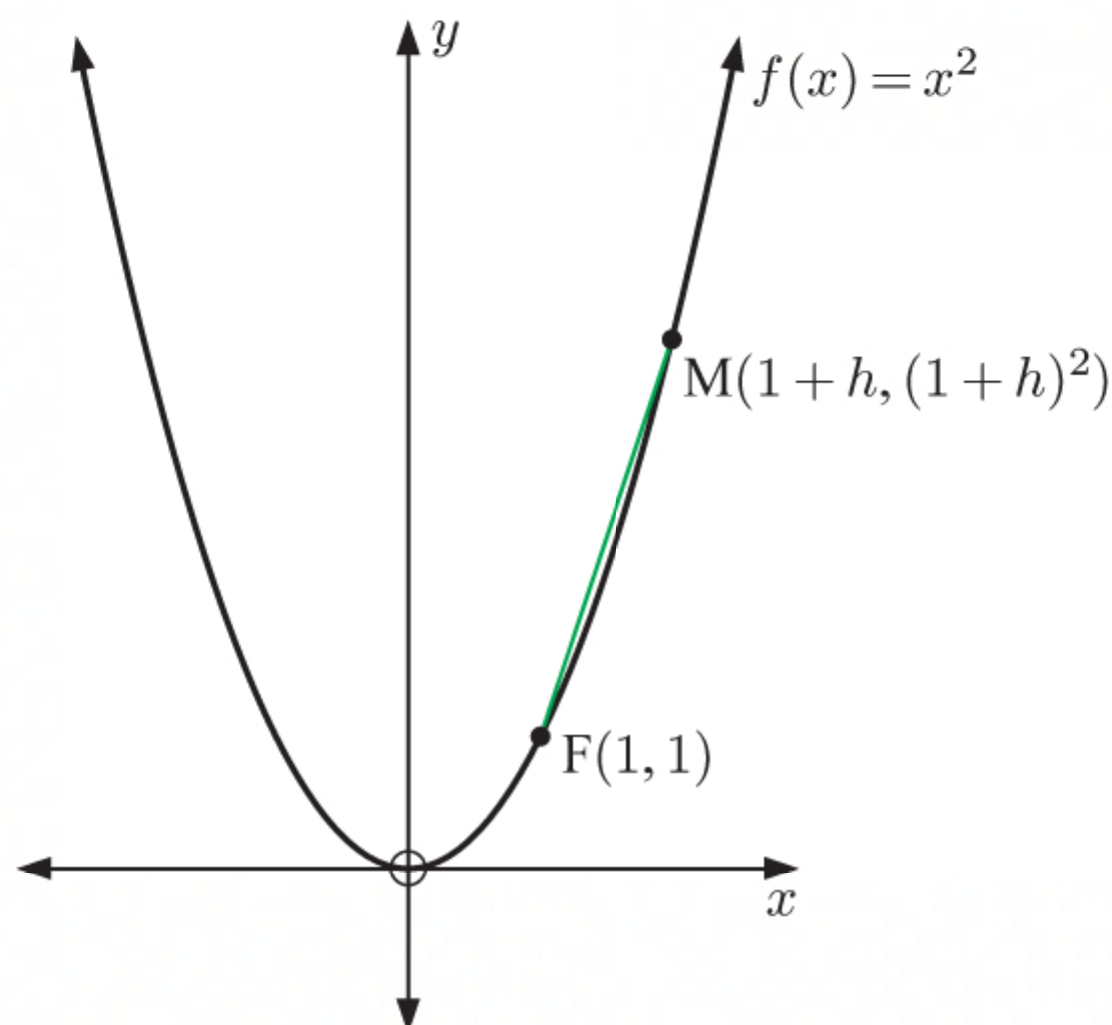
$$6 + 0.01 = 6.01.$$

d Using limit theory, the gradient of the tangent to $f(x) = x^2$ at the point $(3, 9)$ is

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} &= \lim_{h \rightarrow 0} \frac{(3+h)^2 - 3^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{9} + 6h + h^2 - \cancel{9}}{h} \\ &= \lim_{h \rightarrow 0} \frac{6h + h^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{h}(6+h)}{\cancel{h}} \\ &= \lim_{h \rightarrow 0} (6+h) \quad \{\text{as } h \neq 0\} \\ &= 6 \end{aligned}$$

2 a i At $x = 1$, $f(1) = 1^2 = 1$.

Let F be the point $(1, 1)$ and M have x -coordinate $1 + h$, so M is $(1 + h, (1 + h)^2)$.

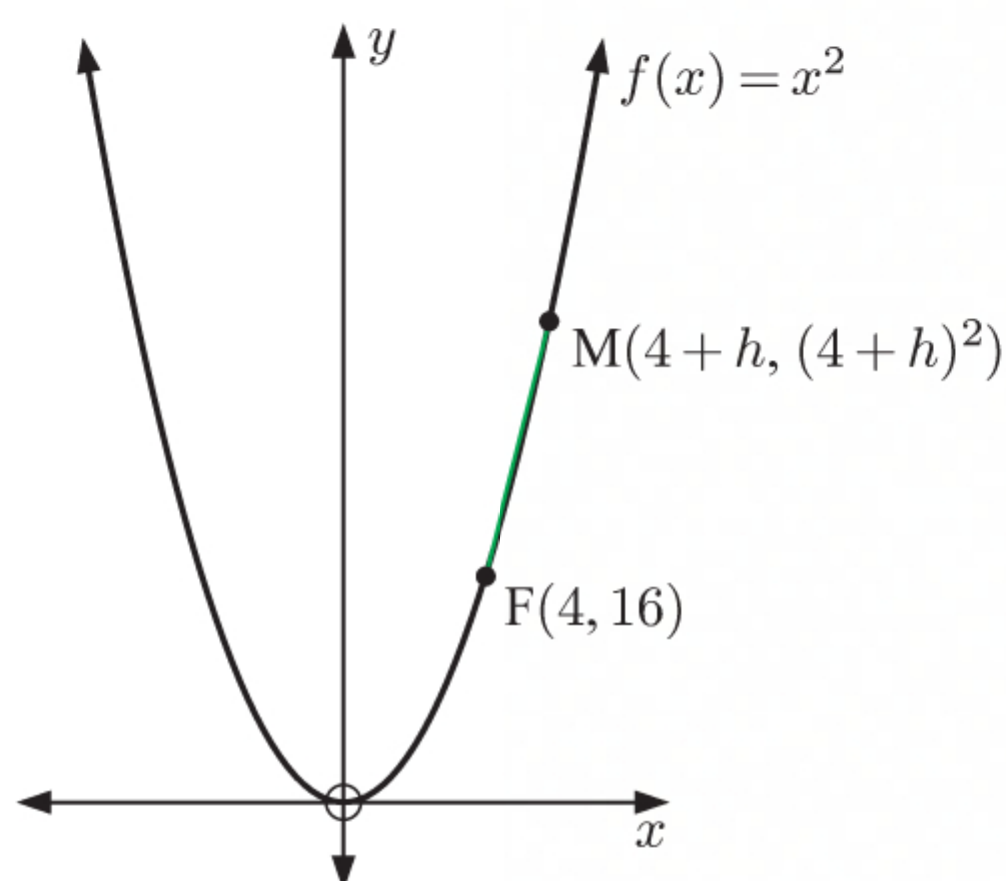


The gradient of the tangent at F

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(1+h)^2 - 1^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{1} + 2h + h^2 - \cancel{1}}{h} \\ &= \lim_{h \rightarrow 0} \frac{2h + h^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{h}(2+h)}{\cancel{h}} \\ &= \lim_{h \rightarrow 0} (2+h) \quad \{\text{as } h \neq 0\} \\ &= 2 \end{aligned}$$

ii At $x = 4$, $f(4) = 4^2 = 16$.

Let F be the point $(4, 16)$ and M have x -coordinate $4 + h$, so M is $(4 + h, (4 + h)^2)$.



The gradient of the tangent at F

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{f(4+h) - f(4)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(4+h)^2 - 4^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{16} + 8h + h^2 - \cancel{16}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{8h + h^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(8+h)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (8+h) \quad \{\text{as } h \neq 0\} \\
 &= 8
 \end{aligned}$$

b The gradient of the tangent to $f(x) = x^2$ at the point where $x = 2$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(2+h)^2 - 2^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{4} + 4h + h^2 - \cancel{4}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{4h + h^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(4+h)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (4+h) \quad \{\text{as } h \neq 0\} \\
 &= 4
 \end{aligned}$$

The gradient of the tangent to $f(x) = x^2$ at the point where $x = 3$

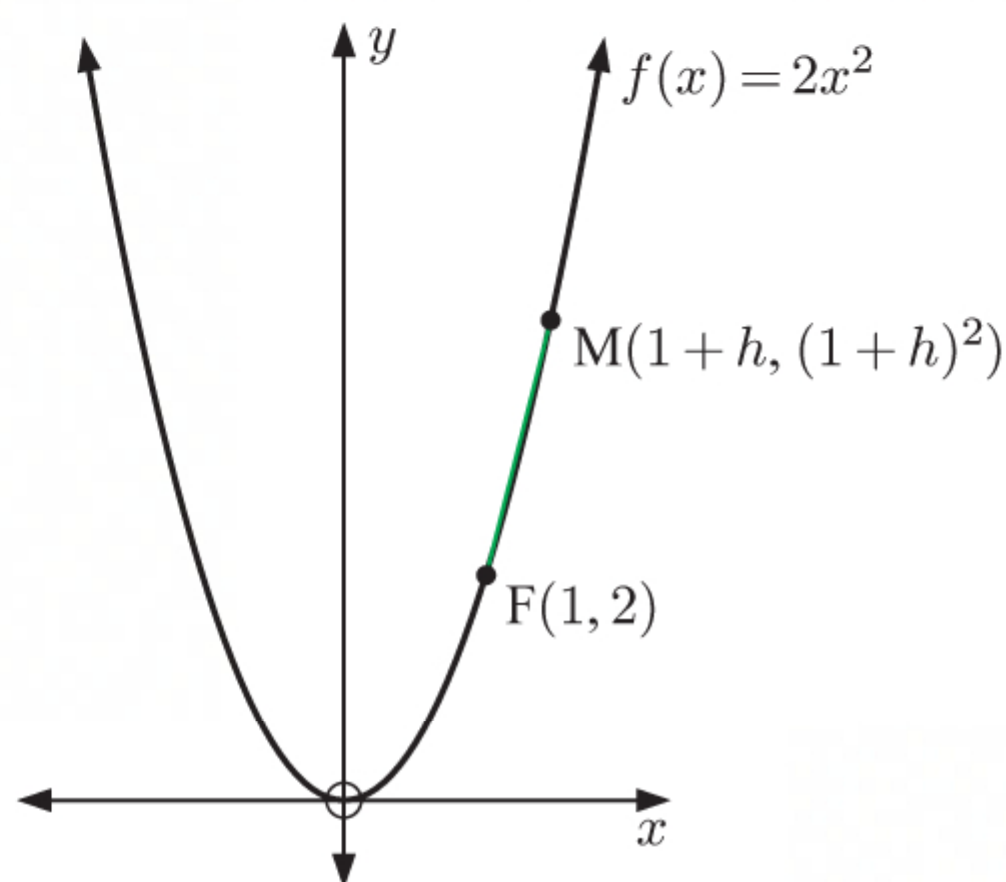
$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(3+h)^2 - 3^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{9} + 6h + h^2 - \cancel{9}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{6h + h^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(6+h)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (6+h) \quad \{\text{as } h \neq 0\} \\
 &= 6
 \end{aligned}$$

Using the results from a, the table is:

x -coordinate	Gradient of tangent
1	2
2	4
3	6
4	8

c The gradient of the tangent is equal to twice the x -coordinate in each case in b. So, we predict the gradient of the tangent to $f(x) = x^2$ at the point where $x = a$ will be $2a$.

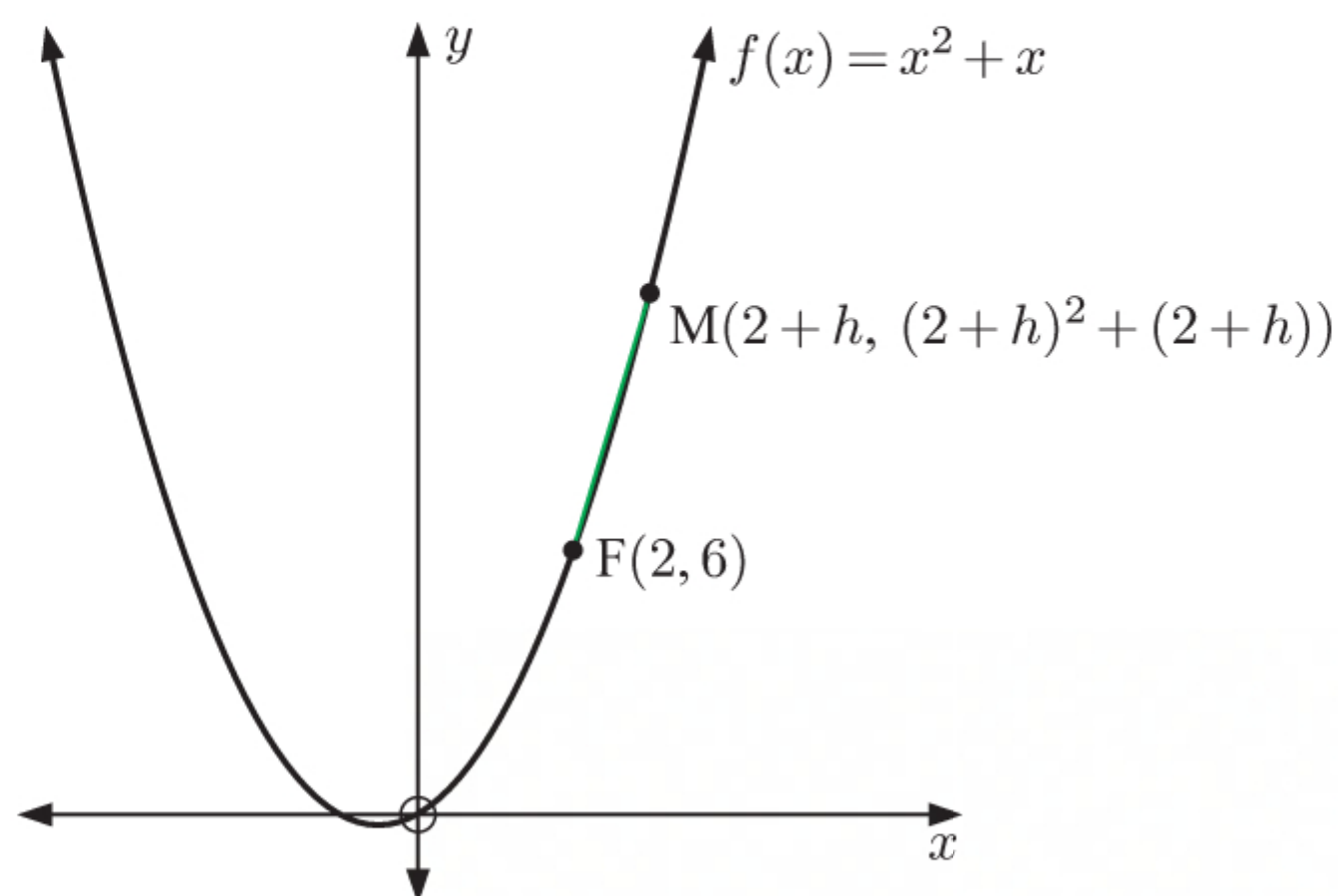
- 3 a** Let F be the point (1, 2) and M have x -coordinate $1 + h$, so M is $(1 + h, 2(1 + h)^2)$.



The gradient of the tangent at F

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{f(1 + h) - f(1)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2(1 + h)^2 - 2(1)^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2(1 + 2h + h^2) - 2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{2} + 4h + 2h^2 - \cancel{2}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{4h + 2h^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(4 + 2h)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (4 + 2h) \quad \{\text{as } h \neq 0\} \\
 &= 4
 \end{aligned}$$

- b** Let F be the point (2, 6) and M have x -coordinate $2 + h$, so M is $(2 + h, (2 + h)^2 + (2 + h))$.



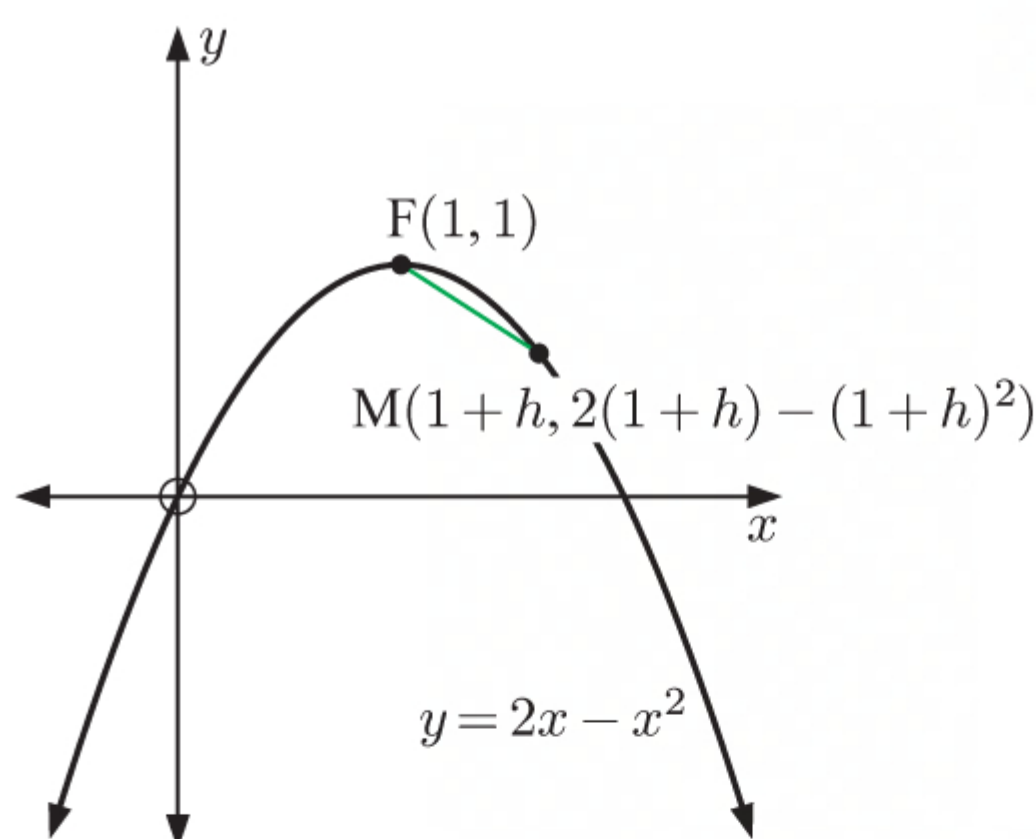
The gradient of the tangent at F

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{f(2 + h) - f(2)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(2 + h)^2 + (2 + h) - (2^2 + 2)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{4 + 4h + h^2 + 2 + h - 6}{h} \\
 &= \lim_{h \rightarrow 0} \frac{5h + h^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(5 + h)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (5 + h) \quad \{\text{as } h \neq 0\} \\
 &= 5
 \end{aligned}$$

- c At $x = 1$, $f(1) = 2(1) - 1^2 = 1$.

Let F be the point $(1, 1)$ and M have x -coordinate $1 + h$,

so M is $(1 + h, 2(1 + h) - (1 + h)^2)$.



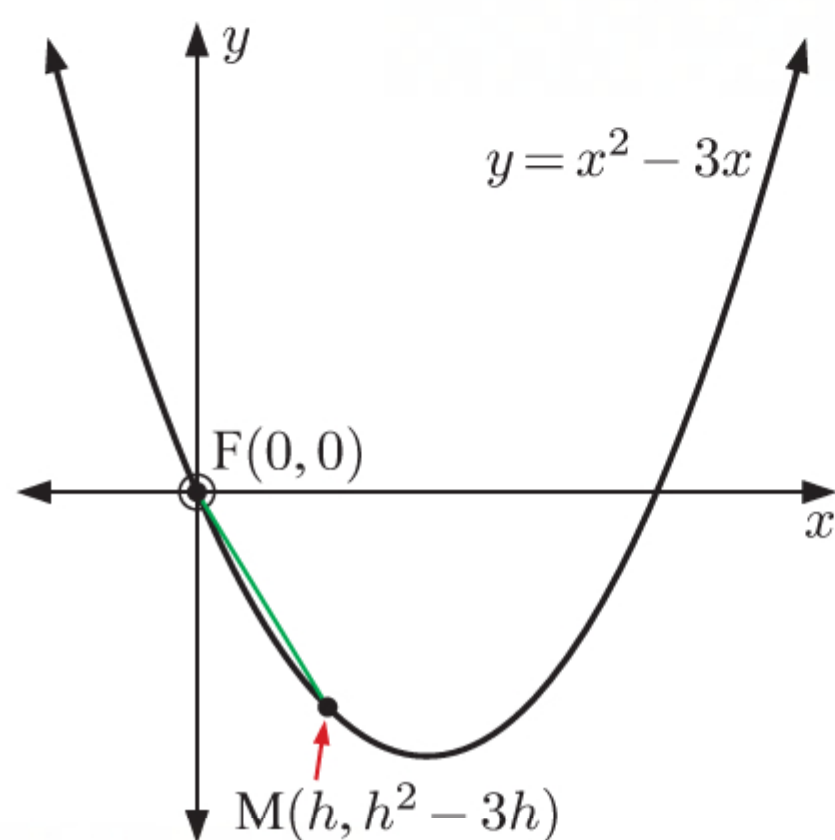
The gradient of the tangent at F

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{f(1 + h) - f(1)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2(1 + h) - (1 + h)^2 - 1}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2 + 2h - (1 + 2h + h^2) - 1}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2 + 2h - 1 - 2h - h^2 - 1}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-h^2}{h} \\
 &= \lim_{h \rightarrow 0} -h \quad \{\text{as } h \neq 0\} \\
 &= 0
 \end{aligned}$$

- d At $x = 0$, $f(0) = 0^2 - 3(0) = 0$.

Let F be the point $(0, 0)$ and M have x -coordinate $0 + h$,

so M is $(h, h^2 - 3h)$.



The gradient of the tangent at F

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h^2 - 3h - 0}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h^2 - 3h}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h(h - 3)}{h} \\
 &= \lim_{h \rightarrow 0} (h - 3) \quad \{\text{as } h \neq 0\} \\
 &= -3
 \end{aligned}$$

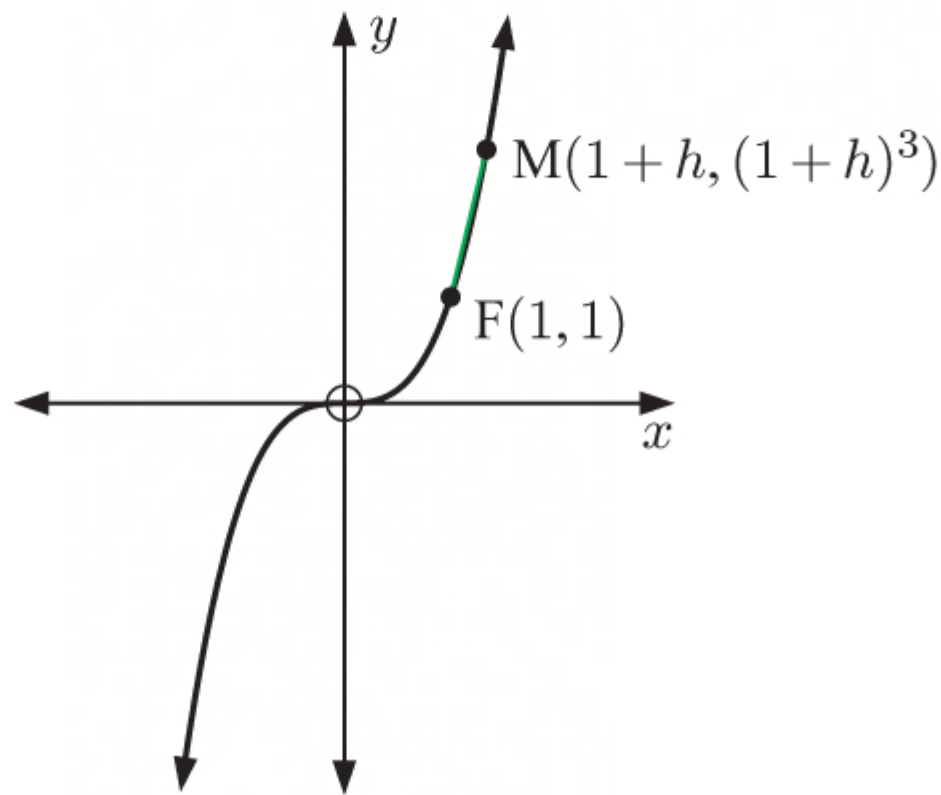
4 a $(1 + h)^3 = (1 + h)^2(1 + h)$

$$\begin{aligned}
 &= (1 + 2h + h^2)(1 + h) \\
 &= 1 + h + 2h + 2h^2 + h^2 + h^3 \\
 &= 1 + 3h + 3h^2 + h^3
 \end{aligned}$$

- b i** M has x -coordinate $1 + h$ and lies on the graph of $y = x^3$, so M is $(1 + h, (1 + h)^3)$.

$$\begin{aligned} \text{The gradient of [FM]} &= \frac{y_M - y_F}{x_M - x_F} \\ &= \frac{(1 + h)^3 - 1}{(1 + h) - 1} \\ &= \frac{1 + 3h + 3h^2 + h^3 - 1}{h} \quad \{\text{using a}\} \\ &= \frac{3h + 3h^2 + h^3}{h} \end{aligned}$$

ii



The gradient of the tangent at F

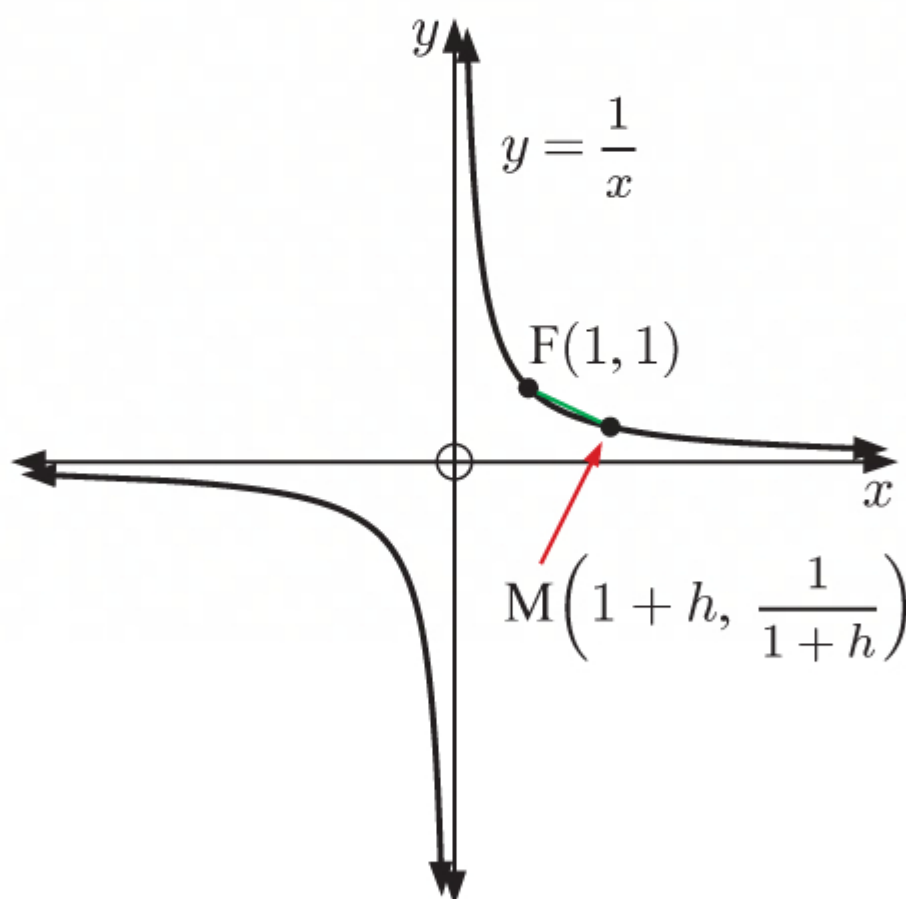
$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{f(1 + h) - f(1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{3h + 3h^2 + h^3}{h} \quad \{\text{from b}\} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{h}(3 + 3h + h^2)}{\cancel{h}} \\ &= \lim_{h \rightarrow 0} (3 + 3h + h^2) \quad \{\text{as } h \neq 0\} \\ &= 3 \end{aligned}$$

5 a $\frac{1}{x + h} - \frac{1}{x} = \frac{\cancel{x} - (\cancel{x} + h)}{x(x + h)}$

$$= \frac{-h}{x(x + h)}$$

- b i** At $x = 1$, $y = \frac{1}{1} = 1$.

Let F be the point $(1, 1)$ and M have x -coordinate $1 + h$, so M is $(1 + h, \frac{1}{1 + h})$.

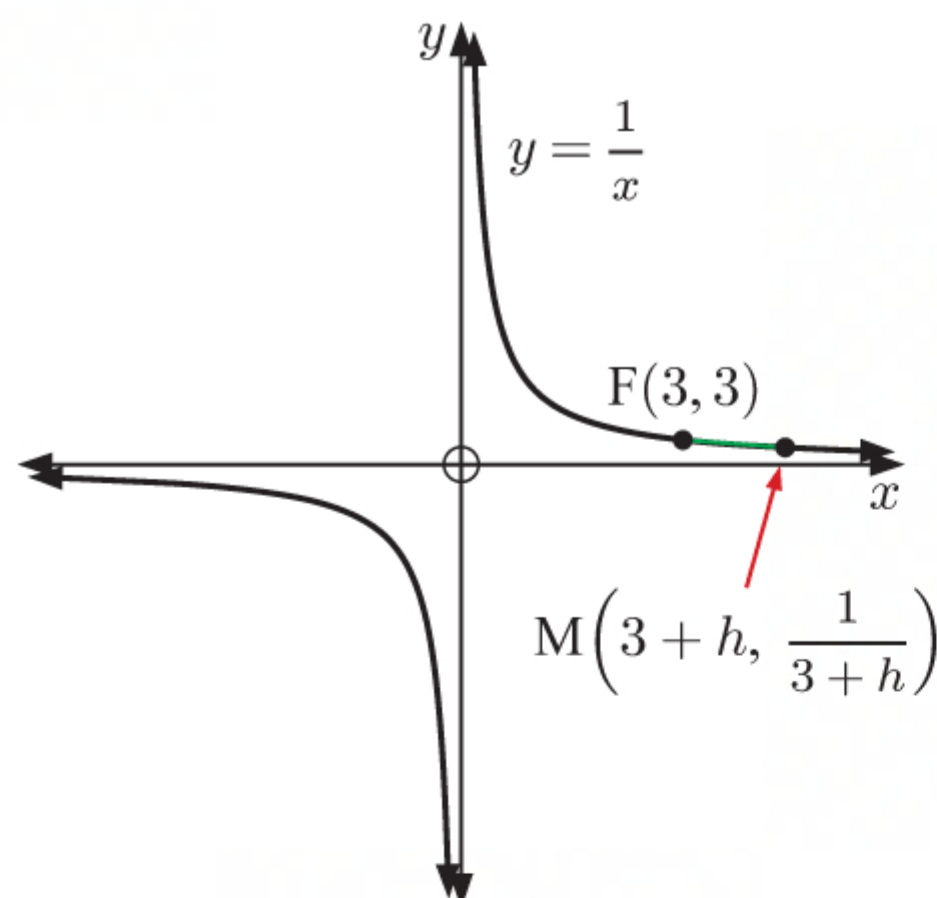


The gradient of the tangent at F

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{y_M - y_F}{x_M - x_F} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1}{1 + h} - 1}{1 + h - 1} \\ &= \lim_{h \rightarrow 0} \frac{\frac{-h}{1(1 + h)}}{h} \quad \{\text{using a with } x = 1\} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{-h}}{\cancel{h}(1 + h)} \\ &= \lim_{h \rightarrow 0} \frac{-1}{1 + h} \quad \{\text{as } h \neq 0\} \\ &= \frac{-1}{1} \\ &= -1 \end{aligned}$$

ii At $x = 3$, $y = \frac{1}{3}$.

Let F be the point $(3, \frac{1}{3})$ and M have x -coordinate $3 + h$, so M is $(3 + h, \frac{1}{3 + h})$.



The gradient of the tangent at F

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{y_M - y_F}{x_M - x_F} \\
 &= \lim_{h \rightarrow 0} \frac{\frac{1}{3+h} - \frac{1}{3}}{3+h-3} \\
 &= \lim_{h \rightarrow 0} \frac{\frac{-h}{3(3+h)}}{h} \quad \{\text{using a with } x = 3\} \\
 &= \lim_{h \rightarrow 0} \frac{-h}{3h(3+h)} \\
 &= \lim_{h \rightarrow 0} \frac{-1}{3(3+h)} \quad \{\text{as } h \neq 0\} \\
 &= -\frac{1}{9}
 \end{aligned}$$

EXERCISE 10E

1 a $f(0) = 4$

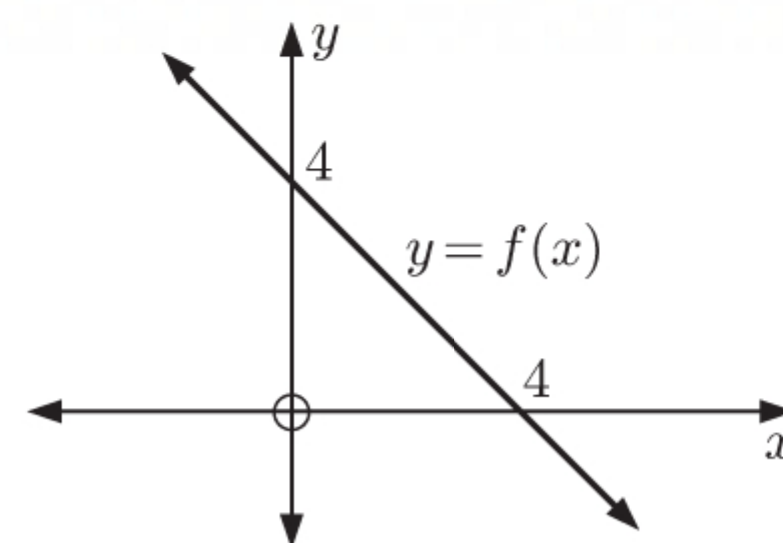
b $f'(0)$ is the gradient of the tangent to $f(x)$ at the point where $x = 0$.

Since $f(x)$ is a straight line, this is the same as the gradient of $f(x)$ itself.

$f(x)$ goes through $(0, 4)$ and $(4, 0)$, so it has

$$\text{gradient} = \frac{0 - 4}{4 - 0} = -1$$

$$\therefore f'(0) = -1$$

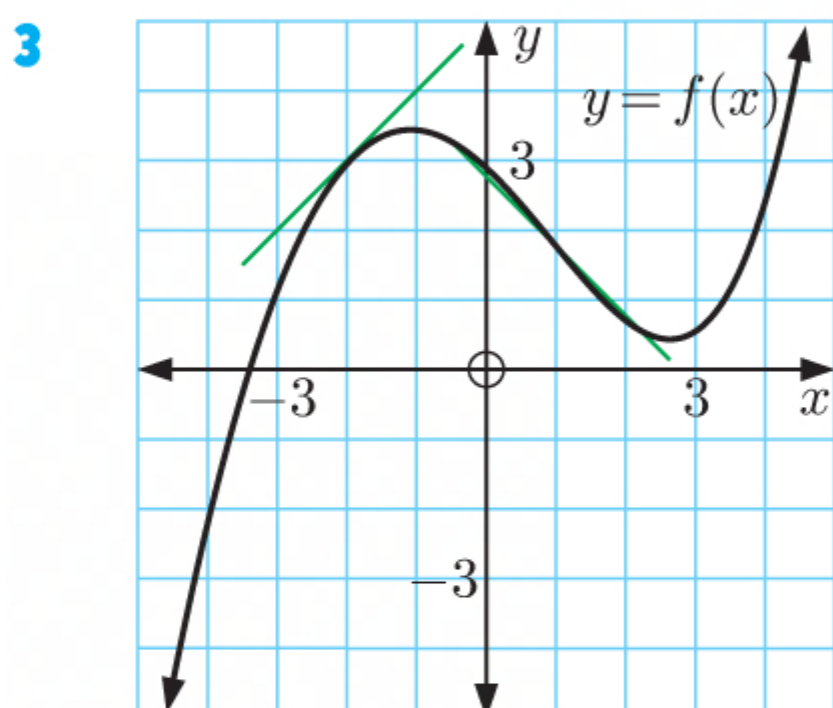
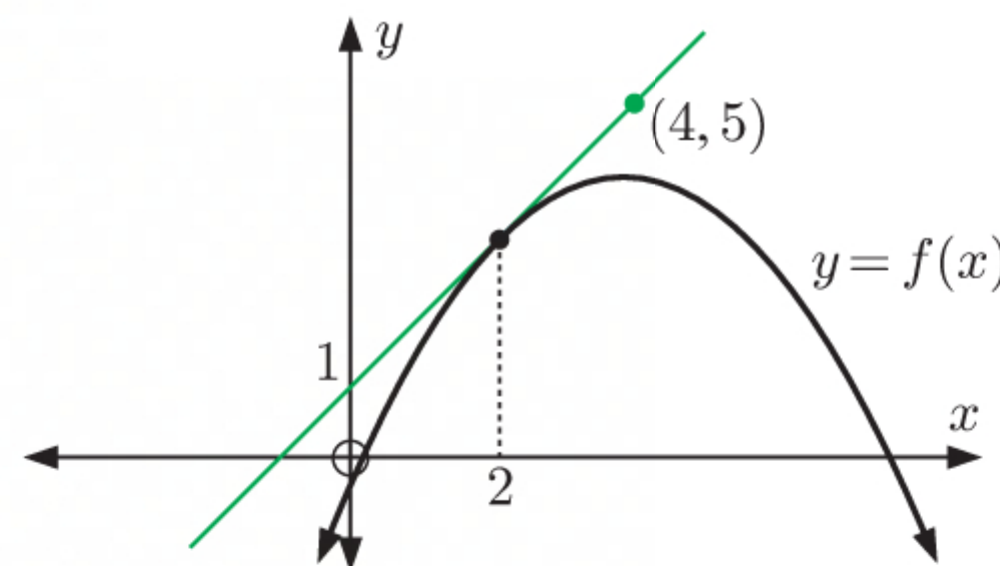


2 The graph shows the tangent to the curve $y = f(x)$ at the point where $x = 2$.

The tangent passes through $(0, 1)$ and $(4, 5)$.

$\therefore f'(2) = \text{gradient of the tangent}$

$$\begin{aligned}
 &= \frac{5 - 1}{4 - 0} \\
 &= 1
 \end{aligned}$$



a $f(3)$ is above the x -axis, so $f(3)$ is positive.

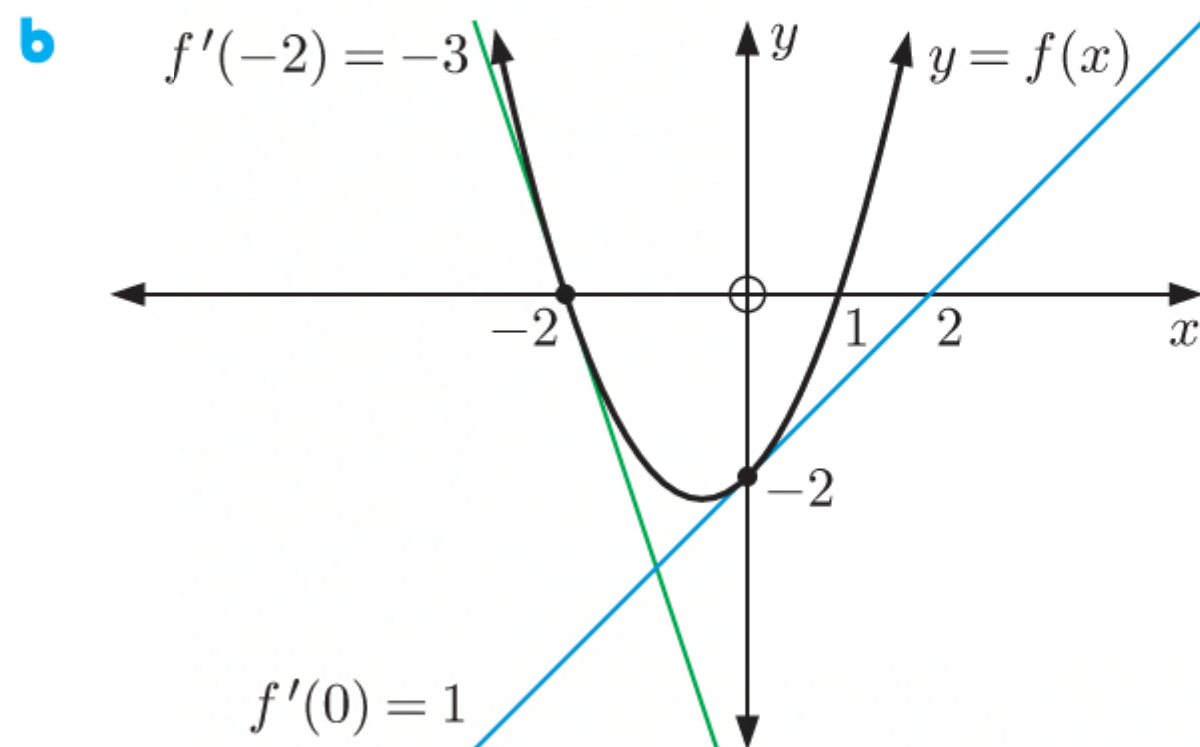
b $f'(1)$ is the gradient of the tangent to $f(x)$ at the point where $x = 1$. Since the curve is decreasing at $x = 1$, then $f'(1)$ is negative.

c $f(-4)$ is below the x -axis, so $f(-4)$ is negative.

d $f'(-2)$ is the gradient of the tangent to $f(x)$ at the point where $x = -2$. Since the curve is increasing at $x = -2$, then $f'(-2)$ is positive.

4 a i $f'(x) = 2x + 1$
 $f'(-2) = 2(-2) + 1$
 $= -3$

The gradient of the tangent to $y = f(x)$ at the point where $x = -2$ is -3 .



ii $f'(0) = 2(0) + 1$
 $= 1$

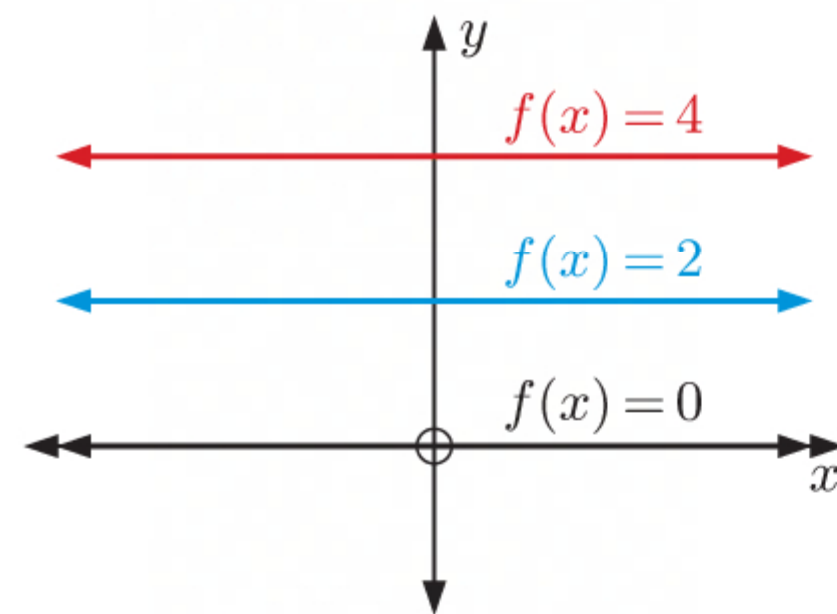
The gradient of the tangent to $y = f(x)$ at the point where $x = 0$ is 1 .

INVESTIGATION 2

GRADIENT FUNCTIONS

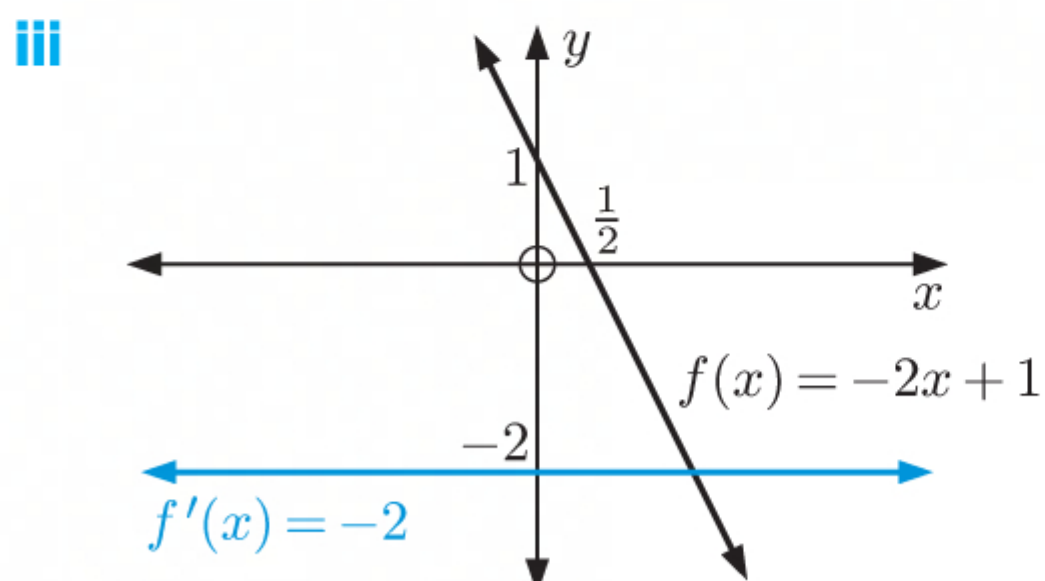
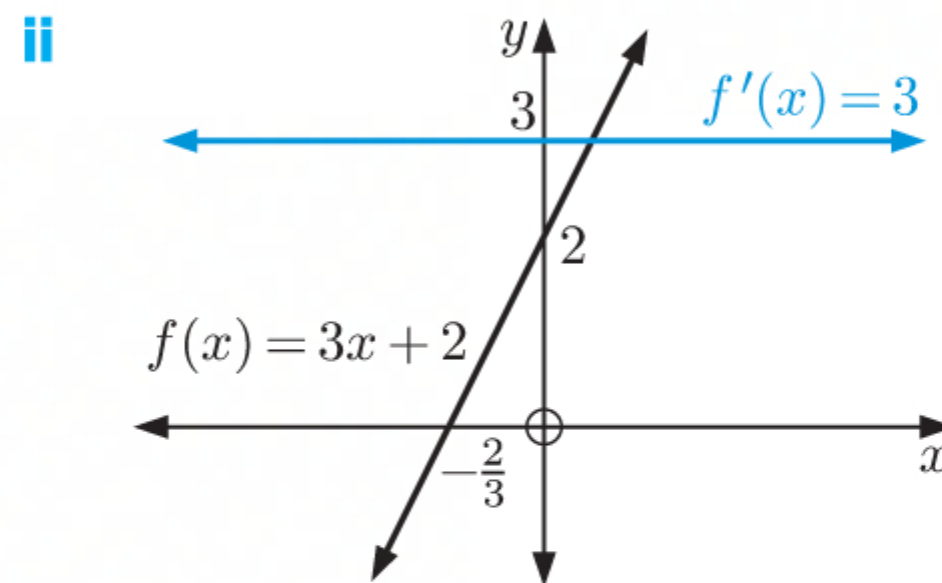
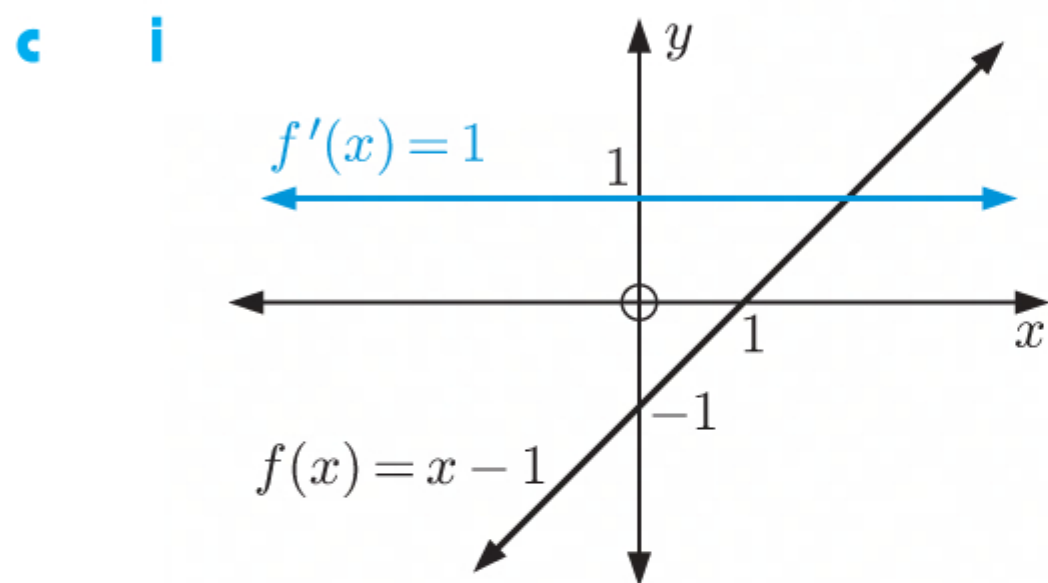
1 a $f(x) = 0$, $f(x) = 2$, and $f(x) = 4$ are all horizontal lines and hence all have gradient 0 .

b Yes, the gradient is constant for all values of x .



2 a The gradient of $f(x) = mx + c$ is m .

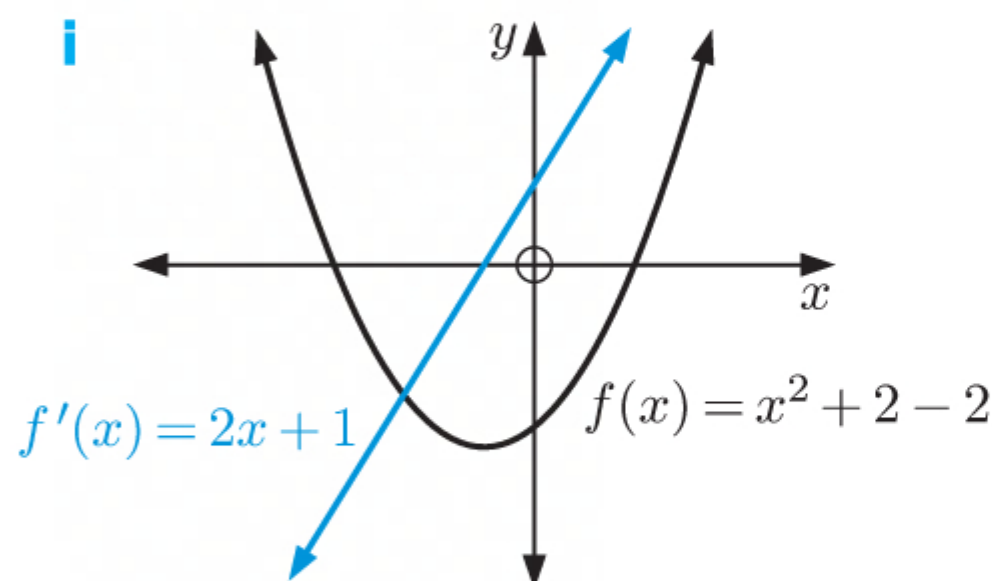
b The gradient m is constant for all values of x .



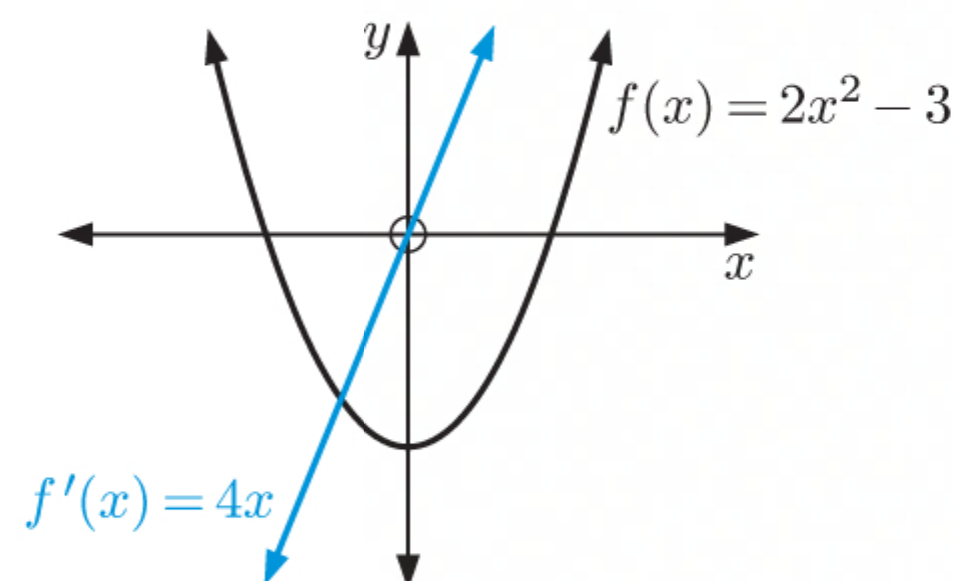
$f'(x)$ is constant for all x .

3 a $f'(x)$ is a linear function.

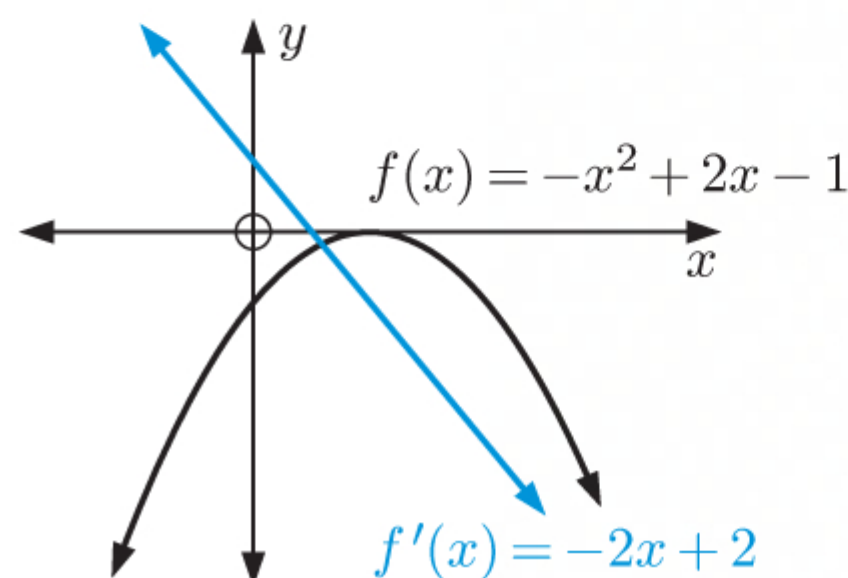
b i



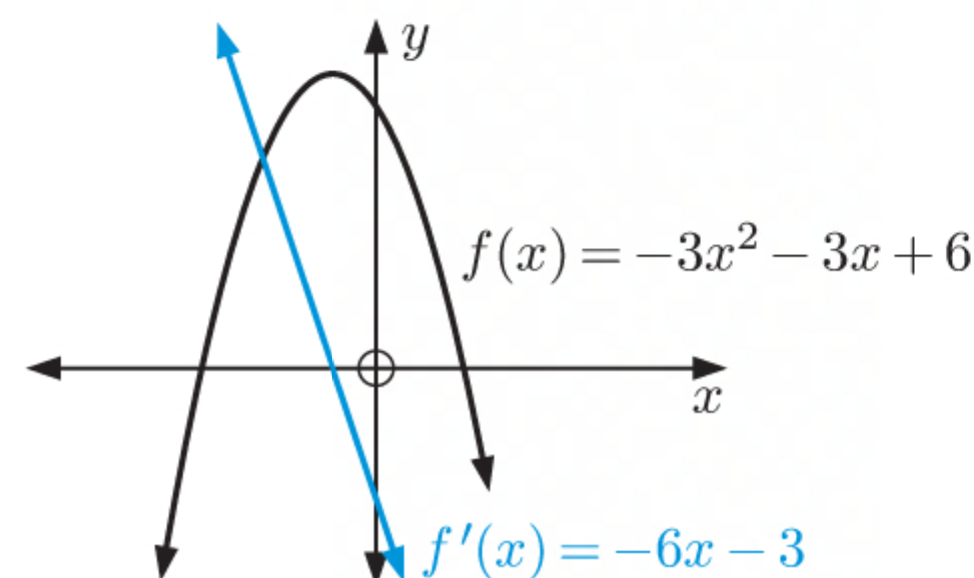
ii



iii



iv



c The gradient functions $f'(x)$ in **b** are all linear functions.

EXERCISE 10F

1 a $f(x) = 1$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1 - 1}{h} \\ &= \lim_{h \rightarrow 0} \frac{0}{h} \\ &= \lim_{h \rightarrow 0} 0 \quad \{\text{as } h \neq 0\} \\ &= 0 \end{aligned}$$

c $f(x) = 2x - 1$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[2(x+h) - 1] - [2x - 1]}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{2x} + 2h - \cancel{1} - \cancel{2x} + \cancel{1}}{h} \\ &= \lim_{h \rightarrow 0} \frac{2\cancel{h}}{\cancel{h}} \\ &= \lim_{h \rightarrow 0} 2 \quad \{\text{as } h \neq 0\} \\ &= 2 \end{aligned}$$

b $f(x) = x$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x+h) - x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{x} + h - \cancel{x}}{\cancel{h}} \\ &= \lim_{h \rightarrow 0} 1 \quad \{\text{as } h \neq 0\} \\ &= 1 \end{aligned}$$

d $f(x) = 3 - x$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[3 - (x+h)] - [3 - x]}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{3} - \cancel{x} - h - \cancel{3} + \cancel{x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{-\cancel{h}}{\cancel{h}} \\ &= \lim_{h \rightarrow 0} -1 \quad \{\text{as } h \neq 0\} \\ &= -1 \end{aligned}$$

2 a $y = f(x) = x^2 + 2$

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\&= \lim_{h \rightarrow 0} \frac{[(x+h)^2 + 2] - [x^2 + 2]}{h} \\&= \lim_{h \rightarrow 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{2} - \cancel{x^2} - \cancel{2}}{h} \\&= \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} \\&= \lim_{h \rightarrow 0} \frac{\cancel{h}(2x + h)}{\cancel{h}} \\&= \lim_{h \rightarrow 0} (2x + h) \quad \{\text{as } h \neq 0\} \\&= 2x\end{aligned}$$

b $y = f(x) = 3 - x^2$

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\&= \lim_{h \rightarrow 0} \frac{[3 - (x+h)^2] - [3 - x^2]}{h} \\&= \lim_{h \rightarrow 0} \frac{3 - (x^2 + 2xh + h^2) - 3 + x^2}{h} \\&= \lim_{h \rightarrow 0} \frac{\cancel{3} - \cancel{x^2} - 2xh - h^2 - \cancel{3} + \cancel{x^2}}{h} \\&= \lim_{h \rightarrow 0} \frac{-2xh - h^2}{h} \\&= \lim_{h \rightarrow 0} \frac{\cancel{h}(-2x - h)}{\cancel{h}} \\&= \lim_{h \rightarrow 0} (-2x - h) \quad \{\text{as } h \neq 0\} \\&= -2x\end{aligned}$$

c $y = f(x) = 2x^2 + x$

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\&= \lim_{h \rightarrow 0} \frac{[2(x+h)^2 + (x+h)] - [2x^2 + x]}{h} \\&= \lim_{h \rightarrow 0} \frac{2(x^2 + 2xh + h^2) + \cancel{x} + h - 2x^2 - \cancel{x}}{h} \\&= \lim_{h \rightarrow 0} \frac{\cancel{2x^2} + 4xh + 2h^2 + h - \cancel{2x^2}}{h} \\&= \lim_{h \rightarrow 0} \frac{4xh + 2h^2 + h}{h} \\&= \lim_{h \rightarrow 0} \frac{\cancel{h}(4x + 2h + 1)}{\cancel{h}} \\&= \lim_{h \rightarrow 0} (4x + 2h + 1) \quad \{\text{as } h \neq 0\} \\&= 4x + 1\end{aligned}$$

d $y = f(x) = -x^2 + 5x - 3$

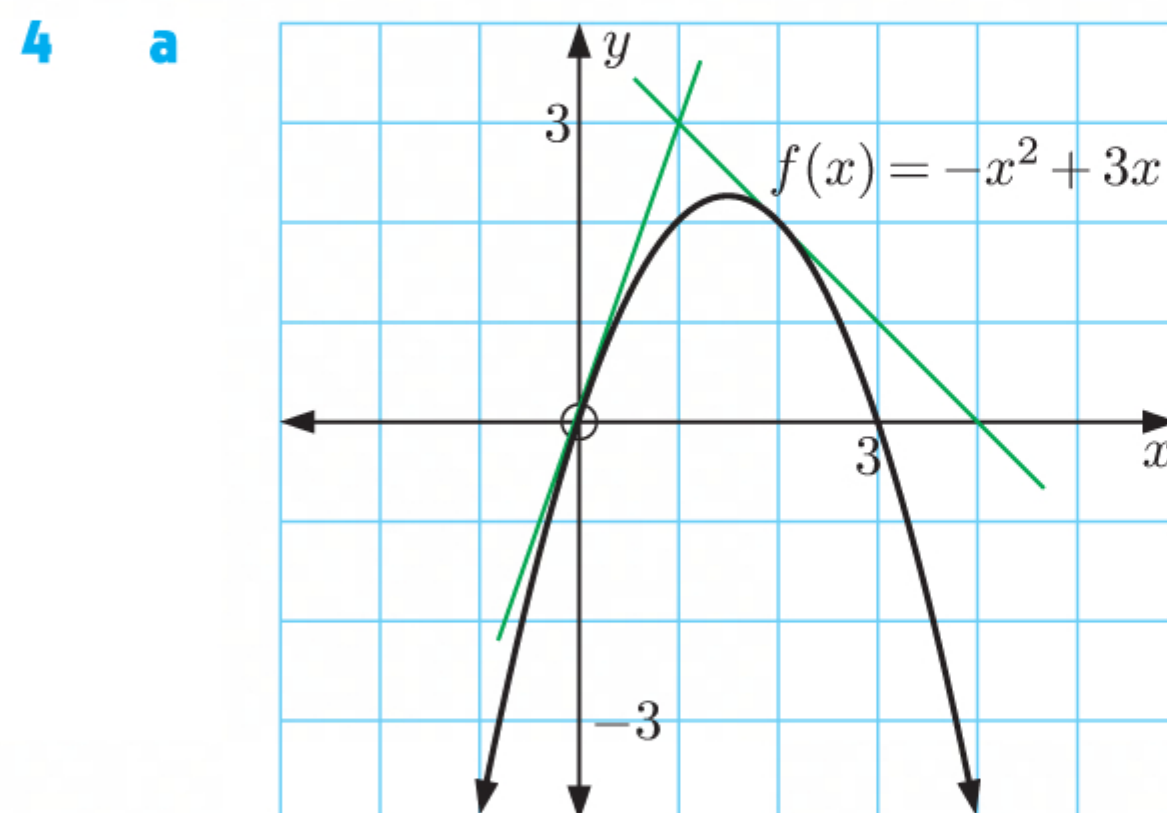
$$\begin{aligned}\frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\&= \lim_{h \rightarrow 0} \frac{[-(x+h)^2 + 5(x+h) - 3] - [-x^2 + 5x - 3]}{h} \\&= \lim_{h \rightarrow 0} \frac{-(x^2 + 2xh + h^2) + \cancel{5x} + 5h - \cancel{3} + x^2 - \cancel{5x} + \cancel{3}}{h} \\&= \lim_{h \rightarrow 0} \frac{-\cancel{x^2} - 2xh - h^2 + 5h + \cancel{x^2}}{h} \\&= \lim_{h \rightarrow 0} \frac{-2xh - h^2 + 5h}{h} \\&= \lim_{h \rightarrow 0} \frac{\cancel{h}(-2x - h + 5)}{\cancel{h}} \\&= \lim_{h \rightarrow 0} (-2x - h + 5) \quad \{\text{as } h \neq 0\} \\&= -2x + 5\end{aligned}$$

3 a $f(x) = 3x^2 - 1$

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[3(x+h)^2 - 1] - [3x^2 - 1]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{3(x^2 + 2xh + h^2) - \cancel{1} - 3x^2 + \cancel{1}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{3x^2} + 6xh + 3h^2 - \cancel{3x^2}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{6xh + 3h^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(6x + 3h)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (6x + 3h) \quad \{\text{as } h \neq 0\} \\
 &= 6x
 \end{aligned}$$

b $f'(2) = 6 \times 2$
 $= 12$

The gradient of the tangent to $y = f(x)$ at the point where $x = 2$ is 12.



i The tangent to $f(x) = -x^2 + 3x$ at the point where $x = 0$ has gradient $\approx \frac{3-0}{1-0} \approx 3$.

ii The tangent to $f(x) = -x^2 + 3x$ at the point where $x = 2$ has gradient $\approx \frac{0-3}{4-1} \approx -1$.

b $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{-(x+h)^2 + 3(x+h) - (-x^2 + 3x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-\cancel{x^2} - 2xh - h^2 + \cancel{3x} + 3h + \cancel{x^2} - \cancel{3x}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-2xh - h^2 + 3h}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(-2x - h + 3)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (-2x - h + 3) \quad \{\text{as } h \neq 0\} \\
 &= -2x + 3
 \end{aligned}$$

c $f'(0) = -2(0) + 3 = 3$ $f'(2) = -2(2) + 3 = -1$

Both values are the same as the estimates in **a**.

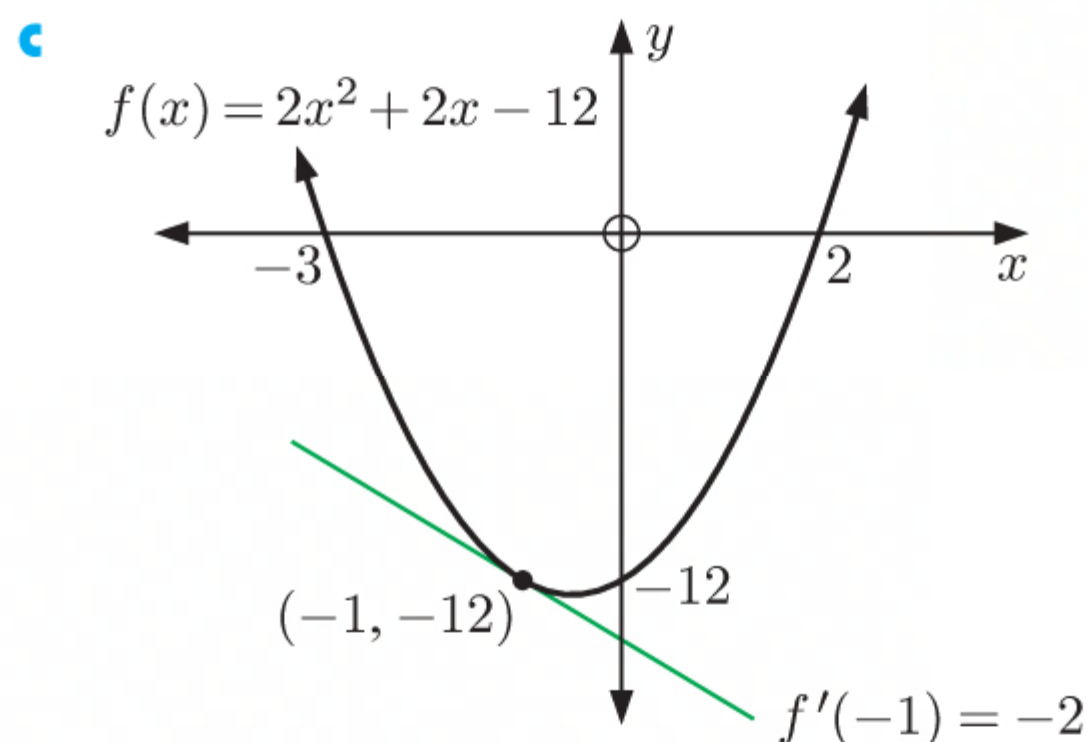
5 a $f(x) = 2x^2 + 2x - 12$

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[2(x+h)^2 + 2(x+h) - 12] - [2x^2 + 2x - 12]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{2x^2} + 4xh + 2h^2 + \cancel{2x} + 2h - \cancel{12} - \cancel{2x^2} - \cancel{2x} + \cancel{12}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{4xh + 2h^2 + 2h}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(4x + 2h + 2)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (4x + 2h + 2) \quad \{\text{as } h \neq 0\} \\
 &= 4x + 2
 \end{aligned}$$

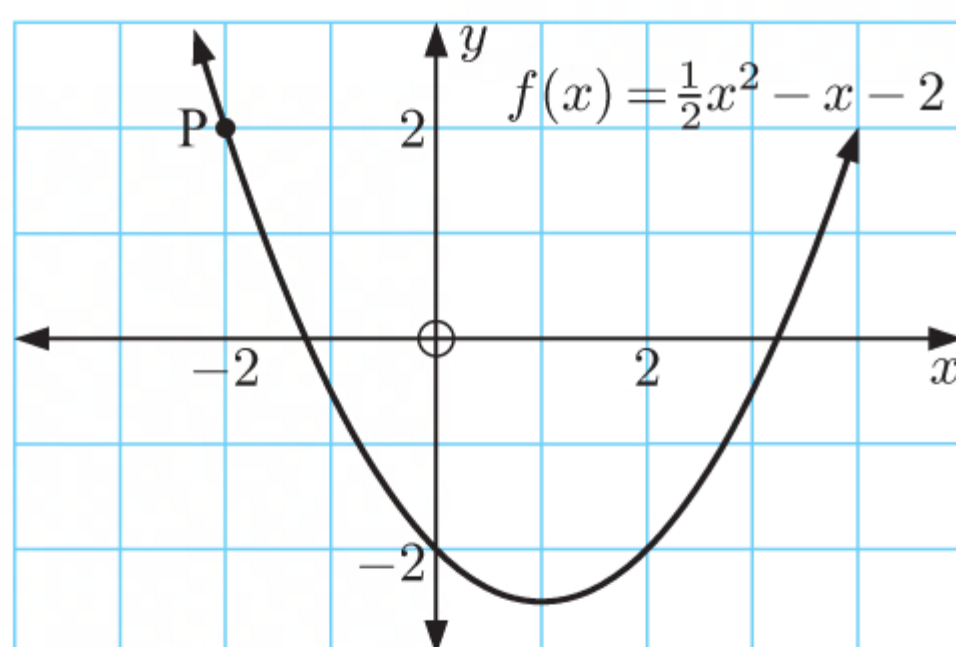
b The tangent has gradient -2 when $f'(x) = -2$
 $\therefore 4x + 2 = -2$
 $\therefore 4x = -4$
 $\therefore x = -1$

Now, $f(-1) = 2(-1)^2 + 2(-1) - 12$
 $= 2 - 2 - 12$
 $= -12$

So, the tangent has gradient -2 at $(-1, -12)$.



6



a $f(x) = \frac{1}{2}x^2 - x - 2$

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\left[\frac{1}{2}(x+h)^2 - (x+h) - 2 \right] - \left[\frac{1}{2}x^2 - x - 2 \right]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\frac{1}{2}(x^2 + 2xh + h^2) - \cancel{x} - h - \cancel{2} - \frac{1}{2}x^2 + \cancel{x} + \cancel{2}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{\frac{1}{2}x^2} + xh + \frac{1}{2}h^2 - h - \cancel{\frac{1}{2}x^2}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{xh + \frac{1}{2}h^2 - h}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(x + \frac{1}{2}h - 1)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (x + \frac{1}{2}h - 1) \quad \{\text{as } h \neq 0\} \\
 &= x - 1
 \end{aligned}$$

b i P is $(-2, 2)$, so the gradient of the tangent at P is $f'(-2) = -2 - 1 = -3$.

ii The tangent has gradient 4 when $f'(x) = 4$
 $\therefore x - 1 = 4$
 $\therefore x = 5$

$$\begin{aligned}
 \text{and } f(5) &= \frac{1}{2}(5)^2 - 5 - 2 \\
 &= \frac{25}{2} - 7 \\
 &= \frac{11}{2}
 \end{aligned}$$

The tangent has gradient 4 at the point $(5, \frac{11}{2})$.

7 $y = x^3$

a Using technology:

x	-3	-2	-1	0	1	2	3
$\frac{dy}{dx}$	27	12	3	0	3	12	27

b $(x+h)^3 = (x+h)(x+h)^2$
 $= (x+h)(x^2 + 2xh + h^2)$
 $= x^3 + 2x^2h + xh^2 + x^2h + 2xh^2 + h^3$
 $= x^3 + 3x^2h + 3xh^2 + h^3$

c $y = f(x) = x^3$

$$\begin{aligned}
 \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}}{h} \quad \{\text{using b}\} \\
 &= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(3x^2 + 3xh + h^2)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2) \quad \{\text{as } h \neq 0\} \\
 &= 3x^2
 \end{aligned}$$

Check:

When $x = -3$, $\frac{dy}{dx} = 3(-3)^2 = 27$ ✓

When $x = -2$, $\frac{dy}{dx} = 3(-2)^2 = 12$ ✓

When $x = -1$, $\frac{dy}{dx} = 3(-1)^2 = 3$ ✓

When $x = 0$, $\frac{dy}{dx} = 3(0)^2 = 0$ ✓

When $x = 1$, $\frac{dy}{dx} = 3(1)^2 = 3$ ✓

When $x = 2$, $\frac{dy}{dx} = 3(2)^2 = 12$ ✓

When $x = 3$, $\frac{dy}{dx} = 3(3)^2 = 27$ ✓

8 $y = \frac{1}{x}$

a Using technology:

x	-3	-2	-1	0	1	2	3
$\frac{dy}{dx}$	$-\frac{1}{9}$	$-\frac{1}{4}$	-1	undefined	-1	$-\frac{1}{4}$	$-\frac{1}{9}$

b $y = f(x) = \frac{1}{x}$

$$\begin{aligned}
 \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\frac{-h}{x(x+h)}}{h} \quad \{\text{using the result in Exercise 10D, question 5 a}\} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{-h}}{\cancel{h}x(x+h)} \\
 &= \lim_{h \rightarrow 0} \frac{-1}{x(x+h)} \quad \{\text{as } h \neq 0\} \\
 &= -\frac{1}{x^2}
 \end{aligned}$$

Check:

$$\text{When } x = -3, \quad \frac{dy}{dx} = -\frac{1}{(-3)^2} = -\frac{1}{9} \quad \checkmark$$

$$\text{When } x = -2, \quad \frac{dy}{dx} = -\frac{1}{(-2)^2} = -\frac{1}{4} \quad \checkmark$$

$$\text{When } x = -1, \quad \frac{dy}{dx} = -\frac{1}{(-1)^2} = -1 \quad \checkmark$$

$$\text{When } x = 0, \quad \frac{dy}{dx} = -\frac{1}{0^2} \text{ which is undefined} \quad \checkmark$$

$$\text{When } x = 1, \quad \frac{dy}{dx} = -\frac{1}{1^2} = -1 \quad \checkmark$$

$$\text{When } x = 2, \quad \frac{dy}{dx} = -\frac{1}{2^2} = -\frac{1}{4} \quad \checkmark$$

$$\text{When } x = 3, \quad \frac{dy}{dx} = -\frac{1}{3^2} = -\frac{1}{9} \quad \checkmark$$

9 a

$f(x)$	$f'(x)$
x^1	1
x^2	$2x$
x^3	$3x^2$
x^{-1}	$-\frac{1}{x^2}$
x^0	0

b If $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

INVESTIGATION 3

RULES FOR DIFFERENTIATION

1 a i

$$y = f(x) = 3x^2$$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{3(x+h)^2 - 3x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{3(x^2 + 2xh + h^2) - 3x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{3x^2} + 6xh + 3h^2 - \cancel{3x^2}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{h}(6x + 3h)}{\cancel{h}} \\ &= \lim_{h \rightarrow 0} (6x + 3h) \quad \{\text{as } h \neq 0\} \\ &= 6x \end{aligned}$$

ii

$$y = f(x) = 5x^2$$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{5(x+h)^2 - 5x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{5(x^2 + 2xh + h^2) - 5x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{5x^2} + 10xh + 5h^2 - \cancel{5x^2}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{h}(10x + 5h)}{\cancel{h}} \\ &= \lim_{h \rightarrow 0} (10x + 5h) \quad \{\text{as } h \neq 0\} \\ &= 10x \end{aligned}$$

iii $y = f(x) = -2x^2$

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-2(x+h)^2 - (-2x^2)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-2(x^2 + 2xh + h^2) + 2x^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-\cancel{2x^2} - 4xh - 2h^2 + \cancel{2x^2}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(-4x - 2h)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (-4x - 2h) \quad \{\text{as } h \neq 0\} \\
 &= -4x
 \end{aligned}$$

b If $f(x) = cx^n$ where c is a constant, then $f'(x) = cnx^{n-1}$.

c i For $y = 4x^4$, using the software we find:

x	-3	-2	-1	0	1	2	3
$\frac{dy}{dx}$	-432	-128	-16	0	16	128	432

Now, using b, $\frac{dy}{dx} = 4 \times 4x^{4-1} = 16x^3$.

Check:

When $x = -3$, $\frac{dy}{dx} = 16 \times (-3)^3 = -432$ ✓

When $x = -2$, $\frac{dy}{dx} = 16 \times (-2)^3 = -128$ ✓

When $x = -1$, $\frac{dy}{dx} = 16 \times (-1)^3 = -16$ ✓

When $x = 0$, $\frac{dy}{dx} = 16 \times 0^3 = 0$ ✓

When $x = 1$, $\frac{dy}{dx} = 16 \times 1^3 = 16$ ✓

When $x = 2$, $\frac{dy}{dx} = 16 \times 2^3 = 128$ ✓

When $x = 3$, $\frac{dy}{dx} = 16 \times 3^3 = 432$ ✓

- ii For $y = -\frac{2}{x} = -2x^{-1}$, using the software we find:

x	-3	-2	-1	0	1	2	3
$\frac{dy}{dx}$	$0.\overline{2} = \frac{2}{9}$	0.5	2	undefined	2	0.5	$0.\overline{2} = \frac{2}{9}$

Now, using **b**, $\frac{dy}{dx} = -2 \times (-1)x^{-1-1} = 2x^{-2} = \frac{2}{x^2}$.

Check:

When $x = -3$, $\frac{dy}{dx} = \frac{2}{(-3)^2} = \frac{2}{9}$ ✓

When $x = -2$, $\frac{dy}{dx} = \frac{2}{(-2)^2} = \frac{2}{4} = \frac{1}{2}$ ✓

When $x = -1$, $\frac{dy}{dx} = \frac{2}{(-1)^2} = \frac{2}{1} = 2$ ✓

When $x = 0$, $\frac{dy}{dx} = \frac{2}{0^2}$ which is undefined ✓

When $x = 1$, $\frac{dy}{dx} = \frac{2}{1^2} = 2$ ✓

When $x = 2$, $\frac{dy}{dx} = \frac{2}{2^2} = \frac{2}{4} = \frac{1}{2}$ ✓

When $x = 3$, $\frac{dy}{dx} = \frac{2}{3^2} = \frac{2}{9}$ ✓

- iii For $y = \frac{3}{x^2} = 3x^{-2}$, using the software we find:

x	-3	-2	-1	0	1	2	3
$\frac{dy}{dx}$	$0.\overline{2} = \frac{2}{9}$	0.75	6	undefined	-6	-0.75	$-0.\overline{2} = -\frac{2}{9}$

Now, using **b**, $\frac{dy}{dx} = 3 \times (-2)x^{-2-1} = -6x^{-3} = -\frac{6}{x^3}$.

Check:

When $x = -3$, $\frac{dy}{dx} = -\frac{6}{(-3)^3} = -\frac{6}{-27} = \frac{2}{9}$ ✓

When $x = -2$, $\frac{dy}{dx} = -\frac{6}{(-2)^3} = -\frac{6}{-8} = \frac{3}{4}$ ✓

When $x = -1$, $\frac{dy}{dx} = -\frac{6}{(-1)^3} = -\frac{6}{-1} = 6$ ✓

When $x = 0$, $\frac{dy}{dx} = -\frac{6}{0^3}$ which is undefined ✓

When $x = 1$, $\frac{dy}{dx} = -\frac{6}{1^3} = -\frac{6}{1} = -6$ ✓

When $x = 2$, $\frac{dy}{dx} = -\frac{6}{2^3} = -\frac{6}{8} = -\frac{3}{4}$ ✓

When $x = 3$, $\frac{dy}{dx} = -\frac{6}{3^3} = -\frac{6}{27} = -\frac{2}{9}$ ✓

2 a In **Exercise 10F**, question **1 c**, we have $f(x) = 2x - 1$ and $f'(x) = 2$.

The derivative of $2x$ is 2, and the derivative of -1 is 0. $2 + 0 = 2$ ✓

In **Exercise 10F**, question **1 d**, we have $f(x) = 3 - x$ and $f'(x) = -1$.

The derivative of 3 is 0 and the derivative of $-x$ is -1 . $0 + (-1) = -1$ ✓

In **Exercise 10F**, question **2 a**, we have $y = x^2 + 2$ and $\frac{dy}{dx} = 2x$.

The derivative of x^2 is $2x$ and the derivative of 2 is 0. $2x + 0 = 2x$ ✓

In **Exercise 10F**, question **2 b**, we have $y = 3 - x^2$ and $\frac{dy}{dx} = -2x$.

The derivative of 3 is 0 and the derivative of $-x^2$ is $-2x$. $0 + (-2x) = -2x$ ✓

In **Exercise 10F**, question **2 c**, we have $y = 2x^2 + x$ and $\frac{dy}{dx} = 4x + 1$.

The derivative of $2x^2$ is $4x$ and the derivative of x is 1. $4x + 1 = 4x + 1$ ✓

In **Exercise 10F**, question **2 d**, we have $y = -x^2 + 5x - 3$ and $\frac{dy}{dx} = -2x + 5$.

The derivative of $-x^2$ is $-2x$, the derivative of $5x$ is 5, and the derivative of -3 is 0.
 $-2x + 5 + 0 = -2x + 5$ ✓

In **Exercise 10F**, question **3 a**, we have $f(x) = 3x^2 - 1$ and $f'(x) = 6x$.

The derivative of $3x^2$ is $6x$ and the derivative of -1 is 0. $6x + 0 = 6x$ ✓

In each case, the derivative of the function is equal to the sum of the derivatives of each term.

If $f(x) = u(x) + v(x)$, then $f'(x) = u'(x) + v'(x)$.

b For $u(x) = x^3$, using the software we find:

x	-3	-2	-1	0	1	2	3
$u'(x)$	27	12	3	0	3	12	27

For $v(x) = -2x^2$, using the software we find:

x	-3	-2	-1	0	1	2	3
$v'(x)$	12	8	4	0	-4	-8	-12

For $f(x) = u(x) + v(x) = x^3 - 2x^2$, using the software we find:

x	-3	-2	-1	0	1	2	3
$f'(x)$	39	20	7	0	-1	4	15

Now, $u'(-3) + v'(-3) = 27 + 12 = 39 = f'(-3)$ ✓

$u'(-2) + v'(-2) = 12 + 8 = 20 = f'(-2)$ ✓

$u'(-1) + v'(-1) = 3 + 4 = 7 = f'(-1)$ ✓

$u'(0) + v'(0) = 0 + 0 = 0 = f'(0)$ ✓

$u'(1) + v'(1) = 3 + (-4) = -1 = f'(1)$ ✓

$u'(2) + v'(2) = 12 + (-8) = 4 = f'(2)$ ✓

$u'(3) + v'(3) = 27 + (-12) = 15 = f'(3)$ ✓

EXERCISE 10G

$$\begin{aligned} 1 \quad a \quad f(x) &= x^3 \\ \therefore f'(x) &= 3x^2 \end{aligned}$$

$$\begin{aligned} d \quad f(x) &= 6x \\ \therefore f'(x) &= 6(1) \\ &= 6 \end{aligned}$$

$$\begin{aligned} g \quad f(x) &= 3x^5 \\ \therefore f'(x) &= 3(5x^4) \\ &= 15x^4 \end{aligned}$$

$$\begin{aligned} j \quad f(x) &= x^2 + 3x - 5 \\ \therefore f'(x) &= 2x + 3(1) \\ &= 2x + 3 \end{aligned}$$

$$\begin{aligned} m \quad f(x) &= 2x^2 + x - 1 \\ \therefore f'(x) &= 2(2x) + 1 \\ &= 4x + 1 \end{aligned}$$

$$\begin{aligned} p \quad f(x) &= \frac{1}{2}x^4 - 6x^2 \\ \therefore f'(x) &= \frac{1}{2}(4x^3) - 6(2x) \\ &= 2x^3 - 12x \end{aligned}$$

$$\begin{aligned} r \quad f(x) &= 7 - x - 4x^3 \\ \therefore f'(x) &= -1 - 4(3x^2) \\ &= -1 - 12x^2 \end{aligned}$$

$$\begin{aligned} t \quad f(x) &= (2x - 1)^2 \\ &= 4x^2 - 4x + 1 \\ \therefore f'(x) &= 4(2x) - 4(1) \\ &= 8x - 4 \end{aligned}$$

$$\begin{aligned} 2 \quad a \quad \text{Let } f(x) &= \frac{1}{x^2} \\ &= x^{-2} \\ \therefore f'(x) &= -2x^{-3} \\ &= -\frac{2}{x^3} \end{aligned}$$

$$\begin{aligned} d \quad \text{Let } f(x) &= \frac{3}{x} \\ &= 3x^{-1} \\ \therefore f'(x) &= 3(-1x^{-2}) \\ &= -3x^{-2} \\ &= -\frac{3}{x^2} \end{aligned}$$

$$\begin{aligned} b \quad f(x) &= x^8 \\ \therefore f'(x) &= 8x^7 \end{aligned}$$

$$\begin{aligned} e \quad f(x) &= 2x^3 \\ \therefore f'(x) &= 2(3x^2) \\ &= 6x^2 \end{aligned}$$

$$\begin{aligned} h \quad f(x) &= 5x^6 \\ \therefore f'(x) &= 5(6x^5) \\ &= 30x^5 \end{aligned}$$

$$\begin{aligned} k \quad f(x) &= 5x - 2 \\ \therefore f'(x) &= 5(1) \\ &= 5 \end{aligned}$$

$$\begin{aligned} n \quad f(x) &= 3x^2 - 7x + 8 \\ \therefore f'(x) &= 3(2x) - 7(1) \\ &= 6x - 7 \end{aligned}$$

$$\begin{aligned} q \quad f(x) &= x^3 - 4x^2 + 6x \\ \therefore f'(x) &= 3x^2 - 4(2x) + 6(1) \\ &= 3x^2 - 8x + 6 \end{aligned}$$

$$\begin{aligned} s \quad f(x) &= \frac{1}{5}x^3 - \frac{7}{2}x^2 - 2 \\ \therefore f'(x) &= \frac{1}{5}(3x^2) - \frac{7}{2}(2x) \\ &= \frac{3}{5}x^2 - 7x \end{aligned}$$

$$\begin{aligned} c \quad f(x) &= x^{11} \\ \therefore f'(x) &= 11x^{10} \end{aligned}$$

$$\begin{aligned} f \quad f(x) &= 7x^2 \\ \therefore f'(x) &= 7(2x) \\ &= 14x \end{aligned}$$

$$\begin{aligned} i \quad f(x) &= x^2 + x \\ \therefore f'(x) &= 2x + 1 \end{aligned}$$

$$\begin{aligned} l \quad f(x) &= x^2 + 3 \\ \therefore f'(x) &= 2x \end{aligned}$$

$$\begin{aligned} o \quad f(x) &= 4 - 2x^2 \\ \therefore f'(x) &= -2(2x) \\ &= -4x \end{aligned}$$

$$\begin{aligned} c \quad \text{Let } f(x) &= \frac{1}{x^8} \\ &= x^{-8} \\ \therefore f'(x) &= -8x^{-9} \\ &= -\frac{8}{x^9} \end{aligned}$$

$$\begin{aligned} b \quad \text{Let } f(x) &= \frac{1}{x^5} \\ &= x^{-5} \\ \therefore f'(x) &= -5x^{-6} \\ &= -\frac{5}{x^6} \end{aligned}$$

$$\begin{aligned} e \quad \text{Let } f(x) &= \frac{4}{x^3} \\ &= 4x^{-3} \\ \therefore f'(x) &= 4(-3x^{-4}) \\ &= -12x^{-4} \\ &= -\frac{12}{x^4} \end{aligned}$$

$$\begin{aligned} f \quad \text{Let } f(x) &= -\frac{7}{x^4} \\ &= -7x^{-4} \\ \therefore f'(x) &= -7(-4x^{-5}) \\ &= 28x^{-5} \\ &= \frac{28}{x^5} \end{aligned}$$

$$\begin{aligned}
 \text{g Let } f(x) &= 2x + \frac{3}{x^2} \\
 &= 2x + 3x^{-2} \\
 \therefore f'(x) &= 2(1) + 3(-2x^{-3}) \\
 &= 2 - 6x^{-3} \\
 &= 2 - \frac{6}{x^3}
 \end{aligned}$$

$$\begin{aligned}
 \text{i Let } f(x) &= 9 - \frac{2}{x^3} \\
 &= 9 - 2x^{-3} \\
 \therefore f'(x) &= -2(-3x^{-4}) \\
 &= 6x^{-4} \\
 &= \frac{6}{x^4}
 \end{aligned}$$

$$\begin{aligned}
 \text{k Let } f(x) &= \frac{2}{x^2} + \frac{9}{x^4} \\
 &= 2x^{-2} + 9x^{-4} \\
 \therefore f'(x) &= 2(-2x^{-3}) + 9(-4x^{-5}) \\
 &= -4x^{-3} - 36x^{-5} \\
 &= -\frac{4}{x^3} - \frac{36}{x^5}
 \end{aligned}$$

$$\begin{aligned}
 \text{m Let } f(x) &= 5 - \frac{8}{x^2} + \frac{4}{x^3} \\
 &= 5 - 8x^{-2} + 4x^{-3} \\
 \therefore f'(x) &= -8(-2x^{-3}) + 4(-3x^{-4}) \\
 &= 16x^{-3} - 12x^{-4} \\
 &= \frac{16}{x^3} - \frac{12}{x^4}
 \end{aligned}$$

$$\begin{aligned}
 \text{o Let } f(x) &= 4x - \frac{1}{4x} \\
 &= 4x - \frac{1}{4}x^{-1} \\
 \therefore f'(x) &= 4(1) - \frac{1}{4}(-1x^{-2}) \\
 &= 4 + \frac{1}{4}x^{-2} \\
 &= 4 + \frac{1}{4x^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{h Let } f(x) &= x^2 - \frac{6}{x} \\
 &= x^2 - 6x^{-1} \\
 \therefore f'(x) &= 2x - 6(-1x^{-2}) \\
 &= 2x + 6x^{-2} \\
 &= 2x + \frac{6}{x^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{j Let } f(x) &= \frac{1}{x} - \frac{5}{x^3} \\
 &= x^{-1} - 5x^{-3} \\
 \therefore f'(x) &= -1x^{-2} - 5(-3x^{-4}) \\
 &= -x^{-2} + 15x^{-4} \\
 &= -\frac{1}{x^2} + \frac{15}{x^4}
 \end{aligned}$$

$$\begin{aligned}
 \text{l Let } f(x) &= 3x - \frac{1}{x} + \frac{2}{x^2} \\
 &= 3x - x^{-1} + 2x^{-2} \\
 \therefore f'(x) &= 3(1) - (-1x^{-2}) + 2(-2x^{-3}) \\
 &= 3 + x^{-2} - 4x^{-3} \\
 &= 3 + \frac{1}{x^2} - \frac{4}{x^3}
 \end{aligned}$$

$$\begin{aligned}
 \text{n Let } f(x) &= \frac{1}{5x^2} \\
 &= \frac{1}{5}x^{-2} \\
 \therefore f'(x) &= \frac{1}{5}(-2x^{-3}) \\
 &= -\frac{2}{5}x^{-3} \\
 &= -\frac{2}{5x^3}
 \end{aligned}$$

$$\begin{aligned}
 \text{p Let } f(x) &= \frac{x^2 - 3}{x} \\
 &= \frac{x^2}{x} - \frac{3}{x} \\
 &= x - 3x^{-1} \\
 \therefore f'(x) &= 1 - 3(-1x^{-2}) \\
 &= 1 + 3x^{-2} \\
 &= 1 + \frac{3}{x^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{q Let } f(x) &= \frac{x^3 + 4}{x} \\
 &= \frac{x^3}{x} + \frac{4}{x} \\
 &= x^2 + 4x^{-1} \\
 \therefore f'(x) &= 2x + 4(-1x^{-2}) \\
 &= 2x - 4x^{-2} \\
 &= 2x - \frac{4}{x^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{r Let } f(x) &= \frac{2x - 5}{x^2} \\
 &= \frac{2x}{x^2} - \frac{5}{x^2} \\
 &= \frac{2}{x} - \frac{5}{x^2} \\
 &= 2x^{-1} - 5x^{-2} \\
 \therefore f'(x) &= 2(-1x^{-2}) - 5(-2x^{-3}) \\
 &= -2x^{-2} + 10x^{-3} \\
 &= -\frac{2}{x^2} + \frac{10}{x^3}
 \end{aligned}$$

$$3 \quad f(x) = 4x^3 - x$$

$$\begin{aligned}
 \text{a } f'(x) &= 4(3x^2) - 1 \\
 &= 12x^2 - 1
 \end{aligned}$$

$$\begin{aligned}
 \text{b } f'(2) &= 12(2)^2 - 1 \\
 &= 47
 \end{aligned}$$

$$\begin{aligned}
 \text{c } f'(0) &= 12(0)^2 - 1 \\
 &= -1
 \end{aligned}$$

$$\begin{aligned}
 4 \quad g(x) &= \frac{x^2 + 1}{x} \\
 &= \frac{x^2}{x} + \frac{1}{x} \\
 &= x + x^{-1}
 \end{aligned}$$

$$\begin{aligned}
 \text{a } g'(x) &= 1 + (-1x^{-2}) \\
 &= 1 - x^{-2} \\
 &= 1 - \frac{1}{x^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } g'(3) &= 1 - \frac{1}{3^2} \\
 &= 1 - \frac{1}{9} \\
 &= \frac{8}{9}
 \end{aligned}$$

$$\begin{aligned}
 \text{c } g'(-2) &= 1 - \frac{1}{(-2)^2} \\
 &= 1 - \frac{1}{4} \\
 &= \frac{3}{4}
 \end{aligned}$$

$$5 \quad \text{a} \quad y = 100x$$

$$\begin{aligned}
 \therefore \frac{dy}{dx} &= 100(1) \\
 &= 100
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad y &= 6x + \frac{5}{x} \\
 &= 6x + 5x^{-1}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \frac{dy}{dx} &= 6(1) + 5(-1x^{-2}) \\
 &= 6 - 5x^{-2} \\
 &= 6 - \frac{5}{x^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad y &= 10(x + 1) \\
 &= 10x + 10 \\
 \therefore \frac{dy}{dx} &= 10(1) \\
 &= 10
 \end{aligned}$$

$$\text{b} \quad y = \pi x^2$$

$$\begin{aligned}
 \therefore \frac{dy}{dx} &= \pi(2x) \\
 &= 2\pi x
 \end{aligned}$$

$$\text{d} \quad y = 2.5x^3 - 1.4x^2 - 1.3$$

$$\begin{aligned}
 \therefore \frac{dy}{dx} &= 2.5(3x^2) - 1.4(2x) \\
 &= 7.5x^2 - 2.8x
 \end{aligned}$$

$$\text{f} \quad y = 4\pi x^3$$

$$\begin{aligned}
 \therefore \frac{dy}{dx} &= 4\pi(3x^2) \\
 &= 12\pi x^2
 \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad y &= (x+1)(x-2) \\ &= x^2 - x - 2 \\ \therefore \frac{dy}{dx} &= 2x - 1 \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad y &= (3+x)(2-x) \\ &= 6 - x - x^2 \\ \therefore \frac{dy}{dx} &= -1 - 2x \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad y &= (5-x)^2 \\ &= 25 - 10x + x^2 \\ \therefore \frac{dy}{dx} &= -10(1) + 2x \\ &= 2x - 10 \end{aligned}$$

$$\begin{aligned} \mathbf{6} \quad \mathbf{a} \quad y &= \frac{1}{2}t^4 - \frac{1}{3}t \\ \therefore \frac{dy}{dt} &= \frac{1}{2}(4t^3) - \frac{1}{3}(1) \\ &= 2t^3 - \frac{1}{3} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad y &= 7 - \frac{1}{2t} \\ &= 7 - \frac{1}{2}t^{-1} \\ \therefore \frac{dy}{dt} &= -\frac{1}{2}(-1t^{-2}) \\ &= \frac{1}{2}t^{-2} \\ &= \frac{1}{2t^2} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad V &= \frac{4}{3}\pi r^3 \\ \therefore \frac{dV}{dr} &= \frac{4}{3}\pi(3r^2) \\ &= 4\pi r^2 \end{aligned}$$

$$\begin{aligned} \mathbf{7} \quad \mathbf{a} \quad y &= x^2 \\ \therefore \frac{dy}{dx} &= 2x \\ \text{When } x &= 2, \frac{dy}{dx} = 2(2) = 4 \\ \text{So, the tangent has gradient } &4. \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad y &= x^3 - 5x + 2 \\ \therefore \frac{dy}{dx} &= 3x^2 - 5(1) \\ &= 3x^2 - 5 \\ \text{At the point } (3, 14), \\ \frac{dy}{dx} &= 3(3)^2 - 5 = 22 \\ \text{So, the tangent has gradient } &22. \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad y &= \frac{8}{x^2} \\ &= 8x^{-2} \\ \therefore \frac{dy}{dx} &= 8(-2x^{-3}) \\ &= -16x^{-3} \\ &= -\frac{16}{x^3} \end{aligned}$$

At the point $(9, \frac{8}{81})$,

$$\frac{dy}{dx} = -\frac{16}{9^3} = -\frac{16}{729}.$$

So, the tangent has gradient $-\frac{16}{729}$.

$$\begin{aligned} \mathbf{d} \quad y &= 2x^2 - 3x + 7 \\ \therefore \frac{dy}{dx} &= 2(2x) - 3(1) \\ &= 4x - 3 \\ \text{When } x &= -1, \\ \frac{dy}{dx} &= 4(-1) - 3 \\ &= -7 \end{aligned}$$

So, the tangent has gradient -7 .

$$\begin{aligned}
 \text{e} \quad y &= 2x - \frac{5}{x} \\
 &= 2x - 5x^{-1} \\
 \therefore \frac{dy}{dx} &= 2(1) - 5(-1x^{-2}) \\
 &= 2 + \frac{5}{x^2}
 \end{aligned}$$

At the point $(2, \frac{3}{2})$,

$$\begin{aligned}
 \frac{dy}{dx} &= 2 + \frac{5}{2^2} \\
 &= 2 + \frac{5}{4} \\
 &= \frac{13}{4}
 \end{aligned}$$

So, the tangent has gradient $\frac{13}{4}$.

$$8 \quad f(x) = x^2 + (b+1)x + 2c, \quad f(2) = 4, \quad \text{and} \quad f'(-1) = 2$$

$$\therefore f'(x) = 2x + (b+1)$$

$$\text{But } f'(-1) = 2, \text{ so } 2(-1) + b + 1 = 2$$

$$\therefore -1 + b = 2$$

$$\therefore b = 3$$

$$\begin{aligned}
 \text{So, } f(x) &= x^2 + (3+1)x + 2c \\
 &= x^2 + 4x + 2c
 \end{aligned}$$

$$\text{But } f(2) = 4, \text{ so } 2^2 + 4(2) + 2c = 4$$

$$\therefore 2c = -8$$

$$\therefore c = -4$$

$$\begin{aligned}
 9 \quad y &= 4x - \frac{3}{x} \\
 &= 4x - 3x^{-1} \\
 \therefore \frac{dy}{dx} &= 4 + 3x^{-2} \\
 &= 4 + \frac{3}{x^2}
 \end{aligned}$$

$\frac{dy}{dx}$ is the gradient function of $y = 4x - \frac{3}{x}$ from which the gradient of the tangent at any point can be found. It is also the instantaneous rate of change of y with respect to x .

$$\begin{aligned}
 10 \quad \text{a} \quad S &= 2t^2 + 4t \text{ m} \\
 \therefore \frac{dS}{dt} &= 4t + 4 \text{ m s}^{-1}
 \end{aligned}$$

$\frac{dS}{dt}$ is the instantaneous rate of change in position at time t . It gives the speed of the car at time t , in metres per second.

$$\begin{aligned}
 \text{b} \quad \text{When } t = 3, \quad \frac{dS}{dt} &= 4(3) + 4 \\
 &= 16 \text{ m s}^{-1}
 \end{aligned}$$

The instantaneous rate of change in position at time $t = 3$ seconds is 16 m s^{-1} , or the speed of the car after 3 seconds is 16 m s^{-1} .

$$\begin{aligned}
 \text{f} \quad y &= \frac{x^3 - 4x - 8}{x^2} \\
 &= \frac{x^3}{x^2} - \frac{4x}{x^2} - \frac{8}{x^2} \\
 &= x - 4x^{-1} - 8x^{-2} \\
 \therefore \frac{dy}{dx} &= 1 - 4(-1x^{-2}) - 8(-2x^{-3}) \\
 &= 1 + \frac{4}{x^2} + \frac{16}{x^3}
 \end{aligned}$$

When $x = -1$,

$$\begin{aligned}
 \frac{dy}{dx} &= 1 + \frac{4}{(-1)^2} + \frac{16}{(-1)^3} \\
 &= -11
 \end{aligned}$$

So, the tangent has gradient -11 .

11 $C = 1785 + 3x + 0.002x^2$ pounds

$$\therefore \frac{dC}{dx} = 3 + 0.002(2x)$$

$$= 3 + 0.004x \text{ pounds per toaster}$$

When $x = 1000$, $\frac{dC}{dx} = 3 + 0.004(1000)$

$$= 7$$

When 1000 toasters are being produced each week, the cost of production is increasing by £7 per toaster.

12 $y = x^2 - 4x + 7$

$$\therefore \frac{dy}{dx} = 2x - 4$$

\therefore the tangent has gradient 2 when $2x - 4 = 2$

$$\therefore 2x = 6$$

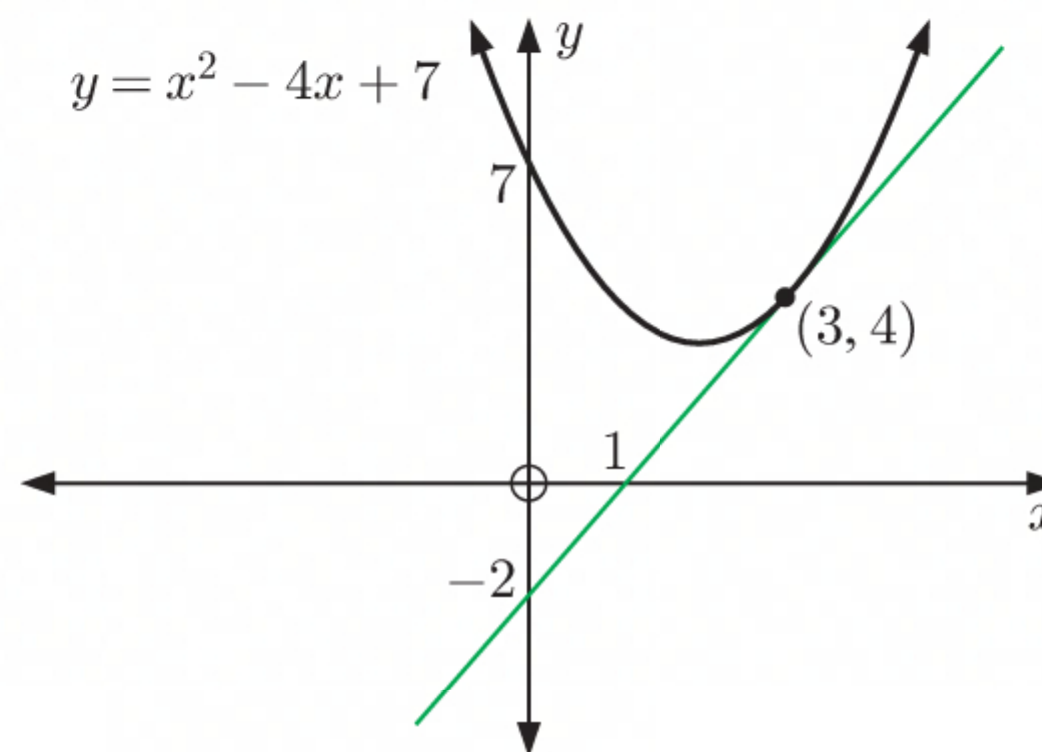
$$\therefore x = 3$$

When $x = 3$, $y = 3^2 - 4(3) + 7$

$$= 9 - 12 + 7$$

$$= 4$$

So, the tangent has gradient 2 at the point $(3, 4)$.



13 a $y = x^2 + 5x + 1$

$$\therefore \frac{dy}{dx} = 2x + 5$$

\therefore the tangent has gradient 3 when

$$2x + 5 = 3$$

$$\therefore 2x = -2$$

$$\therefore x = -1$$

When $x = -1$, $y = (-1)^2 + 5(-1) + 1$

$$= 1 - 5 + 1$$

$$= -3$$

So, the tangent has gradient 3 at the point $(-1, -3)$.

b $y = 3x^2 + 11x + 5$

$$\therefore \frac{dy}{dx} = 6x + 11$$

\therefore the tangent has gradient -7 when

$$6x + 11 = -7$$

$$\therefore 6x = -18$$

$$\therefore x = -3$$

When $x = -3$, $y = 3(-3)^2 + 11(-3) + 5$

$$= 27 - 33 + 5$$

$$= -1$$

So, the tangent has gradient -7 at the point $(-3, -1)$.

$$\text{c} \quad f(x) = 2x^{-2} + x$$

$$= \frac{2}{x^2} + x$$

$$\therefore f'(x) = -4x^{-3} + 1$$

$$= -\frac{4}{x^3} + 1$$

\therefore the tangent has gradient $\frac{1}{2}$ when

$$-\frac{4}{x^3} + 1 = \frac{1}{2}$$

$$\therefore -\frac{4}{x^3} = -\frac{1}{2}$$

$$\therefore x^3 = 8$$

$$\therefore x = \sqrt[3]{8} = 2$$

$$f(2) = \frac{2}{2^2} + 2$$

$$= \frac{2}{4} + 2$$

$$= \frac{1}{2} + 2$$

$$= 2\frac{1}{2} = \frac{5}{2}$$

So, the tangent has gradient $\frac{1}{2}$ at the point $(2, \frac{5}{2})$.

$$\text{e} \quad f(x) = ax^2 + bx + c$$

$$\therefore f'(x) = 2ax + b$$

The tangent is horizontal when the gradient is 0.

\therefore the tangent is horizontal when

$$2ax + b = 0$$

$$\therefore 2ax = -b$$

$$\therefore x = -\frac{b}{2a}$$

$$f\left(-\frac{b}{2a}\right) = a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c$$

$$= \frac{ab^2}{4a^2} - \frac{b^2}{2a} + c$$

$$= \frac{b^2}{4a} - \frac{b^2}{2a} + c$$

So, the tangent is horizontal at the point

$$\left(-\frac{b}{2a}, \frac{b^2}{4a} - \frac{b^2}{2a} + c\right).$$

$$\text{d} \quad f(x) = 3x^3 - 5x + 2$$

$$\therefore f'(x) = 9x^2 - 5$$

\therefore the tangent has gradient 4 when

$$9x^2 - 5 = 4$$

$$\therefore 9x^2 = 9$$

$$\therefore x^2 = 1$$

$$\therefore x = \pm 1$$

$$f(1) = 3(1)^3 - 5(1) + 2$$

$$= 3 - 5 + 2$$

$$= 0$$

$$f(-1) = 3(-1)^3 - 5(-1) + 2$$

$$= -3 + 5 + 2$$

$$= 4$$

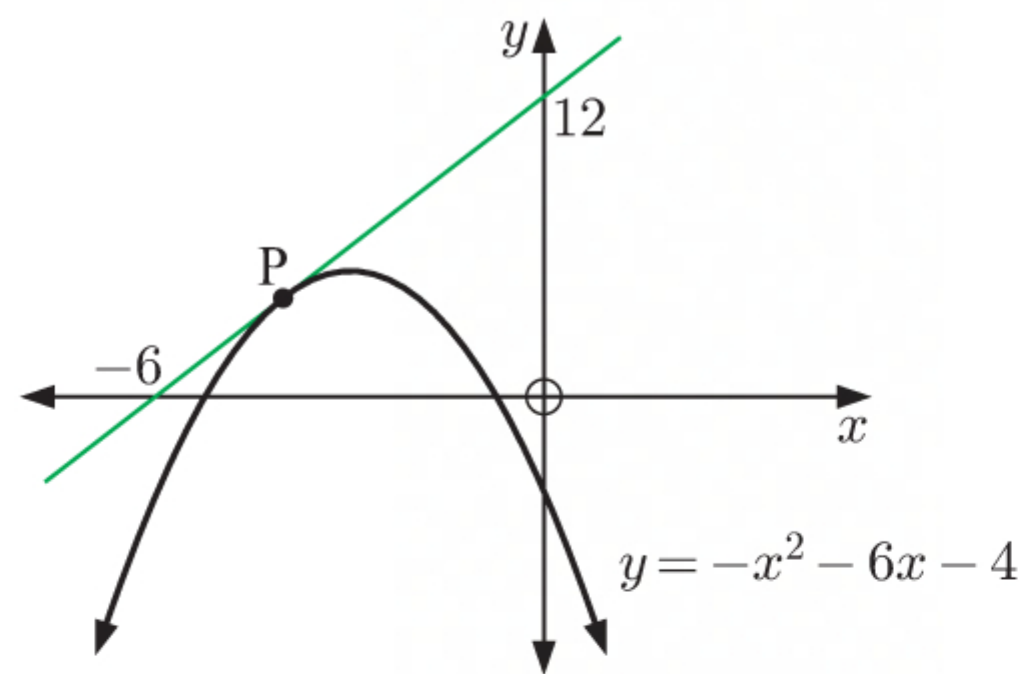
So, the tangent has gradient 4 at the points $(1, 0)$ and $(-1, 4)$.

14 $y = -x^2 - 6x - 4$

$$\therefore \frac{dy}{dx} = -2x - 6$$

The tangent at P passes through $(-6, 0)$ and $(0, 12)$.

$$\begin{aligned} \text{The gradient of the tangent} &= \frac{y\text{-step}}{x\text{-step}} \\ &= \frac{12 - 0}{0 - (-6)} \\ &= \frac{12}{6} \\ &= 2 \end{aligned}$$



$$\begin{aligned} \text{The tangent has gradient 2 when } -2x - 6 &= 2 \\ \therefore -2x &= 8 \\ \therefore x &= -4 \end{aligned}$$

$$\begin{aligned} \text{When } x = -4, \quad y &= -(-4)^2 - 6(-4) - 4 \\ &= -16 + 24 - 4 \\ &= 4 \end{aligned}$$

So, P has coordinates $(-4, 4)$.

15 $f(x) = x^3 + ax + 5$

$$\therefore f'(x) = 3x^2 + a$$

$$\begin{aligned} \text{Now } f'(1) &= 10, \quad \text{so } 3(1)^2 + a = 10 \\ \therefore 3 + a &= 10 \\ \therefore a &= 7 \end{aligned}$$

16 $f(x) = -3x^2 + ax + b$

$$\therefore f'(x) = -6x + a$$

$$\begin{aligned} \text{Now } f'(-3) &= 9, \quad \text{so } -6(-3) + a = 9 \\ \therefore 18 + a &= 9 \\ \therefore a &= -9 \end{aligned}$$

$$\begin{aligned} \text{Also, } f(-3) &= 8, \quad \text{so } -3(-3)^2 + (-9)(-3) + b = 8 \\ \therefore -27 + 27 + b &= 8 \\ \therefore b &= 8 \end{aligned}$$

$$\begin{aligned}
 17 \quad f(x) &= 2x^2 + a + \frac{b}{x} \\
 &= 2x^2 + a + bx^{-1} \\
 \therefore f'(x) &= 4x - bx^{-2} \\
 &= 4x - \frac{b}{x^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } f'(1) &= -2, \quad \text{so } 4(1) - \frac{b}{1^2} = -2 \\
 \therefore 4 - b &= -2 \\
 \therefore b &= 6
 \end{aligned}$$

$$\begin{aligned}
 \text{Also, } f(1) &= 11, \quad \text{so } 2(1)^2 + a + \frac{6}{1} = 11 \\
 \therefore 2 + a + 6 &= 11 \\
 \therefore 8 + a &= 11 \\
 \therefore a &= 3
 \end{aligned}$$

- 18 a The tangent line L passes through $(-2, -5)$ and $(3, 0)$.

The gradient of the tangent

$$\begin{aligned}
 &= \frac{y\text{-step}}{x\text{-step}} \\
 &= \frac{0 - (-5)}{3 - (-2)} \\
 &= \frac{5}{5} \\
 &= 1
 \end{aligned}$$

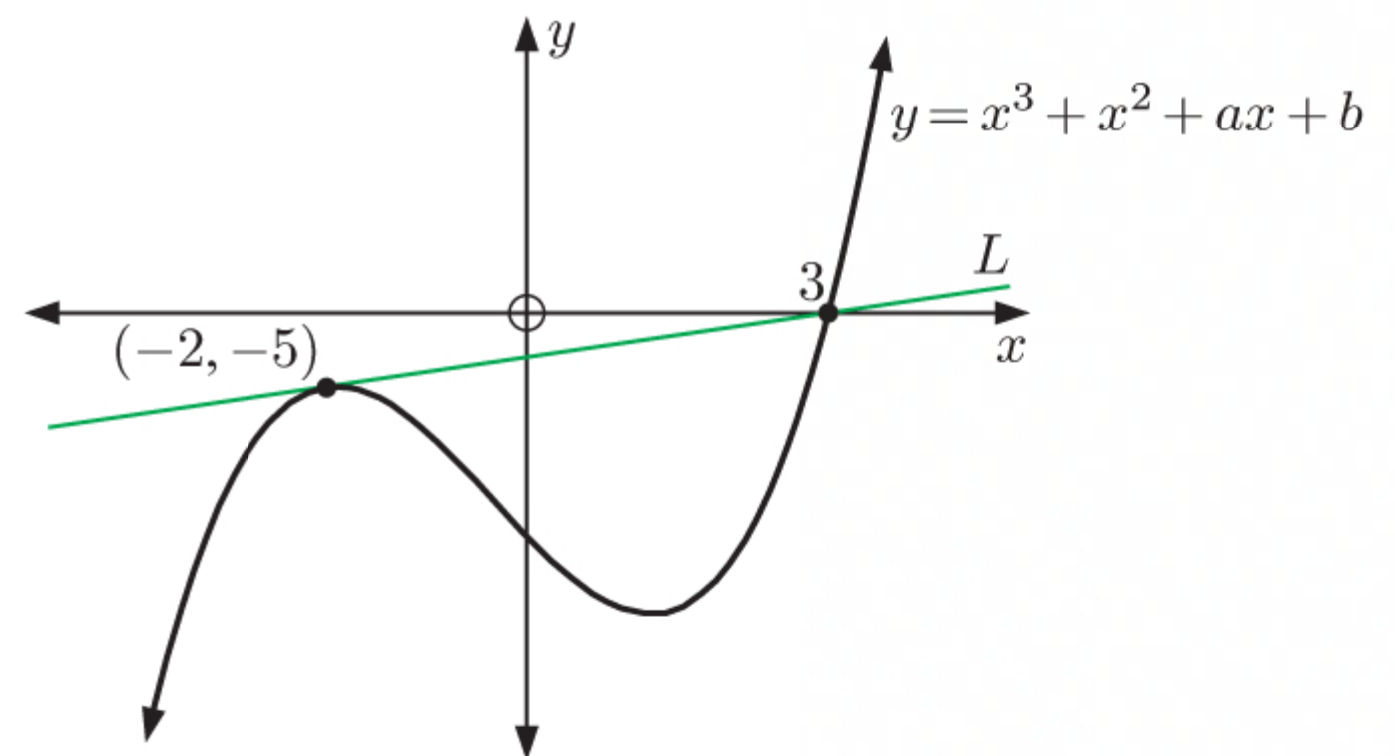
So, the gradient of the tangent line L is 1.

b $y = x^3 + x^2 + ax + b$

$$\therefore \frac{dy}{dx} = 3x^2 + 2x + a$$

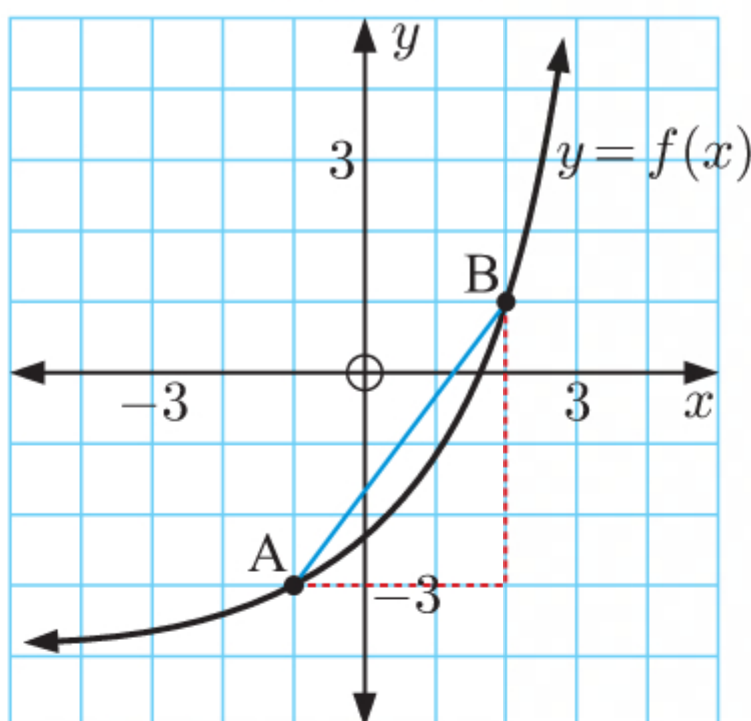
$$\begin{aligned}
 \text{Now, when } x = -2, \quad \frac{dy}{dx} &= 1, \quad \text{so } 3(-2)^2 + 2(-2) + a = 1 \\
 \therefore 12 - 4 + a &= 1 \\
 \therefore 8 + a &= 1 \\
 \therefore a &= -7
 \end{aligned}$$

$$\begin{aligned}
 \text{Also, when } x = -2, \quad y &= -5, \quad \text{so } (-2)^3 + (-2)^2 + (-7)(-2) + b = -5 \\
 \therefore -8 + 4 + 14 + b &= -5 \\
 \therefore 10 + b &= -5 \\
 \therefore b &= -15
 \end{aligned}$$



REVIEW SET 10A

1

average rate of change in $f(x)$ from A to B

$$\begin{aligned}
 &= \frac{f(b) - f(a)}{b - a} \\
 &= \frac{1 - (-3)}{2 - (-1)} \\
 &= \frac{4}{3}
 \end{aligned}$$

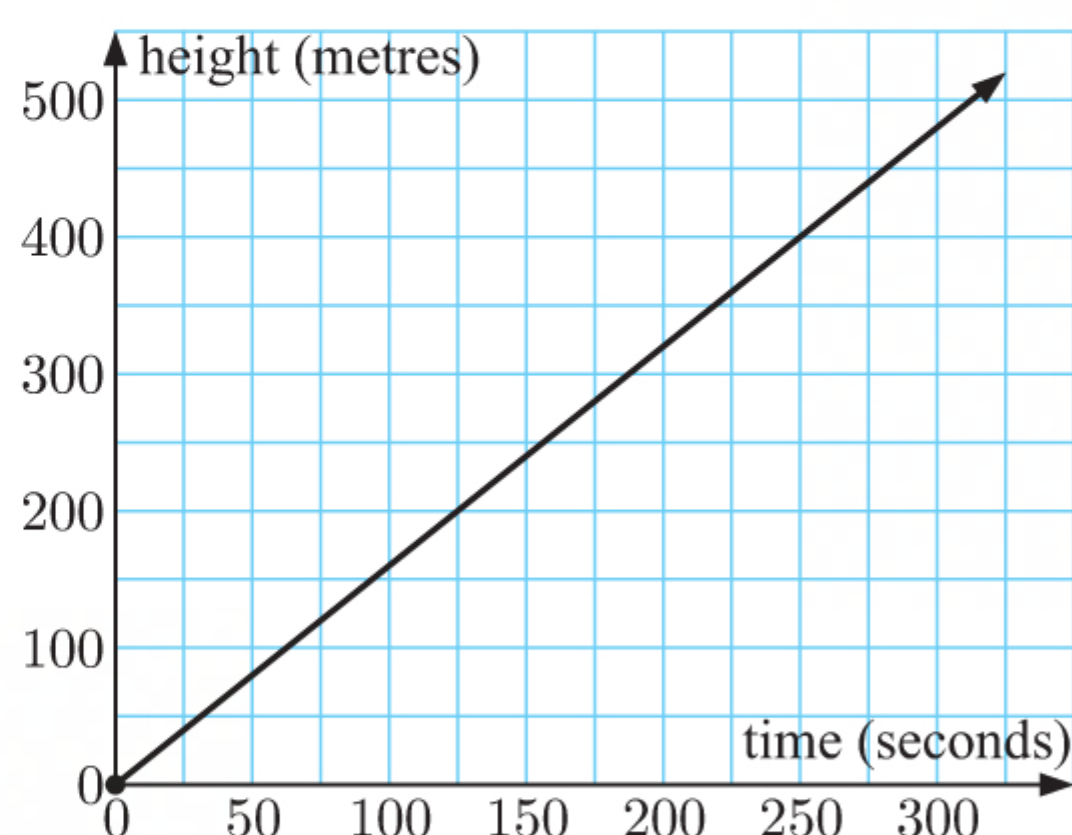
2

a The graph of height against time is a straight line.

\therefore the height increases by the same amount each time interval.

\therefore the ski-lift is increasing in height at a constant rate.

b rate of change $= \frac{(400 - 0) \text{ m}}{(250 - 0) \text{ s}}$
 $= 1.6 \text{ m s}^{-1}$



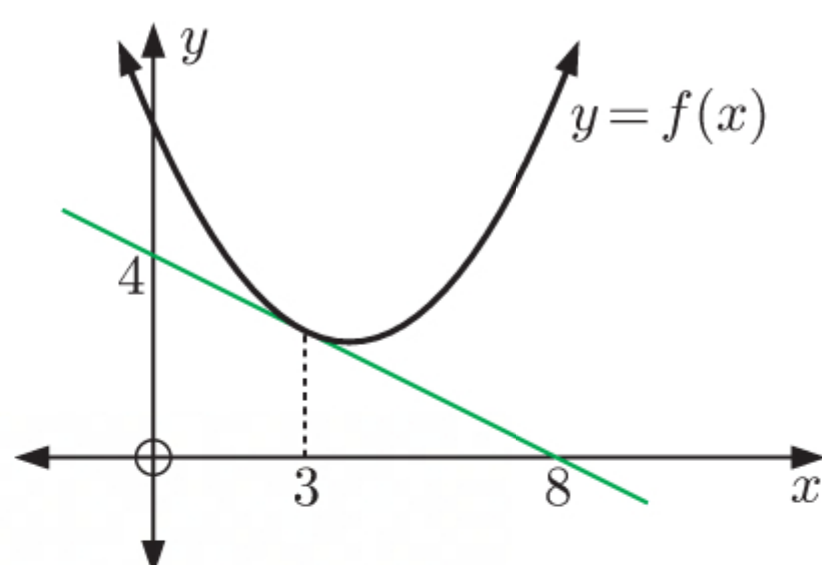
3

a We can make $6x - 7$ as close as we like to -1 by making x sufficiently close to 1.

$$\therefore \lim_{x \rightarrow 1} (6x - 7) = -1$$

b $\lim_{h \rightarrow 0} \frac{2h^2 - h}{h} = \lim_{h \rightarrow 0} \frac{h(2h - 1)}{h}$
 $= \lim_{h \rightarrow 0} (2h - 1) \quad \{\text{as } h \neq 0\}$
 $= -1$

4



The tangent to $y = f(x)$ at the point where $x = 3$ has

$$\text{gradient } \frac{4 - 0}{0 - 8} = -\frac{1}{2}.$$

$$\therefore f'(3) = -\frac{1}{2}$$

5 a $f(x) = x^2 + 2x$

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[(x+h)^2 + 2(x+h)] - [x^2 + 2x]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{2x} + 2h - \cancel{x^2} - \cancel{2x}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2xh + h^2 + 2h}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(2x + h + 2)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (2x + h + 2) \quad \{\text{as } h \neq 0\} \\
 &= 2x + 2
 \end{aligned}$$

b $y = f(x) = 4 - 3x^2$

$$\begin{aligned}
 \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[4 - 3(x+h)^2] - [4 - 3x^2]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{4 - 3(x^2 + 2xh + h^2) - 4 + 3x^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{4} - \cancel{3x^2} - 6xh - 3h^2 - \cancel{4} + \cancel{3x^2}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(-6x - 3h)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (-6x - 3h) \quad \{\text{as } h \neq 0\} \\
 &= -6x
 \end{aligned}$$

6 a $y = f(x) = 2x^2 - 1$

$$\begin{aligned}
 \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[2(x+h)^2 - 1] - [2x^2 - 1]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2(x^2 + 2xh + h^2) - 1 - 2x^2 + 1}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{2x^2} + 4xh + 2h^2 - \cancel{1} - \cancel{2x^2} + \cancel{1}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{4xh + 2h^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(4x + 2h)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (4x + 2h) \quad \{\text{as } h \neq 0\} \\
 &= 4x
 \end{aligned}$$

b The gradient of the tangent to $y = 2x^2 - 1$ at the point where $x = 4$ is $4 \times 4 = 16$.

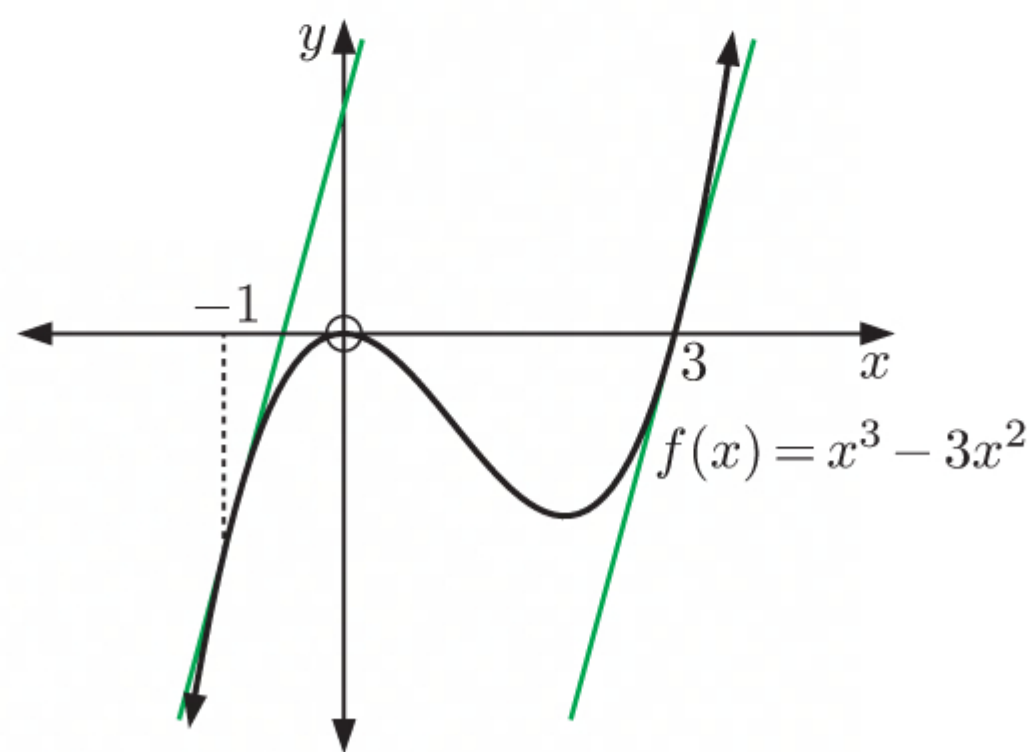
c The gradient of the tangent is equal to -12 when $4x = -12$

$$\therefore x = -3$$

7 a $f(x) = x^3 - 3x^2$

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[(x+h)^3 - 3(x+h)^2] - [x^3 - 3x^2]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{3x^2} - 6xh - 3h^2 - \cancel{x^3} + \cancel{3x^2}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3 - 6xh - 3h^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(3x^2 + 3xh + h^2 - 6x - 3h)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2 - 6x - 3h) \quad \{\text{as } h \neq 0\} \\
 &= 3x^2 - 6x
 \end{aligned}$$

b



The illustrated tangents pass through the point where $x = -1$ and the point where $x = 3$.

The gradient of the tangent at $x = -1$ is

$$\begin{aligned}
 f'(-1) &= 3(-1)^2 - 6(-1) \\
 &= 9
 \end{aligned}$$

and the gradient of the tangent at $x = 3$ is

$$\begin{aligned}
 f'(3) &= 3(3)^2 - 6(3) \\
 &= 9
 \end{aligned}$$

Since $f'(-1) = f'(3)$, the gradients of the tangents are equal.

This means the tangents are parallel.

8 a $f(x) = 5x^3$
 $\therefore f'(x) = 15x^2$

c $f(x) = 7x^2 - \frac{3}{x}$
 $= 7x^2 - 3x^{-1}$
 $\therefore f'(x) = 7(2x) - 3(-x^{-2})$
 $= 14x + 3x^{-2}$
 $= 14x + \frac{3}{x^2}$

b $f(x) = x^6 - 5x$
 $\therefore f'(x) = 6x^5 - 5$

d $f(x) = 3x - \frac{4}{x^2}$
 $= 3x - 4x^{-2}$
 $\therefore f'(x) = 3 - 4(-2x^{-3})$
 $= 3 + 8x^{-3}$
 $= 3 + \frac{8}{x^3}$

9 $f(x) = -x^2 + 4x - 2$
 $\therefore f'(x) = -2x + 4$

At the point $(-3, -23)$, $f'(-3) = -2(-3) + 4$
 $= 10$

So, the gradient of the tangent is 10.

$$\begin{aligned}
 \text{10 a} \quad f(x) &= 7 + x - 3x^2 \\
 \therefore f(3) &= 7 + 3 - 3(3)^2 \\
 &= 10 - 27 \\
 &= -17
 \end{aligned}$$

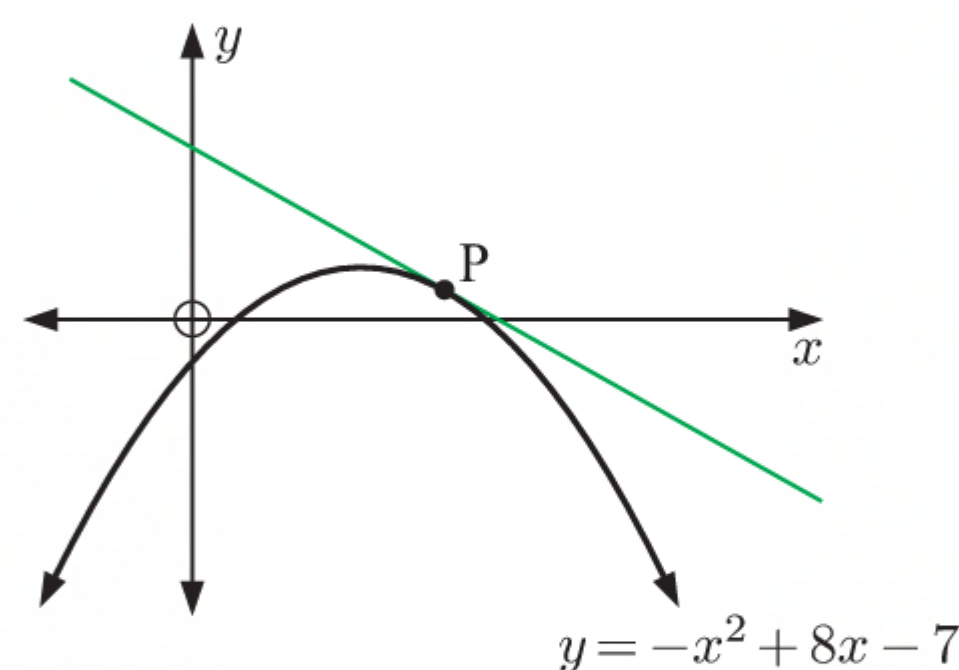
$$\begin{aligned}
 \text{b} \quad f(x) &= 7 + x - 3x^2 \\
 \therefore f'(x) &= 1 - 6x \\
 \therefore f'(3) &= 1 - 6(3) \\
 &= 1 - 18 \\
 &= -17
 \end{aligned}$$

$$\begin{aligned}
 \text{11} \quad y &= -x^2 + 8x - 7 \\
 \therefore \frac{dy}{dx} &= -2x + 8
 \end{aligned}$$

$$\begin{aligned}
 \text{The tangent has gradient } -4 \text{ when } -2x + 8 &= -4 \\
 \therefore -2x &= -12 \\
 \therefore x &= 6
 \end{aligned}$$

$$\begin{aligned}
 \text{When } x = 6, \quad y &= -(6)^2 + 8(6) - 7 \\
 &= -36 + 48 - 7 \\
 &= 5
 \end{aligned}$$

So, P has coordinates (6, 5).



$$\begin{aligned}
 \text{12} \quad y &= ax^3 - 3x + 3 \\
 \therefore \frac{dy}{dx} &= 3ax^2 - 3
 \end{aligned}$$

$$\begin{aligned}
 \text{Now, when } x = 2, \quad \frac{dy}{dx} &= 21, \quad \text{so } 3a(2)^2 - 3 = 21 \\
 \therefore 12a - 3 &= 21 \\
 \therefore 12a &= 24 \\
 \therefore a &= 2
 \end{aligned}$$

$$\begin{aligned}
 \text{13 a} \quad f(x) &= x^2 - 3x \\
 \therefore f'(x) &= 2x - 3 \\
 \therefore f'(-1) &= 2(-1) - 3 \\
 &= -5
 \end{aligned}$$

So, the gradient of $f(x)$ at $x = -1$ is -5 .

$$\begin{aligned}
 \text{b} \quad f(x) &= -3x^2 + 4 \\
 \therefore f'(x) &= -6x \\
 \therefore f'(2) &= -6(2) \\
 &= -12
 \end{aligned}$$

So, the gradient of $f(x)$ at $x = 2$ is -12 .

$$\begin{aligned}
 \text{c} \quad f(x) &= x + \frac{2}{x} \\
 &= x + 2x^{-1} \\
 \therefore f'(x) &= 1 - 2x^{-2} \\
 &= 1 - \frac{2}{x^2} \\
 \therefore f'(3) &= 1 - \frac{2}{3^2} \\
 &= 1 - \frac{2}{9} \\
 &= \frac{7}{9}
 \end{aligned}$$

So, the gradient of $f(x)$ at $x = 3$ is $\frac{7}{9}$.

$$\begin{aligned}
 \text{d} \quad f(x) &= x^3 - x^2 - x - 2 \\
 \therefore f'(x) &= 3x^2 - 2x - 1 \\
 \therefore f'(0) &= 3(0)^2 - 2(0) - 1 \\
 &= -1
 \end{aligned}$$

So, the gradient of $f(x)$ at $x = 0$ is -1 .

- 14** $S(t) = 0.3t^3 - 18t^2 + 550t$ grams
 $\therefore S'(t) = 0.9t^2 - 36t + 550$ grams per second

$S'(t)$ represents the instantaneous rate of change in weight of sand in the bucket, in grams per second, for a given value of t .

- 15 a** The tangent shown on the graph passes through $(0, 5)$ and $(5, 0)$.

\therefore the gradient of the tangent is $\frac{0-5}{5-0} = -1$, so

$$f'(3) = -1.$$

Also, since the tangent passes through $(0, 5)$, it has

equation $\frac{y-5}{x-0} = -1$

$$\therefore y - 5 = -x$$

$$\therefore y = -x + 5$$

So when $x = 3$, $y = -3 + 5 = 2$

\therefore the point of contact is $(3, 2)$, and hence $f(3) = 2$.

- b** $f(x)$ has the form $f(x) = ax^2 + bx + c$

The y -intercept is 14 $\therefore f(0) = 14$

$$\therefore a(0)^2 + b(0) + c = 14$$

$$\therefore c = 14$$

Now $f(3) = 2$ {from **a**}

$$\therefore a(3)^2 + b(3) + 14 = 2$$

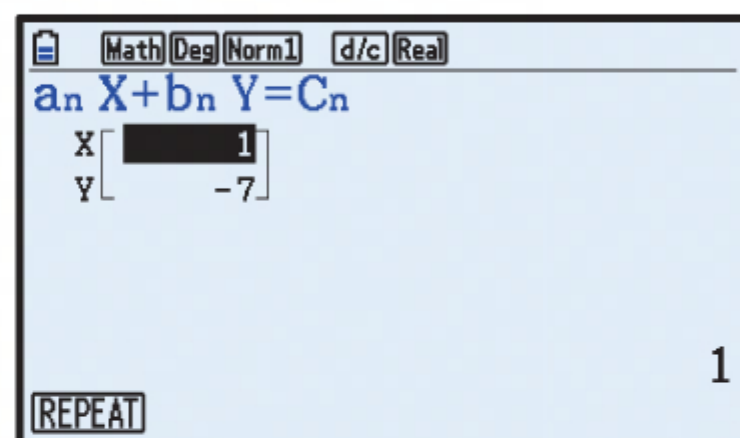
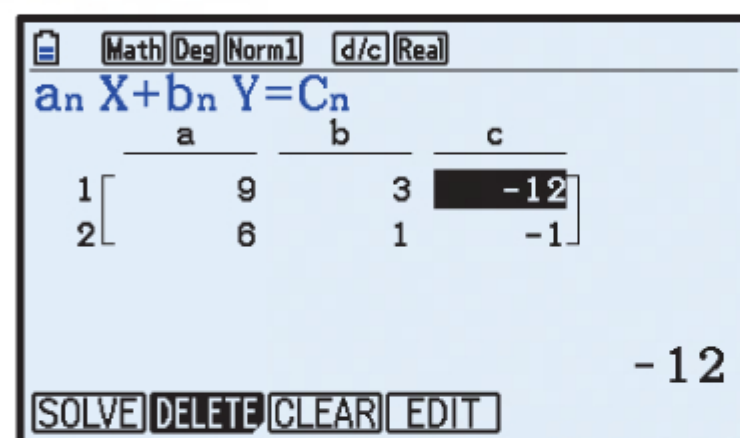
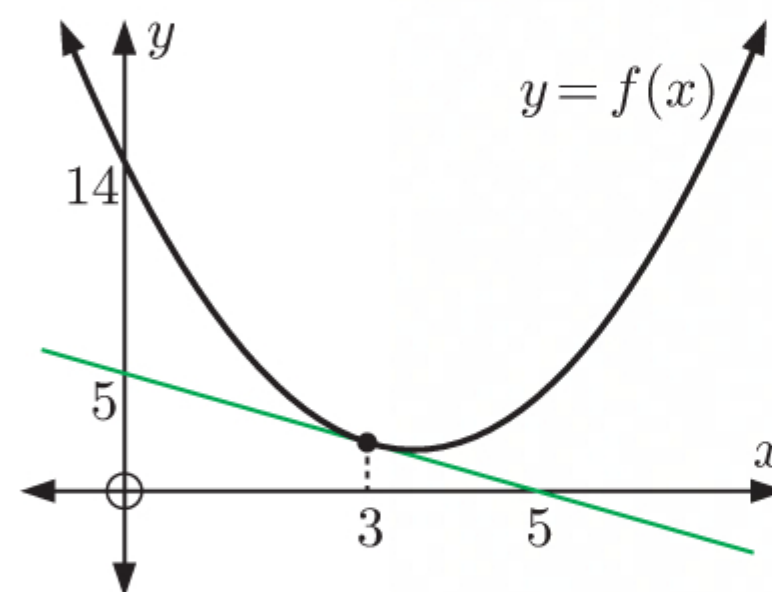
$$\therefore 9a + 3b = -12 \quad \dots (1)$$

Also $f'(3) = -1$

and $f'(x) = 2ax + b$

$$\therefore 2a(3) + b = -1$$

$$\therefore 6a + b = -1 \quad \dots (2)$$

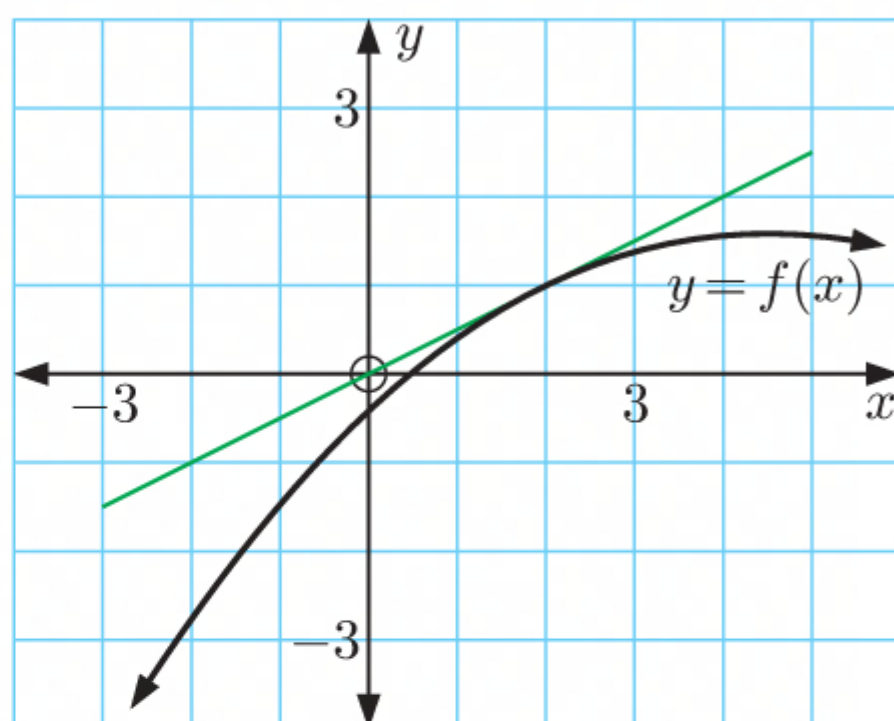


Solving (1) and (2) simultaneously using technology gives $a = 1$ and $b = -7$.

So, $f(x) = x^2 - 7x + 14$.

REVIEW SET 10B

1



The tangent at $x = 2$ has gradient $\frac{1-0}{2-0} = \frac{1}{2}$.

\therefore the instantaneous rate of change in $f(x)$ at $x = 2$ is $\frac{1}{2}$.

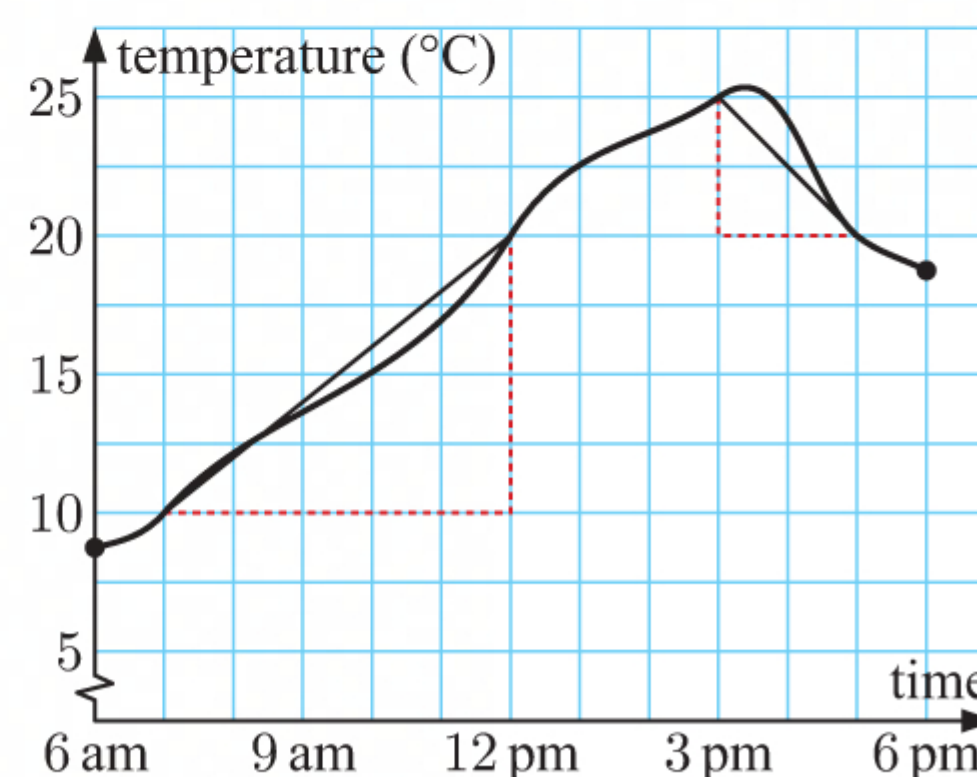
2

- a** average rate of change in temperature from 7 am to noon

$$\begin{aligned} &= \frac{(20 - 10)^\circ\text{C}}{(6 - 1) \text{ h}} \\ &= \frac{10}{5}^\circ\text{C per hour} \\ &= 2^\circ\text{C per hour} \end{aligned}$$

- b** average rate of change in temperature from 3 pm to 5 pm

$$\begin{aligned} &= \frac{(20 - 25)^\circ\text{C}}{(11 - 9) \text{ h}} \\ &= \frac{-5}{2}^\circ\text{C per hour} \\ &= -2.5^\circ\text{C per hour} \end{aligned}$$

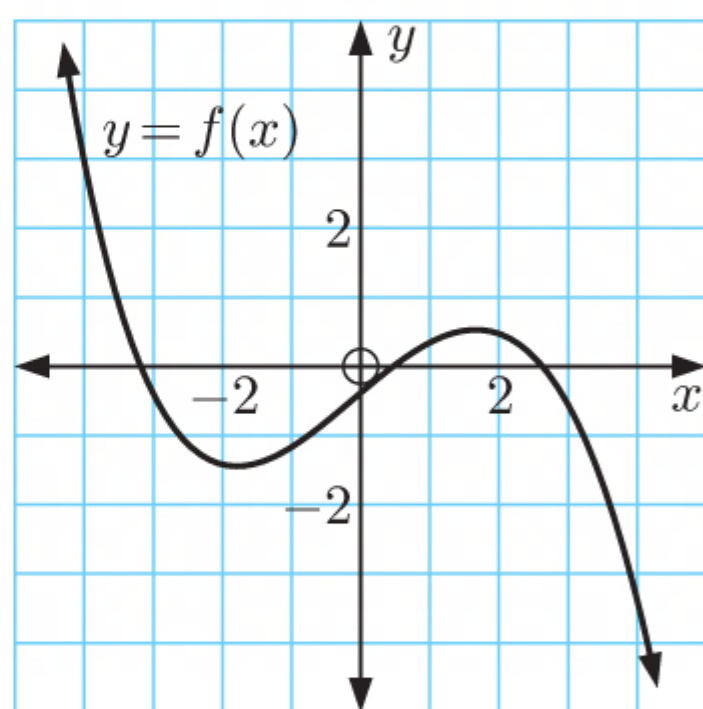


3

$$\begin{aligned} \text{a } \lim_{h \rightarrow 0} \frac{h^3 - 3h}{h} &= \lim_{h \rightarrow 0} \frac{\cancel{h}(h^2 - 3)}{\cancel{h}} \\ &= \lim_{h \rightarrow 0} (h^2 - 3) \quad \{\text{as } h \neq 0\} \\ &= -3 \end{aligned}$$

$$\begin{aligned} \text{b } \lim_{x \rightarrow 1} \frac{3x^2 - 3x}{x - 1} &= \lim_{x \rightarrow 1} \frac{3x(\cancel{x - 1})}{(\cancel{x - 1})} \\ &= \lim_{x \rightarrow 1} 3x \quad \{\text{as } x \neq 1\} \\ &= 3 \end{aligned}$$

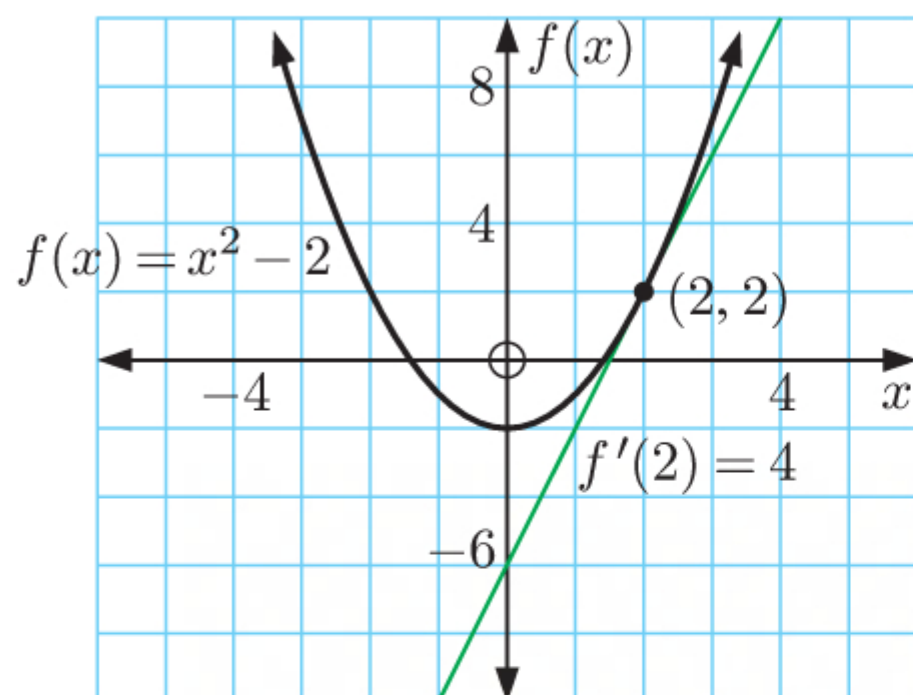
4



- a** $f(-1)$ is below the x -axis, so $f(-1)$ is negative.
- b** $f'(0)$ is the gradient of the tangent to $f(x)$ at the point where $x = 0$. Since the curve is increasing at $x = 0$, $f'(0)$ is positive.
- c** $f(2)$ is above the x -axis, so $f(2)$ is positive.
- d** $f'(3)$ is the gradient of the tangent to $f(x)$ at the point where $x = 3$. Since the curve is decreasing at $x = 3$, $f'(3)$ is negative.

5 a

x	-3	-2	-1	0	1	2	3
$f(x) = x^2 - 2$	7	2	-1	-2	-1	2	7



- b** The tangent to $f(x) = x^2 - 2$ at the point where $x = 2$ has gradient $\frac{6 - (-2)}{3 - 1} = \frac{8}{2} = 4$.
 \therefore the instantaneous rate of change in $f(x) = x^2 - 2$ when $x = 2$ is 4.

$$\begin{aligned}
 \text{c } f'(2) &= \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[(2+h)^2 - 2] - [2^2 - 2]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{4 + 4h + h^2 - 2 - 2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{4h + h^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(4+h)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (4+h) \quad \{\text{as } h \neq 0\} \\
 &= 4 \quad \checkmark
 \end{aligned}$$

6 a $f(x) = x^4 - 2x$

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[(x+h)^4 - 2(x+h)] - [x^4 - 2x]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{x^4} + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - \cancel{2x} - 2h - \cancel{x^4} + \cancel{2x}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4 - 2h}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cancel{h}(4x^3 + 6x^2h + 4xh^2 + h^3 - 2)}{\cancel{h}} \\
 &= \lim_{h \rightarrow 0} (4x^3 + 6x^2h + 4xh^2 + h^3 - 2) \quad \{\text{as } h \neq 0\} \\
 &= 4x^3 - 2
 \end{aligned}$$

b $f'(-2) = 4(-2)^3 - 2$
 $= -34$

The gradient of the tangent to $y = f(x)$ at the point where $x = -2$ is -34 .

7 a $y = f(x) = x^2 + 5x - 2$

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[(x+h)^2 + 5(x+h) - 2] - [x^2 + 5x - 2]}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{5x} + 5h - \cancel{2} - \cancel{x^2} - \cancel{5x} + \cancel{2}}{h} \\ &= \lim_{h \rightarrow 0} \frac{2xh + h^2 + 5h}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{h}(2x + h + 5)}{\cancel{h}} \\ &= \lim_{h \rightarrow 0} (2x + h + 5) \quad \{\text{as } h \neq 0\} \\ &= 2x + 5\end{aligned}$$

b The tangent to $f(x) = x^2 + 5x - 2$ has gradient -3 when $f'(x) = \frac{dy}{dx} = -3$

$$\begin{aligned}\therefore 2x + 5 &= -3 \\ \therefore 2x &= -8 \\ \therefore x &= -4\end{aligned}$$

Now, $f(-4) = (-4)^2 + 5(-4) - 2$
 $= -6$

So, the tangent has gradient -3 at the point $(-4, -6)$.

8 $f(t) = 452 - 4.8t^2$ metres, $0 \leq t \leq 3$ seconds

a i $f(1) = 452 - 4.8(1)^2$
 $= 447.2$

The jumper is 447.2 m above ground level after 1 second.

ii $f(2) = 452 - 4.8(2)^2$
 $= 432.8$

The jumper is 432.8 m above ground level after 2 seconds.

b $f'(t) = \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h}$

$$\begin{aligned}&= \lim_{h \rightarrow 0} \frac{[452 - 4.8(t+h)^2] - [452 - 4.8t^2]}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{452} - \cancel{4.8t^2} - 9.6th - 4.8h^2 - \cancel{452} + \cancel{4.8t^2}}{h} \\ &= \lim_{h \rightarrow 0} \frac{-9.6th - 4.8h^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cancel{h}(-9.6t - 4.8h)}{\cancel{h}} \\ &= \lim_{h \rightarrow 0} (-9.6t - 4.8h) \quad \{\text{as } h \neq 0\} \\ &= -9.6t\end{aligned}$$

- c** The speed of the jumper is equal to the rate of change in the jumper's altitude which is given by $f'(t)$.

$$\begin{aligned}\text{i} \quad f'(1) &= -9.6(1) \\ &= -9.6\end{aligned}$$

The jumper's speed was 9.6 m s^{-1} after 1 second.

(The negative sign indicates the jumper is moving downwards.)

$$\begin{aligned}\text{ii} \quad f'(2) &= -9.6(2) \\ &= -19.2\end{aligned}$$

The jumper's speed was 19.2 m s^{-1} after 2 seconds.

(The negative sign indicates the jumper is moving downwards.)

$$\begin{aligned}\text{9 a} \quad f(x) &= 7x^3 \\ \therefore f'(x) &= 7(3x^2) \\ &= 21x^2\end{aligned}$$

$$\begin{aligned}\text{c} \quad f(x) &= (2x - 3)^2 \\ &= 4x^2 - 12x + 9 \\ \therefore f'(x) &= 4(2x) - 12(1) \\ &= 8x - 12\end{aligned}$$

$$\begin{aligned}\text{b} \quad f(x) &= 3x^2 - x^3 \\ \therefore f'(x) &= 3(2x) - 3x^2 \\ &= 6x - 3x^2\end{aligned}$$

$$\begin{aligned}\text{d} \quad f(x) &= \frac{7x^3 + 2x^4}{x^2} \\ &= \frac{7x^3}{x^2} + \frac{2x^4}{x^2} \\ &= 7x + 2x^2 \\ \therefore f'(x) &= 7(1) + 2(2x) \\ &= 7 + 4x\end{aligned}$$

$$\begin{aligned}\text{10 a} \quad f(x) &= x^4 - 3x - 1 \\ \therefore f'(x) &= 4x^3 - 3\end{aligned}$$

$$\begin{aligned}\text{b} \quad f'(2) &= 4(2)^3 - 3 \\ &= 32 - 3 \\ &= 29\end{aligned}$$

$$\begin{aligned}\text{c} \quad f'(0) &= 4(0)^3 - 3 \\ &= -3\end{aligned}$$

$$\text{11} \quad f(x) = -2x^2 + 5x + 3$$

- a** average rate of change from $x = 2$ to $x = 4$

$$\begin{aligned}&= \frac{f(4) - f(2)}{4 - 2} \\ &= \frac{[-2(4)^2 + 5(4) + 3] - [-2(2)^2 + 5(2) + 3]}{2} \\ &= \frac{-32 + 20 + 3 - (-8 + 10 + 3)}{2} \\ &= \frac{-9 - 5}{2} \\ &= -\frac{14}{2} \\ &= -7\end{aligned}$$

$$\begin{aligned}\text{b} \quad f'(x) &= -4x + 5 \\ \therefore f'(2) &= -4(2) + 5 \\ &= -8 + 5 \\ &= -3\end{aligned}$$

So, the tangent at $x = 2$ has gradient -3 .

\therefore the instantaneous rate of change at $x = 2$ is -3 .

12 a $y = 3x^2 - 7x + 4$

$$\therefore \frac{dy}{dx} = 3(2x) - 7(1) \\ = 6x - 7$$

c $y = \frac{3}{x} - \frac{5}{x^3}$

$$= 3x^{-1} - 5x^{-3}$$

$$\therefore \frac{dy}{dx} = 3(-x^{-2}) - 5(-3x^{-4}) \\ = -3x^{-2} + 15x^{-4} \\ = -\frac{3}{x^2} + \frac{15}{x^4}$$

13 a $f(x) = \frac{x^2 + 2}{x}$

$$= x + 2x^{-1}$$

$$\therefore f'(x) = 1 - 2x^{-2} \\ = 1 - \frac{2}{x^2}$$

b i $f'(1) = 1 - \frac{2}{1^2}$

$$= -1$$

\therefore gradient of tangent $= -1$

ii $f'(-2) = 1 - \frac{2}{(-2)^2}$

$$= 1 - \frac{2}{4} \\ = \frac{1}{2}$$

\therefore gradient of tangent $= \frac{1}{2}$

14 $y = 2x^3 + ax + b$

$$\therefore \frac{dy}{dx} = 6x^2 + a$$

Since the gradient of the tangent at $(-2, 33)$ is 10, then $6(-2)^2 + a = 10$

$$\therefore 24 + a = 10$$

$$\therefore a = -14$$

$$\therefore y = 2x^3 - 14x + b$$

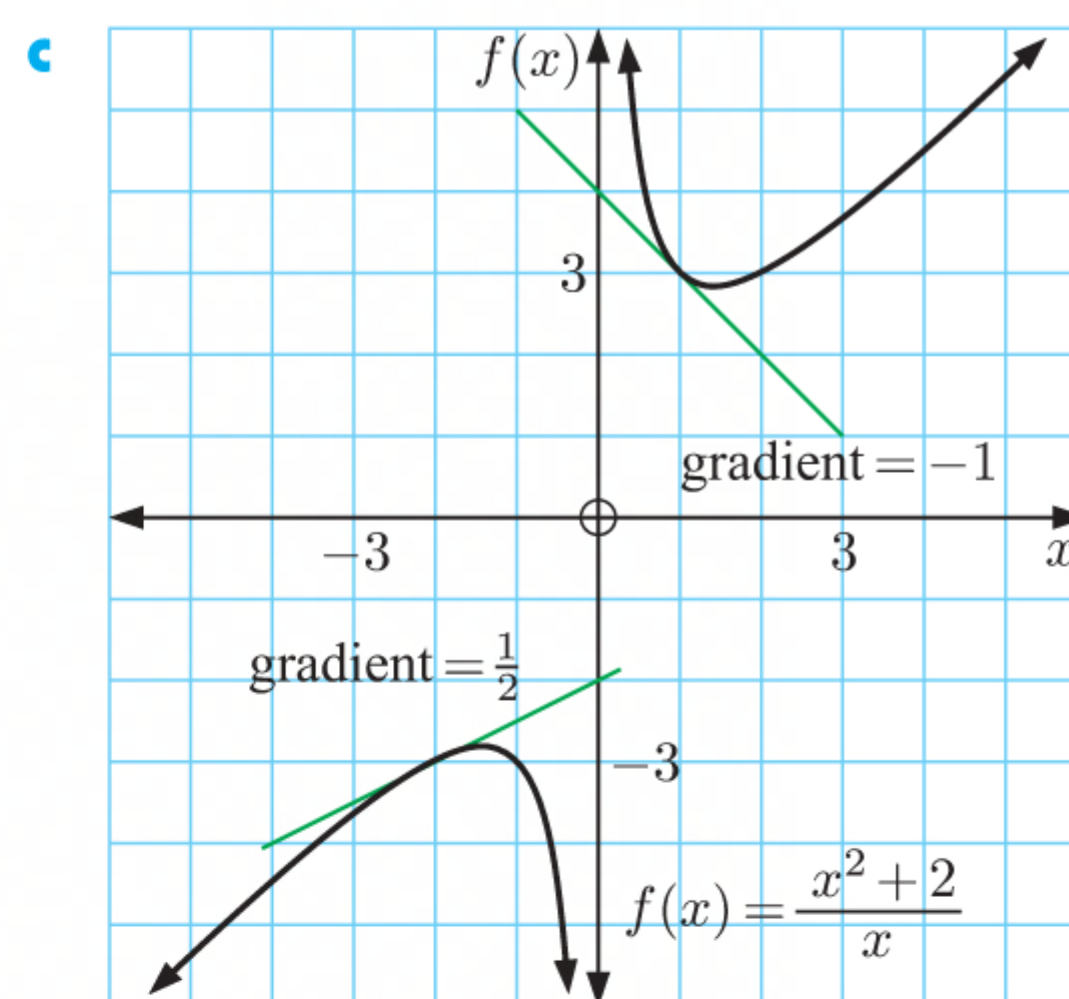
Since the curve passes through $(-2, 33)$, then $33 = 2(-2)^3 - 14(-2) + b$

$$= -16 + 28 + b$$

$$\therefore b = 21$$

b $y = 2x^3 - 6x^2 + 7x - 4$

$$\therefore \frac{dy}{dx} = 2(3x^2) - 6(2x) + 7(1) \\ = 6x^2 - 12x + 7$$



Chapter 11

PROPERTIES OF CURVES

EXERCISE 11A

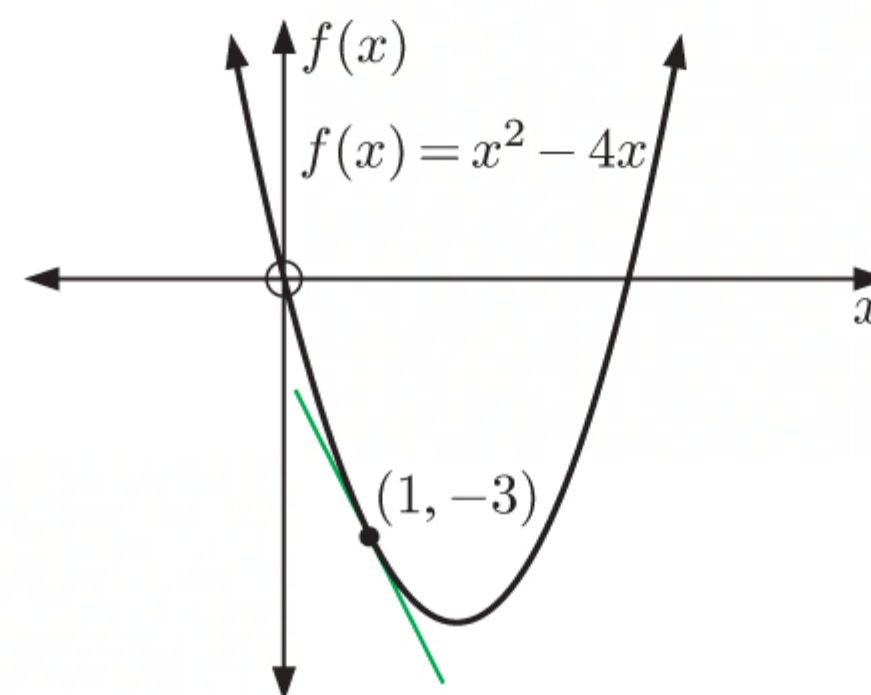
1 a $f(x) = x^2 - 4x$

$$\therefore f'(x) = 2x - 4$$

b The point of contact is $(1, -3)$.

$$\begin{aligned} f'(1) &= 2(1) - 4 \\ &= -2 \end{aligned}$$

So, the tangent has equation $y = -2(x - 1) + (-3)$
 $\therefore y = -2x - 1$



2 a $y = x^2$

When $x = 4$, $y = 4^2 = 16$,
so the point of contact is $(4, 16)$.

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= 2x, \text{ so at } x = 4, \\ \frac{dy}{dx} &= 2(4) = 8 \end{aligned}$$

The tangent has equation

$$\begin{aligned} y &= 8(x - 4) + 16 \\ &= 8x - 32 + 16 \\ &= 8x - 16 \end{aligned}$$

c $y = 3x^{-1} = \frac{3}{x}$

When $x = -1$, $y = \frac{3}{-1} = -3$,
so the point of contact is $(-1, -3)$.

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= -3x^{-2} = -\frac{3}{x^2} \\ \text{so at } x = -1, \frac{dy}{dx} &= -\frac{3}{(-1)^2} = -3 \end{aligned}$$

The tangent has equation

$$\begin{aligned} y &= -3(x - (-1)) + (-3) \\ &= -3(x + 1) - 3 \\ &= -3x - 3 - 3 \\ &= -3x - 6 \end{aligned}$$

b $y = x^3$

When $x = -2$, $y = (-2)^3 = -8$,
so the point of contact is $(-2, -8)$.

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= 3x^2, \text{ so at } x = -2, \\ \frac{dy}{dx} &= 3(-2)^2 = 12 \end{aligned}$$

The tangent has equation

$$\begin{aligned} y &= 12(x - (-2)) + (-8) \\ &= 12(x + 2) - 8 \\ &= 12x + 24 - 8 \\ &= 12x + 16 \end{aligned}$$

d $y = \frac{4}{x^3} = 4x^{-3}$

When $x = 2$, $y = \frac{4}{2^3} = \frac{4}{8} = \frac{1}{2}$,
so the point of contact is $(2, \frac{1}{2})$.

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= -12x^{-4} = -\frac{12}{x^4} \\ \text{so at } x = 2, \frac{dy}{dx} &= -\frac{12}{2^4} = -\frac{12}{16} = -\frac{3}{4} \end{aligned}$$

The tangent has equation

$$\begin{aligned} y &= -\frac{3}{4}(x - 2) + \frac{1}{2} \\ &= -\frac{3}{4}x + \frac{3}{2} + \frac{1}{2} \\ &= -\frac{3}{4}x + 2 \end{aligned}$$

e $y = x^2 + 5x - 4$

When $x = 1$, $y = 1^2 + 5(1) - 4 = 2$,
so the point of contact is $(1, 2)$.

Now $\frac{dy}{dx} = 2x + 5$, so at $x = 1$,

$$\frac{dy}{dx} = 2(1) + 5 = 7$$

The tangent has equation

$$\begin{aligned} y &= 7(x - 1) + 2 \\ &= 7x - 7 + 2 \\ &= 7x - 5 \end{aligned}$$

g $y = x^3 + 2x$

When $x = 0$, $y = 0^3 + 2(0) = 0$,
so the point of contact is $(0, 0)$.

Now $\frac{dy}{dx} = 3x^2 + 2$, so at $x = 0$,

$$\frac{dy}{dx} = 3(0)^2 + 2 = 2$$

The tangent has equation

$$\begin{aligned} y &= 2(x - 0) + 0 \\ &= 2x \end{aligned}$$

i $y = x - 2x^2 + 3$

When $x = 2$, $y = 2 - 2(2)^2 + 3 = -3$,
so, the point of contact is $(2, -3)$.

Now $\frac{dy}{dx} = 1 - 4x$, so at $x = 2$,

$$\frac{dy}{dx} = 1 - 4(2) = -7$$

The tangent has equation

$$\begin{aligned} y &= -7(x - 2) + (-3) \\ &= -7x + 14 - 3 \\ &= -7x + 11 \end{aligned}$$

f $y = 2x^2 + 5x + 3$

When $x = -2$, $y = 2(-2)^2 + 5(-2) + 3 = 1$,

so the point of contact is $(-2, 1)$.

Now $\frac{dy}{dx} = 4x + 5$, so at $x = -2$,

$$\frac{dy}{dx} = 4(-2) + 5 = -3$$

The tangent has equation

$$\begin{aligned} y &= -3(x - (-2)) + 1 \\ &= -3(x + 2) + 1 \\ &= -3x - 6 + 1 \\ &= -3x - 5 \end{aligned}$$

h $y = \frac{x^2 + 4}{x} = x + 4x^{-1}$

When $x = -1$, $y = \frac{(-1)^2 + 4}{-1} = -5$,

so the point of contact is $(-1, -5)$.

Now $\frac{dy}{dx} = 1 - 4x^{-2} = 1 - \frac{4}{x^2}$

so at $x = -1$, $\frac{dy}{dx} = 1 - \frac{4}{(-1)^2} = -3$

The tangent has equation

$$\begin{aligned} y &= -3(x - (-1)) + (-5) \\ &= -3(x + 1) - 5 \\ &= -3x - 3 - 5 \\ &= -3x - 8 \end{aligned}$$

j $y = x^3 - 5x$

When $x = 1$, $y = 1^3 - 5(1) = -4$,
so, the point of contact is $(1, -4)$.

Now $\frac{dy}{dx} = 3x^2 - 5$, so at $x = 1$,

$$\frac{dy}{dx} = 3(1)^2 - 5 = -2$$

The tangent has equation

$$\begin{aligned} y &= -2(x - 1) + (-4) \\ &= -2x + 2 - 4 \\ &= -2x - 2 \end{aligned}$$

$$\begin{aligned}
 \mathbf{k} \quad y &= \frac{3}{x} - \frac{1}{x^2} \\
 &= 3x^{-1} - x^{-2} \\
 \text{Now } \frac{dy}{dx} &= -3x^{-2} + 2x^{-3} \\
 &= -\frac{3}{x^2} + \frac{2}{x^3}, \text{ so at } (-1, -4), \\
 \frac{dy}{dx} &= -\frac{3}{(-1)^2} + \frac{2}{(-1)^3} \\
 &= -3 - 2 \\
 &= -5
 \end{aligned}$$

The tangent has equation

$$\begin{aligned}
 y &= -5(x - (-1)) + (-4) \\
 &= -5(x + 1) - 4 \\
 &= -5x - 5 - 4 \\
 &= -5x - 9
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{3} \quad \mathbf{a} \quad f(x) &= -x^2 + 6x + 4 \\
 \therefore f'(x) &= -2x + 6
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{l} \quad y &= 3x^2 - \frac{1}{x} = 3x^2 - x^{-1} \\
 \text{When } x &= -1, \\
 y &= 3(-1)^2 - \frac{1}{(-1)} = 4, \\
 \text{so, the point of contact is } &(-1, 4). \\
 \text{Now } \frac{dy}{dx} &= 6x + x^{-2} \\
 &= 6x + \frac{1}{x^2}, \text{ so at } x = -1, \\
 \frac{dy}{dx} &= 6(-1) + \frac{1}{(-1)^2} = -5
 \end{aligned}$$

The tangent has equation

$$\begin{aligned}
 y &= -5(x - (-1)) + 4 \\
 &= -5(x + 1) + 4 \\
 &= -5x - 5 + 4 \\
 &= -5x - 1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad f'(x) &= 0 \quad \text{when } -2x + 6 = 0 \\
 \therefore -2x &= -6 \\
 \therefore x &= 3
 \end{aligned}$$

\mathbf{c} The tangent at the point where $x = 3$ is horizontal.

$$\mathbf{4} \quad y = 2x^3 + kx^2 - 3$$

$$\begin{aligned}
 \mathbf{a} \quad \frac{dy}{dx} &= 6x^2 + 2kx \\
 \text{When } x &= 2, \quad \frac{dy}{dx} = 4 \\
 \therefore 6(2)^2 + 2k(2) &= 4 \\
 \therefore 24 + 4k &= 4 \\
 \therefore 4k &= -20 \\
 \therefore k &= -5
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \text{Since } k &= -5, \\
 y &= 2x^3 - 5x^2 - 3 \\
 \text{When } x &= 2, \\
 y &= 2(2)^3 - 5(2)^2 - 3 \\
 &= -7 \\
 \text{So, the point of contact is } &(2, -7). \\
 \text{The tangent has equation} \\
 y &= 4(x - 2) + (-7) \\
 &= 4x - 15
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{5} \quad y &= 1 - 3x + 12x^2 - 8x^3 \\
 \therefore \frac{dy}{dx} &= -3 + 24x - 24x^2 \\
 \text{When } x &= 1, \quad \frac{dy}{dx} = -3 + 24(1) - 24(1)^2 \\
 &= -3
 \end{aligned}$$

So, the tangent at $(1, 2)$ has gradient -3 .

$$\begin{aligned}
 \text{The tangents to the curve have gradient } -3 \text{ when } &-3 + 24x - 24x^2 = -3 \\
 \therefore 24x^2 - 24x &= 0 \\
 \therefore 24x(x - 1) &= 0 \\
 \therefore x &= 0 \text{ or } 1
 \end{aligned}$$

So the other x -value for which the tangent to the curve has gradient -3 is $x = 0$,
and when $x = 0$, $y = 1 - 3(0) + 12(0)^2 - 8(0)^3 = 1$.

\therefore the tangent to the curve at $(0, 1)$ is parallel to the tangent at $(1, 2)$.

This tangent has equation $y = -3(x - 0) + 1$
 $= -3x + 1$

- 6** The tangent to the curve $y = x^2 + ax + b$ at the point where $x = 1$ is $2x + y = 6$ or $y = -2x + 6$.
 \therefore the tangent has gradient -2 , and the point of contact is $(1, -2(1) + 6)$ which is $(1, 4)$.

Now, $y = x^2 + ax + b$

$$\therefore \frac{dy}{dx} = 2x + a$$

When $x = 1$, $\frac{dy}{dx} = -2$

$$\therefore 2(1) + a = -2$$

$$\therefore 2 + a = -2$$

$$\therefore a = -4 \quad \dots (*)$$

and $y = 4$

$$\therefore 1^2 + a(1) + b = 4$$

$$\therefore 1 + (-4) + b = 4 \quad \{\text{using } (*)\}$$

$$\therefore b = 7$$

So, $a = -4$ and $b = 7$.

7 a

$$\begin{aligned} f(x) &= x^2 + \frac{4}{x^2} \\ &= x^2 + 4x^{-2} \\ \therefore f'(x) &= 2x - 8x^{-3} \\ &= 2x - \frac{8}{x^3} \end{aligned}$$

- b** Horizontal tangents have gradient 0, so

$$2x - \frac{8}{x^3} = 0$$

$$\therefore \frac{2x^4 - 8}{x^3} = 0$$

$$\therefore 2x^4 - 8 = 0$$

$$\therefore 2x^4 = 8$$

$$\therefore x^4 = 4$$

$$\therefore x = \pm\sqrt{2}$$

c When $x = \sqrt{2}$, $f(\sqrt{2}) = (\sqrt{2})^2 + \frac{4}{(\sqrt{2})^2}$
 $= 2 + \frac{4}{2}$
 $= 4$

\therefore the horizontal tangent at $(\sqrt{2}, 4)$ is $y = 4$.

When $x = -\sqrt{2}$, $f(-\sqrt{2}) = (-\sqrt{2})^2 + \frac{4}{(-\sqrt{2})^2}$
 $= 2 + \frac{4}{2}$
 $= 4$

\therefore the horizontal tangent at $(-\sqrt{2}, 4)$ is $y = 4$.

The tangents are the same line, $y = 4$.

8 a Let $y = f(x) = x^2 + 1$
 $\therefore f'(x) = 2x$

b When $x = 1$, $y = (1)^2 + 1 = 2$, so the point of contact is $(1, 2)$.

Now $f'(1) = 2(1) = 2$

\therefore the tangent has equation $y = 2(x - 1) + 2$
 $= 2x - 2 + 2$
 $= 2x$

c The tangent has equation $y = 2x$.

When $x = 0$, $y = 2(0) = 0$, so the tangent passes through the origin.

9 a $f(x) = 2x^3 - 5x + 1$

i $f(-1) = 2(-1)^3 - 5(-1) + 1$
 $= -2 + 5 + 1$
 $= 4$

So, the point of contact is $(-1, 4)$.

Now $f'(x) = 6x^2 - 5$

$\therefore f'(-1) = 6(-1)^2 - 5$
 $= 6 - 5$
 $= 1$

\therefore the tangent has equation

$y = 1(x - (-1)) + 4$
 $= x + 1 + 4$
 $= x + 5$

b $y = x^2 + \frac{3}{x} + 2 = x^2 + 3x^{-1} + 2$

i When $x = 3$, $y = 3^2 + \frac{3}{3} + 2$
 $= 9 + 1 + 2$
 $= 12$

So, the point of contact is $(3, 12)$.

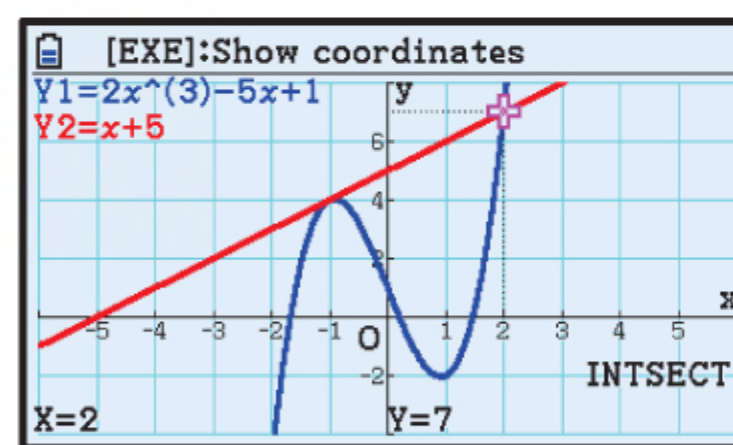
Now $\frac{dy}{dx} = 2x - 3x^{-2}$
 $= 2x - \frac{3}{x^2}$

When $x = 3$, $\frac{dy}{dx} = 2(3) - \frac{3}{3^2}$
 $= 6 - \frac{1}{3} = \frac{17}{3}$

\therefore the tangent has equation

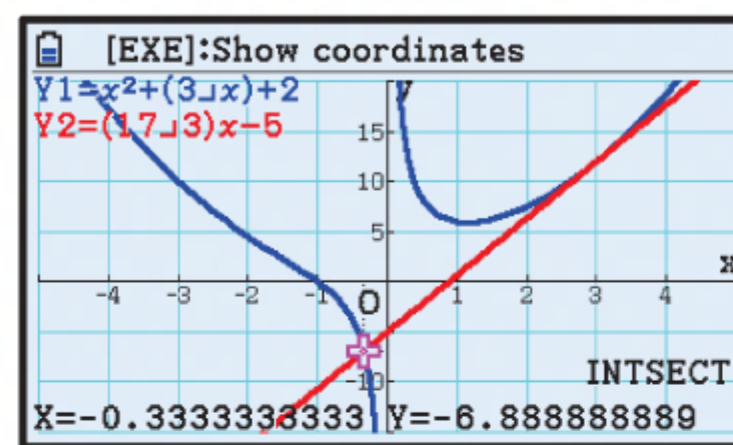
$y = \frac{17}{3}(x - 3) + 12$
 $= \frac{17}{3}x - 17 + 12$
 $= \frac{17}{3}x - 5$

ii We use technology to find where the tangent meets the curve again:



The tangent meets the curve again at $(2, 7)$.

ii We use technology to find where the tangent meets the curve again:



The tangent meets the curve again at about $(-0.333, -6.89)$.

c $f(x) = x^3 + 5$

i $f(1.5) = (1.5)^3 + 5$
 $= 3.375 + 5$
 $= 8.375$

So, the point of contact is $(1.5, 8.375)$.

Now $f'(x) = 3x^2$
 $\therefore f'(1.5) = 3(1.5)^2$
 $= 3 \times 2.25$
 $= 6.75$

\therefore the tangent has equation

$$\begin{aligned} y &= 6.75(x - 1.5) + 8.375 \\ &= 6.75x - 10.125 + 8.375 \\ &= 6.75x - 1.75 \end{aligned}$$

d $y = x^3 + \frac{1}{x} = x^3 + x^{-1}$

i When $x = -1$, $y = (-1)^3 + \frac{1}{-1}$
 $= -1 - 1$
 $= -2$

So, the point of contact is $(-1, -2)$.

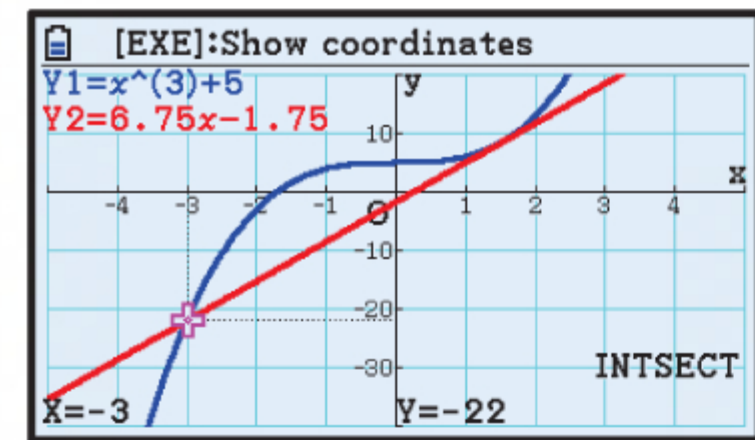
Now $\frac{dy}{dx} = 3x^2 - x^{-2}$
 $= 3x^2 - \frac{1}{x^2}$

When $x = -1$, $\frac{dy}{dx} = 3(-1)^2 - \frac{1}{(-1)^2}$
 $= 3 - 1$
 $= 2$

\therefore the tangent has equation

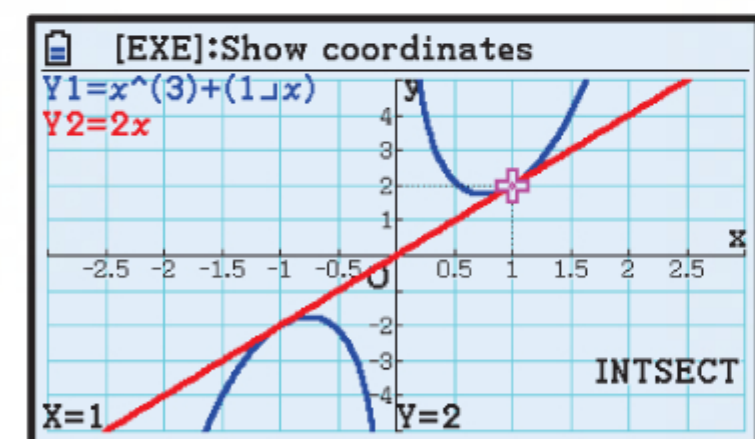
$$\begin{aligned} y &= 2(x - (-1)) - 2 \\ &= 2(x + 1) - 2 \\ &= 2x + 2 - 2 \\ &= 2x \end{aligned}$$

ii We use technology to find where the tangent meets the curve again:



The tangent meets the curve again at $(-3, -22)$.

ii We use technology to find where the tangent meets the curve again:



The tangent meets the curve again at $(1, 2)$.

e $f(x) = 3x^3 + 2x^2 - x + 2$

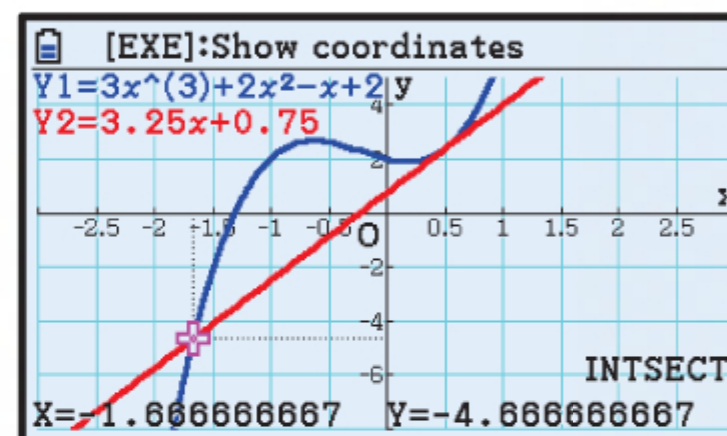
i $f(0.5) = 3(0.5)^3 + 2(0.5)^2 - 0.5 + 2$
 $= 3 \times 0.125 + 2 \times 0.25 + 1.5$
 $= 2.375$

So, the point of contact is $(0.5, 2.375)$.

Now $f'(x) = 9x^2 + 4x - 1$
 $\therefore f'(0.5) = 9(0.5)^2 + 4(0.5) - 1$
 $= 9 \times 0.25 + 2 - 1$
 $= 3.25$

\therefore the tangent has equation
 $y = 3.25(x - 0.5) + 2.375$
 $= 3.25x - 1.625 + 2.375$
 $= 3.25x + 0.75$

ii We use technology to find where the tangent meets the curve again:



The tangent meets the curve again at about $(-1.67, -4.67)$.

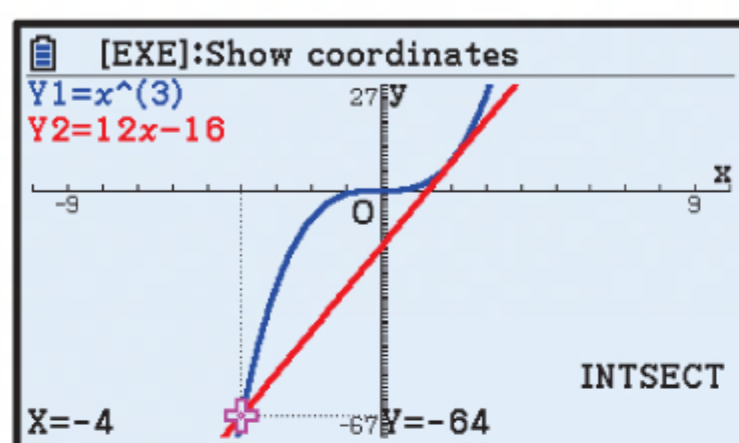
10 a Consider the tangent to $y = x^3$ at $x = 2$.

When $x = 2$, $y = 2^3 = 8$ so the point of contact is $(2, 8)$.

Now $\frac{dy}{dx} = 3x^2$ and so at $x = 2$,
 $\frac{dy}{dx} = 3(2)^2 = 12$

So, the tangent at $(2, 8)$ has gradient 12 and its equation is $y = 12(x - 2) + 8$
 $= 12x - 24 + 8$
 $= 12x - 16$

We use technology to find where the tangent meets the curve again:



The tangent meets the curve again at $(-4, -64)$.

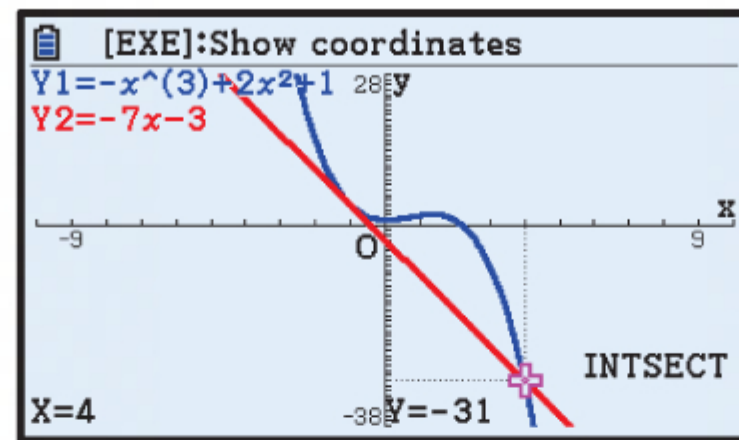
b Consider the tangent to $y = -x^3 + 2x^2 + 1$ at $x = -1$.

When $x = -1$, $y = -(-1)^3 + 2(-1)^2 + 1 = 4$ and so the point of contact is $(-1, 4)$.

Now $\frac{dy}{dx} = -3x^2 + 4x$ and so at $x = -1$,
 $\frac{dy}{dx} = -3(-1)^2 + 4(-1) = -7$

So, the tangent at $(-1, 4)$ has gradient -7 and its equation is $y = -7(x - (-1)) + 4$
 $= -7(x + 1) + 4$
 $= -7x - 7 + 4$
 $= -7x - 3$

We use technology to find where the tangent meets the curve again:



The tangent meets the curve again at $(4, -31)$.

- c** Consider the tangent to $y = \frac{1}{x} - \frac{1}{x^2}$ at $x = 1$.

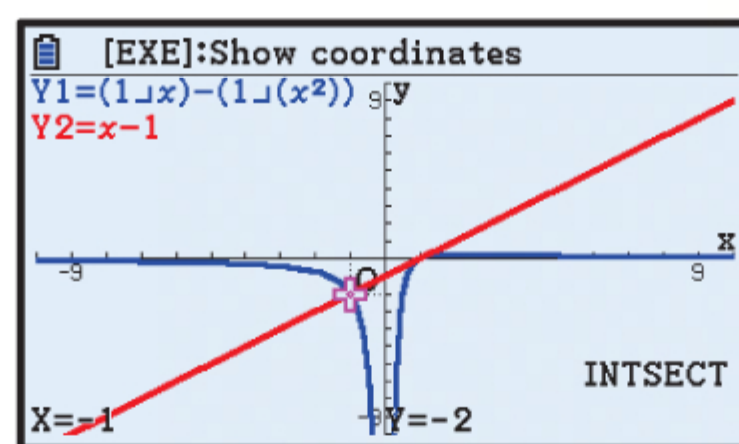
When $x = 1$, $y = \frac{1}{1} - \frac{1}{1^2} = 0$ and so the point of contact is $(1, 0)$.

Now $y = x^{-1} - x^{-2}$

$$\begin{aligned}\therefore \frac{dy}{dx} &= -x^{-2} + 2x^{-3} \\ &= -\frac{1}{x^2} + \frac{2}{x^3} \quad \text{and so at } x = 1, \\ \frac{dy}{dx} &= -\frac{1}{1^2} + \frac{2}{1^3} = 1\end{aligned}$$

So, the tangent at $(1, 0)$ has gradient 1 and its equation is $y = 1(x - 1) + 0$
 $= x - 1$

We use technology to find where the tangent meets the curve again:



The tangent meets the curve again at $(-1, -2)$.

11 a $y = 2x^3 + 3x^2 - x + 4$

$$\begin{aligned}\text{When } x = -1, \quad y &= 2(-1)^3 + 3(-1)^2 - (-1) + 4 \\ &= -2 + 3 + 1 + 4 \\ &= 6\end{aligned}$$

So, the point of contact is $(-1, 6)$.

$$\text{Now } \frac{dy}{dx} = 6x^2 + 6x - 1$$

$$\begin{aligned}\text{When } x = -1, \quad \frac{dy}{dx} &= 6(-1)^2 + 6(-1) - 1 \\ &= 6 - 6 - 1 \\ &= -1\end{aligned}$$

$$\begin{aligned}\therefore \text{ the tangent has equation } y &= -1(x - (-1)) + 6 \\ &= -(x + 1) + 6 \\ &= -x - 1 + 6 \\ &= -x + 5\end{aligned}$$

The tangent meets the x -axis where $y = 0$

$$\therefore -x + 5 = 0$$

$$\therefore x = 5$$

So, the point where the tangent meets the x -axis is $(5, 0)$.

b

$$y = x^3 + 5$$

$$\therefore \frac{dy}{dx} = 3x^2$$

At $(-2, -3)$, $\frac{dy}{dx} = 3(-2)^2 = 12$

$$\begin{aligned} \therefore \text{the tangent has equation } y &= 12(x - (-2)) - 3 \\ &= 12(x + 2) - 3 \\ &= 12x + 24 - 3 \\ &= 12x + 21 \end{aligned}$$

The tangent meets the line $y = 2$ where $12x + 21 = 2$

$$\therefore 12x = -19$$

$$\therefore x = -\frac{19}{12}$$

So, the point where the tangent meets the line $y = 2$ is $(-\frac{19}{12}, 2)$.

c

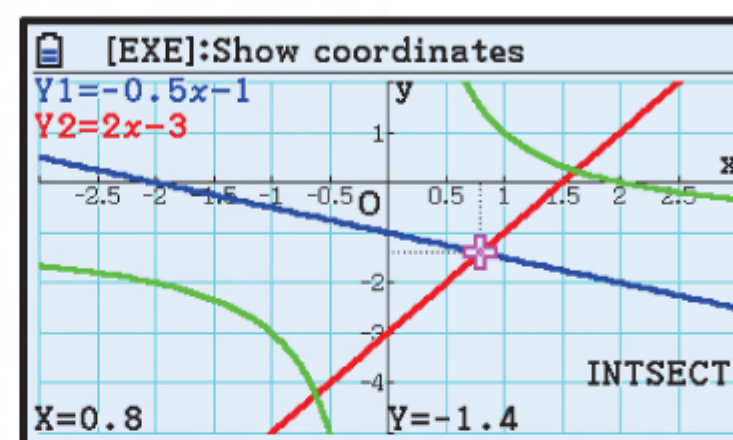
$$y = \frac{2}{x} + 1 = 2x^{-1} + 1$$

$$\therefore \frac{dy}{dx} = -2x^{-2} = -\frac{2}{x^2}$$

At $(-2, 0)$, $\frac{dy}{dx} = -\frac{2}{(-2)^2} = -\frac{2}{4} = -\frac{1}{2}$

$$\begin{aligned} \therefore \text{the tangent has equation } y &= -\frac{1}{2}(x - (-2)) + 0 \\ &= -\frac{1}{2}(x + 2) \\ &= -\frac{1}{2}x - 1 \end{aligned}$$

We use technology to find where the tangent meets the line $y = 2x - 3$:



So, the point where the tangent meets the line $y = 2x - 3$ is $(0.8, -1.4)$.

d $y = 3x^3 - 2x + 1$

$$\begin{aligned}\text{When } x = 1, \quad y &= 3(1)^3 - 2(1) + 1 \\ &= 3 - 2 + 1 \\ &= 2\end{aligned}$$

So, the point of contact is $(1, 2)$.

$$\text{Now } \frac{dy}{dx} = 9x^2 - 2$$

$$\begin{aligned}\text{When } x = 1, \quad \frac{dy}{dx} &= 9(1)^2 - 2 \\ &= 9 - 2 \\ &= 7\end{aligned}$$

$$\begin{aligned}\therefore \text{ the tangent has equation } y &= 7(x - 1) + 2 \\ &= 7x - 7 + 2 \\ &= 7x - 5\end{aligned}$$

The tangent meets the y -axis where $x = 0$

$$\begin{aligned}\therefore y &= 7(0) - 5 \\ &= -5\end{aligned}$$

So, the point where the tangent meets the y -axis is $(0, -5)$.

12 $f(x) = x^4 - 2x^3 - x^2 + 4$

$$\therefore f'(x) = 4x^3 - 6x^2 - 2x$$

$$\begin{aligned}\text{At } P(-1, 6), \quad f'(-1) &= 4(-1)^3 - 6(-1)^2 - 2(-1) \\ &= -4 - 6 + 2 \\ &= -8\end{aligned}$$

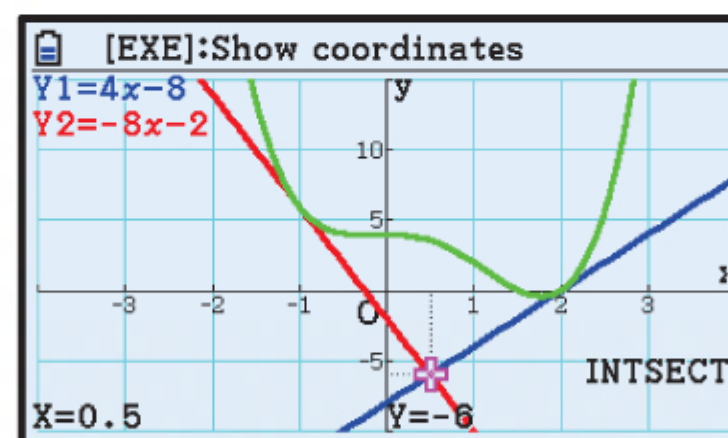
\therefore the tangent at P has equation

$$\begin{aligned}y &= -8(x - (-1)) + 6 \\ &= -8(x + 1) + 6 \\ &= -8x - 8 + 6 \\ &= -8x - 2\end{aligned}$$

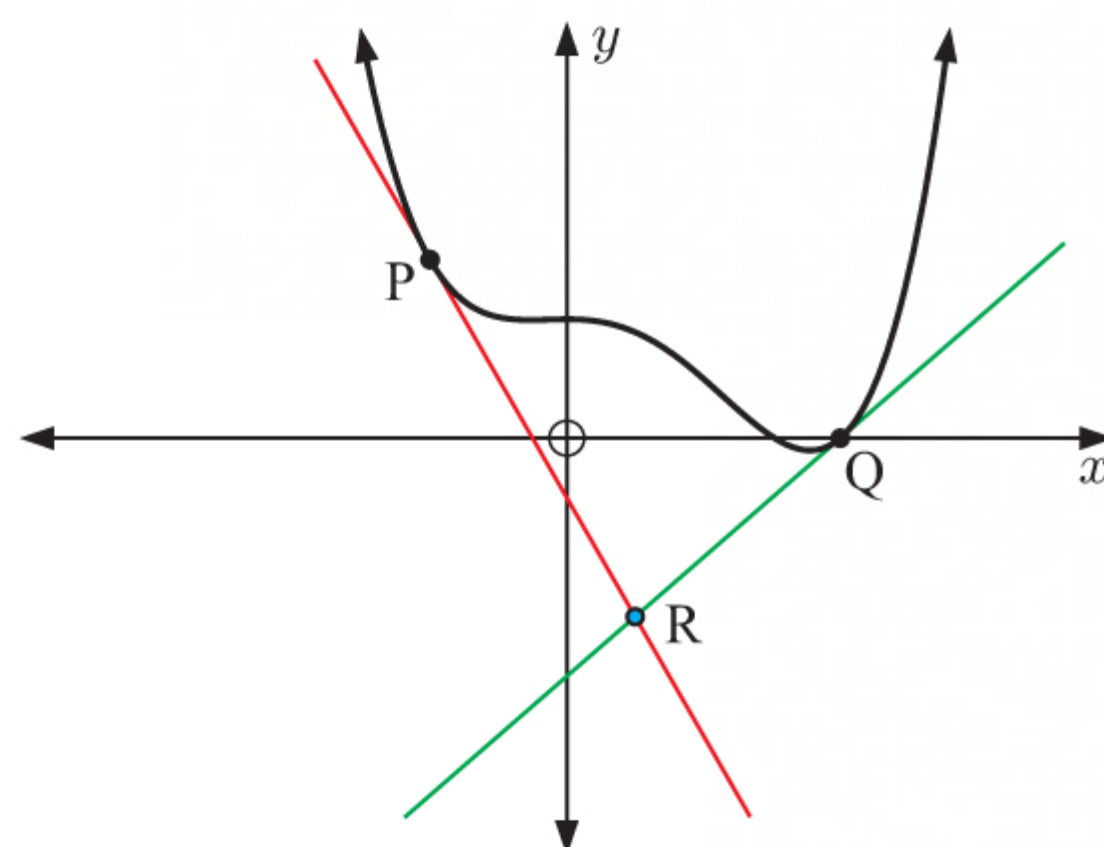
$$\begin{aligned}\text{At } Q(2, 0), \quad f'(2) &= 4(2)^3 - 6(2)^2 - 2(2) \\ &= 32 - 24 - 4 \\ &= 4\end{aligned}$$

$$\begin{aligned}\therefore \text{ the tangent at Q has equation } y &= 4(x - 2) + 0 \\ &= 4x - 8\end{aligned}$$

We use technology to find where the tangents intersect:



So, the tangents intersect at $R(0.5, -6)$.



13 $f(x) = x^4 - 2x^2 + 2x + 3$

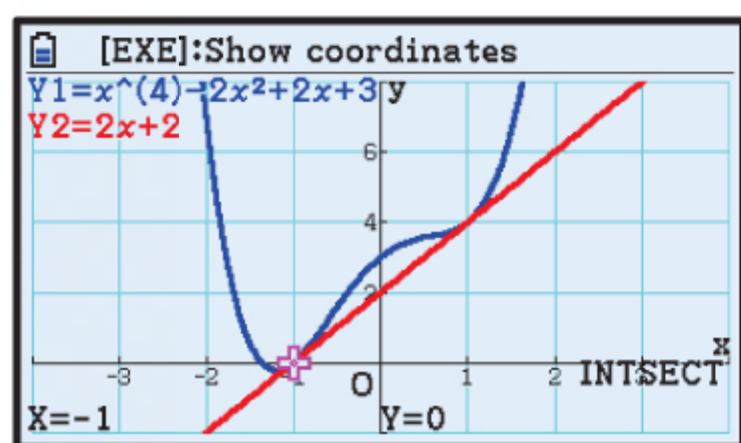
a $f(1) = 1^4 - 2(1)^2 + 2(1) + 3$
 $= 1 - 2 + 2 + 3$
 $= 4$

So, the point of contact is $(1, 4)$.

Now $f'(x) = 4x^3 - 4x + 2$
 $\therefore f'(1) = 4(1)^3 - 4(1) + 2$
 $= 4 - 4 + 2$
 $= 2$

\therefore the tangent has equation $y = 2(x - 1) + 4$
 $= 2x - 2 + 4$
 $= 2x + 2$

b We use technology to find where the tangent meets the curve again:



The tangent meets the curve again at $(-1, 0)$.

c At $(-1, 0)$, $f'(-1) = 4(-1)^3 - 4(-1) + 2$
 $= -4 + 4 + 2$
 $= 2$

\therefore the tangent has equation $y = 2(x - (-1)) + 0$
 $= 2(x + 1)$
 $= 2x + 2$

So, $y = 2x + 2$ is the tangent to the curve at the point $(-1, 0)$ also.

EXERCISE 11B**1 a**

$$y = x^2$$

$$\therefore \frac{dy}{dx} = 2x$$

$$\text{When } x = 3, \quad \frac{dy}{dx} = 2(3) = 6$$

So, the normal at $(3, 9)$ has gradient $-\frac{1}{6}$.

\therefore the normal has equation

$$\begin{aligned} y &= -\frac{1}{6}(x - 3) + 9 \\ &= -\frac{1}{6}x + \frac{1}{2} + 9 \\ &= -\frac{1}{6}x + \frac{19}{2} \end{aligned}$$

c

$$y = \frac{1}{x} + 2$$

$$= x^{-1} + 2$$

$$\therefore \frac{dy}{dx} = -x^{-2}$$

$$= -\frac{1}{x^2}$$

$$\begin{aligned} \text{When } x = -1, \quad \frac{dy}{dx} &= -\frac{1}{(-1)^2} \\ &= -1 \end{aligned}$$

So, the normal at $(-1, 1)$ has gradient 1.

\therefore the normal has equation

$$\begin{aligned} y &= 1(x - (-1)) + 1 \\ &= x + 1 + 1 \\ &= x + 2 \end{aligned}$$

$$\mathbf{b} \quad y = x^3 - 5x + 2$$

$$\begin{aligned} \text{When } x = -2, \quad y &= (-2)^3 - 5(-2) + 2 \\ &= -8 + 10 + 2 \\ &= 4 \end{aligned}$$

So, the point of contact is $(-2, 4)$.

$$\text{Now } \frac{dy}{dx} = 3x^2 - 5$$

$$\begin{aligned} \text{When } x = -2, \quad \frac{dy}{dx} &= 3(-2)^2 - 5 \\ &= 12 - 5 \\ &= 7 \end{aligned}$$

So, the normal at $(-2, 4)$ has gradient $-\frac{1}{7}$.

\therefore the normal has equation

$$\begin{aligned} y &= -\frac{1}{7}(x - (-2)) + 4 \\ &= -\frac{1}{7}(x + 2) + 4 \\ &= -\frac{1}{7}x - \frac{2}{7} + 4 \\ &= -\frac{1}{7}x + \frac{26}{7} \end{aligned}$$

$$\mathbf{d} \quad y = 2x^3 - 3x + 1$$

$$\begin{aligned} \text{When } x = 1, \quad y &= 2(1)^3 - 3(1) + 1 \\ &= 2 - 3 + 1 \\ &= 0 \end{aligned}$$

So, the point of contact is $(1, 0)$.

$$\text{Now } \frac{dy}{dx} = 6x^2 - 3$$

$$\begin{aligned} \text{When } x = 1, \quad \frac{dy}{dx} &= 6(1)^2 - 3 \\ &= 6 - 3 \\ &= 3 \end{aligned}$$

So, the normal at $(1, 0)$ has gradient $-\frac{1}{3}$.

\therefore the normal has equation

$$\begin{aligned} y &= -\frac{1}{3}(x - 1) + 0 \\ &= -\frac{1}{3}x + \frac{1}{3} \end{aligned}$$

e $y = x^2 - 3x + 2$

$$\begin{aligned}\text{When } x = 3, \quad y &= 3^2 - 3(3) + 2 \\ &= 9 - 9 + 2 \\ &= 2\end{aligned}$$

So, the point of contact is $(3, 2)$.

$$\text{Now } \frac{dy}{dx} = 2x - 3$$

$$\begin{aligned}\text{When } x = 3, \quad \frac{dy}{dx} &= 2(3) - 3 \\ &= 6 - 3 \\ &= 3\end{aligned}$$

So, the normal at $(3, 2)$ has gradient $-\frac{1}{3}$.

\therefore the normal has equation

$$\begin{aligned}y &= -\frac{1}{3}(x - 3) + 2 \\ &= -\frac{1}{3}x + 1 + 2 \\ &= -\frac{1}{3}x + 3\end{aligned}$$

f $y = 3x + \frac{1}{x} - 4 = 3x + x^{-1} - 4$

$$\begin{aligned}\text{When } x = 1, \quad y &= 3(1) + \frac{1}{1} - 4 \\ &= 3 + 1 - 4 \\ &= 0\end{aligned}$$

So, the point of contact is $(1, 0)$.

$$\begin{aligned}\text{Now } \frac{dy}{dx} &= 3 - x^{-2} \\ &= 3 - \frac{1}{x^2}\end{aligned}$$

$$\begin{aligned}\text{When } x = 1, \quad \frac{dy}{dx} &= 3 - \frac{1}{1^2} \\ &= 3 - 1 \\ &= 2\end{aligned}$$

So, the normal at $(1, 0)$ has gradient $-\frac{1}{2}$.

\therefore the normal has equation

$$\begin{aligned}y &= -\frac{1}{2}(x - 1) + 0 \\ &= -\frac{1}{2}x + \frac{1}{2}\end{aligned}$$

2 $y = \frac{2}{x} = 2x^{-1}$

$$\begin{aligned}\therefore \frac{dy}{dx} &= -2x^{-2} \\ &= -\frac{2}{x^2}\end{aligned}$$

a i At $P(2, 1)$, $\frac{dy}{dx} = -\frac{2}{2^2}$

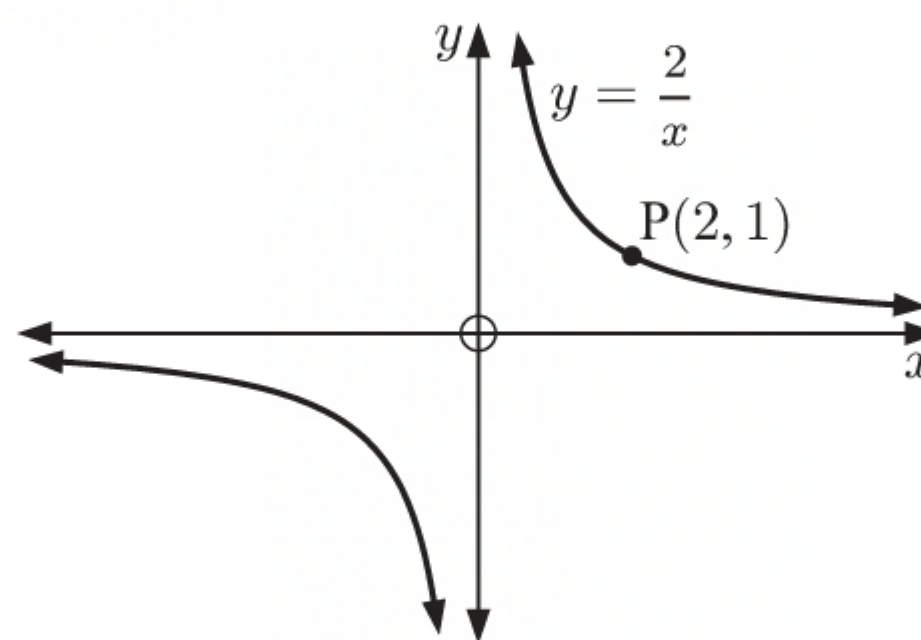
$$= -\frac{2}{4} = -\frac{1}{2}$$

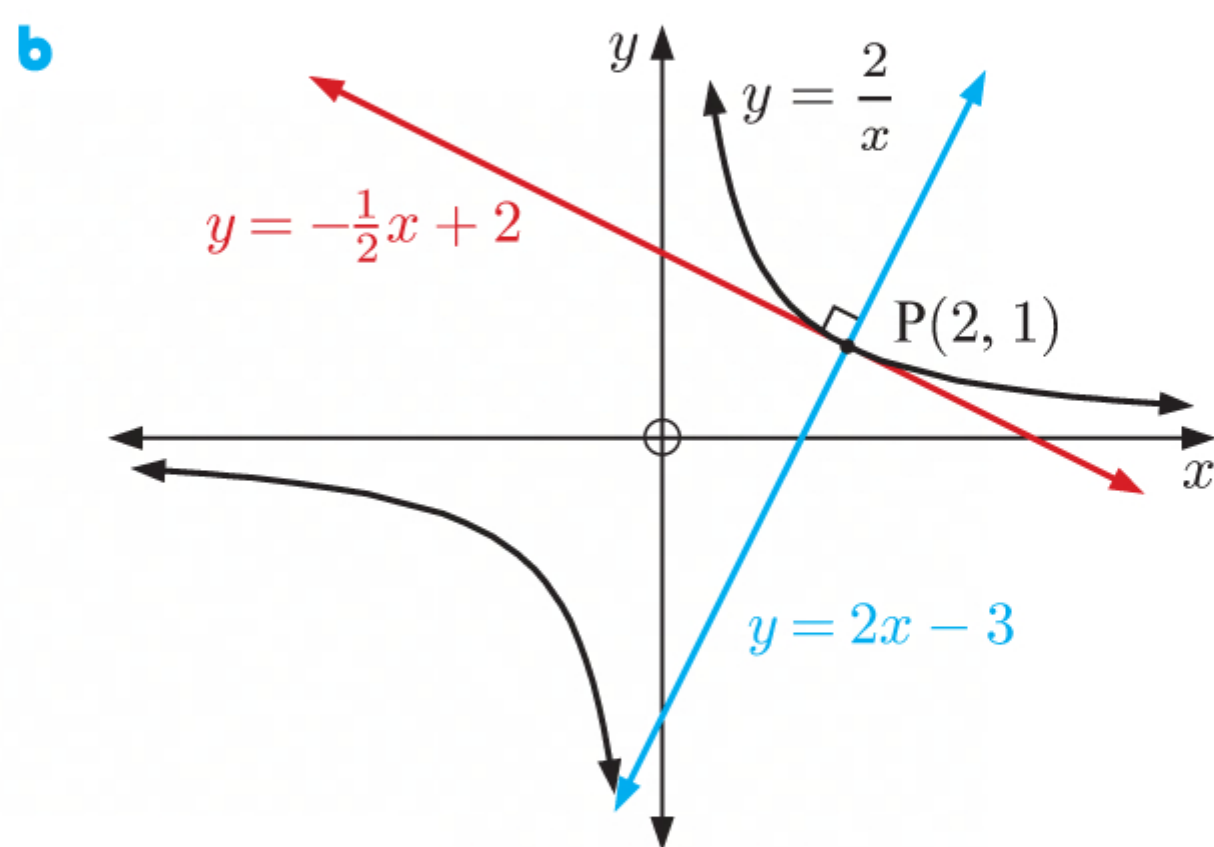
\therefore the tangent at P has equation

$$\begin{aligned}y &= -\frac{1}{2}(x - 2) + 1 \\ &= -\frac{1}{2}x + 1 + 1 \\ &= -\frac{1}{2}x + 2\end{aligned}$$

ii The normal at $P(2, 1)$ has gradient 2.

$$\begin{aligned}\therefore \text{ the normal has equation } y &= 2(x - 2) + 1 \\ &= 2x - 4 + 1 \\ &= 2x - 3\end{aligned}$$





3 $f(x) = x^2 - \frac{8}{x} = x^2 - 8x^{-1}$
 $\therefore f'(x) = 2x + 8x^{-2}$
 $= 2x + \frac{8}{x^2}$

a $f(-2) = (-2)^2 - \frac{8}{(-2)}$
 $= 4 + 4$
 $= 8$

So, the point of contact is $(-2, 8)$.

Now $f'(-2) = 2(-2) + \frac{8}{(-2)^2}$
 $= -4 + 2$
 $= -2$

\therefore the equation of the tangent is

$$\begin{aligned} y &= -2(x - (-2)) + 8 \\ &= -2(x + 2) + 8 \\ &= -2x - 4 + 8 \\ &= -2x + 4 \end{aligned}$$

4 $f(x) = (x - 1)(x - 4)$

a $f(x) = 0$ when $(x - 1)(x - 4) = 0$
 $\therefore x = 1$ or 4

\therefore the x -intercepts are 1 and 4.

Now $f(0) = (0 - 1)(0 - 4) = 4$

\therefore the y -intercept is 4.

b $f(x) = (x - 1)(x - 4)$
 $= x^2 - 5x + 4$
 $\therefore f'(x) = 2x - 5$

b $f(3) = 3^2 - \frac{8}{3}$
 $= 9 - \frac{8}{3}$
 $= \frac{19}{3}$

So, the point of contact is $(3, \frac{19}{3})$.

Now $f'(3) = 2(3) + \frac{8}{3^2}$ {from **a**}
 $= 6 + \frac{8}{9}$
 $= \frac{62}{9}$

So, the normal at $(3, \frac{19}{3})$ has gradient $-\frac{9}{62}$.

\therefore the equation of the normal is

$$\begin{aligned} y &= -\frac{9}{62}(x - 3) + \frac{19}{3} \\ &= -\frac{9}{62}x + \frac{27}{62} + \frac{19}{3} \\ &= -\frac{9}{62}x + \frac{1259}{186} \end{aligned}$$

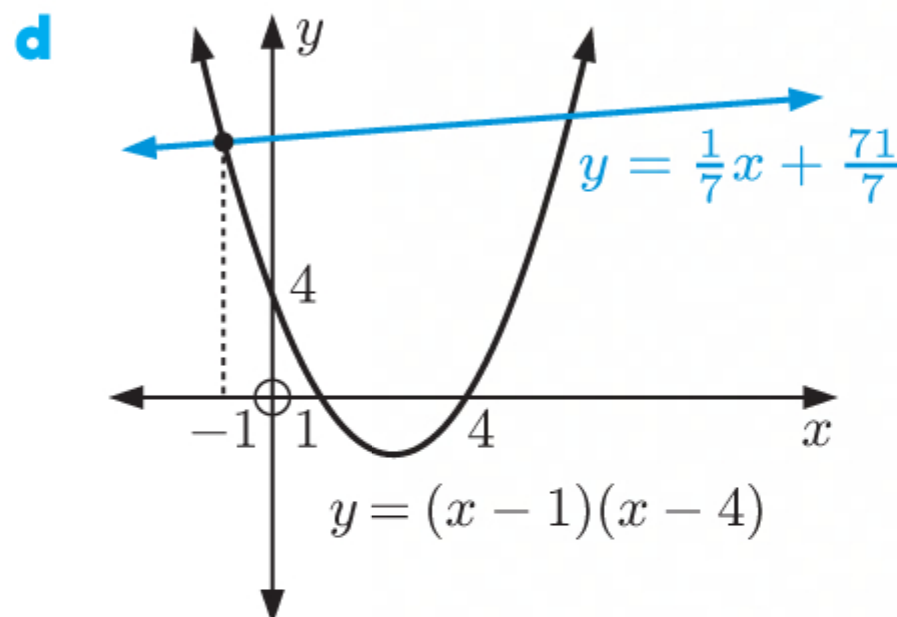
c $f(-1) = (-1 - 1)(-1 - 4) = 10$

So the point of contact is $(-1, 10)$.

$$f'(-1) = 2(-1) - 5 = -7$$

So, the normal at $(-1, 10)$ has gradient $\frac{1}{7}$.

$$\begin{aligned}\therefore \text{ the normal has equation } y &= \frac{1}{7}(x - (-1)) + 10 \\ &= \frac{1}{7}(x + 1) + 10 \\ &= \frac{1}{7}x + \frac{1}{7} + 10 \\ &= \frac{1}{7}x + \frac{71}{7}\end{aligned}$$



5 a $y = x^2$

$$\therefore \frac{dy}{dx} = 2x$$

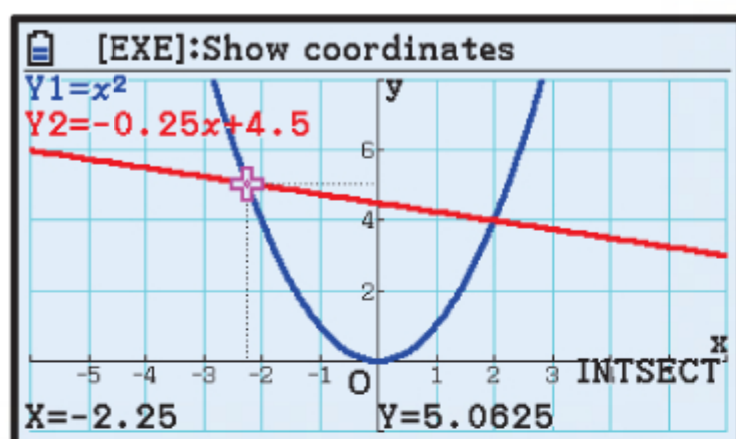
So when $x = 2$, $\frac{dy}{dx} = 2(2) = 4$

So, the normal at $(2, 4)$ has gradient $-\frac{1}{4}$.

\therefore the normal has equation

$$\begin{aligned}y &= -\frac{1}{4}(x - 2) + 4 \\ &= -\frac{1}{4}x + \frac{1}{2} + 4 \\ &= -\frac{1}{4}x + \frac{9}{2}\end{aligned}$$

We use technology to find where the normal meets the curve again:



The normal meets the curve again at $(-2.25, 5.0625)$.

b $y = \frac{1}{x} + 2$

$$= x^{-1} + 2$$

$$\therefore \frac{dy}{dx} = -x^{-2} = -\frac{1}{x^2}$$

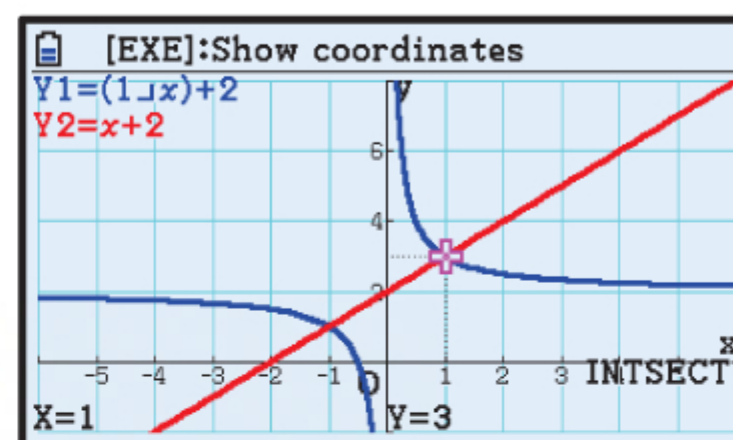
So when $x = -1$, $\frac{dy}{dx} = -\frac{1}{(-1)^2} = -1$

So, the normal at $(-1, 1)$ has gradient 1.

\therefore the normal has equation

$$\begin{aligned}y &= 1(x - (-1)) + 1 \\ &= x + 1 + 1 \\ &= x + 2\end{aligned}$$

We use technology to find where the normal meets the curve again:



The normal meets the curve again at $(1, 3)$.

c $y = x^3$

When $x = -1$, $y = (-1)^3$
 $= -1$

So, the point of contact is $(-1, -1)$.

Now $\frac{dy}{dx} = 3x^2$

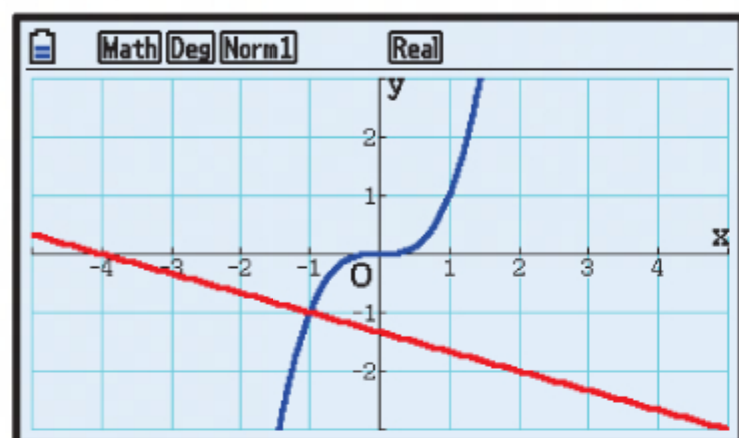
So when $x = -1$, $\frac{dy}{dx} = 3(-1)^2 = 3$

So, the normal at $(-1, -1)$ has gradient $-\frac{1}{3}$.

\therefore the normal has equation

$$\begin{aligned} y &= -\frac{1}{3}(x - (-1)) - 1 \\ &= -\frac{1}{3}(x + 1) - 1 \\ &= -\frac{1}{3}x - \frac{1}{3} - 1 \\ &= -\frac{1}{3}x - \frac{4}{3} \end{aligned}$$

We use technology to find where the normal meets the curve again:



The normal does not meet the curve again.

d $y = x^3 - 12x + 2$

$\therefore \frac{dy}{dx} = 3x^2 - 12$

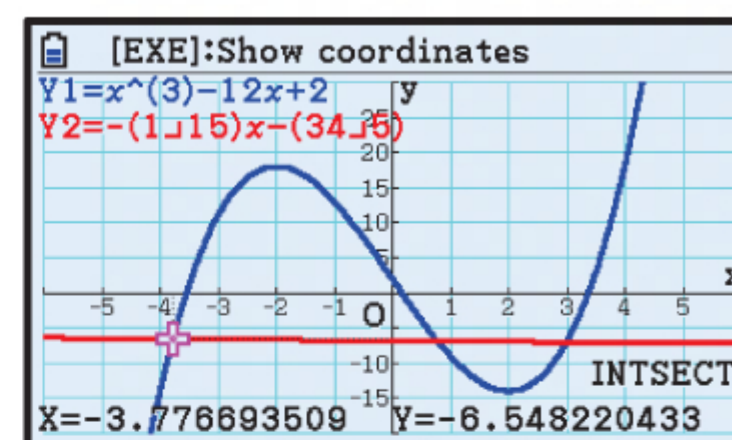
So when $x = 3$, $\frac{dy}{dx} = 3(3)^2 - 12$
 $= 27 - 12$
 $= 15$

So, the normal at $(3, -7)$ has gradient $-\frac{1}{15}$.

\therefore the normal has equation

$$\begin{aligned} y &= -\frac{1}{15}(x - 3) - 7 \\ &= -\frac{1}{15}x + \frac{1}{5} - 7 \\ &= -\frac{1}{15}x - \frac{34}{5} \end{aligned}$$

We use technology to find where the normal meets the curve again:



The normal meets the curve again at about $(-3.78, -6.55)$ and about $(0.777, -6.85)$.

6 a $y = x^3 - 12x + 2$

$$\begin{aligned}\text{When } x = -2, \quad y &= (-2)^3 - 12(-2) + 2 \\ &= -8 + 24 + 2 \\ &= 18\end{aligned}$$

So, the point of contact is $(-2, 18)$.

$$\text{Now } \frac{dy}{dx} = 3x^2 - 12$$

$$\begin{aligned}\text{So when } x = -2, \quad \frac{dy}{dx} &= 3(-2)^2 - 12 \\ &= 12 - 12 \\ &= 0\end{aligned}$$

So, the gradient of the normal at $(-2, 18)$ is undefined.

\therefore the normal must be a vertical line.

Since the normal passes through $(-2, 18)$, it has equation $x = -2$.

The normal meets the x -axis at $(-2, 0)$.

c $y = \frac{1}{x} - 3$
 $= x^{-1} - 3$

$$\therefore \frac{dy}{dx} = -x^{-2} = -\frac{1}{x^2}$$

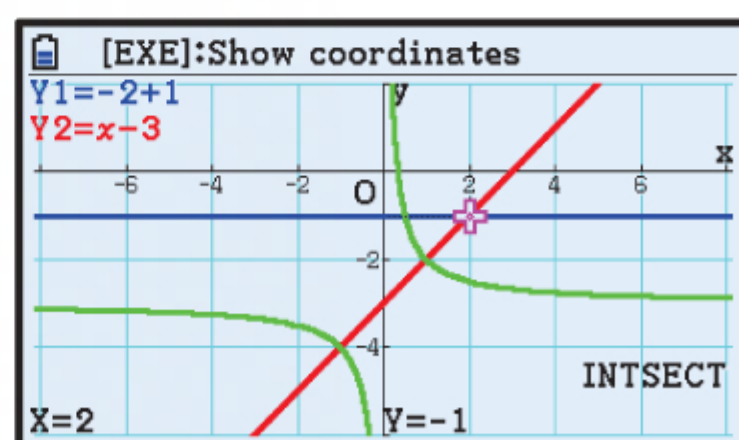
$$\text{So when } x = -1, \quad \frac{dy}{dx} = -\frac{1}{(-1)^2} = -1$$

So, the normal at $(-1, -4)$ has gradient 1.

\therefore the normal has equation

$$\begin{aligned}y &= 1(x - (-1)) - 4 \\ &= x + 1 - 4 \\ &= x - 3\end{aligned}$$

We use technology to find where the normal meets the line $y = -2x + 1$:



The normal meets the line $y = -2x + 1$ at about $(1.33, -1.67)$.

b $y = x^3$

$$\therefore \frac{dy}{dx} = 3x^2$$

$$\begin{aligned}\text{So when } x = -1, \quad \frac{dy}{dx} &= 3(-1)^2 \\ &= 3\end{aligned}$$

So, the normal at $(-1, -1)$ has gradient $-\frac{1}{3}$.

\therefore the normal has equation

$$\begin{aligned}y &= -\frac{1}{3}(x - (-1)) - 1 \\ &= -\frac{1}{3}(x + 1) - 1 \\ &= -\frac{1}{3}x - \frac{1}{3} - 1 \\ &= -\frac{1}{3}x - \frac{4}{3}\end{aligned}$$

The normal meets the line $y = 3$ where

$$\begin{aligned}-\frac{1}{3}x - \frac{4}{3} &= 3 \\ \therefore -\frac{1}{3}x &= \frac{13}{3} \\ \therefore x &= -13\end{aligned}$$

The normal meets the line $y = 3$ at $(-13, 3)$.

d $y = 2x^3 - 3x + 1$

$$\begin{aligned}\text{When } x = 1, \quad y &= 2(1)^3 - 3(1) + 1 \\ &= 2 - 3 + 1 \\ &= 0\end{aligned}$$

So, the point of contact is $(1, 0)$.

$$\text{Now } \frac{dy}{dx} = 6x^2 - 3$$

$$\begin{aligned}\text{So when } x = 1, \quad \frac{dy}{dx} &= 6(1)^2 - 3 \\ &= 6 - 3 \\ &= 3\end{aligned}$$

So, the normal at $(1, 0)$ has gradient $-\frac{1}{3}$.

\therefore the normal has equation

$$\begin{aligned}y &= -\frac{1}{3}(x - 1) + 0 \\ &= -\frac{1}{3}x + \frac{1}{3}\end{aligned}$$

The normal meets the y -axis where $x = 0$

$$\begin{aligned}\therefore y &= -\frac{1}{3}(0) + \frac{1}{3} \\ &= \frac{1}{3}\end{aligned}$$

So, the normal meets the y -axis at $(0, \frac{1}{3})$.

$$\begin{aligned}
 7 \quad y &= 5 - \frac{a}{x} \\
 &= 5 - ax^{-1} \\
 \therefore \frac{dy}{dx} &= ax^{-2} \\
 &= \frac{a}{x^2}
 \end{aligned}$$

So when $x = -2$, $\frac{dy}{dx} = \frac{a}{(-2)^2} = \frac{a}{4}$

So, the normal at the point where $x = -2$ has gradient $-\frac{4}{a}$.

$$\begin{aligned}
 \therefore -\frac{4}{a} &= 1 \\
 \therefore a &= -4
 \end{aligned}$$

$$8 \quad \text{a} \quad \text{i} \quad y = \frac{1}{2}x^3 - x^2 + ax + 4$$

$$\therefore \frac{dy}{dx} = \frac{3}{2}x^2 - 2x + a$$

At point P, with x -coordinate 2, the gradient is 1.

$$\therefore \frac{3}{2}(2)^2 - 2(2) + a = 1$$

$$\therefore 6 - 4 + a = 1$$

$$\therefore 2 + a = 1$$

$$\therefore a = -1$$

$$\text{ii} \quad y = \frac{1}{2}x^3 - x^2 - x + 4$$

$$\text{When } x = 2, \quad y = \frac{1}{2}(2)^3 - 2^2 - 2 + 4$$

$$= \frac{8}{2} - 4 - 2 + 4$$

$$= 4 - 2$$

$$= 2$$

So, the coordinates of P are (2, 2).

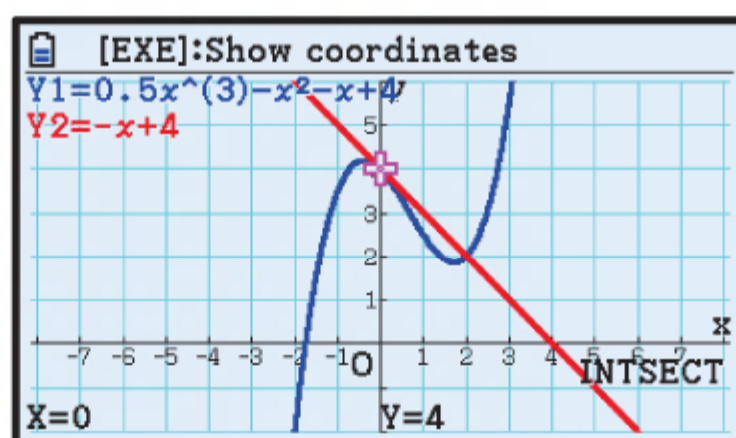
b The normal at P(2, 2) has gradient -1 .

$$\therefore \text{the normal has equation } y = -1(x - 2) + 2$$

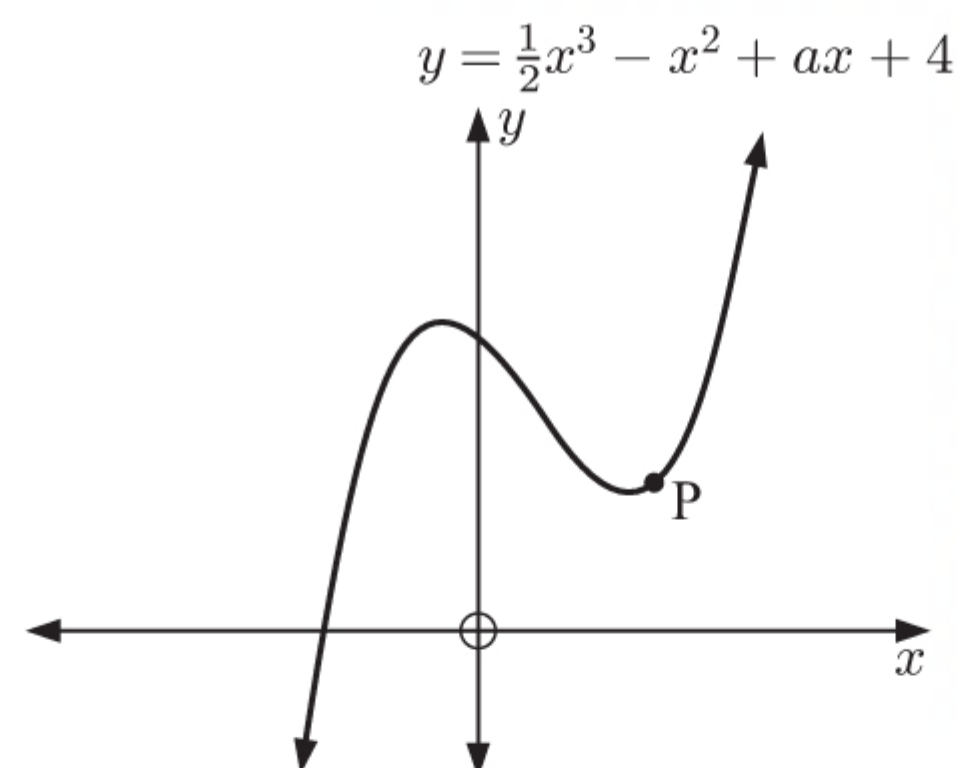
$$= -x + 2 + 2$$

$$= -x + 4$$

c We use technology to find where the normal meets the curve again:



The normal meets the curve again at the point Q(0, 4).



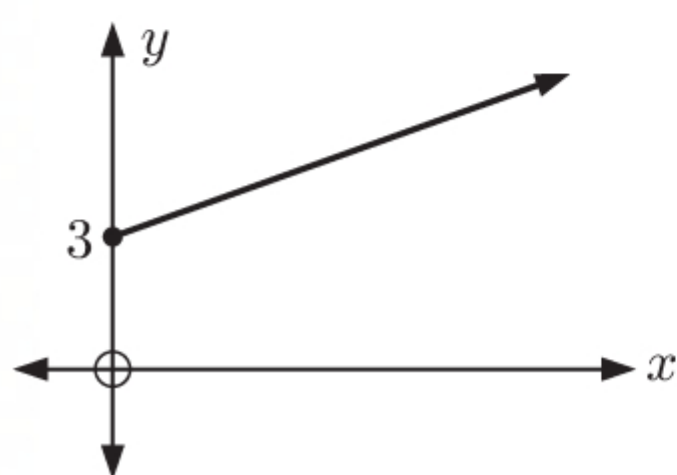
d When $x = 0$, $\frac{dy}{dx} = \frac{3}{2}(0)^2 - 2(0) - 1$
 $= -1$

\therefore the tangent has equation $y = -1(x - 0) + 4$
 $= -x + 4$

This is the same line as the normal to the curve at P.

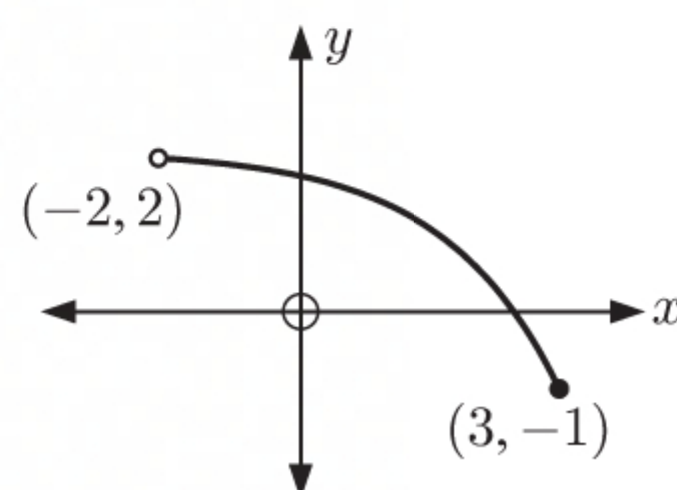
EXERCISE 11C

1 a



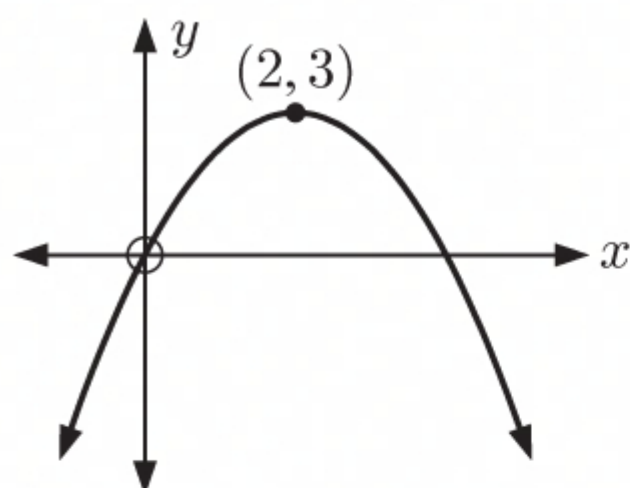
- i** The graph is increasing for $x \geq 0$.
- ii** The graph is never decreasing.

b



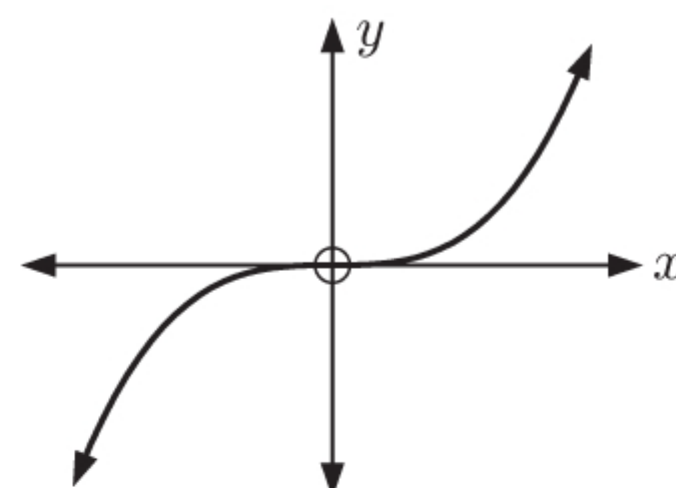
- i** The graph is never increasing.
- ii** The graph is decreasing for $-2 < x \leq 3$.

c



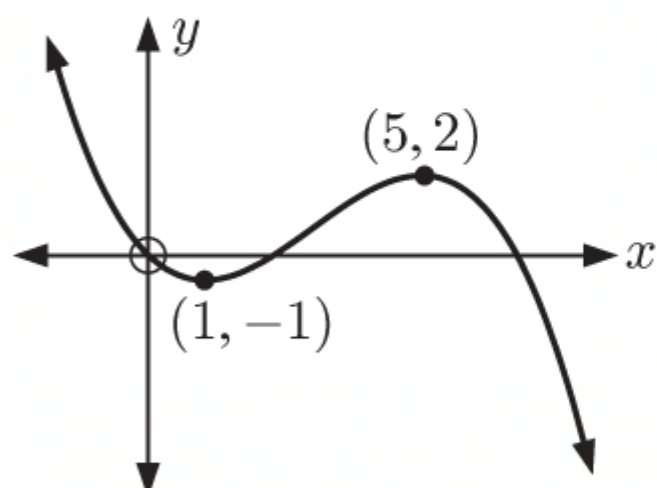
- i** The graph is increasing for $x \leq 2$.
- ii** The graph is decreasing for $x \geq 2$.

d



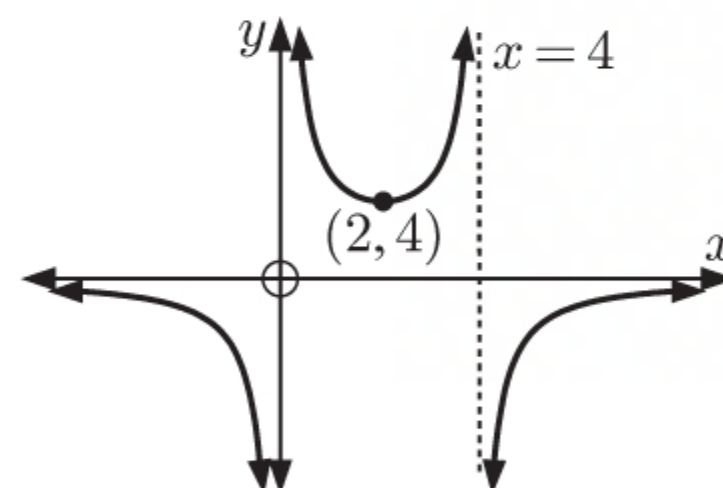
- i** The graph is increasing for all real x .
- ii** The graph is never decreasing.

e

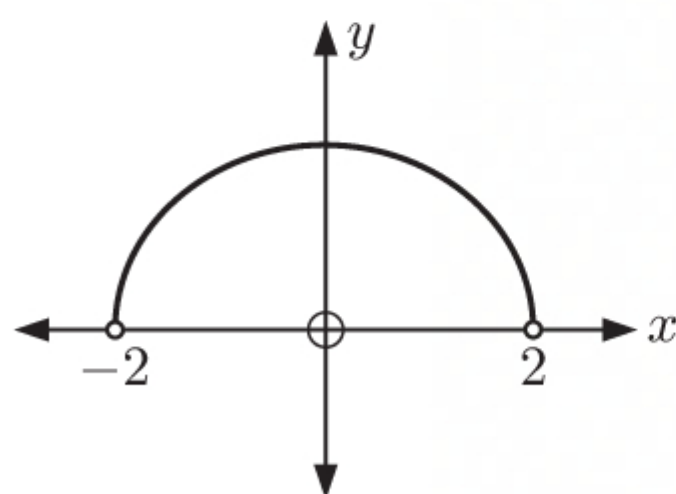


- i** The graph is increasing for $1 \leq x \leq 5$.
- ii** The graph is decreasing for $x \leq 1$ and $x \geq 5$.

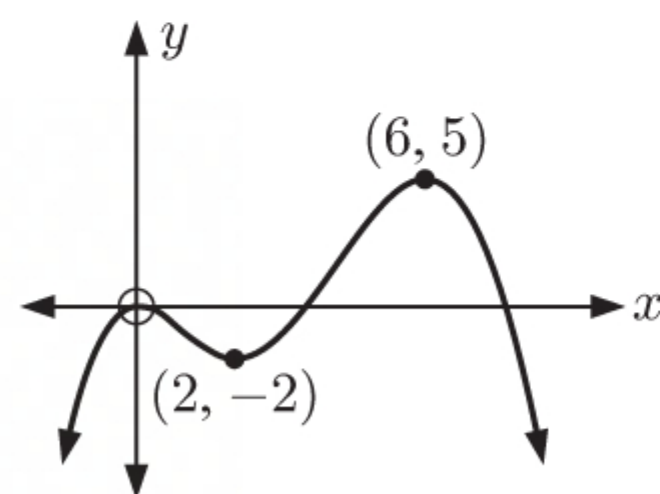
f



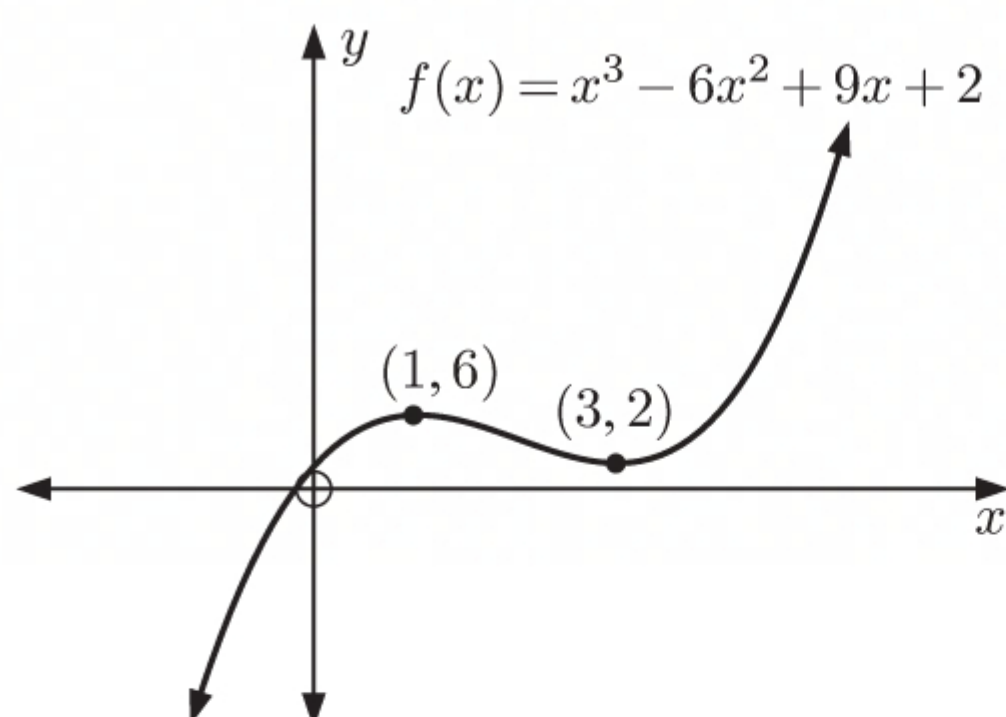
- i** The graph is increasing for $2 \leq x < 4$ and $x > 4$.
- ii** The graph is decreasing for $x < 0$ and $0 < x \leq 2$.

g

- i** The graph is increasing for $-2 < x \leq 0$.
- ii** The graph is decreasing for $0 \leq x < 2$.

h

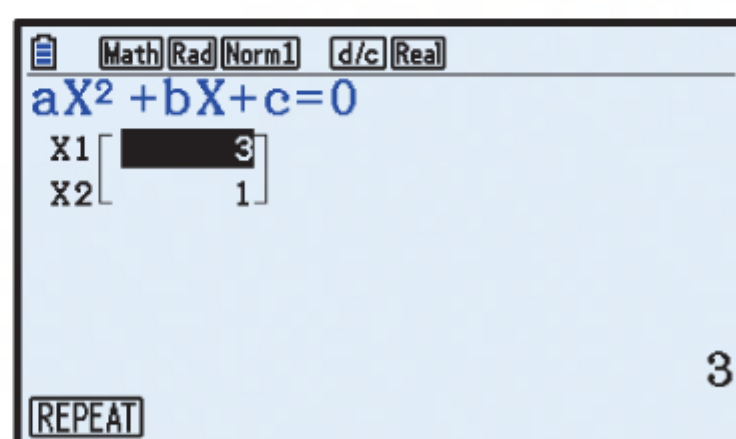
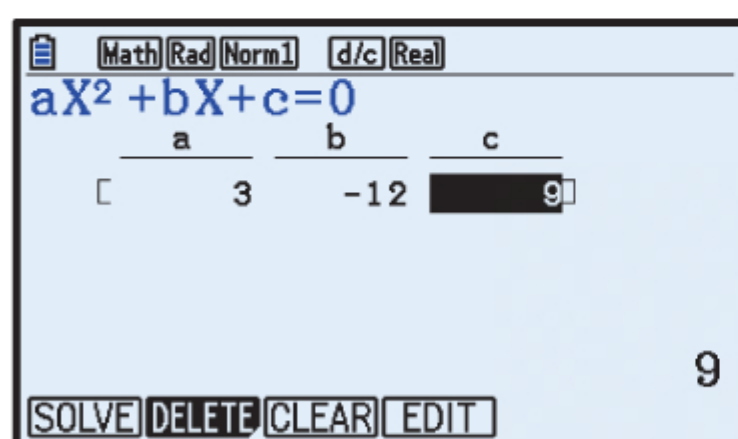
- i** The graph is increasing for $x \leq 0$ and $2 \leq x \leq 6$.
- ii** The graph is decreasing for $0 \leq x \leq 2$ and $x \geq 6$.

2 a

- i** The function is increasing for $x \leq 1$ and $x \geq 3$.
- ii** The function is decreasing for $1 \leq x \leq 3$.

b $f(x) = x^3 - 6x^2 + 9x + 2$
 $\therefore f'(x) = 3x^2 - 12x + 9$

We find the zeros of $f'(x)$ using technology:



So, $f'(x) = 0$ when $x = 1$ or 3 .

$f'(x)$ has sign diagram: $\begin{array}{c} + \quad - \quad + \\ \leftarrow \quad 1 \quad \quad 3 \quad \rightarrow x \end{array}$

So, $f(x)$ is increasing for $x \leq 1$ and $x \geq 3$, and decreasing for $1 \leq x \leq 3$.

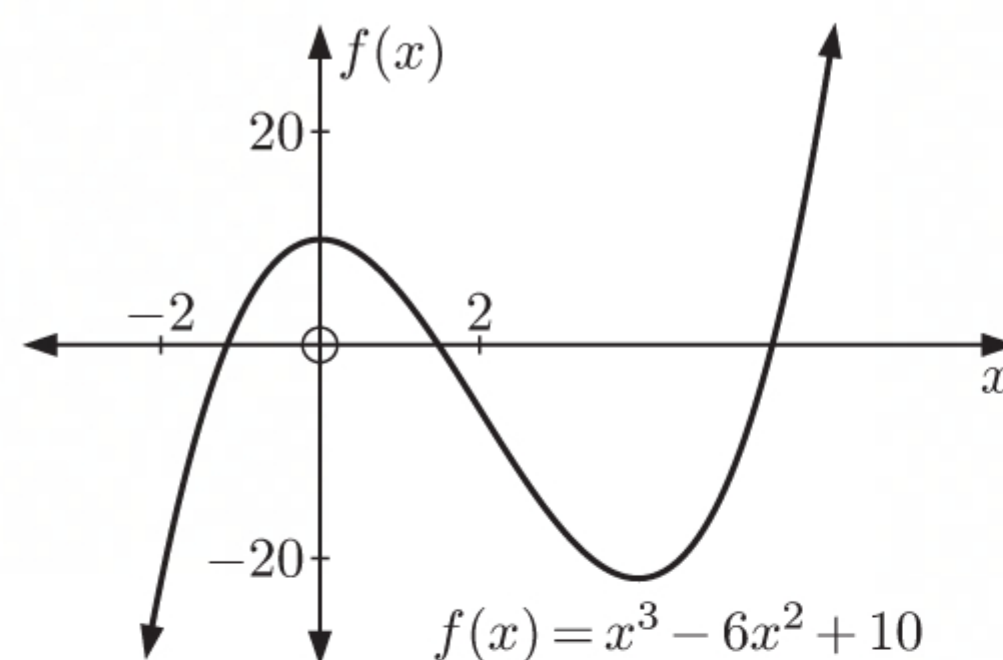
3 a

$f(x) = x^3 - 6x^2 + 10$
 $\therefore f'(x) = 3x^2 - 12x$
 $= 3x(x - 4)$

which has sign diagram:

$\begin{array}{c} + \quad - \quad + \\ \leftarrow \quad 0 \quad \quad 4 \quad \rightarrow x \end{array}$

- b** $f(x)$ is increasing for $x \leq 0$ and for $x \geq 4$.
 $f(x)$ is decreasing for $0 \leq x \leq 4$.



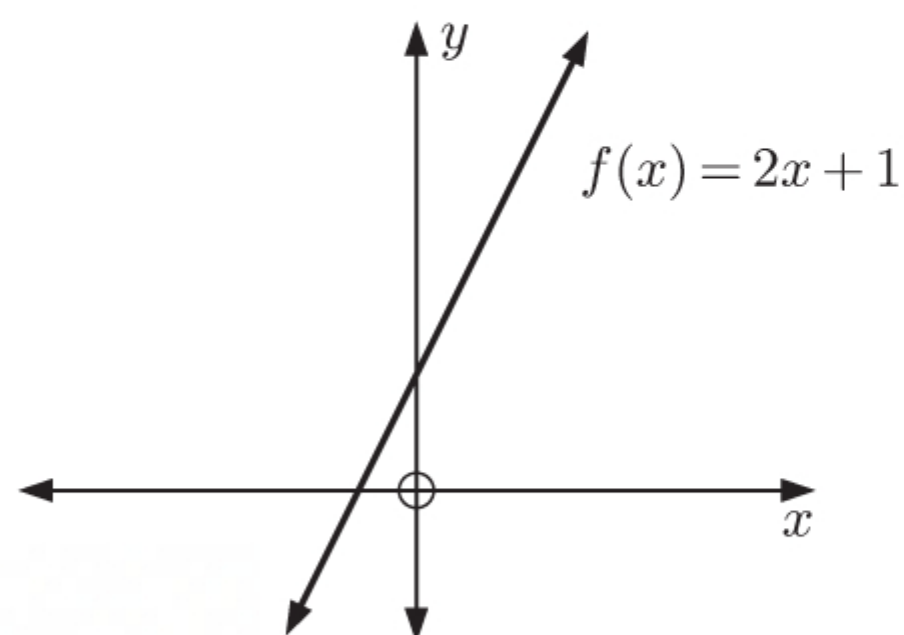
4 a $f(x) = 2x + 1$

$\therefore f'(x) = 2$

which has sign diagram:



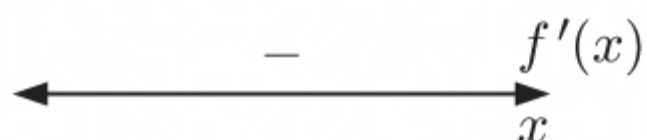
So, $f(x)$ is increasing for all $x \in \mathbb{R}$.



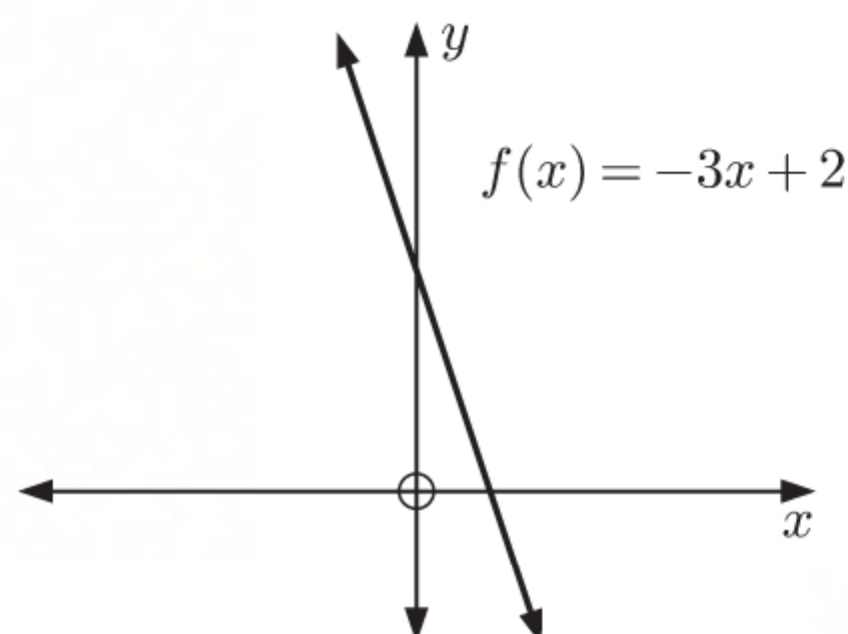
b $f(x) = -3x + 2$

$\therefore f'(x) = -3$

which has sign diagram:



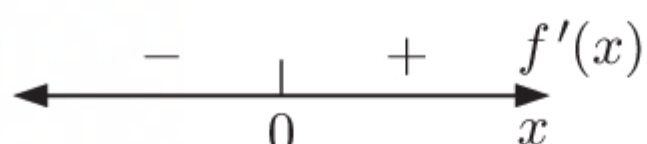
So, $f(x)$ is decreasing for all $x \in \mathbb{R}$.



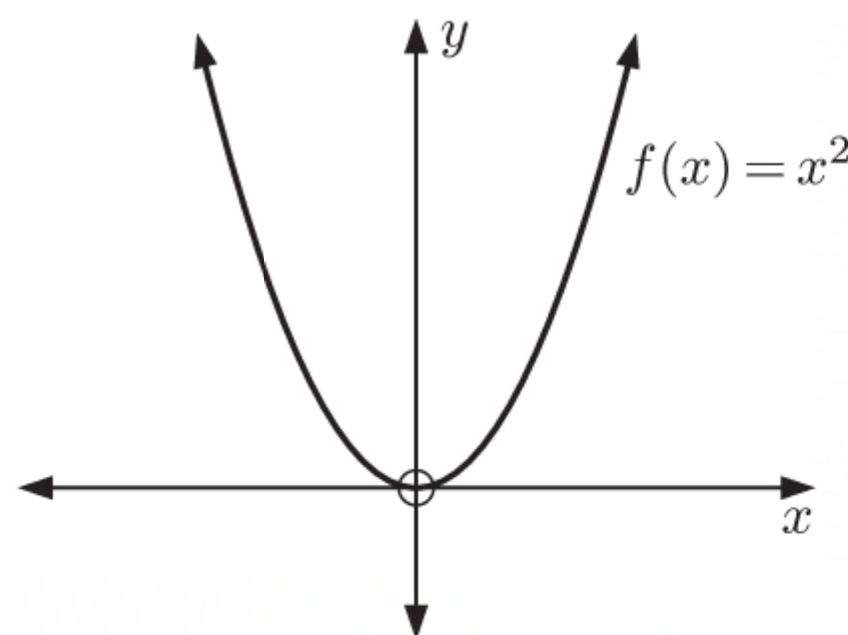
c $f(x) = x^2$

$\therefore f'(x) = 2x$

which has sign diagram:



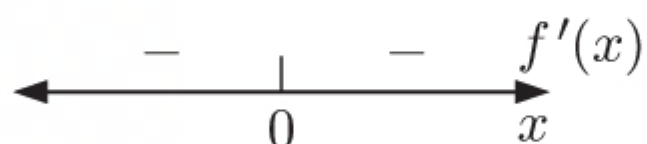
So, $f(x)$ is increasing for $x \geq 0$,
and decreasing for $x \leq 0$.



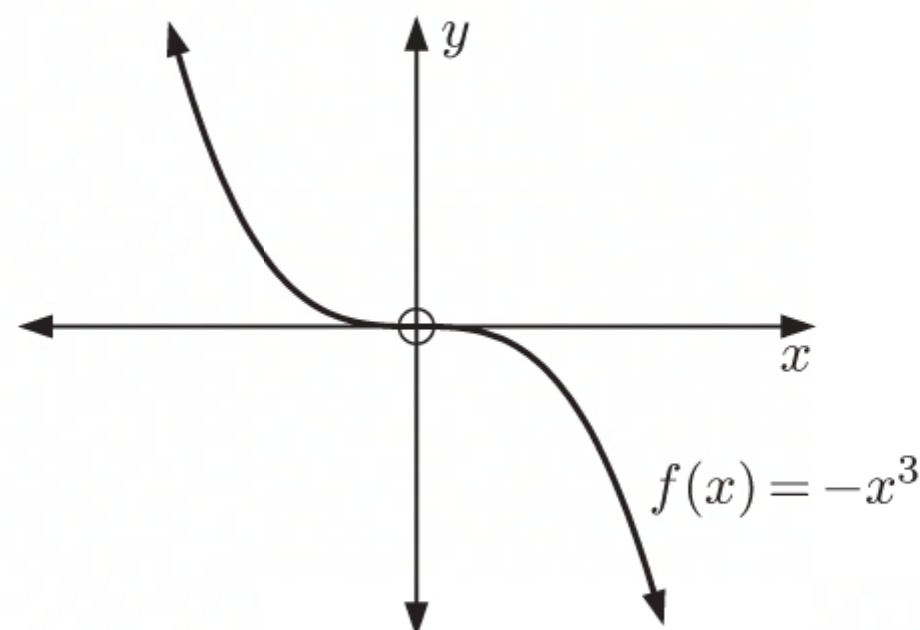
d $f(x) = -x^3$

$\therefore f'(x) = -3x^2$

which has sign diagram:



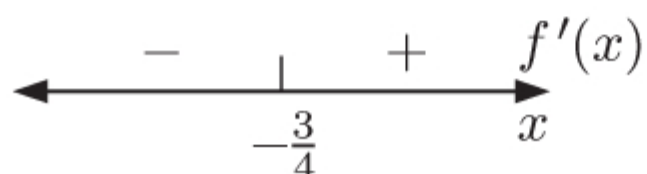
So, $f(x)$ is decreasing for all $x \in \mathbb{R}$.



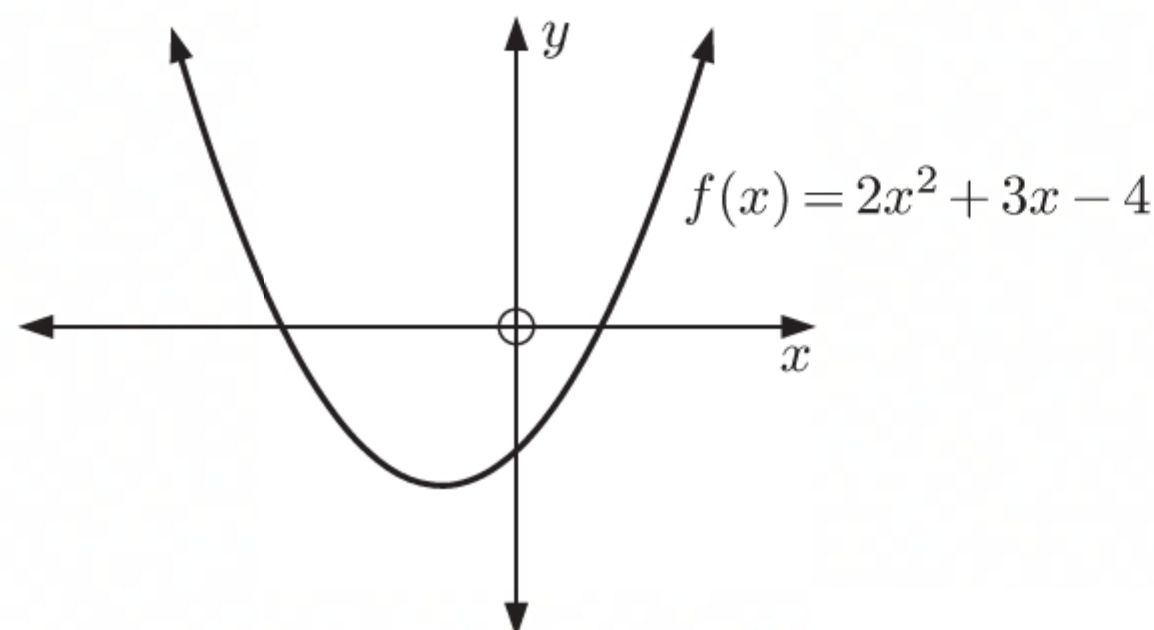
e $f(x) = 2x^2 + 3x - 4$

$\therefore f'(x) = 4x + 3$

which has sign diagram:

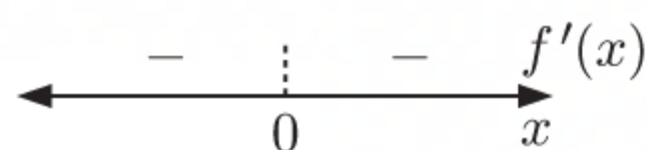


So, $f(x)$ is increasing for $x \geq -\frac{3}{4}$,
and decreasing for $x \leq -\frac{3}{4}$.

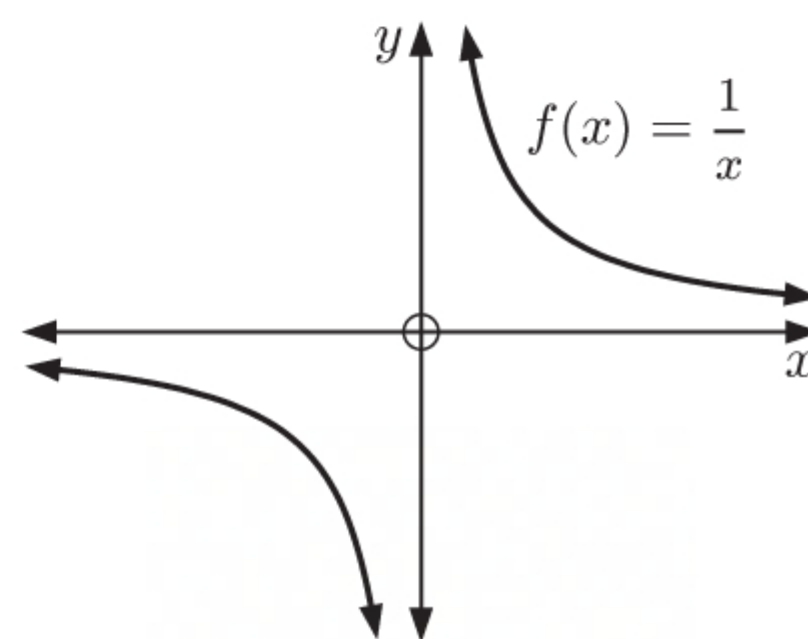


f $f(x) = \frac{1}{x} = x^{-1}$
 $\therefore f'(x) = -x^{-2} = -\frac{1}{x^2}$

which has sign diagram:

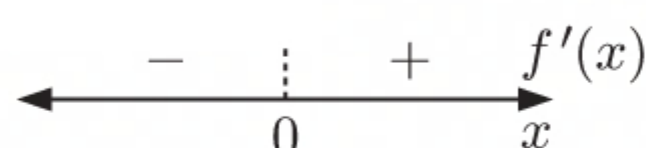


So, $f(x)$ is decreasing for all $x \neq 0$.

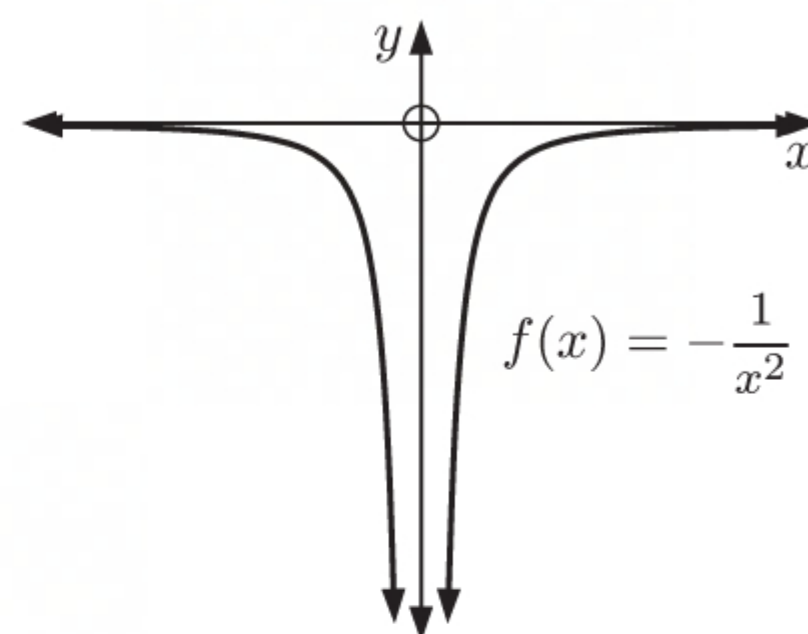


g $f(x) = -\frac{1}{x^2} = -x^{-2}$
 $\therefore f'(x) = 2x^{-3} = \frac{2}{x^3}$

which has sign diagram:

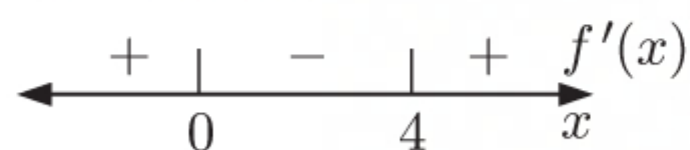


So, $f(x)$ is increasing for $x > 0$,
and decreasing for $x < 0$.

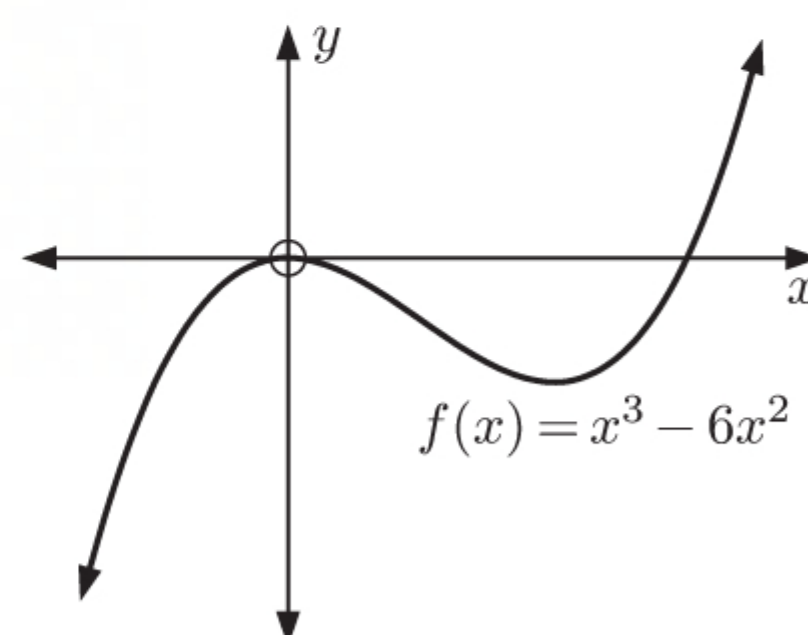


h $f(x) = x^3 - 6x^2$
 $\therefore f'(x) = 3x^2 - 12x$
 $= 3x(x - 4)$

which has sign diagram:

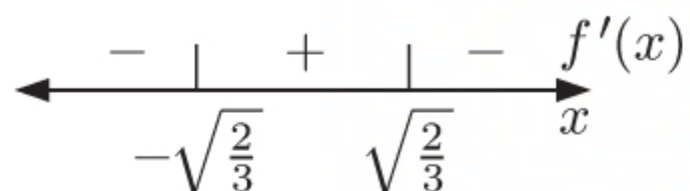


So, $f(x)$ is increasing for $x \leq 0$ and
 $x \geq 4$, and decreasing for $0 \leq x \leq 4$.

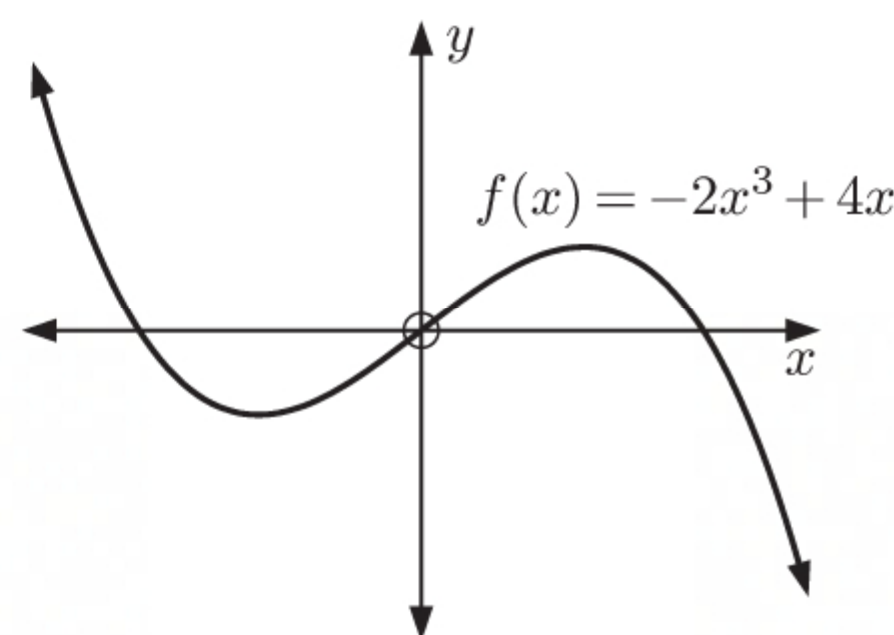


i $f(x) = -2x^3 + 4x$
 $\therefore f'(x) = -6x^2 + 4$
 Now $f'(x) = 0$ when $-6x^2 + 4 = 0$
 $\therefore -6x^2 = -4$
 $\therefore x^2 = \frac{2}{3}$
 $\therefore x = \pm\sqrt{\frac{2}{3}}$

$f'(x)$ has sign diagram:

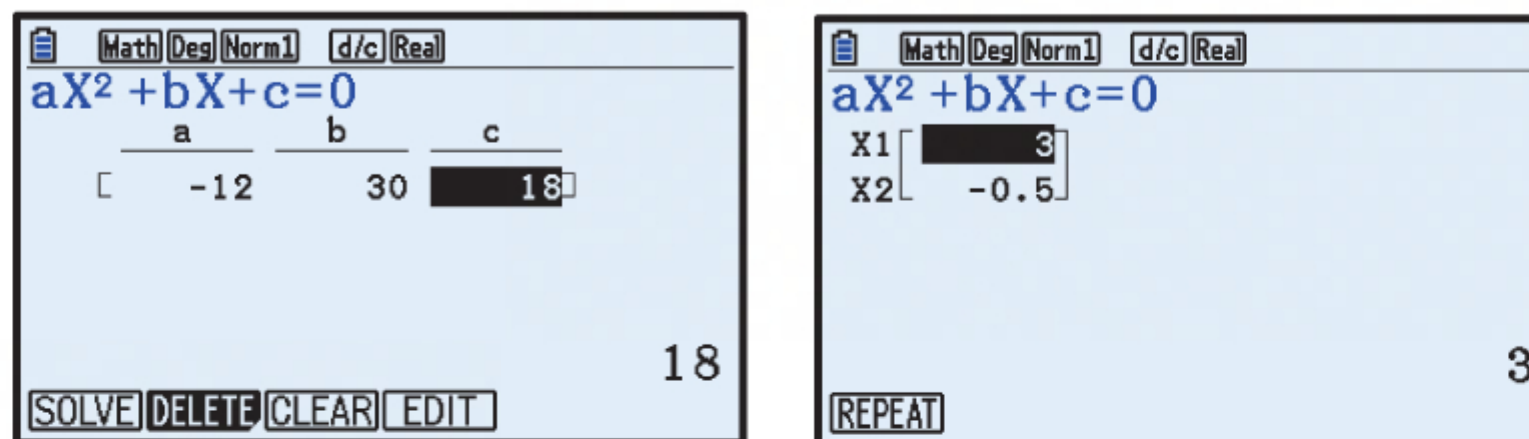


So, $f(x)$ is increasing for $-\sqrt{\frac{2}{3}} \leq x \leq \sqrt{\frac{2}{3}}$, and decreasing for $x \leq -\sqrt{\frac{2}{3}}$ and $x \geq \sqrt{\frac{2}{3}}$.



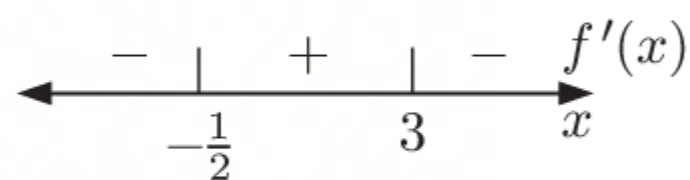
5 a i $f(x) = -4x^3 + 15x^2 + 18x + 3$
 $\therefore f'(x) = -12x^2 + 30x + 18$

ii We find the zeros of $f'(x)$ using technology:



So, $f'(x) = 0$ when $x = -\frac{1}{2}$ or $x = 3$.

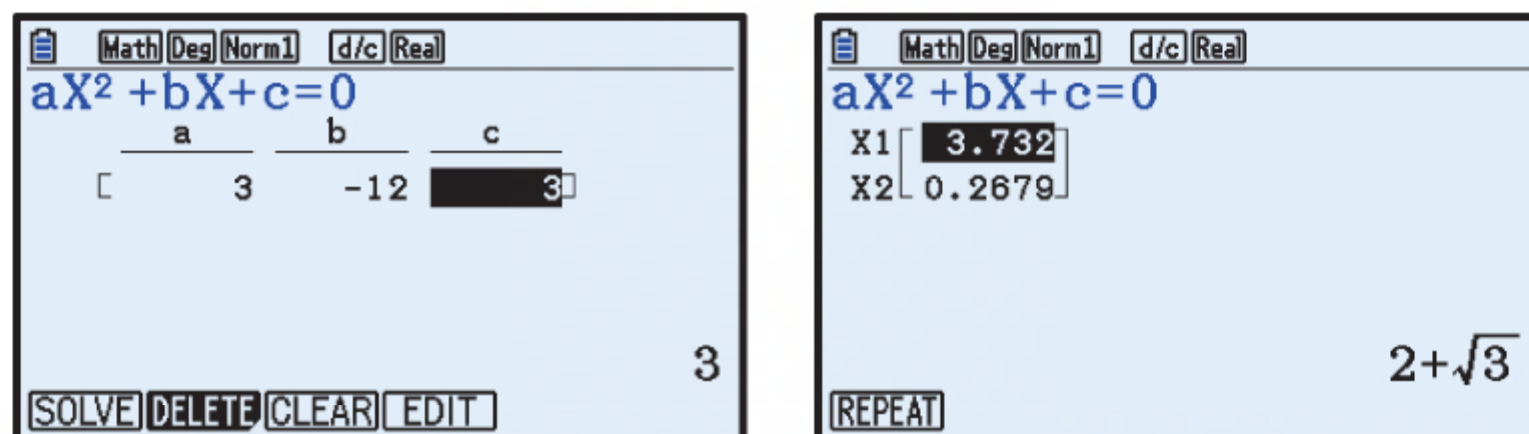
iii $f'(x)$ has sign diagram:



So, $f(x)$ is increasing for $-\frac{1}{2} \leq x \leq 3$, and decreasing for $x \leq -\frac{1}{2}$ and $x \geq 3$.

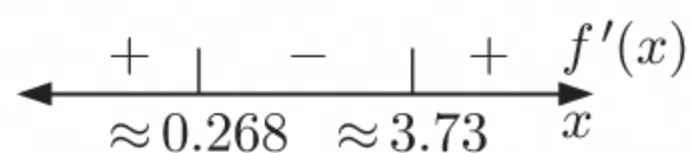
b i $f(x) = x^3 - 6x^2 + 3x - 1$
 $\therefore f'(x) = 3x^2 - 12x + 3$

ii We find the zeros of $f'(x)$ using technology:



So, $f'(x) = 0$ when $x \approx 0.268$ or 3.73 .

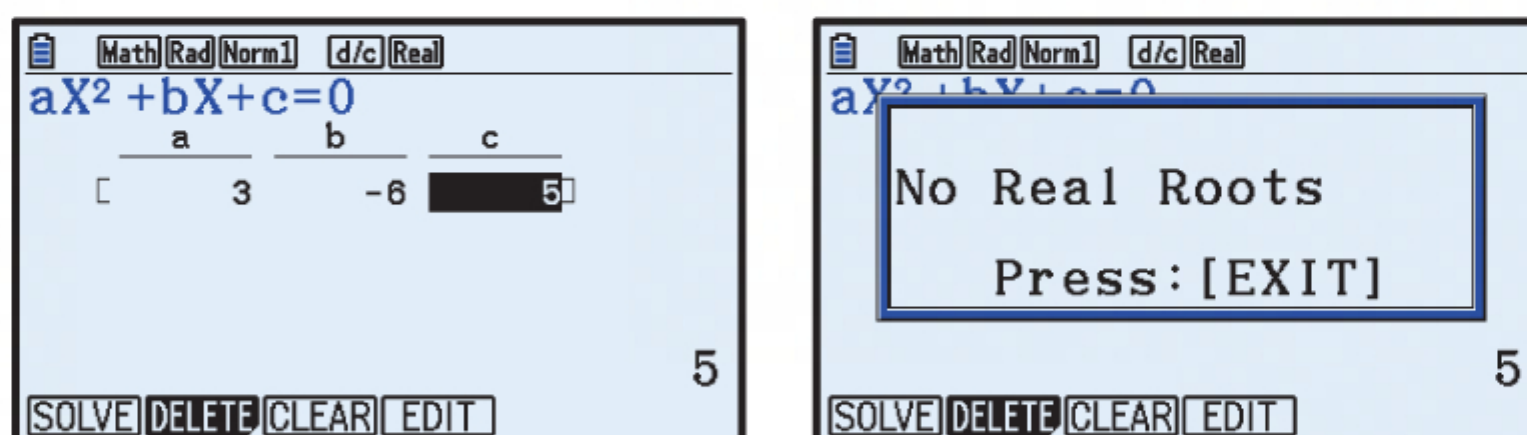
iii $f'(x)$ has sign diagram:



So, $f(x)$ is increasing for $x \leq 0.268$ and $x \geq 3.73$, and decreasing for $0.268 \leq x \leq 3.73$.

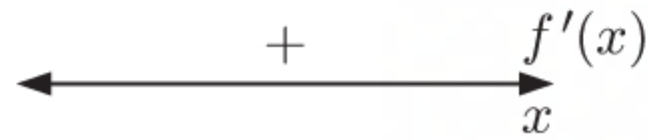
6 a $f(x) = x^3 - 3x^2 + 5x + 2$
 $\therefore f'(x) = 3x^2 - 6x + 5$

b We find the zeros of $f'(x)$ using technology:

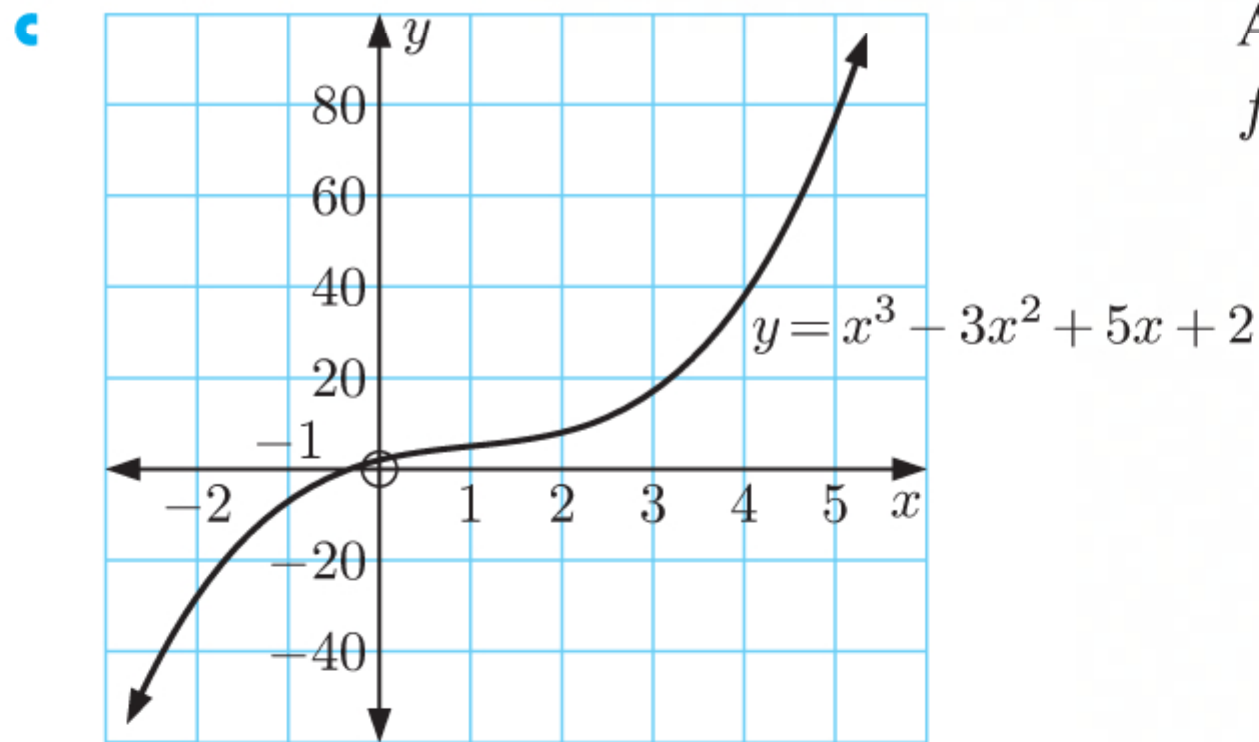


So, $f'(x) \neq 0$ for all $x \in \mathbb{R}$.

$f'(x)$ has sign diagram:



So, $f'(x) > 0$ for all $x \in \mathbb{R}$ which means that $f(x)$ is increasing for all $x \in \mathbb{R}$.



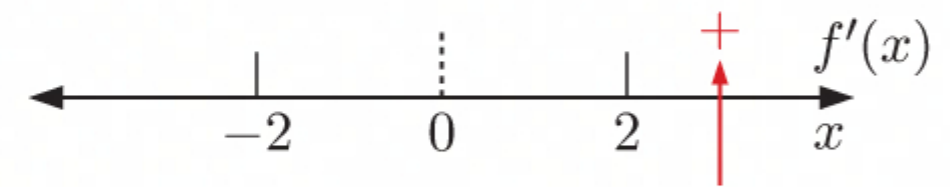
As we can see from the graph of $y = f(x)$, $f(x)$ is increasing for all $x \in \mathbb{R}$.

7 a
$$\begin{aligned} f(x) &= 2x + \frac{8}{x} \\ &= 2x + 8x^{-1} \\ \therefore f'(x) &= 2 - 8x^{-2} \\ &= 2 - \frac{8}{x^2} \\ &= 2 \times \frac{x^2}{x^2} - \frac{8}{x^2} \\ &= \frac{2x^2 - 8}{x^2} \\ &= \frac{2(x^2 - 4)}{x^2} \end{aligned}$$

Now $(x+2)(x-2) = x^2 - 2x + 2x - 4 = x^2 - 4$

So, $f'(x) = \frac{2(x+2)(x-2)}{x^2}$

b $f'(x) = \frac{2(x+2)(x-2)}{x^2}$ is zero when $x = -2$ or 2 , and undefined when $x = 0$.

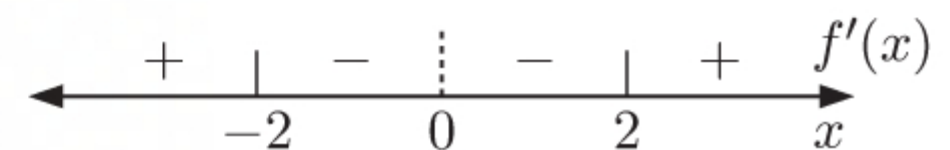


When $x = 4$, $f'(x) = \frac{2(6)(2)}{4^2} = \frac{3}{2} > 0$

Since $(x+2)$ and $(x-2)$ are single factors, the signs alternate about $x = -2$ and $x = 2$.

Since x^2 is a squared factor, the sign stays the same about $x = 0$.

$f'(x)$ has sign diagram:

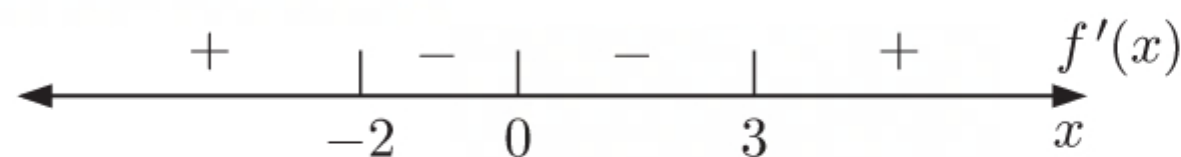


c $f(x)$ is increasing for $x \leq -2$ and $x \geq 2$, and decreasing for $-2 \leq x < 0$ and $0 < x \leq 2$.

EXERCISE 11D

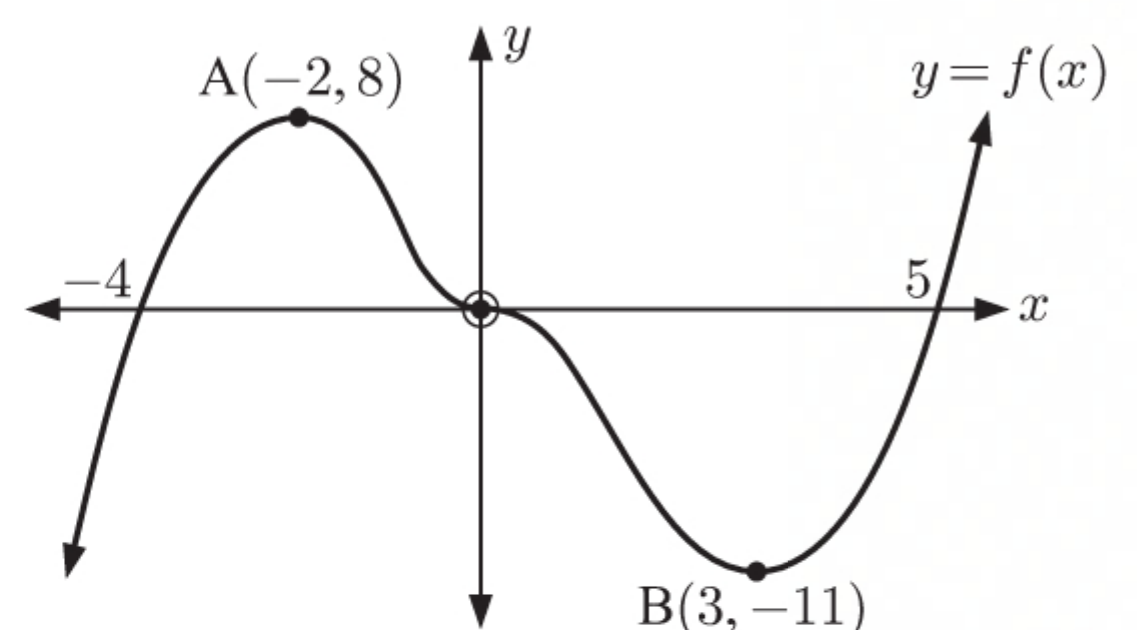
1 a A is a local maximum, O is a stationary inflection, B is a local minimum.

b $f'(x)$ has sign diagram:

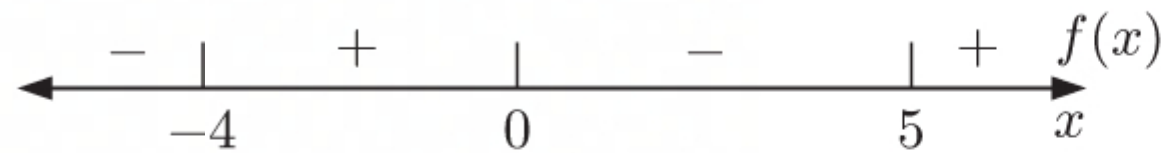


c i $f(x)$ is increasing for $x \leq -2$ and $x \geq 3$.

ii $f(x)$ is decreasing for $-2 \leq x \leq 3$.



d $f(x)$ has sign diagram:



2 a The graph has four stationary points at $(-5, -4)$, $(-2, 9)$, $(2, -6)$, and $(4, -10)$.

b i The local maximum is at $(-2, 9)$.

ii The horizontal inflection is at $(2, -6)$.

c i The greatest value (global maximum) of $f(x)$ on $-8 \leq x \leq 6$ is 11 when $x = 6$.

ii The least value (global minimum) of $f(x)$ on $-8 \leq x \leq 6$ is -10 when $x = 4$.

d The greatest value of $f(x)$ on $-8 \leq x \leq 4$ is 9 when $x = -2$.

e The least value of $f(x)$ on $-5 \leq x \leq 2$ is -6 when $x = 2$.

3 $f(x) = 2x^2 - 5x + 1$ has $a = 2$, $b = -5$, and $c = 1$.

a $-\frac{b}{2a} = -\frac{-5}{2(2)} = \frac{5}{4}$

\therefore the axis of symmetry has equation $x = \frac{5}{4}$.

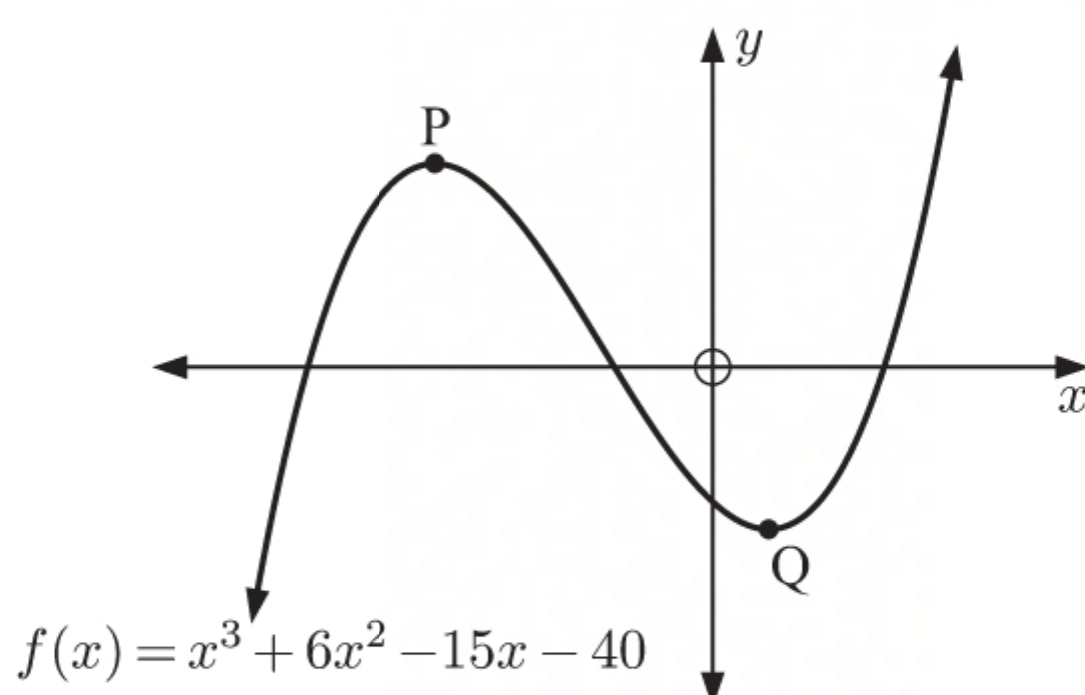
b $f'(x) = 4x - 5$ and $f'(x) = 0$ when $4x - 5 = 0$

$\therefore 4x = 5$

$\therefore x = \frac{5}{4}$

The vertex of a quadratic (a local minimum in this case) is always on the axis of symmetry.

4



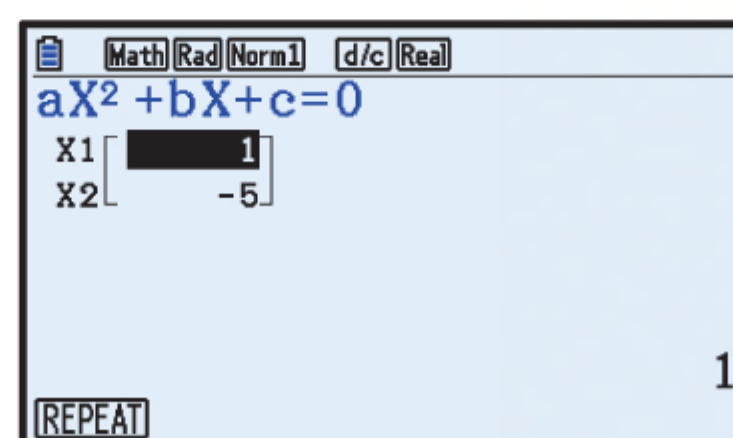
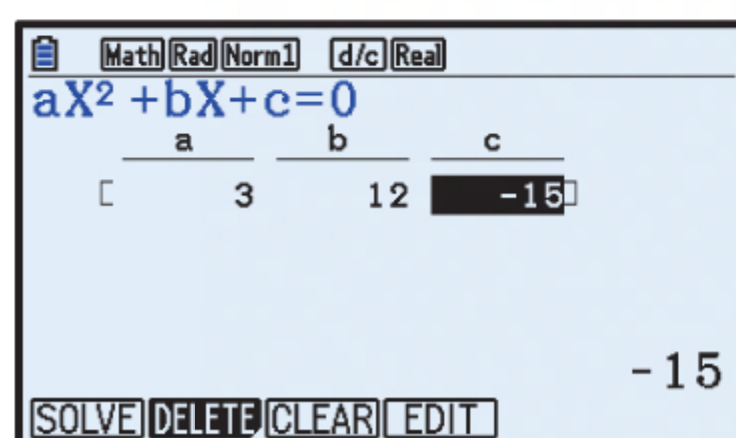
a P is a local maximum.

Q is a local minimum.

b $f(x) = x^3 + 6x^2 - 15x - 40$

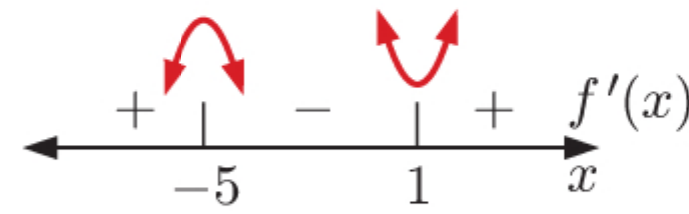
$\therefore f'(x) = 3x^2 + 12x - 15$

c



Using technology, $f'(x) = 0$ when $x = -5$ or 1 .

\therefore the sign diagram for $f'(x)$ is:



\therefore there is a local maximum at $x = -5$, and a local minimum at $x = 1$.

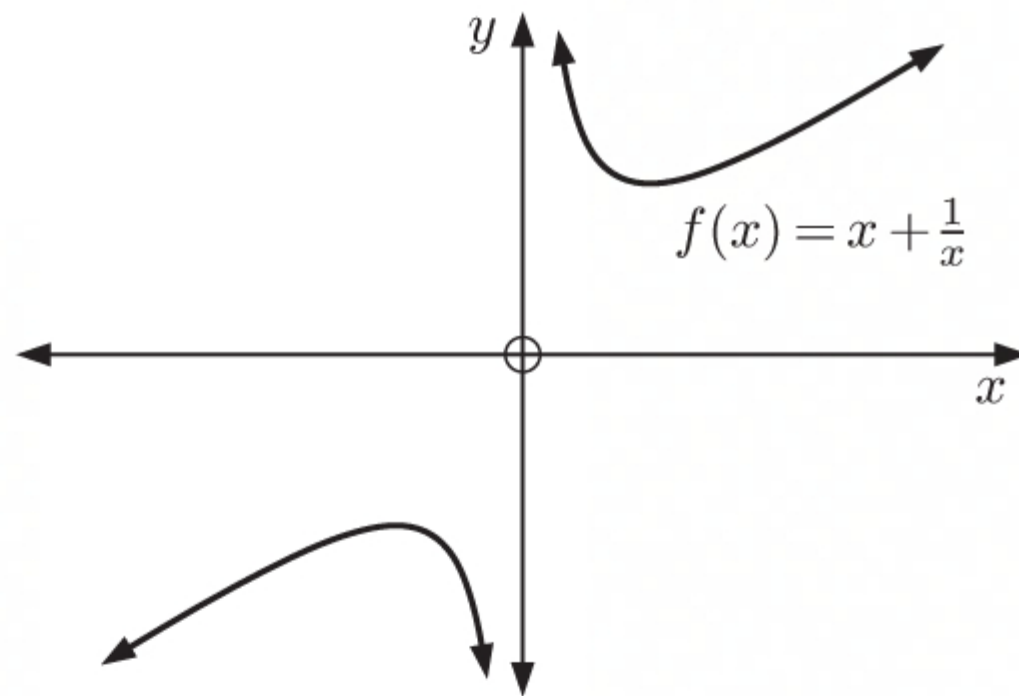
So, P has x -coordinate -5 , and Q has x -coordinate 1 .

$$\begin{aligned} f(-5) &= (-5)^3 + 6(-5)^2 - 15(-5) - 40 \\ &= 60 \end{aligned}$$

$$\begin{aligned} f(1) &= 1^3 + 6(1)^2 - 15(1) - 40 \\ &= -48 \end{aligned}$$

So, P is $(-5, 60)$ and Q is $(1, -48)$.

5 a



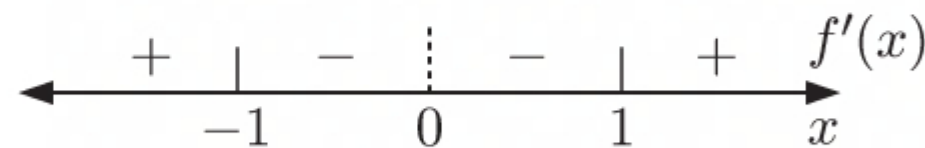
b

$$\begin{aligned} f(x) &= x + \frac{1}{x} \\ &= x + x^{-1} \\ \therefore f'(x) &= 1 - x^{-2} \\ &= 1 - \frac{1}{x^2} \end{aligned}$$

$$\begin{aligned} \text{c } f'(x) = 0 \quad \text{when} \quad 1 - \frac{1}{x^2} &= 0 \\ \therefore 1 &= \frac{1}{x^2} \\ \therefore x^2 &= 1 \\ \therefore x &= \pm 1 \end{aligned}$$

$f'(x)$ is undefined when $x = 0$.

So, $f'(x)$ has sign diagram:



$$\text{d } f(-1) = -1 + \frac{1}{-1} = -2$$

So, there is a local maximum at the point $(-1, -2)$.

$$f(1) = 1 + \frac{1}{1} = 2$$

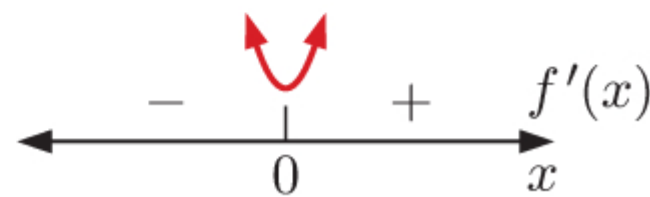
So, there is a local minimum at the point $(1, 2)$.

6 a $f(x) = x^2 - 2$

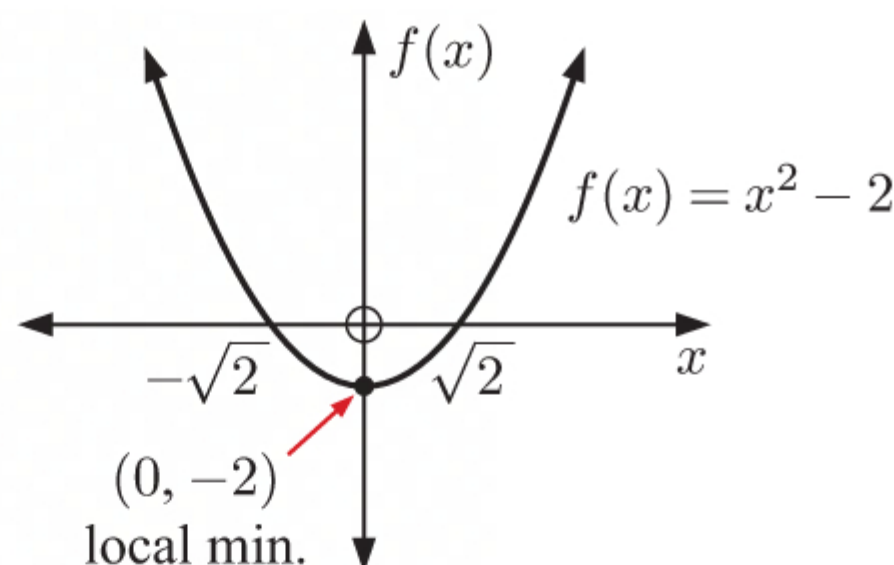
$$\therefore f'(x) = 2x$$

which has

sign diagram:



Now $f(0) = -2$, so there is a local minimum at $(0, -2)$.



b $f(x) = x^3 + 1$

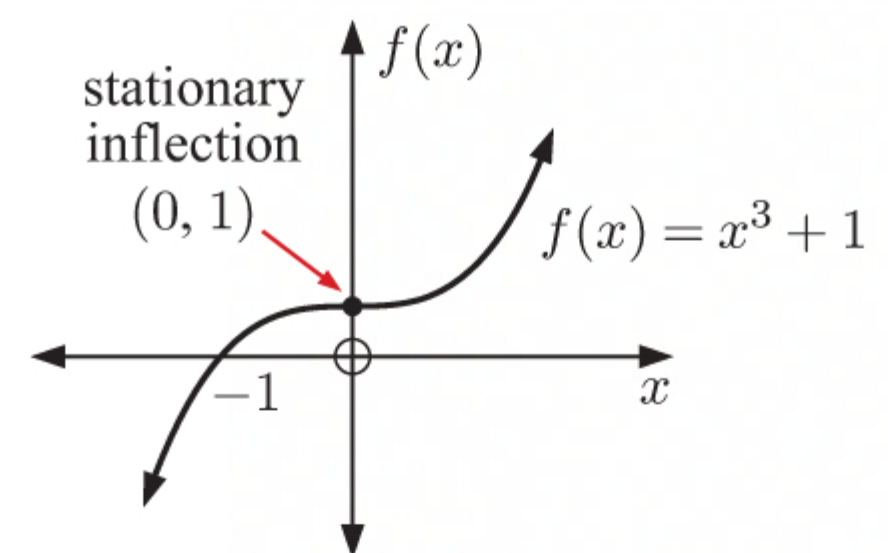
$$\therefore f'(x) = 3x^2$$

which has

sign diagram:



Now $f(0) = 1$, so there is a stationary inflection at $(0, 1)$.



c $f(x) = x^3 - 3x + 2$

$\therefore f'(x) = 3x^2 - 3$

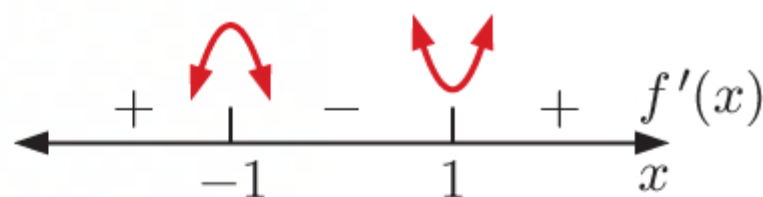
Now $f'(x) = 0$ when $3x^2 - 3 = 0$

$\therefore 3x^2 = 3$

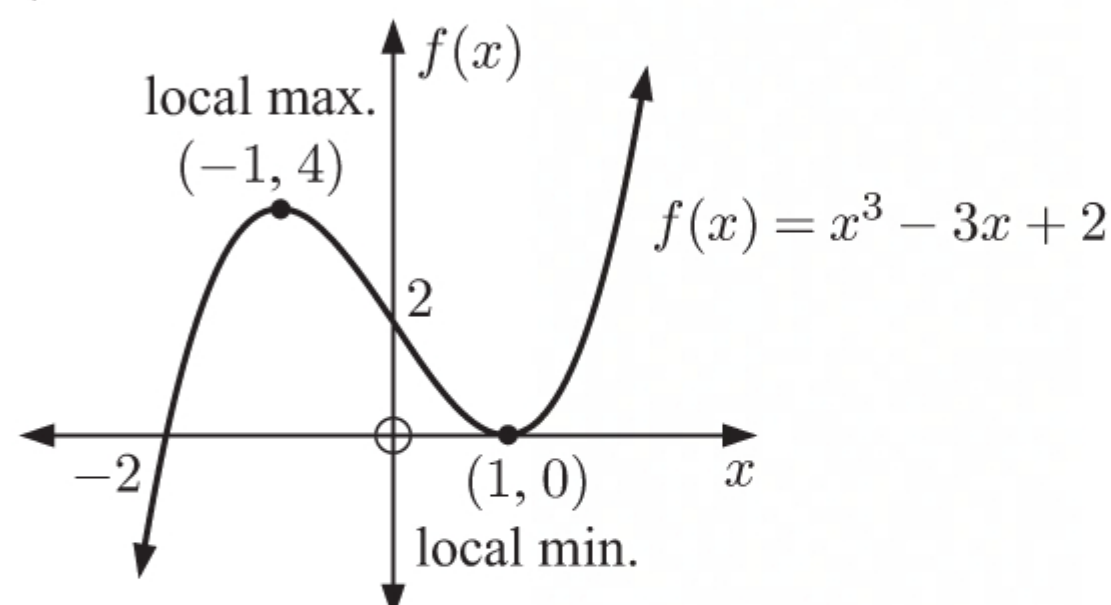
$\therefore x^2 = 1$

$\therefore x = \pm 1$

\therefore the sign diagram for $f'(x)$ is:

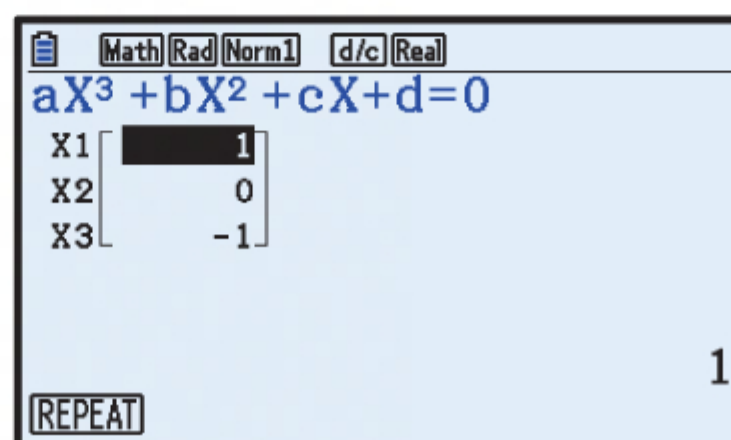
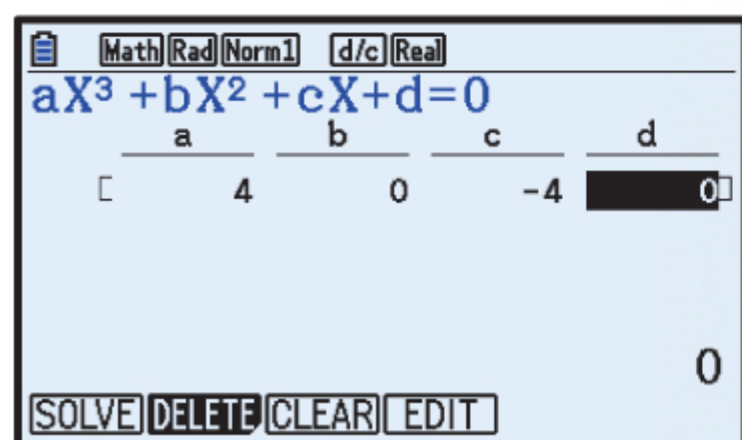


Now $f(-1) = 4$, $f(1) = 0$, so there is a local maximum at $(-1, 4)$, and a local minimum at $(1, 0)$.



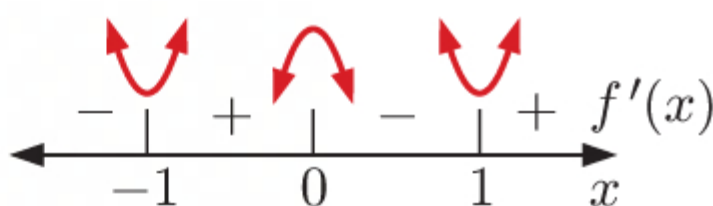
d $f(x) = x^4 - 2x^2$

$\therefore f'(x) = 4x^3 - 4x$

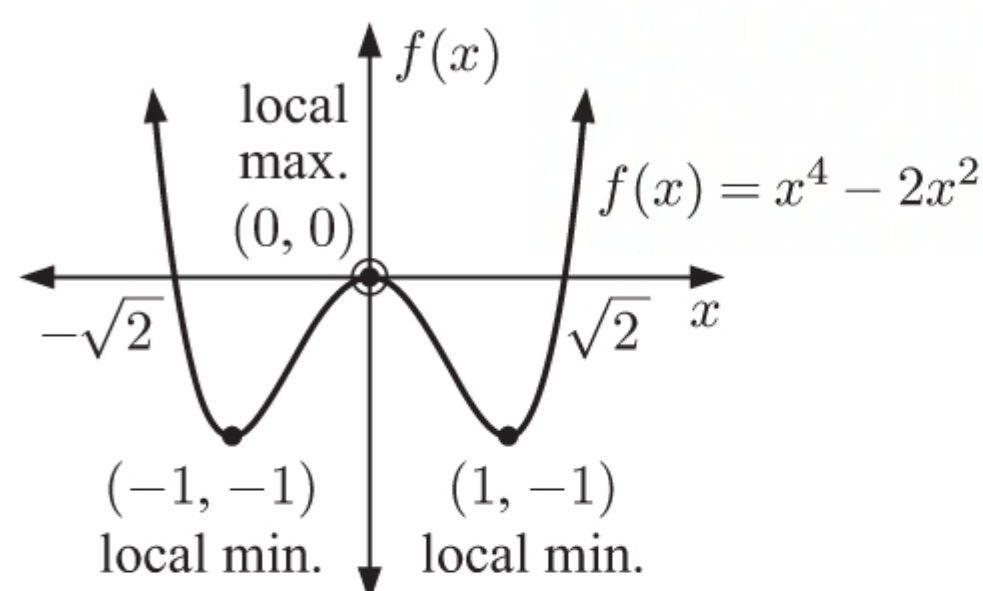


Using technology, $f'(x) = 0$ when $x = -1, 0$, or 1 .

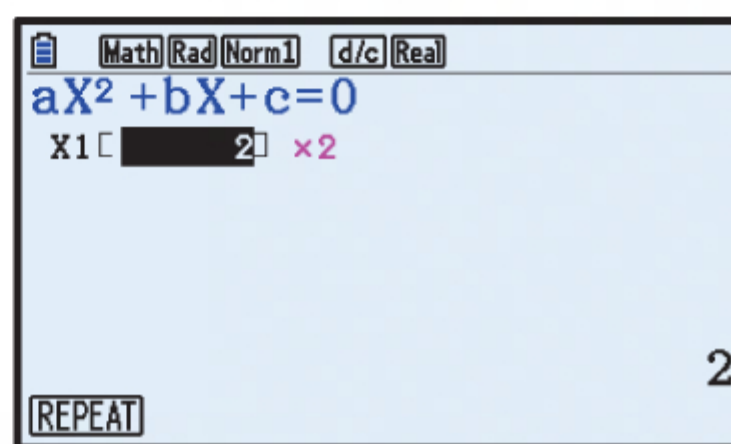
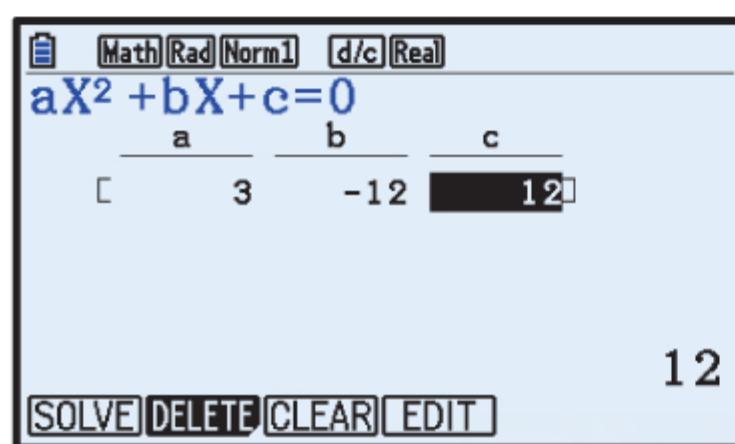
\therefore the sign diagram for $f'(x)$ is:



Now $f(-1) = -1$, $f(1) = -1$, $f(0) = 0$, so there are local minima at $(-1, -1)$ and $(1, -1)$, and a local maximum at $(0, 0)$.



e $f(x) = x^3 - 6x^2 + 12x + 1$
 $\therefore f'(x) = 3x^2 - 12x + 12$

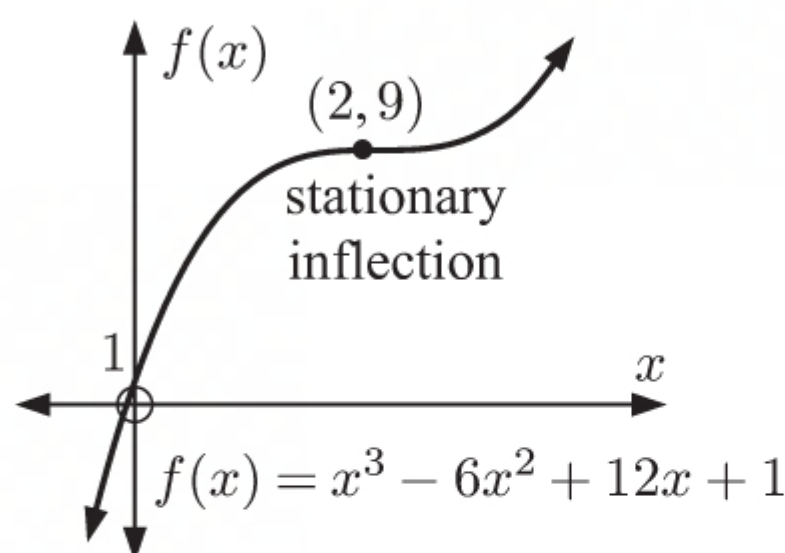


Using technology, $f'(x) = 0$ when $x = 2$.

\therefore the sign diagram for $f'(x)$ is:



Now $f(2) = 9$, so there is a stationary inflection at $(2, 9)$.



f $f(x) = 4x - x^3$
 $\therefore f'(x) = 4 - 3x^2$

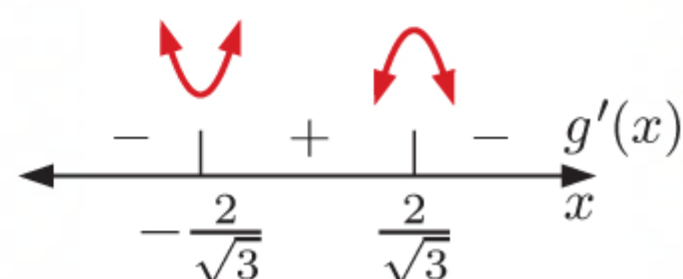
Now $f'(x) = 0$ when $4 - 3x^2 = 0$

$$\therefore 3x^2 = 4$$

$$\therefore x^2 = \frac{4}{3}$$

$$\therefore x = \pm \frac{2}{\sqrt{3}}$$

\therefore the sign diagram for $f'(x)$ is:

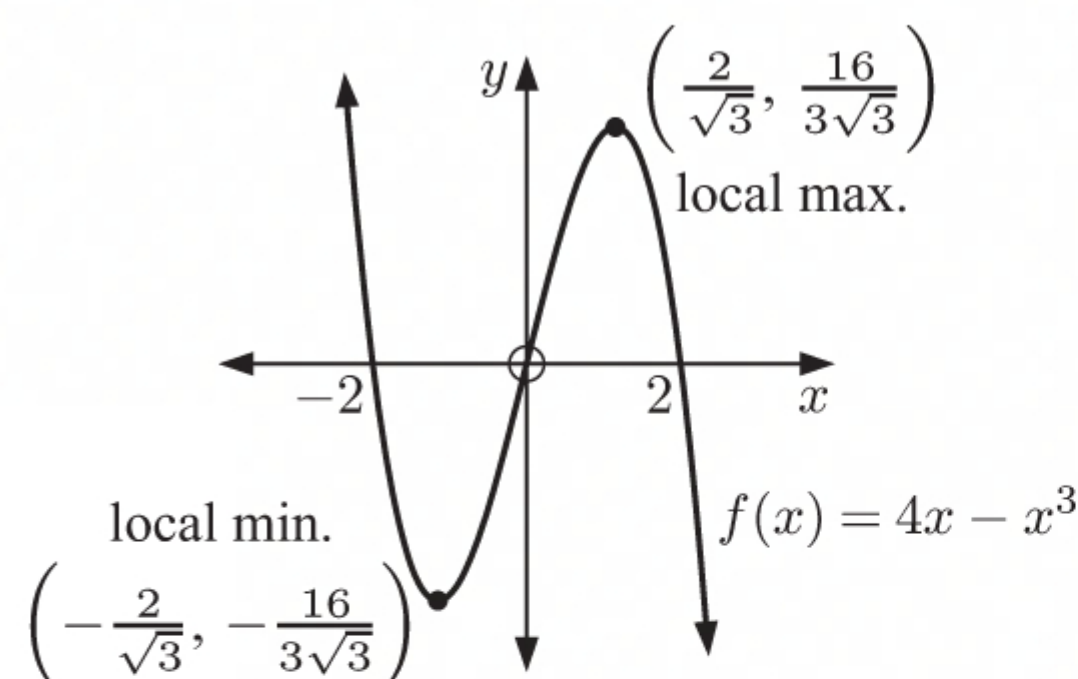


$$\begin{aligned} \text{Now } f\left(-\frac{2}{\sqrt{3}}\right) &= 4\left(-\frac{2}{\sqrt{3}}\right) - \left(-\frac{2}{\sqrt{3}}\right)^3 \\ &= -\frac{8}{\sqrt{3}} + \frac{8}{3\sqrt{3}} \\ &= -\frac{16}{3\sqrt{3}} \end{aligned}$$

So there is a local minimum at $\left(-\frac{2}{\sqrt{3}}, -\frac{16}{3\sqrt{3}}\right)$.

$$\begin{aligned} \text{Now } f\left(\frac{2}{\sqrt{3}}\right) &= 4\left(\frac{2}{\sqrt{3}}\right) - \left(\frac{2}{\sqrt{3}}\right)^3 \\ &= \frac{8}{\sqrt{3}} - \frac{8}{3\sqrt{3}} \\ &= \frac{16}{3\sqrt{3}} \end{aligned}$$

So there is a local maximum at $\left(\frac{2}{\sqrt{3}}, \frac{16}{3\sqrt{3}}\right)$.



$$\begin{aligned} \mathbf{g} \quad f(x) &= 2x + \frac{1}{x^2} = 2x + x^{-2} \\ \therefore f'(x) &= 2 - 2x^{-3} \\ &= 2 - \frac{2}{x^3} \end{aligned}$$

$$f'(x) = 0 \quad \text{when} \quad 2 - \frac{2}{x^3} = 0$$

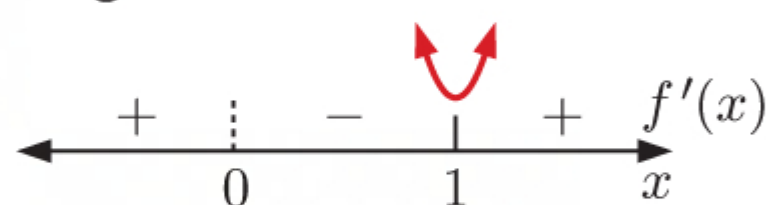
$$\therefore 2 = \frac{2}{x^3}$$

$$\therefore 2x^3 = 2$$

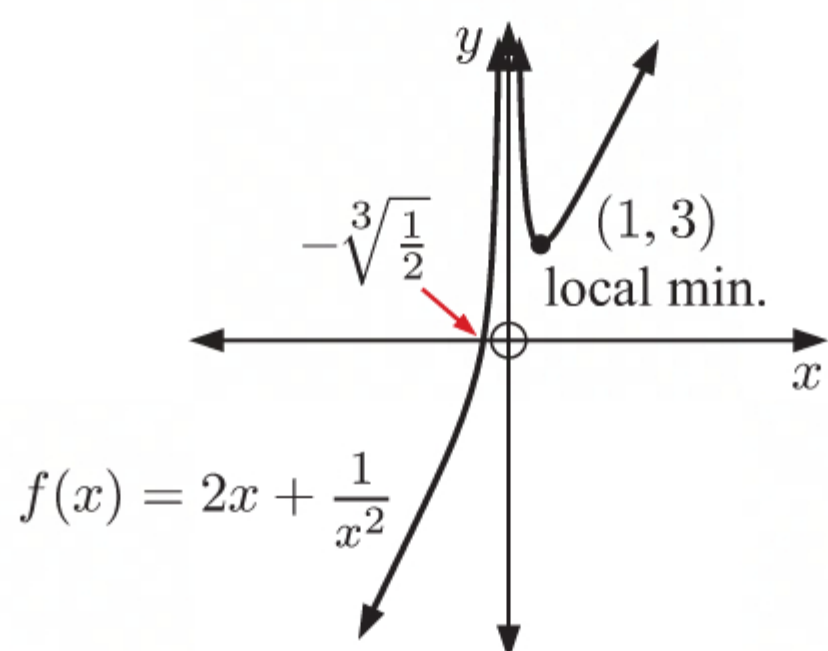
$$\therefore x^3 = 1$$

$$\therefore x = 1$$

$f'(x)$ is undefined when $x = 0$, so the sign diagram is:



Now $f(1) = 3$, so there is a local minimum at $(1, 3)$.



$$\begin{aligned} \mathbf{h} \quad f(x) &= -x - \frac{9}{x} = -x - 9x^{-1} \\ \therefore f'(x) &= -1 + 9x^{-2} \\ &= -1 + \frac{9}{x^2} \end{aligned}$$

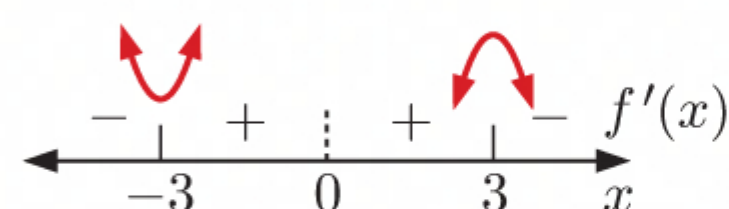
$$f'(x) = 0 \quad \text{when} \quad -1 + \frac{9}{x^2} = 0$$

$$\therefore -1 = -\frac{9}{x^2}$$

$$\therefore x^2 = 9$$

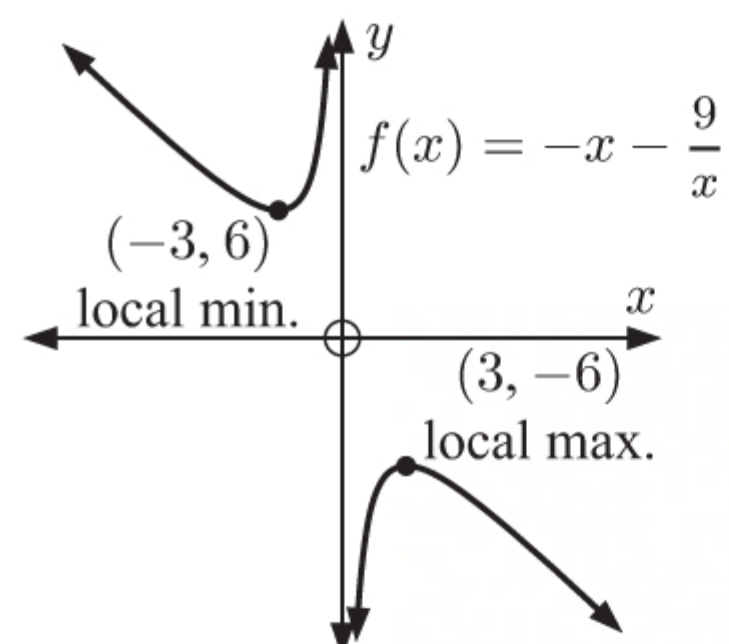
$$\therefore x = \pm 3$$

$f'(x)$ is undefined when $x = 0$, so the sign diagram is:



Now $f(-3) = 6$, so there is a local minimum at $(-3, 6)$.

Now $f(3) = -6$, so there is a local maximum at $(3, -6)$.



i $f(x) = x^2 + \frac{16}{x} = x^2 + 16x^{-1}$

$$\therefore f'(x) = 2x - 16x^{-2}$$

$$= 2x - \frac{16}{x^2}$$

$$f'(x) = 0 \quad \text{when} \quad 2x - \frac{16}{x^2} = 0$$

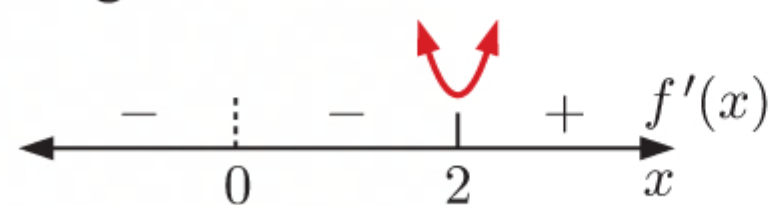
$$\therefore 2x = \frac{16}{x^2}$$

$$\therefore 2x^3 = 16$$

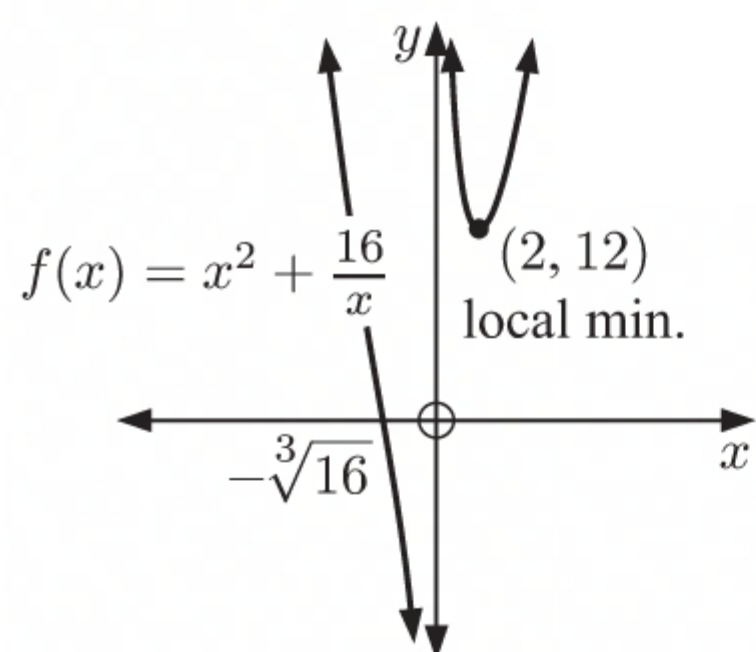
$$\therefore x^3 = 8$$

$$\therefore x = 2$$

$f'(x)$ is undefined when $x = 0$, so the sign diagram is:



Now $f(2) = 12$, so there is a local minimum at $(2, 12)$.



7 a $f(x) = ax^2 + bx + c, \quad a \neq 0$

$$\therefore f'(x) = 2ax + b$$

$f(x)$ has a stationary point when $f'(x) = 0$

$$\therefore 2ax + b = 0$$

$$\therefore x = -\frac{b}{2a}$$

b When $a < 0$, $f(x)$ opens downwards , so there is a local maximum when $a < 0$.

When $a > 0$, $f(x)$ opens upwards , so there is a local minimum when $a > 0$.

8 a $f(x) = 2x^3 + ax^2 - 24x + 1$

$$\therefore f'(x) = 6x^2 + 2ax - 24$$

But $f'(-4) = 0$, so $6(-4)^2 + 2a(-4) - 24 = 0$

$$\therefore 96 - 8a - 24 = 0$$

$$\therefore 72 = 8a$$

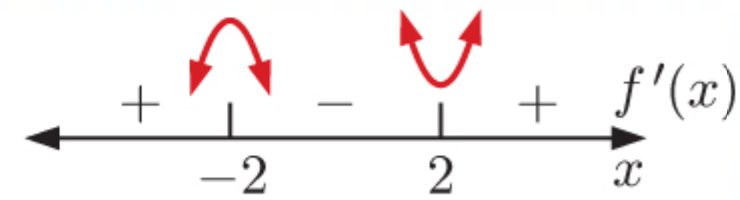
$$\therefore a = 9$$

- b** Since $a = 9$, then $f(x) = 2x^3 + 9x^2 - 24x + 1$
 $\therefore f(-4) = 2(-4)^3 + 9(-4)^2 - 24(-4) + 1$
 $= 113$
 \therefore the local maximum is at $(-4, 113)$.

9 a $f(x) = x^3 + ax + b$
 $\therefore f'(x) = 3x^2 + a$
 But $f'(-2) = 0$
 $\therefore 3(-2)^2 + a = 0$
 $\therefore 12 + a = 0$
 $\therefore a = -12$
 Also, $f(-2) = 3$
 $\therefore (-2)^3 - 12(-2) + b = 3$
 $\therefore -8 + 24 + b = 3$
 $\therefore b = -13$

b Now $f(x) = x^3 - 12x - 13$
 $\therefore f'(x) = 3x^2 - 12$
c Now $f'(x) = 0$ when $3x^2 - 12 = 0$
 $\therefore 3x^2 = 12$
 $\therefore x^2 = 4$
 $\therefore x = \pm 2$

\therefore the sign diagram for $f'(x)$ is:



Now $f(2) = -29$, $f(-2) = 3$, so there is a local minimum at $(2, -29)$ and a local maximum at $(-2, 3)$.

10 $P(x) = ax^3 + bx^2 + cx + d$
 $\therefore P'(x) = 3ax^2 + 2bx + c \dots (1)$
 Now $(0, 2)$ lies on $P(x)$, so $P(0) = 2$
 $\therefore a(0) + b(0) + c(0) + d = 2$
 $\therefore d = 2$
 The tangent at $(0, 2)$ is $y = 9x + 2$, so $P'(0) = 9$
 $\therefore 3a(0) + 2b(0) + c = 9$
 $\therefore c = 9 \dots (2)$

There is a stationary point at $(-1, -7)$, so $P'(-1) = 0$.

$\therefore 3a(-1)^2 + 2b(-1) + c = 0$ {using (1)}
 $\therefore 3a - 2b + c = 0$

So, using (2), $3a - 2b = -9 \dots (3)$

Finally, $(-1, -7)$ lies on $P(x)$

$\therefore a(-1)^3 + b(-1)^2 + c(-1) + d = -7$
 $\therefore -a + b - 9 + 2 = -7$
 $\therefore b - a = 0$
 $\therefore a = b$

So, using (3), $3a - 2a = -9$
 $\therefore a = -9$
 $\therefore a = b = -9$

$\therefore P(x) = -9x^3 - 9x^2 + 9x + 2$

11 a Let $f(x) = x^3 - 12x - 2$, for $-3 \leq x \leq 5$
 $\therefore f'(x) = 3x^2 - 12$

Now $f'(x) = 0$ when $3x^2 - 12 = 0$
 $\therefore 3x^2 = 12$
 $\therefore x^2 = 4$
 $\therefore x = \pm 2$

\therefore the sign diagram for $f'(x)$ is:

\therefore there is a local maximum at $x = -2$, and a local minimum at $x = 2$.

Critical value (x)	$f(x)$
-3 (end point)	7
-2 (local maximum)	14
2 (local minimum)	-18
5 (end point)	63

The greatest of these values is 63 when $x = 5$.

The least of these values is -18 when $x = 2$.

b Let $f(x) = 4 - 3x^2 + x^3$, for $-2 \leq x \leq 3$
 $\therefore f'(x) = -6x + 3x^2$
 $= 3x(x - 2)$

which is 0 when $x = 0$ or 2

\therefore the sign diagram for $f'(x)$ is:

\therefore there is a local maximum at $x = 0$, and a local minimum at $x = 2$.

Critical value (x)	$f(x)$
-2 (end point)	-16
0 (local maximum)	4
2 (local minimum)	0
3 (end point)	4

The greatest of these values is 4 when $x = 0$ or $x = 3$.

The least of these values is -16 when $x = -2$.

c Let $f(x) = x^2 + \frac{16}{x} = x^2 + 16x^{-1}$, for $1 \leq x \leq 4$

$$\begin{aligned}\therefore f'(x) &= 2x - 16x^{-2} \\ &= 2x - \frac{16}{x^2}\end{aligned}$$

Now $f'(x) = 0$ when $2x - \frac{16}{x^2} = 0$

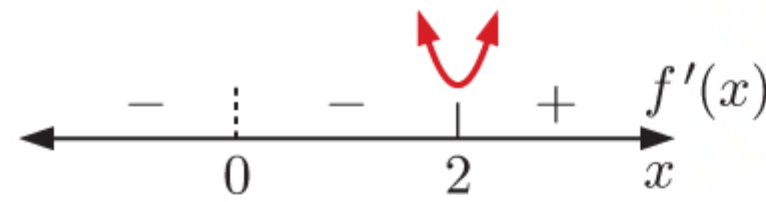
$$\therefore 2x = \frac{16}{x^2}$$

$$\therefore 2x^3 = 16$$

$$\therefore x^3 = 8$$

$$\therefore x = 2$$

The sign diagram of $f'(x)$ is:



\therefore there is a local minimum at $x = 2$.

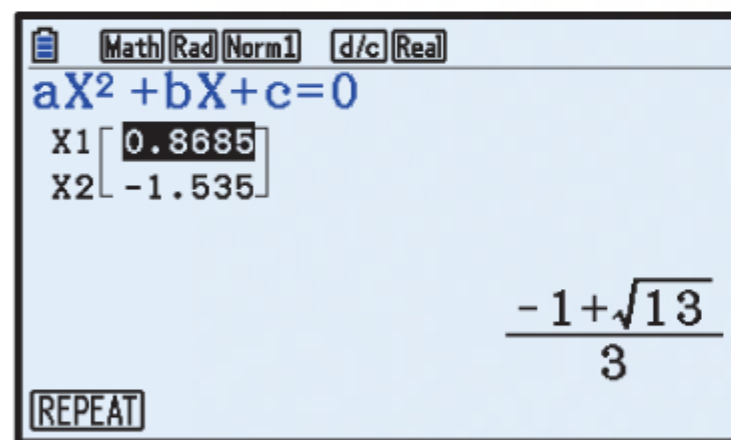
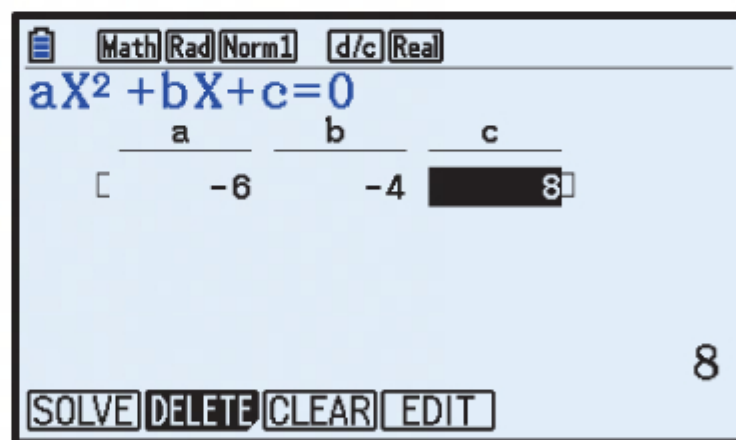
Critical value (x)	$f(x)$
1 (end point)	17
2 (local minimum)	12
4 (end point)	20

The greatest of these values is 20 when $x = 4$.

The least of these values is 12 when $x = 2$.

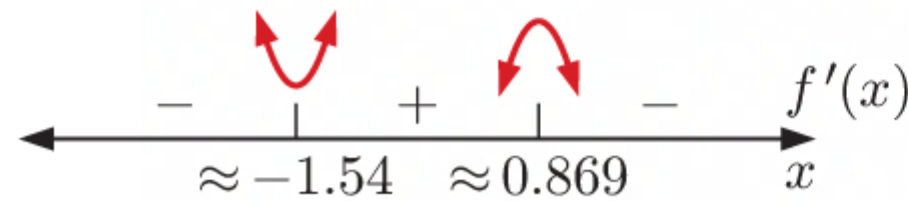
d Let $f(x) = -2x^3 - 2x^2 + 8x + 3$, for $-2 \leq x \leq 2$

$$\therefore f'(x) = -6x^2 - 4x + 8$$



Using technology, $f'(x) = 0$ when $x \approx -1.54$ or ≈ 0.869 .

The sign diagram of $f'(x)$ is:



\therefore there is a local minimum at $x \approx -1.54$, and a local maximum at $x \approx 0.869$.

We use technology to obtain the table of values:

Critical value (x)	$f(x)$
-2 (end point)	7
≈ -1.54 (local minimum)	≈ -6.76
≈ 0.869 (local maximum)	≈ 7.13
2 (end point)	-5

The greatest of these values is ≈ 7.13 when $x \approx 0.869$.

The least of these values is ≈ -6.76 when $x \approx -1.54$.

REVIEW SET 11A

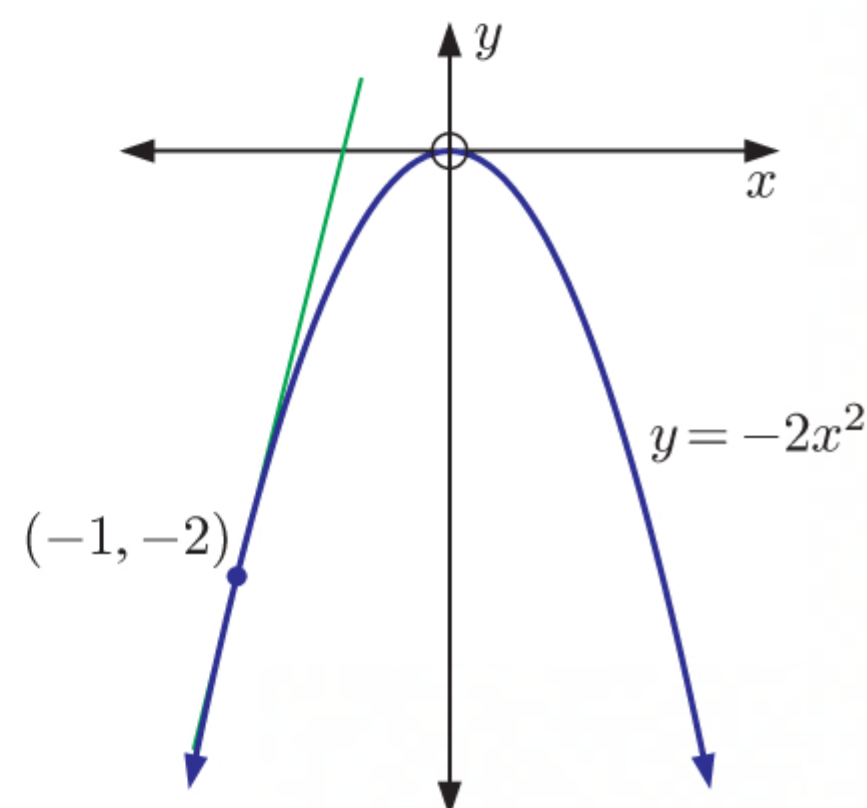
1 a $y = -2x^2$

When $x = -1$, $y = -2(-1)^2 = -2$,
so the point of contact is $(-1, -2)$.

Now $\frac{dy}{dx} = -4x$, so at $x = -1$,

$$\frac{dy}{dx} = -4(-1) = 4$$

The tangent has equation $y = 4(x - (-1)) - 2$
 $= 4(x + 1) - 2$
 $= 4x + 4 - 2$
 $= 4x + 2$

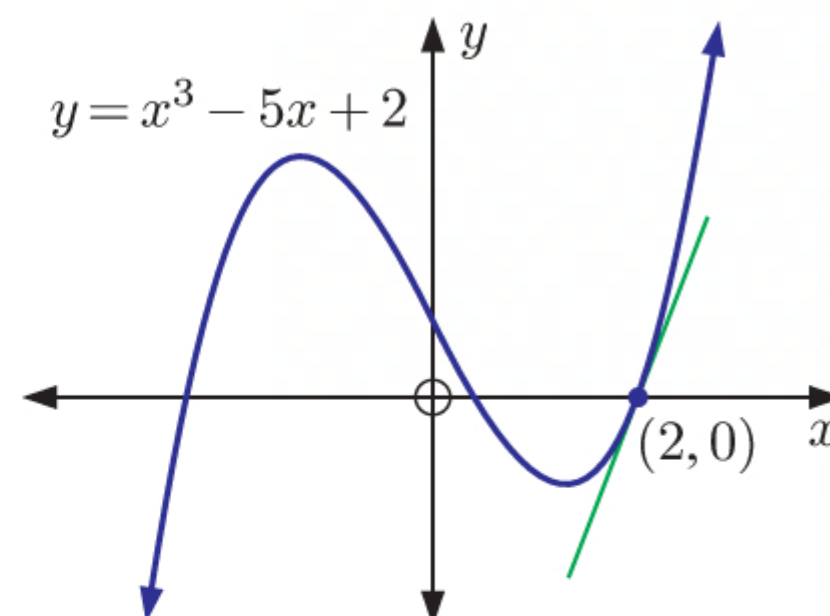


b $y = x^3 - 5x + 2$

$\therefore \frac{dy}{dx} = 3x^2 - 5$, so at $(2, 0)$,

$$\begin{aligned}\frac{dy}{dx} &= 3(2)^2 - 5 \\ &= 12 - 5 \\ &= 7\end{aligned}$$

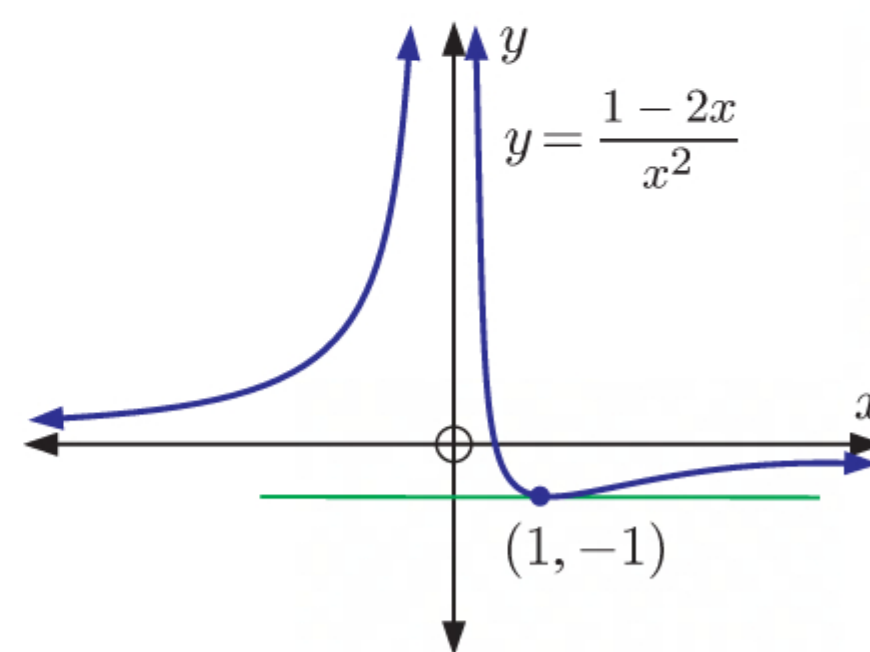
The tangent has equation $y = 7(x - 2) + 0$
 $= 7x - 14$



c $y = \frac{1 - 2x}{x^2}$
 $= \frac{1}{x^2} - \frac{2x}{x^2}$
 $= x^{-2} - 2x^{-1}$

$$\begin{aligned}\therefore \frac{dy}{dx} &= -2x^{-3} + 2x^{-2} \\ &= -\frac{2}{x^3} + \frac{2}{x^2}, \text{ so at } (1, -1), \\ \frac{dy}{dx} &= -\frac{2}{1^3} + \frac{2}{1^2} \\ &= -2 + 2 \\ &= 0\end{aligned}$$

The tangent has equation $y = 0(x - 1) - 1$
 $= -1$



- 2** The tangent to the curve $f(x) = a - \frac{b}{x^2}$ at $(-1, -1)$ is $y = -6x - 7$.

\therefore the tangent has gradient -6 , and the point of contact is $(-1, -1)$.

Now, $f(x) = a - \frac{b}{x^2} = a - bx^{-2}$

$$\therefore f'(x) = 2bx^{-3} = \frac{2b}{x^3}$$

$$f'(-1) = -6 \quad \text{and} \quad f(-1) = -1$$

$$\therefore \frac{2b}{(-1)^3} = -6 \quad \therefore a - \frac{b}{(-1)^2} = -1$$

$$\therefore -2b = -6 \quad \therefore a - 3 = -1 \quad \{\text{using (*)}\}$$

$$\therefore b = 3 \quad \dots (*) \quad \therefore a = 2$$

So, $a = 2$ and $b = 3$.

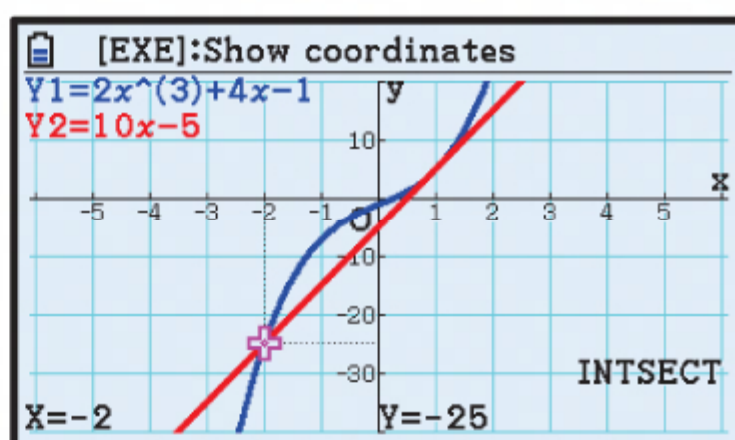
- 3** $y = 2x^3 + 4x - 1$

$$\therefore \frac{dy}{dx} = 6x^2 + 4, \text{ so at } (1, 5),$$

$$\begin{aligned} \frac{dy}{dx} &= 6(1)^2 + 4 \\ &= 10 \end{aligned}$$

$$\begin{aligned} \therefore \text{the tangent has equation } y &= 10(x - 1) + 5 \\ &= 10x - 10 + 5 \\ &= 10x - 5 \end{aligned}$$

We use technology to find where the tangent meets the curve again:



The tangent meets the curve again at $(-2, -25)$.

4 a $y = x^3 + 3x - 2$

$$\begin{aligned}\text{When } x = 2, \quad y &= 2^3 + 3(2) - 2 \\ &= 8 + 6 - 2 \\ &= 12\end{aligned}$$

So, the point of contact is $(2, 12)$.

$$\text{Now } \frac{dy}{dx} = 3x^2 + 3$$

$$\begin{aligned}\text{When } x = 2, \quad \frac{dy}{dx} &= 3(2)^2 + 3 \\ &= 12 + 3 \\ &= 15\end{aligned}$$

So, the normal at $(2, 12)$ has gradient $-\frac{1}{15}$.

\therefore the normal has equation

$$\begin{aligned}y &= -\frac{1}{15}(x - 2) + 12 \\ &= -\frac{1}{15}x + \frac{2}{15} + 12 \\ &= -\frac{1}{15}x + \frac{182}{15}\end{aligned}$$

5 $y = x^2 - 7x - 44$

$$\begin{aligned}\text{When } x = -3, \quad y &= (-3)^2 - 7(-3) - 44 \\ &= 9 + 21 - 44 \\ &= -14\end{aligned}$$

So, the point of contact is $(-3, -14)$.

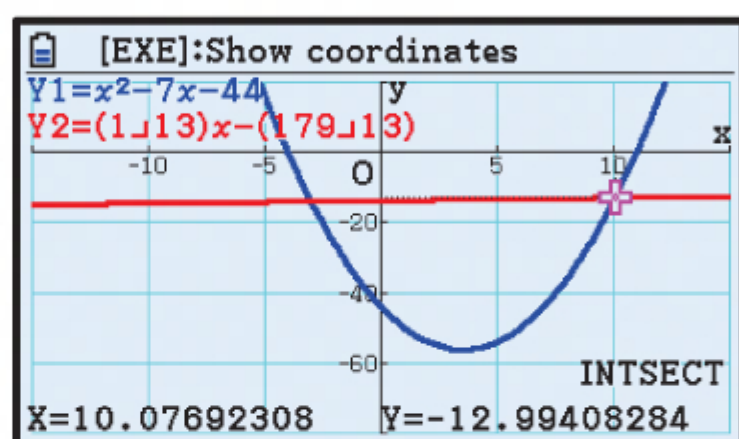
$$\text{Now } \frac{dy}{dx} = 2x - 7$$

$$\begin{aligned}\text{When } x = -3, \quad \frac{dy}{dx} &= 2(-3) - 7 \\ &= -6 - 7 \\ &= -13\end{aligned}$$

So, the normal at $(-3, -14)$ has gradient $\frac{1}{13}$.

$$\begin{aligned}\therefore \text{ the normal has equation } y &= \frac{1}{13}(x - (-3)) - 14 \\ &= \frac{1}{13}(x + 3) - 14 \\ &= \frac{1}{13}x + \frac{3}{13} - 14 \\ &= \frac{1}{13}x - \frac{179}{13}\end{aligned}$$

We use technology to find where the normal meets the curve again:



The normal meets the curve again at about $(10.1, -13.0)$.

b $y = 2 + \frac{1}{x} + 3x = 2 + x^{-1} + 3x$

$$\begin{aligned}\text{When } x = 1, \quad y &= 2 + \frac{1}{1} + 3(1) \\ &= 2 + 1 + 3 \\ &= 6\end{aligned}$$

So, the point of contact is $(1, 6)$.

$$\begin{aligned}\text{Now } \frac{dy}{dx} &= -x^{-2} + 3 \\ &= -\frac{1}{x^2} + 3\end{aligned}$$

$$\begin{aligned}\text{When } x = 1, \quad \frac{dy}{dx} &= -\frac{1}{1^2} + 3 \\ &= -1 + 3 \\ &= 2\end{aligned}$$

So, the normal at $(1, 6)$ has gradient $-\frac{1}{2}$.

\therefore the normal has equation

$$\begin{aligned}y &= -\frac{1}{2}(x - 1) + 6 \\ &= -\frac{1}{2}x + \frac{1}{2} + 6 \\ &= -\frac{1}{2}x + \frac{13}{2}\end{aligned}$$

6 a $y = 2x + \frac{4}{x^2} = 2x + 4x^{-2}$

$$\therefore \frac{dy}{dx} = 2 - 8x^{-3} = 2 - \frac{8}{x^3}$$

The tangent has a gradient of 1.

$$\therefore \frac{dy}{dx} = 2 - \frac{8}{x^3} = 1$$

$$\therefore 1 = \frac{8}{x^3}$$

$$\therefore x^3 = 8$$

$$\therefore x = 2$$

$$\begin{aligned} \text{When } x = 2, \quad y &= 2(2) + \frac{4}{2^2} \\ &= 4 + 1 \\ &= 5 \end{aligned}$$

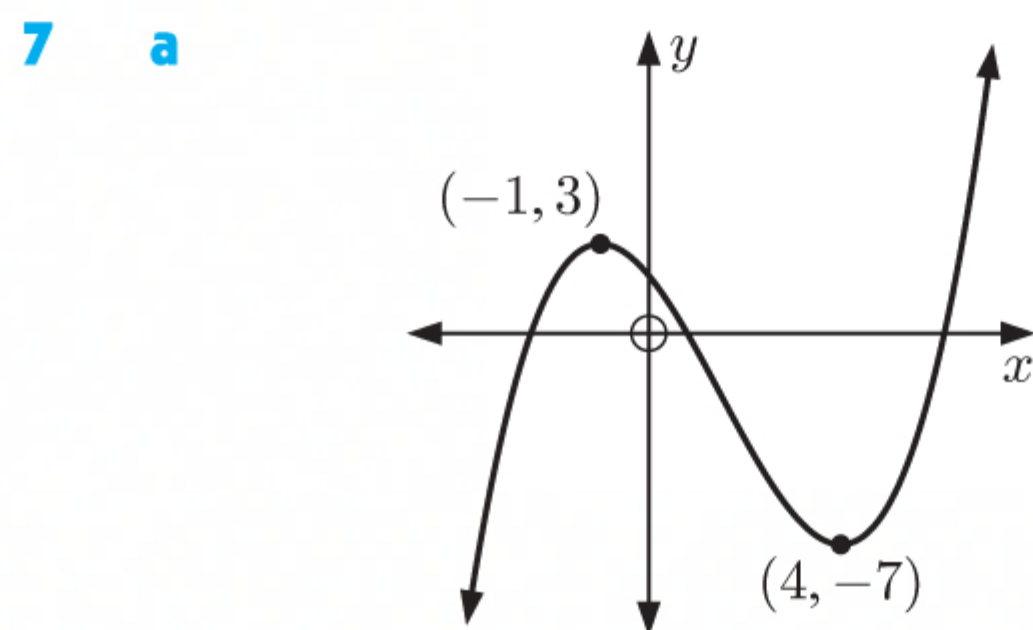
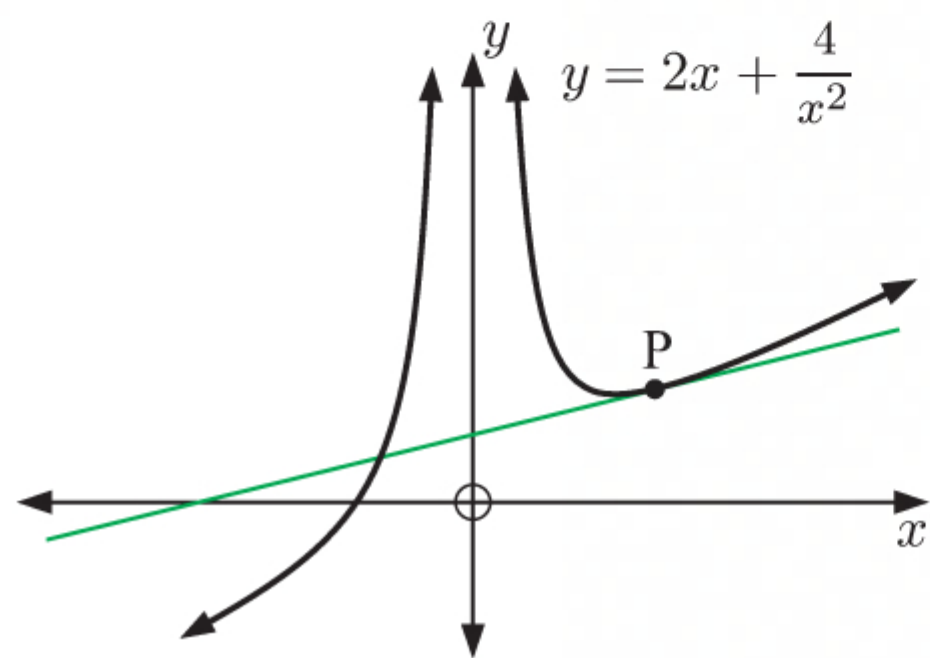
So, the coordinates of P are (2, 5).

b The tangent has equation $y = 1(x - 2) + 5$
 $= x - 2 + 5$
 $= x + 3$

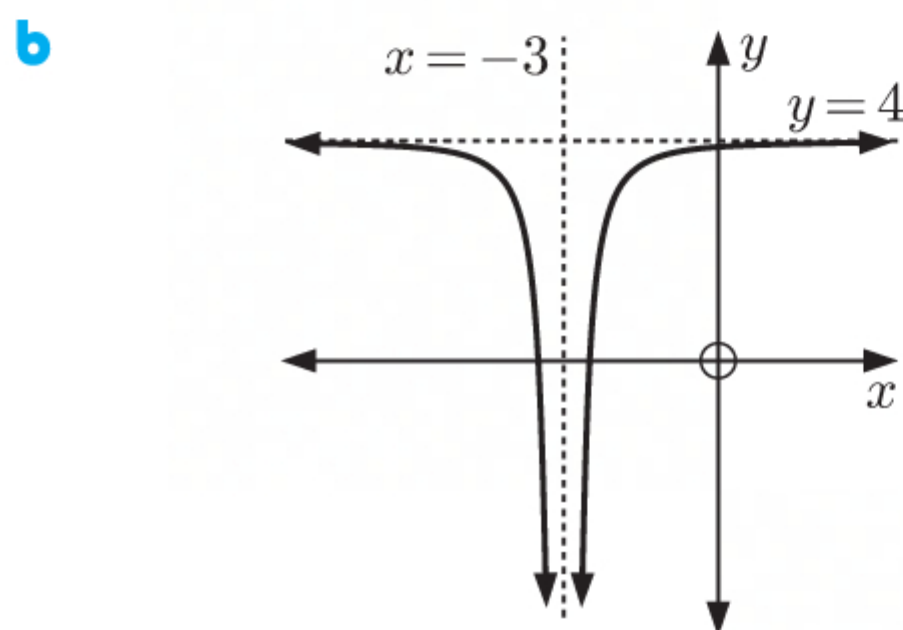
c The tangent cuts the x -axis where $y = 0$
 $\therefore x + 3 = 0$
 $\therefore x = -3$

So, the tangent cuts the x -axis at $(-3, 0)$.

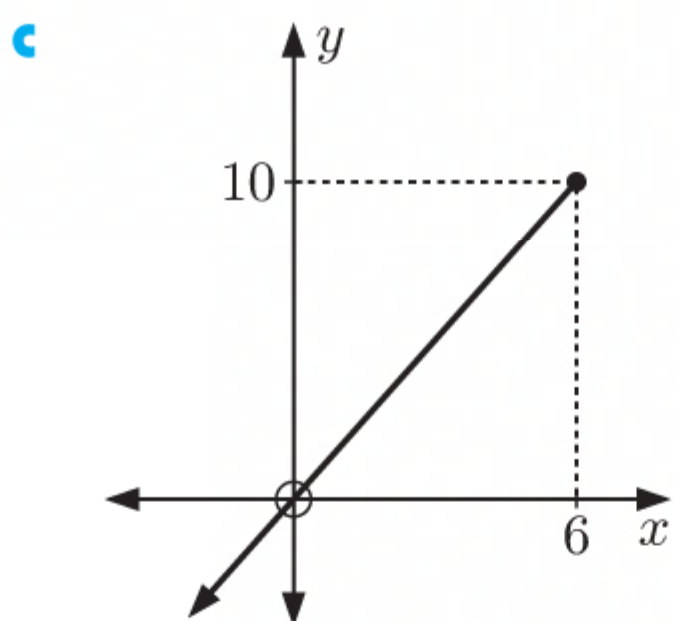
d The normal at P(2, 5) has gradient -1 .
 \therefore the normal has equation $y = -1(x - 2) + 5$
 $= -x + 2 + 5$
 $= -x + 7$



The function is increasing for $x \leq -1$ and $x \geq 4$, and decreasing for $-1 \leq x \leq 4$.



The function is increasing for $x > -3$, and decreasing for $x < -3$.



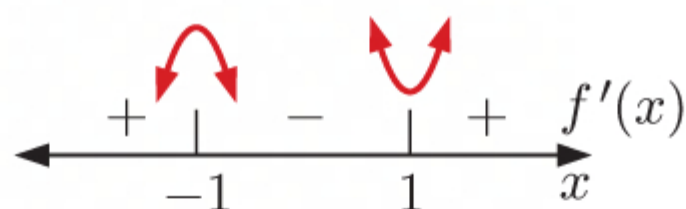
The function is increasing for $x \leq 6$, and never decreasing.

8 $f(x) = x^3 - 3x$

a $f(0) = 0^3 - 3(0) = 0$
 \therefore the y -intercept is 0.

c $f'(x) = 0$ when $3x^2 - 3 = 0$
 $\therefore 3x^2 = 3$
 $\therefore x^2 = 1$
 $\therefore x = \pm 1$

So, $f'(x)$ has sign diagram:

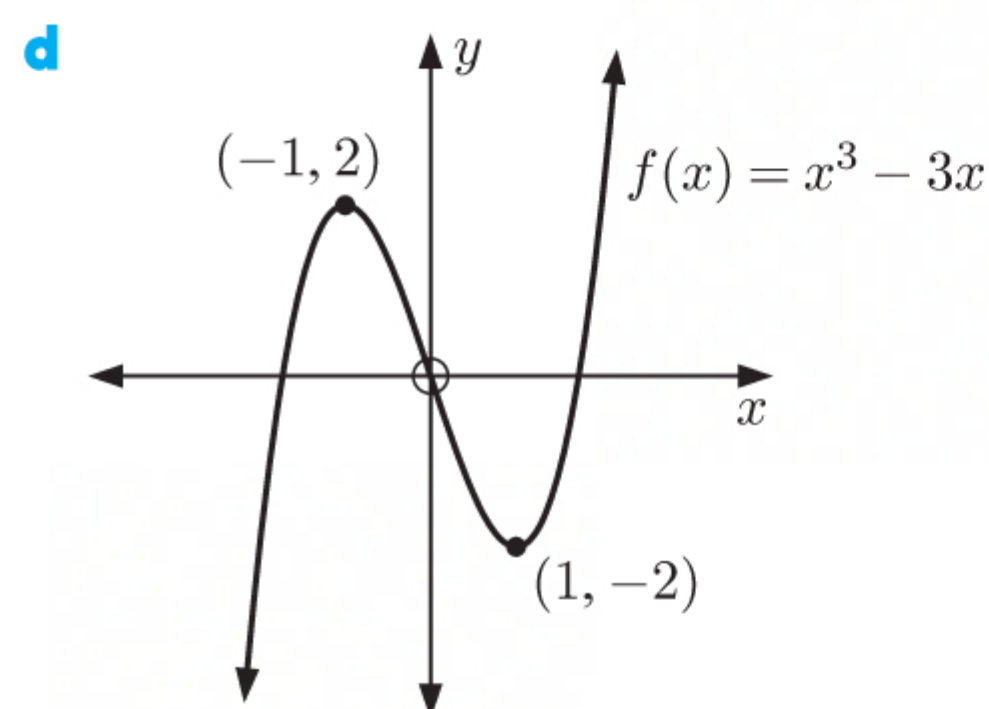


Now $f(-1) = (-1)^3 - 3(-1)$
 $= -1 + 3$
 $= 2$

and $f(1) = 1^3 - 3(1)$
 $= 1 - 3$
 $= -2$

So, $f(x)$ has a local maximum at $(-1, 2)$ and a local minimum at $(1, -2)$.

b $f'(x) = 3x^2 - 3$

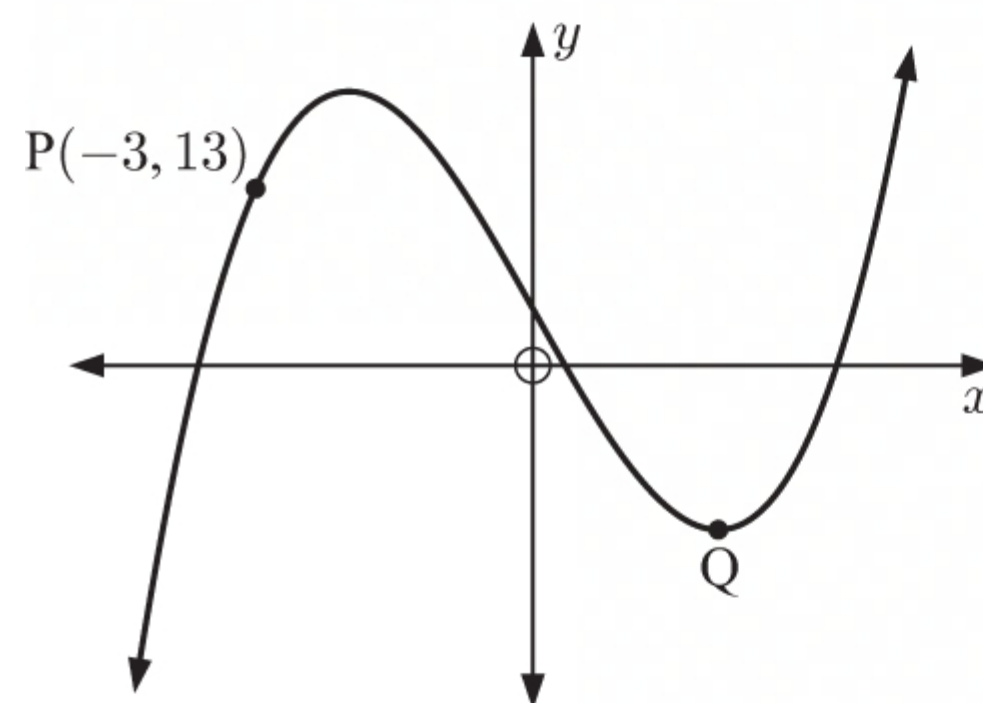


9 a $f(x) = x^3 - 12x + 4$
 $\therefore f'(x) = 3x^2 - 12$

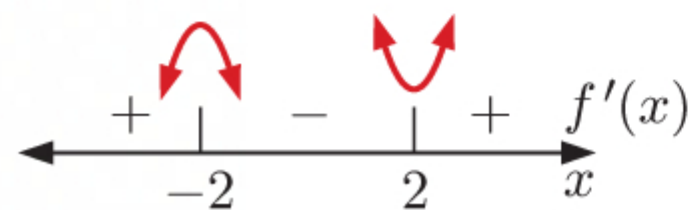
b $f'(-3) = 3(-3)^2 - 12$
 $= 27 - 12$
 $= 15$

So, the tangent at P has gradient 15.

c $f'(x) = 0$ when $3x^2 - 12 = 0$
 $\therefore 3x^2 = 12$
 $\therefore x^2 = 4$
 $\therefore x = \pm 2$



So, $f'(x)$ has sign diagram:



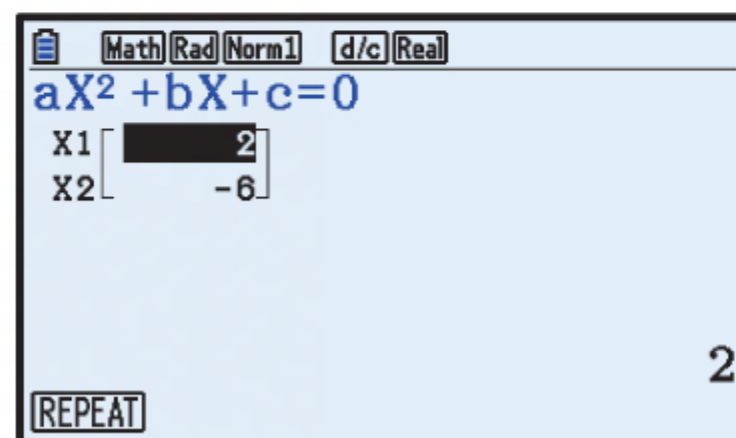
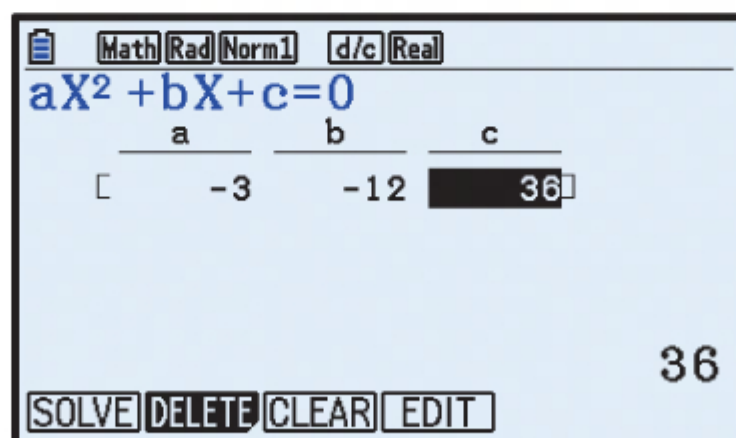
So, the local minimum occurs when $x = 2$.

$$\begin{aligned} f(2) &= 2^3 - 12(2) + 4 \\ &= 8 - 24 + 4 \\ &= -12 \end{aligned}$$

So, Q is at $(2, -12)$.

10 $f(x) = -x^3 - 6x^2 + 36x - 17$
 $\therefore f'(x) = -3x^2 - 12x + 36$

We find the zeros of $f'(x)$ using technology:



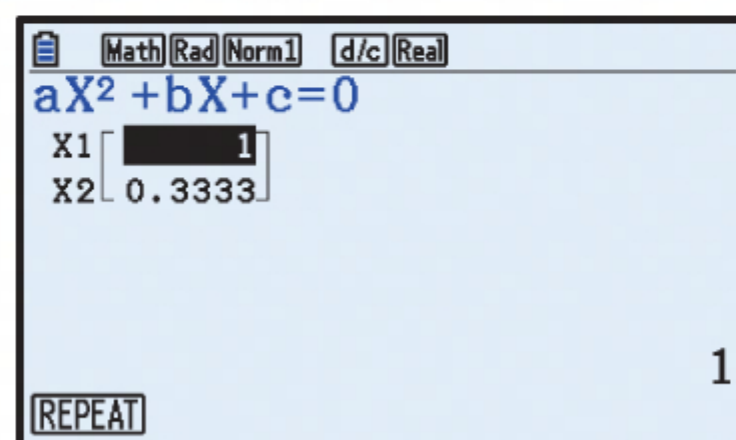
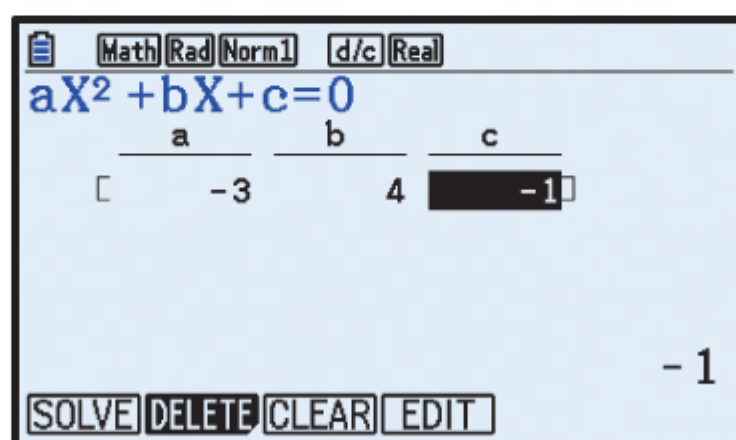
So, $f'(x) = 0$ when $x = -6$ or 2 .

$f'(x)$ has sign diagram:

- a** $f(x)$ is increasing for $-6 \leq x \leq 2$.
- b** $f(x)$ is decreasing for $x \leq -6$ and $x \geq 2$.

11 $f(x) = -x^3 + 2x^2 - x + 3$
 $\therefore f'(x) = -3x^2 + 4x - 1$

We find the zeros of $f'(x)$ using technology:



So, $f'(x) = 0$ when $x = \frac{1}{3}$ or 1 .

$f'(x)$ has sign diagram:

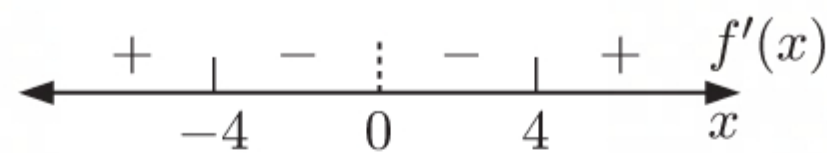
$$\begin{aligned} \text{Now } f\left(\frac{1}{3}\right) &= -\left(\frac{1}{3}\right)^3 + 2\left(\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right) + 3 & \text{and } f(1) &= -(1)^3 + 2(1)^2 - (1) + 3 \\ &= -\frac{1}{27} + \frac{2}{9} - \frac{1}{3} + 3 & &= -1 + 2 - 1 + 3 \\ &= \frac{77}{27} & &= 3 \end{aligned}$$

$\therefore \left(\frac{1}{3}, \frac{77}{27}\right)$ is a local minimum, and $(1, 3)$ is a local maximum.

12 a $f(x) = 3x + 2 + \frac{48}{x}$
 $= 3x + 2 + 48x^{-1}$
 $\therefore f'(x) = 3 - 48x^{-2}$
 $= 3 - \frac{48}{x^2}$
 $f'(x) = 0$ when $3 - \frac{48}{x^2} = 0$
 $\therefore 3 = \frac{48}{x^2}$
 $\therefore 3x^2 = 48$
 $\therefore x^2 = 16$
 $\therefore x = \pm 4$

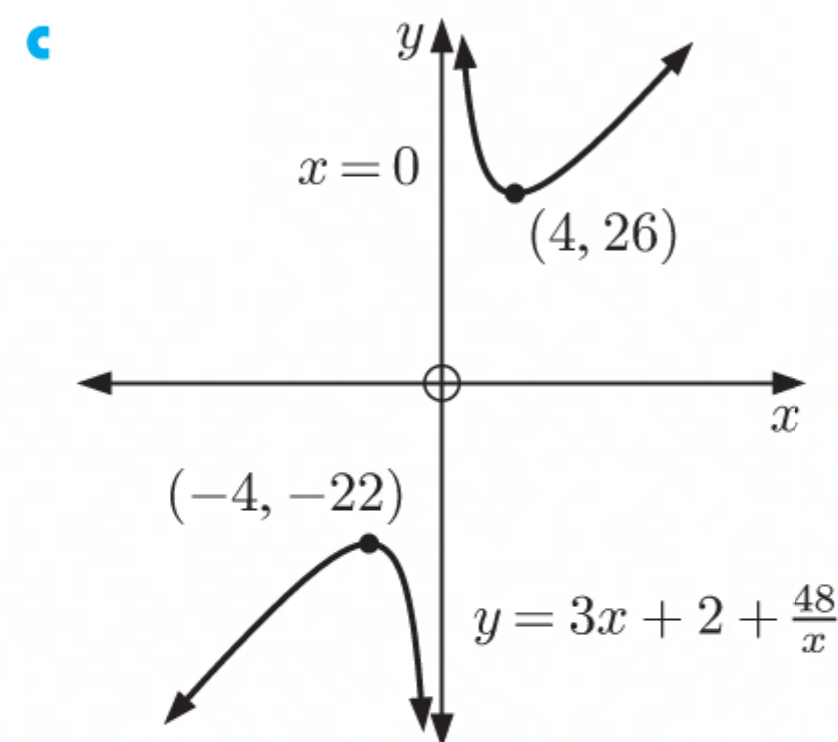
Also, $f'(x)$ is undefined when $x = 0$.

So, $f'(x)$ has sign diagram:



b $f(-4) = 3(-4) + 2 + \frac{48}{-4}$
 $= -12 + 2 - 12$
 $= -22$
 $f(4) = 3(4) + 2 + \frac{48}{4}$
 $= 12 + 2 + 12$
 $= 26$

\therefore there is a local maximum at $(-4, -22)$ and a local minimum at $(4, 26)$.

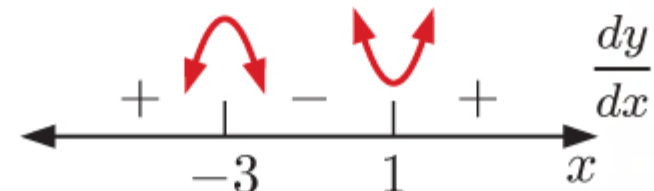


13 a $y = \frac{1}{3}x^3 + x^2 - 3x$, for $-4 \leq x \leq 4$

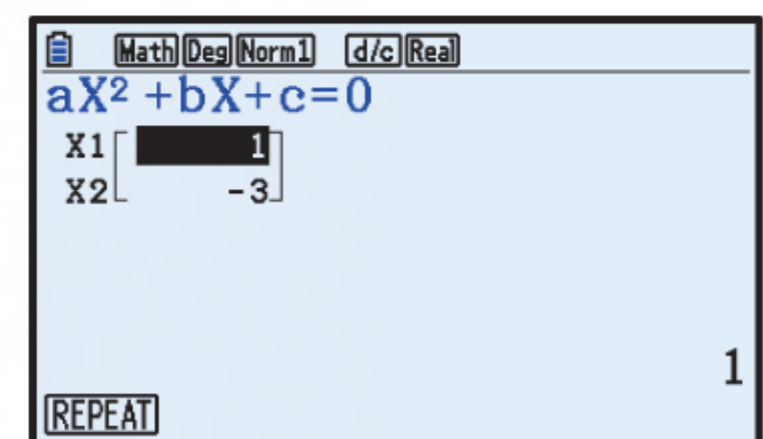
$\therefore \frac{dy}{dx} = x^2 + 2x - 3$

Using technology, $\frac{dy}{dx} = 0$ when $x = -3$ or 1 .

The sign diagram of $\frac{dy}{dx}$ is:



\therefore there is a local maximum at $x = -3$, and a local minimum at $x = 1$.



Critical value (x)	$f(x)$
-4 (end point)	$6\frac{2}{3}$
-3 (local maximum)	9
1 (local minimum)	$-1\frac{2}{3}$
4 (end point)	$25\frac{1}{3}$

The greatest of these values is $25\frac{1}{3}$ when $x = 4$.

The least of these values is $-1\frac{2}{3}$ when $x = 1$.

b Let $y = x + \frac{32}{x^2} = x + 32x^{-2}$, $2 \leq x \leq 10$

$$\therefore \frac{dy}{dx} = 1 - 64x^{-3} = 1 - \frac{64}{x^3}$$

$$\frac{dy}{dx} = 0 \text{ when } 1 - \frac{64}{x^3} = 0$$

$$\therefore 1 = \frac{64}{x^3}$$

$$\therefore x^3 = 64$$

$$\therefore x = \sqrt[3]{64} = 4$$

$\frac{dy}{dx}$ has sign diagram:

\therefore there is a local minimum at $x = 4$.

Critical value (x)	y
2 (end point)	10
4 (local minimum)	6
10 (end point)	$10\frac{8}{25}$

The greatest of these values is $10\frac{8}{25}$ when $x = 10$.

The least of these values is 6 when $x = 4$.

REVIEW SET 11B

1 a $y = x^3 - 3x + 5$

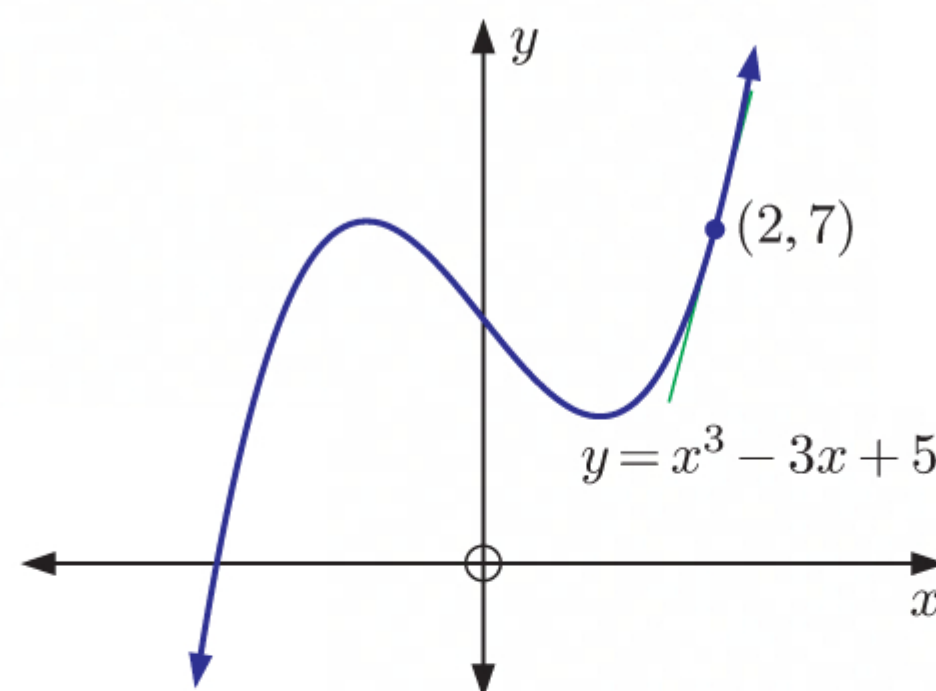
$$\begin{aligned} \text{When } x = 2, \quad y &= 2^3 - 3(2) + 5 \\ &= 8 - 6 + 5 \\ &= 7, \end{aligned}$$

so the point of contact is $(2, 7)$.

Now $\frac{dy}{dx} = 3x^2 - 3$, so at $x = 2$,

$$\begin{aligned} \frac{dy}{dx} &= 3(2)^2 - 3 \\ &= 12 - 3 \\ &= 9 \end{aligned}$$

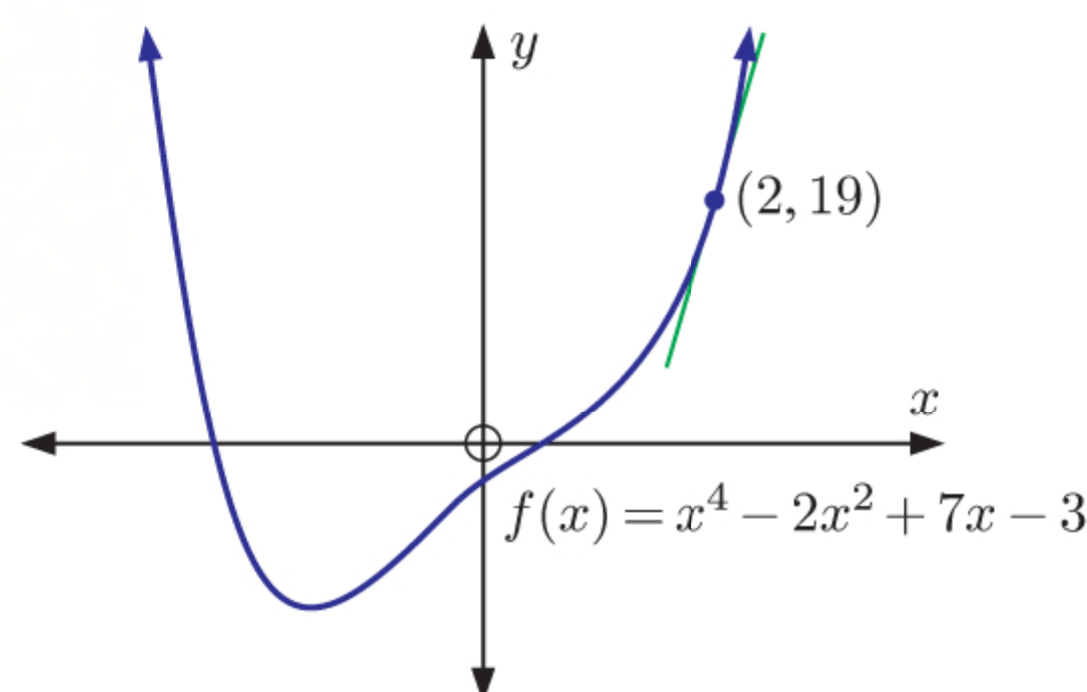
$$\begin{aligned} \text{The tangent has equation } y &= 9(x - 2) + 7 \\ &= 9x - 18 + 7 \\ &= 9x - 11 \end{aligned}$$



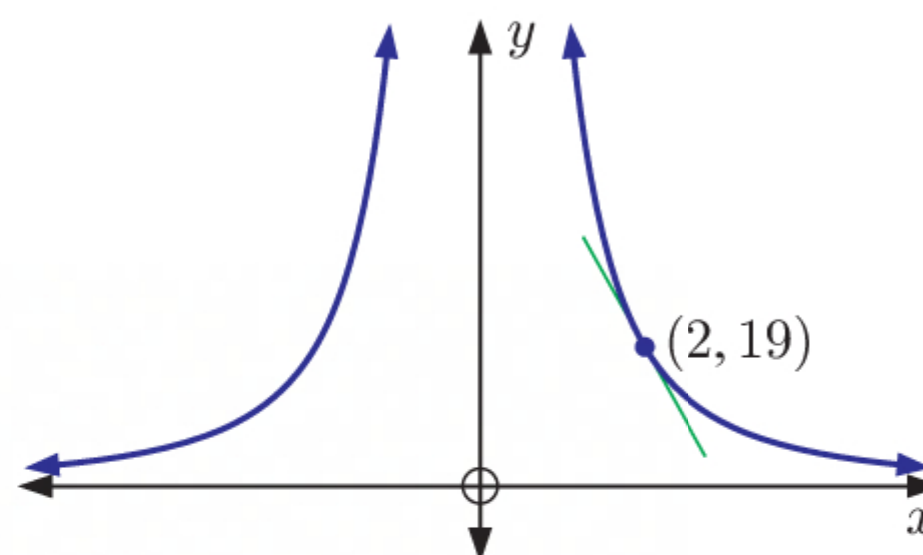
$$\begin{aligned}
 \text{b} \quad f(x) &= x^4 - 2x^2 + 7x - 3 \\
 \therefore f'(x) &= 4x^3 - 4x + 7 \\
 \therefore f'(2) &= 4(2)^3 - 4(2) + 7 \\
 &= 32 - 8 + 7 \\
 &= 31
 \end{aligned}$$

The tangent has equation

$$\begin{aligned}
 y &= 31(x - 2) + 19 \\
 &= 31x - 62 + 19 \\
 &= 31x - 43
 \end{aligned}$$



$$\begin{aligned}
 \text{c} \quad y &= \frac{12}{x^2} = 12x^{-2} \\
 \therefore \frac{dy}{dx} &= -24x^{-3} \\
 &= -\frac{24}{x^3}, \text{ so at } (1, 12), \\
 \frac{dy}{dx} &= -\frac{24}{1^3} = -24
 \end{aligned}$$



$$\begin{aligned}
 \text{So, the tangent has equation } y &= -24(x - 1) + 12 \\
 &= -24x + 24 + 12 \\
 &= -24x + 36
 \end{aligned}$$

$$\begin{aligned}
 \text{2 a} \quad y &= 3 + 2x - x^2 \\
 \therefore \frac{dy}{dx} &= 2 - 2x
 \end{aligned}$$

$$\begin{aligned}
 \text{When } x = 2, \quad \frac{dy}{dx} &= 2 - 2(2) \\
 &= 2 - 4 \\
 &= -2
 \end{aligned}$$

So, the normal at (2, 3) has gradient $\frac{1}{2}$.

$$\begin{aligned}
 \therefore \text{ the normal has equation } y &= \frac{1}{2}(x - 2) + 3 \\
 &= \frac{1}{2}x - 1 + 3 \\
 &= \frac{1}{2}x + 2
 \end{aligned}$$

$$\text{b} \quad y = \frac{1}{x^2} - \frac{2}{x} = x^{-2} - 2x^{-1}$$

$$\text{When } x = 1, \quad y = \frac{1}{1^2} - \frac{2}{1} = 1 - 2 = -1$$

So, the point of contact is (1, -1).

$$\begin{aligned}
 \text{Now } \frac{dy}{dx} &= -2x^{-3} + 2x^{-2} \\
 &= -\frac{2}{x^3} + \frac{2}{x^2}
 \end{aligned}$$

$$\text{When } x = 1, \quad \frac{dy}{dx} = -\frac{2}{1^3} + \frac{2}{1^2} = -2 + 2 = 0$$

\therefore the gradient of the normal at (1, -1) is undefined.

So, the normal must be a vertical line.

Since the normal passes through (1, -1), the equation of the normal is $x = 1$.

3 $y = x^3 + ax^2 - 4x + 3$

a $\frac{dy}{dx} = 3x^2 + 2ax - 4$

The tangent at $x = 1$ is parallel to the line $y = 3x$, and $y = 3x$ has gradient 3.

\therefore the tangent at $x = 1$ has gradient 3.

$$\therefore 3(1)^2 + 2a(1) - 4 = 3$$

$$\therefore 3 + 2a - 4 = 3$$

$$\therefore 2a = 4$$

$$\therefore a = 2$$

b Since $a = 2$, $y = x^3 + 2x^2 - 4x + 3$ and $\frac{dy}{dx} = 3x^2 + 4x - 4$

When $x = 1$, $y = 1^3 + 2(1)^2 - 4(1) + 3 = 2$

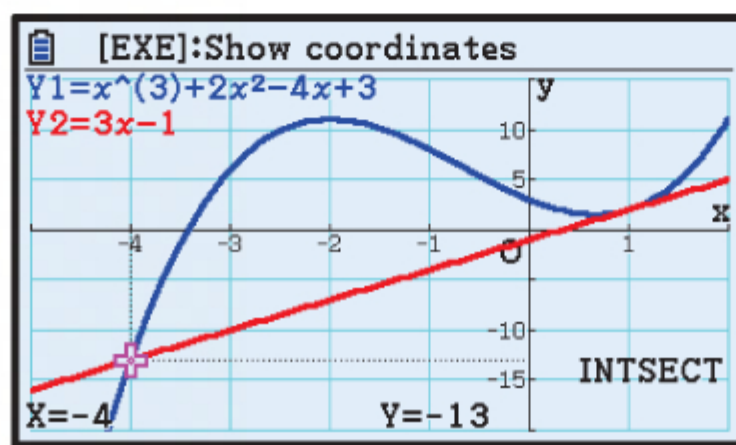
and $\frac{dy}{dx} = 3(1)^2 + 4(1) - 4 = 3$

So, the point of contact is $(1, 2)$, and the tangent at $(1, 2)$ has gradient 3.

\therefore the tangent has equation $y = 3(x - 1) + 2$

$$\text{which is } y = 3x - 1$$

c We use technology to find where the tangent meets the curve again:



The tangent meets the curve again at $(-4, -13)$.

4 $y = x^2 - 4x + 2$

$$\therefore \frac{dy}{dx} = 2x - 4$$

When $x = 3$, $y = (3)^2 - 4(3) + 2$ and $\frac{dy}{dx} = 2(3) - 4$
 $= 9 - 12 + 2$ $= 6 - 4$
 $= -1$ $= 2$

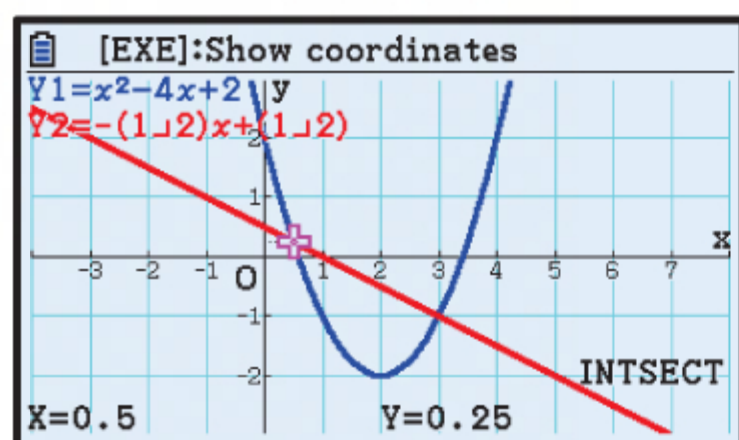
So, the point of contact is $(3, -1)$ and the normal at $(3, -1)$ has gradient $-\frac{1}{2}$.

\therefore the normal has equation $y = -\frac{1}{2}(x - 3) + (-1)$

$$= -\frac{1}{2}x + \frac{3}{2} - 1$$

$$= -\frac{1}{2}x + \frac{1}{2}$$

We use technology to find where the normal meets the curve again:



The normal meets the curve again at $(\frac{1}{2}, \frac{1}{4})$.

5 $y = x^3 - 2x^2 + ax - b$

$$\therefore \frac{dy}{dx} = 3x^2 - 4x + a$$

The tangent to $y = x^3 - 2x^2 + ax - b$ at $(2, -1)$ is $y = 7x - 15$.

\therefore the tangent has gradient 7, and the point of contact is $(2, -1)$.

$$\begin{aligned} \text{When } x = 2, \quad \frac{dy}{dx} &= 7 \\ \therefore 3(2)^2 - 4(2) + a &= 7 \\ \therefore 12 - 8 + a &= 7 \\ \therefore a &= 3 \quad \dots (*) \end{aligned}$$

$$\begin{aligned} \text{When } x = 2, \quad y &= -1 \\ \therefore 2^3 - 2(2)^2 + 2a - b &= -1 \\ \therefore 8 - 8 + 2(3) - b &= -1 \quad \{\text{using } (*)\} \\ \therefore 6 - b &= -1 \\ \therefore b &= 7 \end{aligned}$$

So, $a = 3$ and $b = 7$.

6 $y = -3x^3 + 5x - 1$

$$\therefore \frac{dy}{dx} = -9x^2 + 5$$

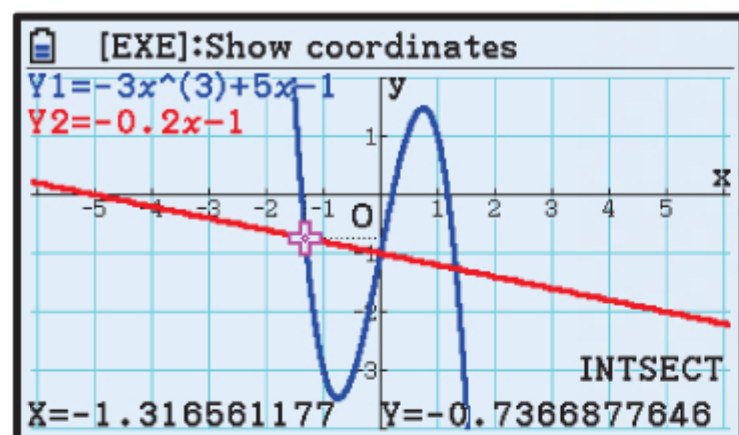
$$\begin{aligned} \text{When } x = 0, \quad y &= -3(0)^3 + 5(0) - 1 = -1 \quad \text{and} \quad \frac{dy}{dx} = -9(0)^2 + 5 = 5 \end{aligned}$$

So, the point of contact is $(0, -1)$ and the normal at $(0, -1)$ has gradient $-\frac{1}{5}$.

\therefore the normal has equation $y = -\frac{1}{5}(x - 0) - 1$

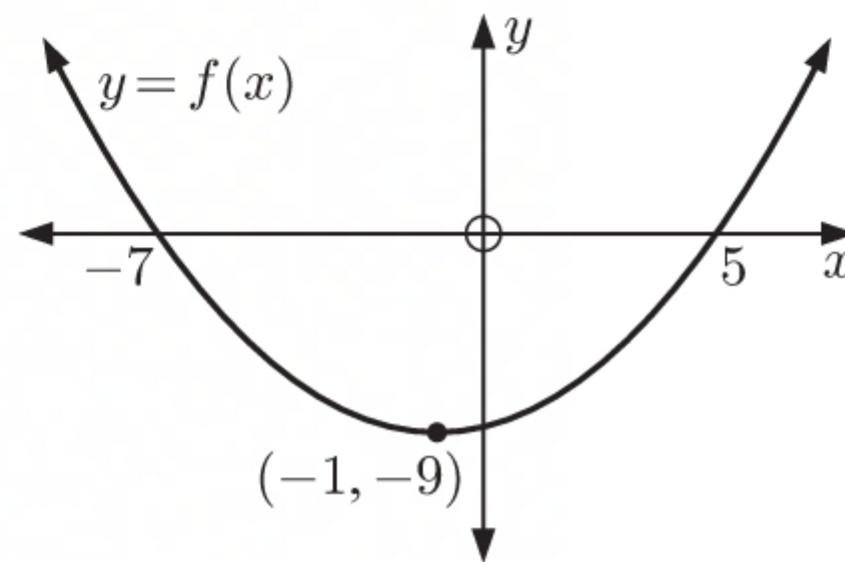
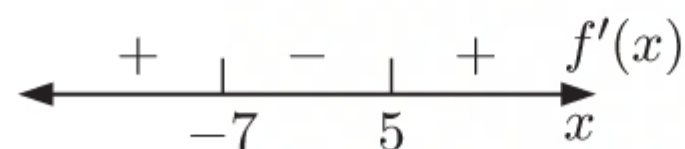
$$\therefore y = -\frac{1}{5}x - 1$$

We use technology to find where the normal meets the curve again:



The normal meets the curve again at about $(-1.32, -0.737)$ and about $(1.32, -1.26)$.

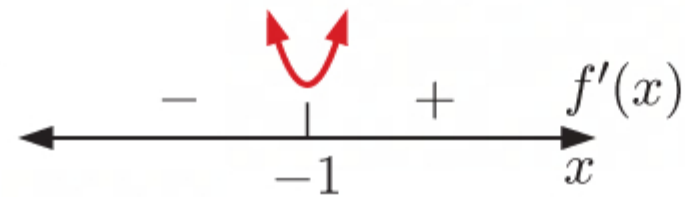
- 7 a $f(x)$ has sign diagram:



- b $f(x)$ is increasing for $x \geq -1$.

$f(x)$ is decreasing for $x \leq -1$.

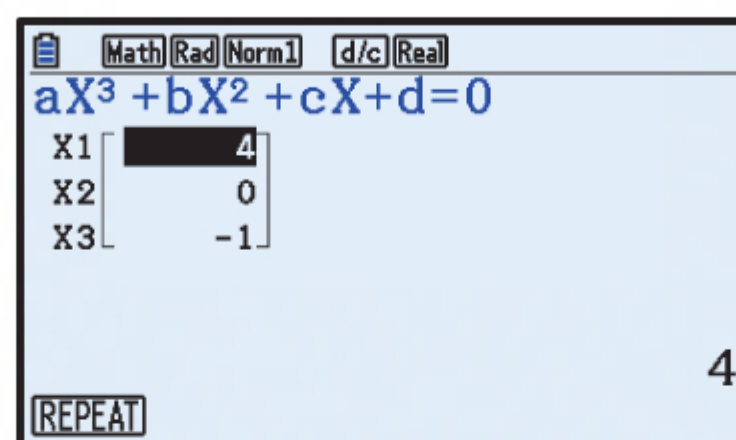
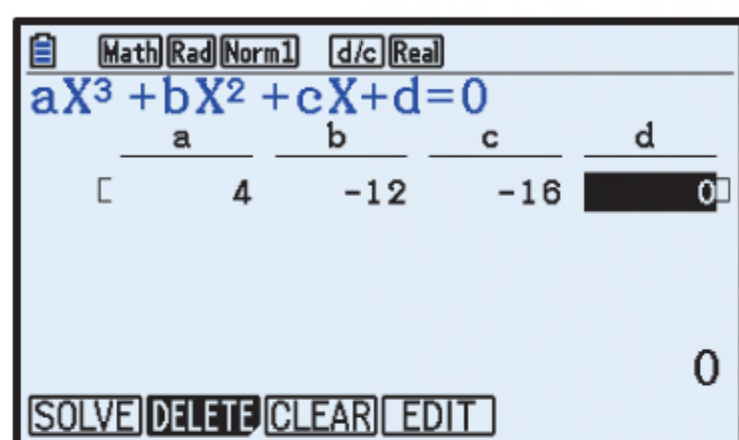
$f'(x)$ has sign diagram:



- 8 $f(x) = x^4 - 4x^3 - 8x^2 + 5$

$$\therefore f'(x) = 4x^3 - 12x^2 - 16x$$

We find the zeros of $f'(x)$ using technology:



So, $f'(x) = 0$ when $x = -1$, 0 , or 4 .

$f'(x)$ has sign diagram:

- a $f(x)$ is increasing for $-1 \leq x \leq 0$ and $x \geq 4$.

- b $f(x)$ is decreasing for $x \leq -1$ and $0 \leq x \leq 4$.

- 9 $f(x) = x^3 - 3x^2 + ax + 50$

$$\therefore f'(x) = 3x^2 - 6x + a$$

- a $f(x)$ has a stationary point at $x = 3$

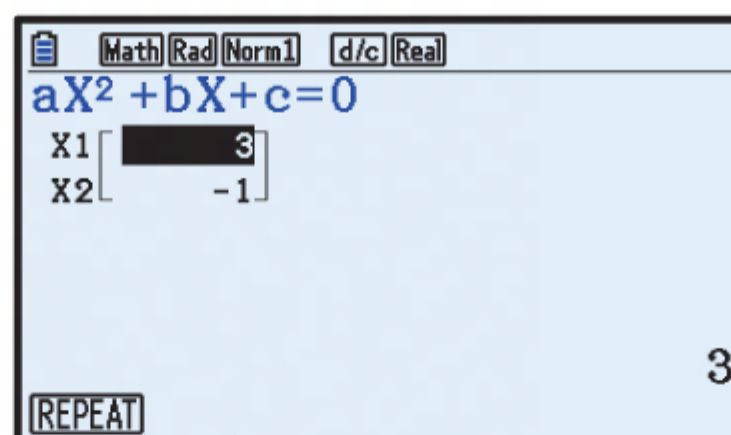
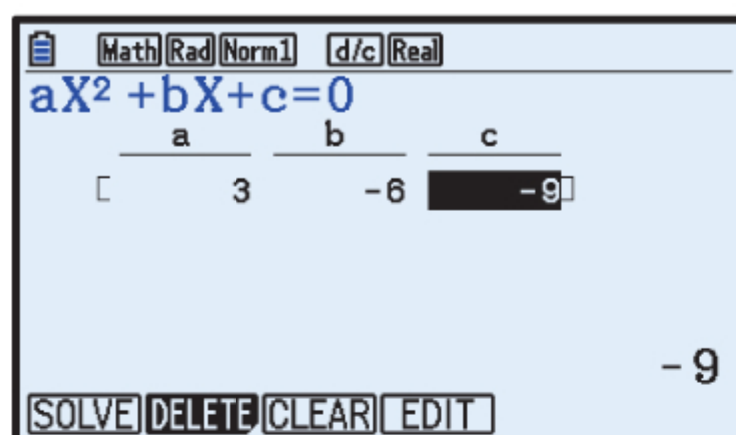
$$\therefore f'(3) = 0$$

$$\therefore 3(3)^2 - 6(3) + a = 0$$

$$\therefore 27 - 18 + a = 0$$

$$\therefore a = -9$$

- b Since $a = -9$, then $f(x) = x^3 - 3x^2 - 9x + 50$
and $f'(x) = 3x^2 - 6x - 9$



Using technology, $f'(x) = 0$ when $x = -1$ or 3 .

\therefore the sign diagram for $f'(x)$ is:

Now $f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 50 = 55$

and $f(3) = 3^3 - 3(3)^2 - 9(3) + 50 = 23$

So, there is a local maximum at $(-1, 55)$ and a local minimum at $(3, 23)$.

10 $f(x) = x^3 + Ax + B$

$\therefore f'(x) = 3x^2 + A$

a $f(x)$ has a stationary point at $(1, 5)$

$\therefore f'(1) = 0$

$\therefore 3(1)^2 + A = 0$

$\therefore 3 + A = 0$

$\therefore A = -3$

Also, $f(1) = 5 \quad \therefore (1)^3 - 3(1) + B = 5$

$\therefore 1 - 3 + B = 5$

$\therefore B = 7$

b Since $A = -3$ and $B = 7$, $f(x) = x^3 - 3x + 7$
and $f'(x) = 3x^2 - 3$

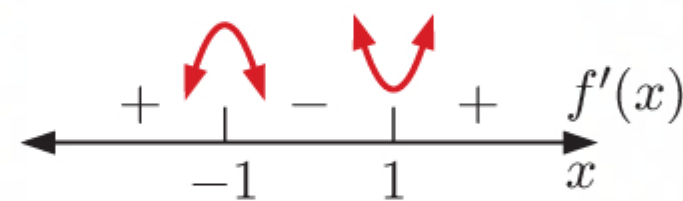
Now $f'(x) = 0$ when $3x^2 - 3 = 0$

$\therefore 3x^2 = 3$

$\therefore x^2 = 1$

$\therefore x = \pm 1$

So $f'(x)$ has sign diagram:



Now $f(-1) = (-1)^3 - 3(-1) + 7 = 9$

and $f(3) = 1^3 - 3(1) + 7 = 5$

So, there is a local maximum at $(-1, 9)$ and a local minimum at $(1, 5)$.

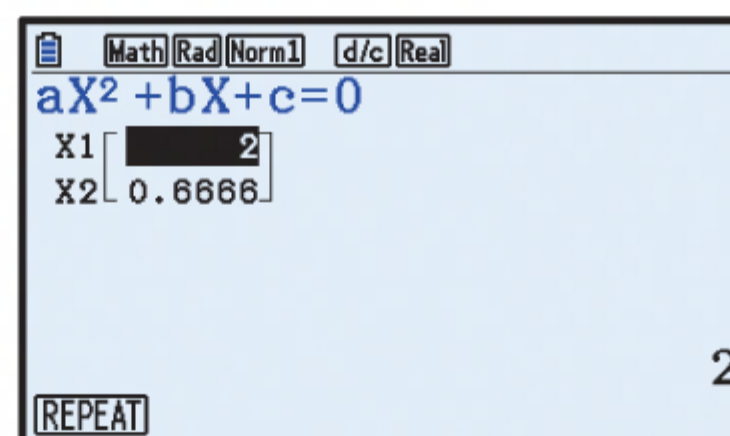
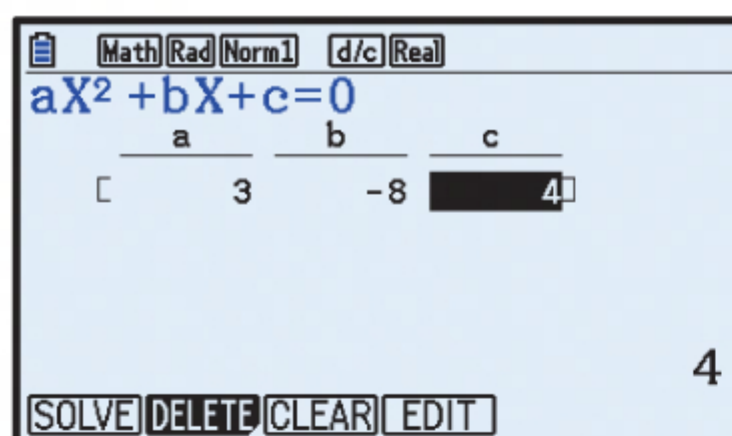
11 $f(x) = x^3 - 4x^2 + 4x$

a $f(0) = 0^3 - 4(0)^2 + 4(0) = 0$

\therefore the y -intercept is 0.

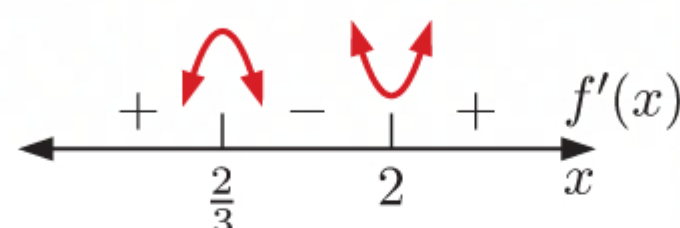
b $f'(x) = 3x^2 - 8x + 4$

We find the zeros of $f'(x)$ using technology:



So, $f'(x) = 0$ when $x = \frac{2}{3}$ or 2

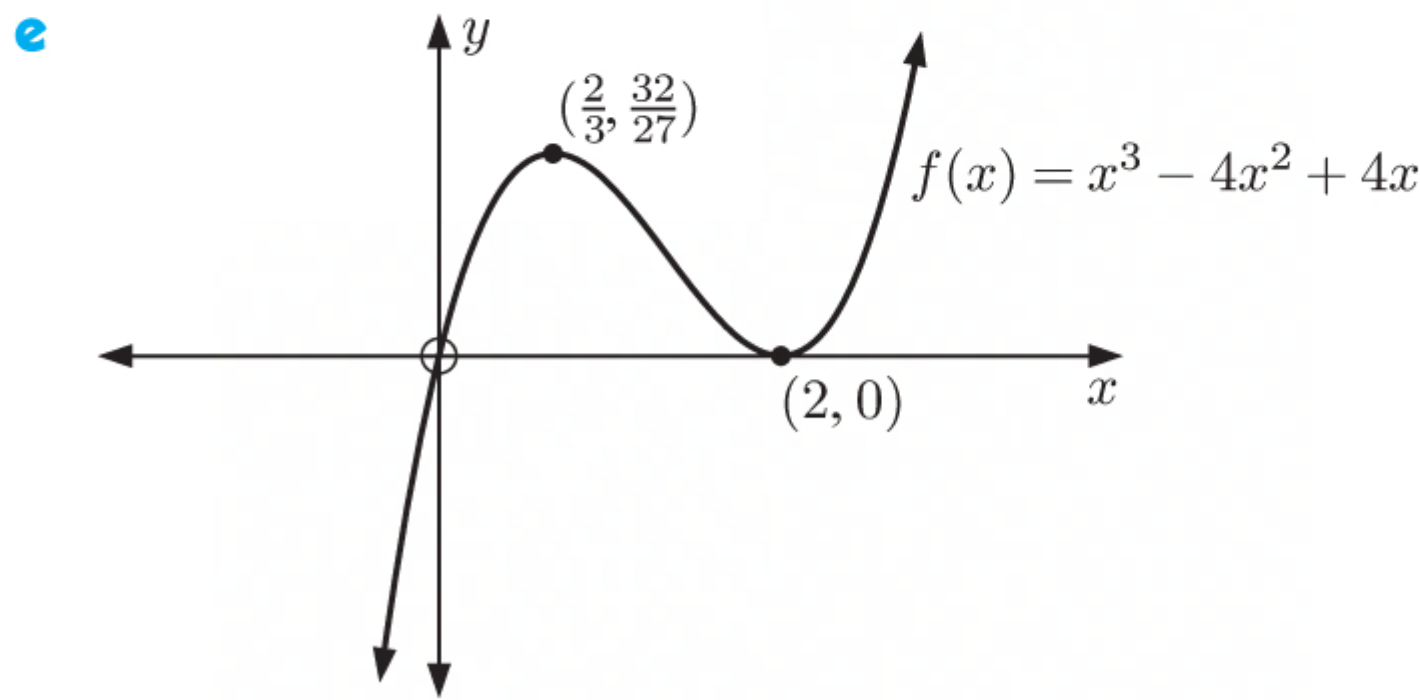
$f'(x)$ has sign diagram:



- $f(x)$ is increasing for $x \leq \frac{2}{3}$ and $x \geq 2$.
 $f(x)$ is decreasing for $\frac{2}{3} \leq x \leq 2$.

d Now $f(\frac{2}{3}) = (\frac{2}{3})^3 - 4(\frac{2}{3})^2 + 4(\frac{2}{3})$
 $= \frac{8}{27} - \frac{16}{9} + \frac{8}{3}$
 $= \frac{32}{27}$
and $f(2) = 2^3 - 4(2)^2 + 4(2)$
 $= 8 - 16 + 8$
 $= 0$

So, $f(x)$ has a local maximum at $(\frac{2}{3}, \frac{32}{27})$ and a local minimum at $(2, 0)$.



12 Let $f(x) = x^3 - 3x^2 + 5$, for $-1 \leq x \leq 4$
 $\therefore f'(x) = 3x^2 - 6x$
 $= 3x(x - 2)$

which is 0 when $x = 0$ or 2

The sign diagram of $f'(x)$ is:

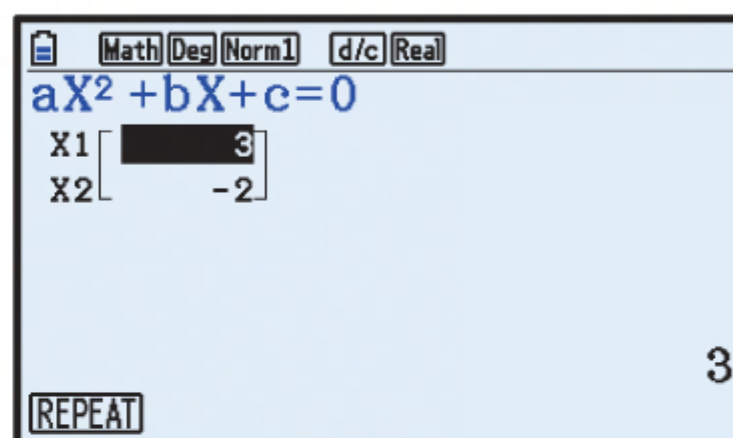
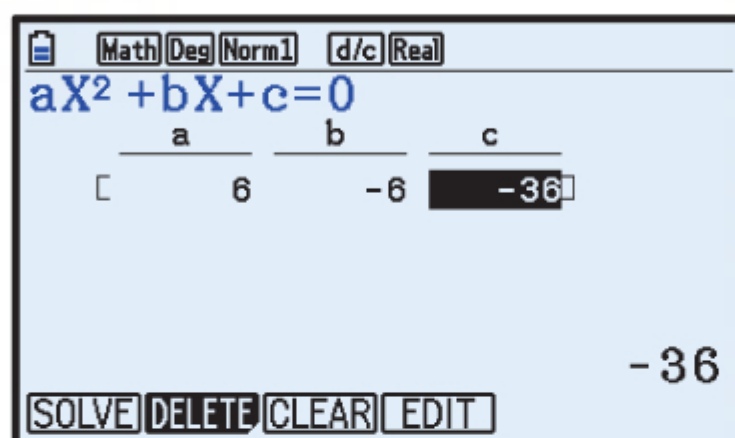
\therefore there is a local maximum at $x = 0$, and a local minimum at $x = 2$.

Critical value (x)	$f(x)$
-1 (end point)	1
0 (local maximum)	5
2 (local minimum)	1
4 (end point)	21

The maximum value is 21 when $x = 4$.

The minimum value is 1 when $x = -1$ or 2 .

13 a $f(x) = 2x^3 - 3x^2 - 36x + 7$
 $\therefore f'(x) = 6x^2 - 6x - 36$

b

Using technology, $f'(x) = 0$ when $x = -2$ or 3 .

\therefore the sign diagram for $f'(x)$ is:

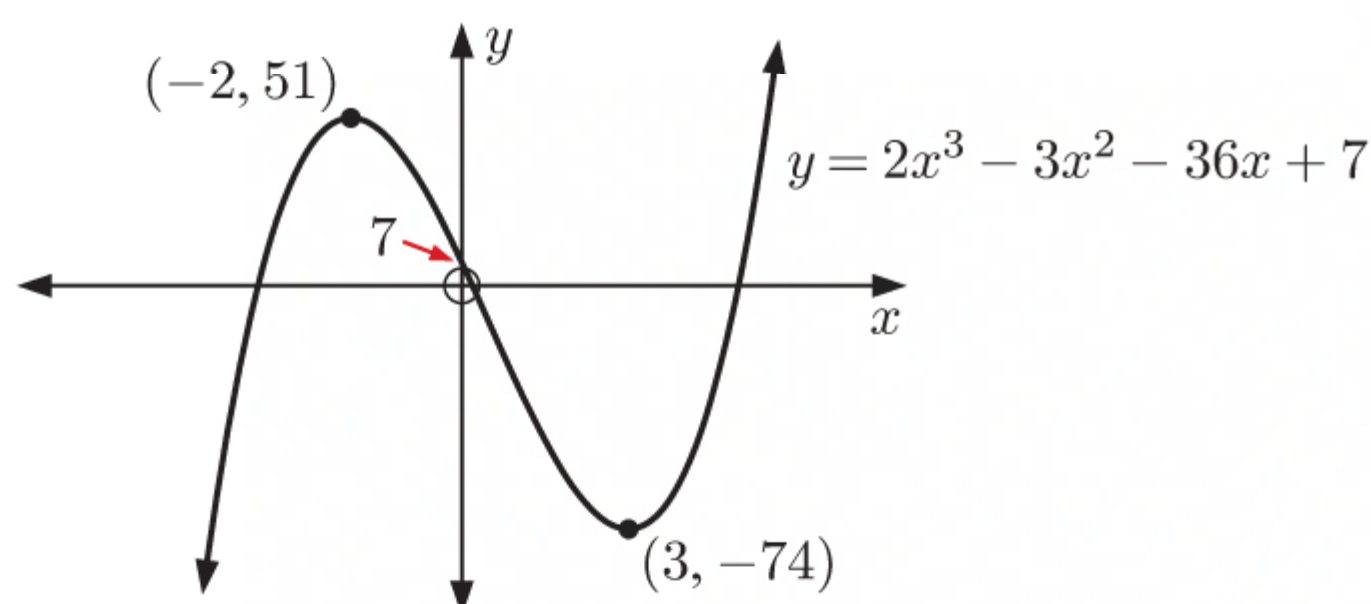


$$\begin{aligned} f(-2) &= 2(-2)^3 - 3(-2)^2 - 36(-2) + 7 \\ &= -16 - 12 + 72 + 7 \\ &= 51 \end{aligned}$$

$$\begin{aligned} f(3) &= 2(3)^3 - 3(3)^2 - 36(3) + 7 \\ &= 54 - 27 - 108 + 7 \\ &= -74 \end{aligned}$$

So, there is a local maximum at $(-2, 51)$ and a local minimum at $(3, -74)$.

- c** $f(x)$ is increasing for $x \leq -2$ and $x \geq 3$.
 $f(x)$ is decreasing for $-2 \leq x \leq 3$.

d

Chapter 12

APPLICATIONS OF DIFFERENTIATION

EXERCISE 12A.1

1 a $M = t^3 - 3t^2 + 1$

$$\therefore \frac{dM}{dt} = 3t^2 - 6t$$

b $R = (2t + 1)^2$

$$= 4t^2 + 4t + 1$$

$$\therefore \frac{dR}{dt} = 8t + 4$$

2 a i Since A is in cm^2 and t is in seconds, $\frac{dA}{dt}$ is in square centimetres per second, or $\text{cm}^2 \text{s}^{-1}$.

ii $\frac{dA}{dt}$ tells us the rate at which the area is changing after t seconds.

b i Since V is in m^3 and t is in minutes, $\frac{dV}{dt}$ is in cubic metres per minute, or $\text{m}^3 \text{min}^{-1}$.

ii $\frac{dV}{dt}$ tells us the rate at which the volume is changing after t minutes.

3 a $P(t) = 2t^2 - 12t + 118$ thousand dollars

$$\begin{aligned} P(0) &= 2(0)^2 - 12(0) + 118 \\ &= 118 \end{aligned}$$

\therefore the current annual profit is \$118 000.

b $P = 2t^2 - 12t + 118$

$$\therefore \frac{dP}{dt} = 4t - 12 \text{ thousand dollars per year}$$

c When $t = 8$, $\frac{dP}{dt} = 4(8) - 12$
 $= 32 - 12$
 $= 20$

This means that in 8 years from now, profits will be increasing at a rate of \$20 000 per year.

4 a $V = 2(50 - t)^2 \text{ m}^3$

$$\begin{aligned} \text{When } t = 0, \quad V &= 2(50)^2 \\ &= 2 \times 2500 \\ &= 5000 \text{ m}^3 \end{aligned}$$

$$\begin{aligned} \text{When } t = 5, \quad V &= 2(50 - 5)^2 \\ &= 2(45)^2 \\ &= 2 \times 2025 \\ &= 4050 \text{ m}^3 \end{aligned}$$

\therefore the average rate at which the water evaporates in the first 5 days is

$$\frac{5000 - 4050}{5} = \frac{950}{5} = 190 \text{ m}^3 \text{ per day.}$$

$$\begin{aligned}
 \text{b} \quad V &= 2(50 - t)^2 \\
 &= 2(250 - 100t + t^2) \\
 &= 500 - 200t + 2t^2
 \end{aligned}$$

$$\therefore \frac{dV}{dt} = 4t - 200$$

$$\begin{aligned}
 \text{When } t = 5, \quad \frac{dV}{dt} &= 4(5) - 200 \\
 &= 20 - 200 \\
 &= -180
 \end{aligned}$$

\therefore the instantaneous rate at which the water is evaporating at $t = 5$ days is 180 m^3 per day.

$$\begin{aligned}
 \text{5 a} \quad B(t) &= 0.3t^3 + 30t + 150 \text{ million} \\
 \therefore B'(t) &= 0.9t^2 + 30 \text{ million per hour}
 \end{aligned}$$

$B'(t)$ is the instantaneous rate of growth of the bacteria.

$$\begin{aligned}
 \text{b} \quad B'(3) &= 0.9(3)^2 + 30 \\
 &= 38.1 \text{ million per hour}
 \end{aligned}$$

After 3 hours, the bacteria are increasing at a rate of 38.1 million per hour.

$$\begin{aligned}
 \text{c} \quad t^2 &\geq 0 \text{ for all } 0 \leq t \leq 10 \\
 \therefore 0.9t^2 &\geq 0 \text{ for all } 0 \leq t \leq 10 \\
 \therefore 0.9t^2 + 30 &> 0 \text{ for all } 0 \leq t \leq 10
 \end{aligned}$$

So, $B'(t) = 0.9t^2 + 30$ is always positive for $0 \leq t \leq 10$.

$\therefore B(t)$ is increasing over the first 10 hours.

$$\text{6 } s(t) = 1.2 + 28.1t - 4.9t^2$$

$$\text{a } s(0) = 1.2$$

The ball was released 1.2 m above the ground.

$$\text{b } s'(t) = 28.1 - 9.8t$$

This is the speed of the ball (in m s^{-1}) after t seconds.

$$\text{c } \text{When } s'(t) = 0, \quad 28.1 - 9.8t = 0$$

$$\therefore 28.1 = 9.8t$$

$$\therefore t = \frac{28.1}{9.8} \approx 2.87$$

After about 2.87 seconds, the ball has reached its maximum height.

$$\begin{aligned}
 \text{d } s(2.87) &\approx 1.2 + 28.1(2.87) - 4.9(2.87)^2 \\
 &\approx 41.5
 \end{aligned}$$

The maximum height reached by the ball is about 41.5 m.

$$\text{e i } s'(0) = 28.1$$

The ball's speed when released is 28.1 m s^{-1} .

$$\begin{aligned}
 \text{ii } s'(2) &= 28.1 - 9.8(2) \\
 &= 8.5
 \end{aligned}$$

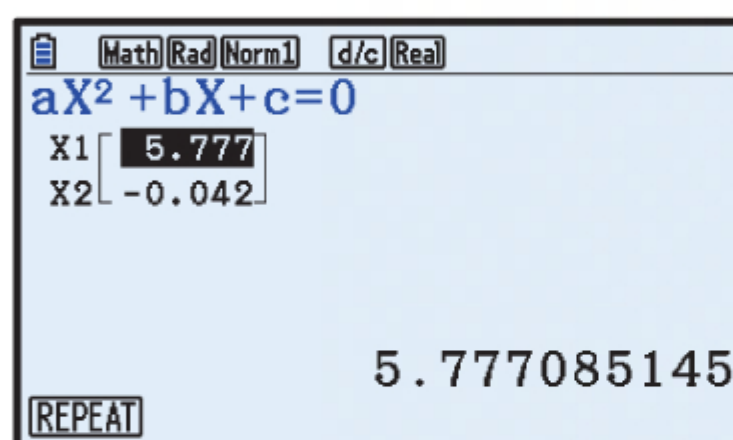
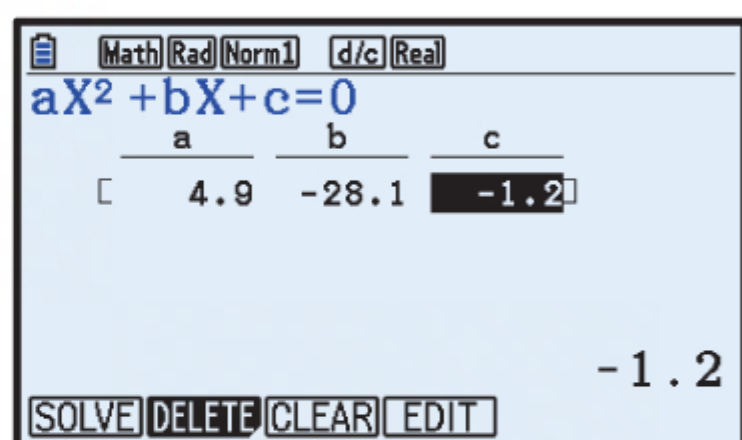
The ball's speed at $t = 2 \text{ s}$ is 8.5 m s^{-1} .

$$\begin{aligned} \text{iii } s'(5) &= 28.1 - 9.8(5) \\ &= -20.9 \end{aligned}$$

The ball's speed at $t = 5$ s is -20.9 m s^{-1} .

The sign tells us whether the ball is travelling upwards (+) or downwards (-).

$$\begin{aligned} \text{f } s(t) &= 0 \quad \text{when} \quad 1.2 + 28.1t - 4.9t^2 = 0 \\ &\therefore 4.9t^2 - 28.1t - 1.2 = 0 \end{aligned}$$



Using technology, $t \approx -0.0424$ or 5.78

But $t > 0$, so the ball hits the ground after about 5.78 seconds.

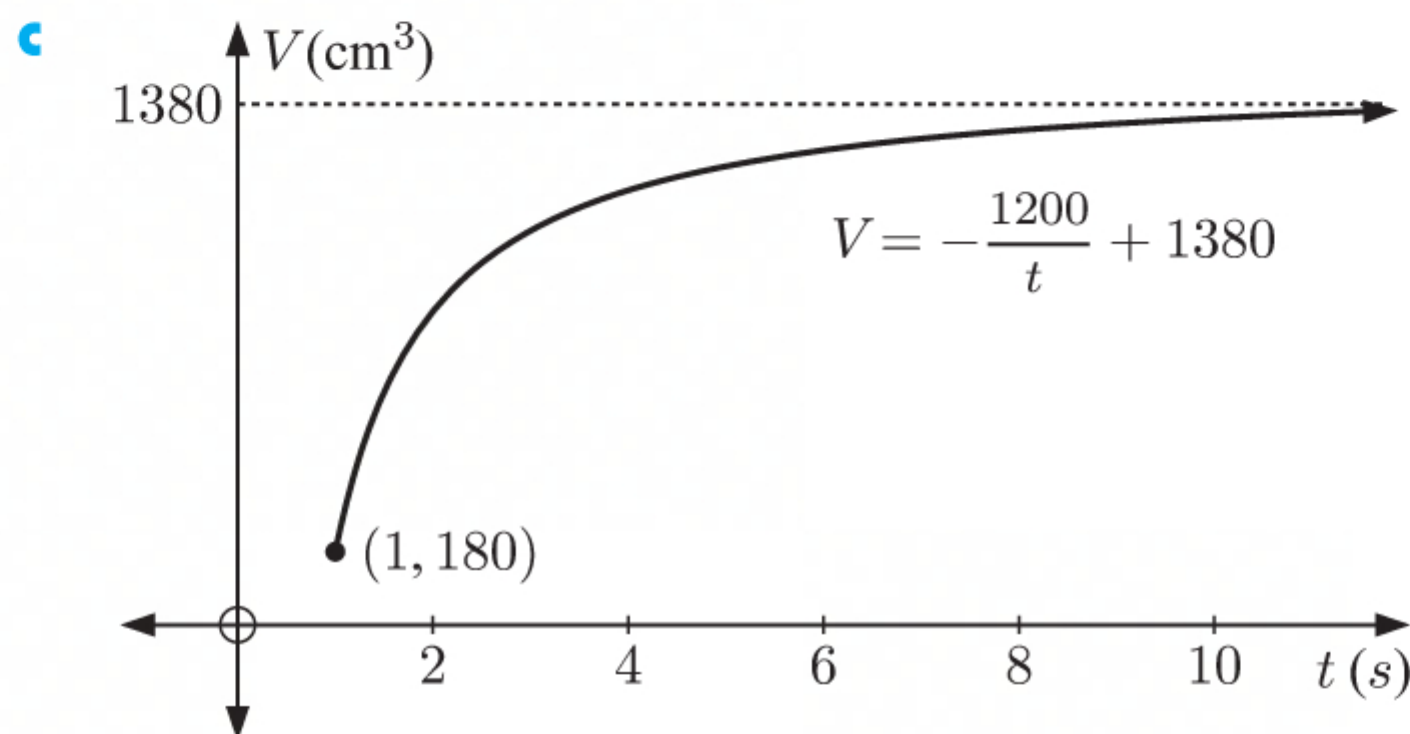
$$\begin{aligned} \text{7 a } V &= -\frac{1200}{t} + 1380 \\ &= -1200t^{-1} + 1380 \\ \therefore \frac{dV}{dt} &= 1200t^{-2} \\ &= \frac{1200}{t^2} \text{ cm}^3 \text{ s}^{-1} \end{aligned}$$

$$\begin{aligned} \text{b i When } t = 2, \quad \frac{dV}{dt} &= \frac{1200}{2^2} \\ &= \frac{1200}{4} \\ &= 300 \text{ cm}^3 \text{ s}^{-1} \end{aligned}$$

Air is being pumped into the tyre at a rate of $300 \text{ cm}^3 \text{ s}^{-1}$ after 2 seconds.

$$\begin{aligned} \text{ii When } t = 6, \quad \frac{dV}{dt} &= \frac{1200}{6^2} \\ &= \frac{1200}{36} \\ &= 33\frac{1}{3} \text{ cm}^3 \text{ s}^{-1} \end{aligned}$$

Air is being pumped into the tyre at a rate of $33\frac{1}{3} \text{ cm}^3 \text{ s}^{-1}$ after 6 seconds.



$$\text{d As } t \rightarrow \infty, \quad \frac{dV}{dt} \rightarrow 0.$$

So, the rate that air is being pumped into the tyre decreases to 0 as the amount of air in the tyre reaches its maximum.

8 a i $h = 0.1t^2 + 0.15t$

$$\therefore \frac{dh}{dt} = 0.2t + 0.15 \text{ metres per year}$$

ii When $t = 1$, $\frac{dh}{dt} = 0.2(1) + 0.15$
 $= 0.35$ metres per year

After 1 year, the tree was growing at a rate of 0.35 metres per year.

iii When $t = 4$, $h = 0.1(4)^2 + 0.15(4)$
 $= 2.2$

After 4 years, the tree was 2.2 m tall.

b $H = 20 - \frac{k}{t}$ metres, $t \geq 4$

i From **a iii**, the tree was 2.2 m tall after 4 years, so when $t = 4$, $H = 2.2$

$$\therefore 20 - \frac{k}{4} = 2.2$$

$$\therefore 17.8 = \frac{k}{4}$$

$$\therefore k = 71.2$$

ii $H = 20 - \frac{71.2}{t}$
 $= 20 - 71.2t^{-1}$

$$\therefore \frac{dH}{dt} = 71.2t^{-2}$$

$$= \frac{71.2}{t^2}$$

iii When $t = 4$, $\frac{dh}{dt} = 0.2(4) + 0.15$ and $\frac{dH}{dt} = \frac{71.2}{4^2}$
 $= 0.95$ metres per year $= 4.45$ metres per year

The rate at which the tree is growing changes when it is taken out of its pot and planted in the ground, since $\frac{dh}{dt} \neq \frac{dH}{dt}$ when $t = 4$.

iv $t^2 > 0$ for all $t \geq 4$

$$\therefore \frac{1}{t^2} > 0 \text{ for all } t \geq 4$$

$$\therefore \frac{dH}{dt} = \frac{71.2}{t^2} > 0 \text{ for all } t \geq 4$$

This means that according to the model, the tree will never stop growing.

$$\begin{aligned} \text{v When } t = 10, \quad H &= 20 - \frac{71.2}{10} \\ &= 12.88 \end{aligned}$$

$$\begin{aligned} \text{When } t = 20, \quad H &= 20 - \frac{71.2}{20} \\ &= 16.44 \end{aligned}$$

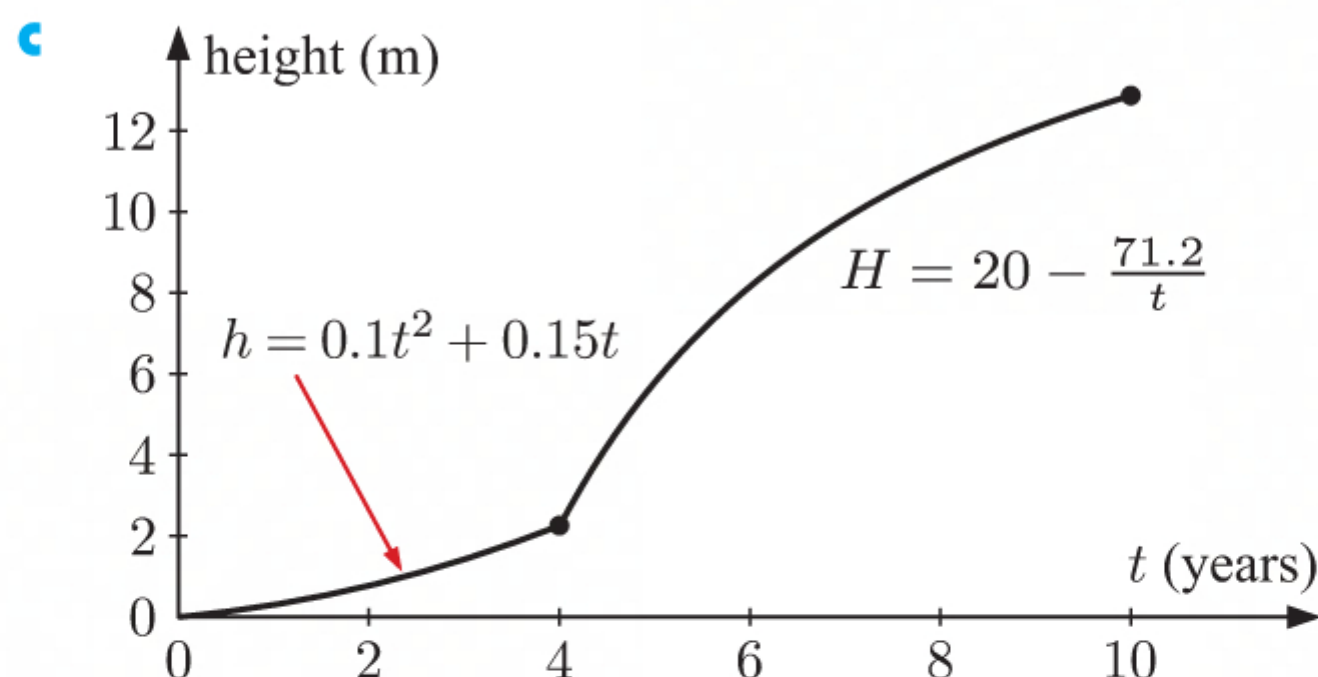
$$\begin{aligned} \text{When } t = 50, \quad H &= 20 - \frac{71.2}{50} \\ &= 18.576 \end{aligned}$$

So, after 10 years, height = 12.88 m

after 20 years, height = 16.44 m

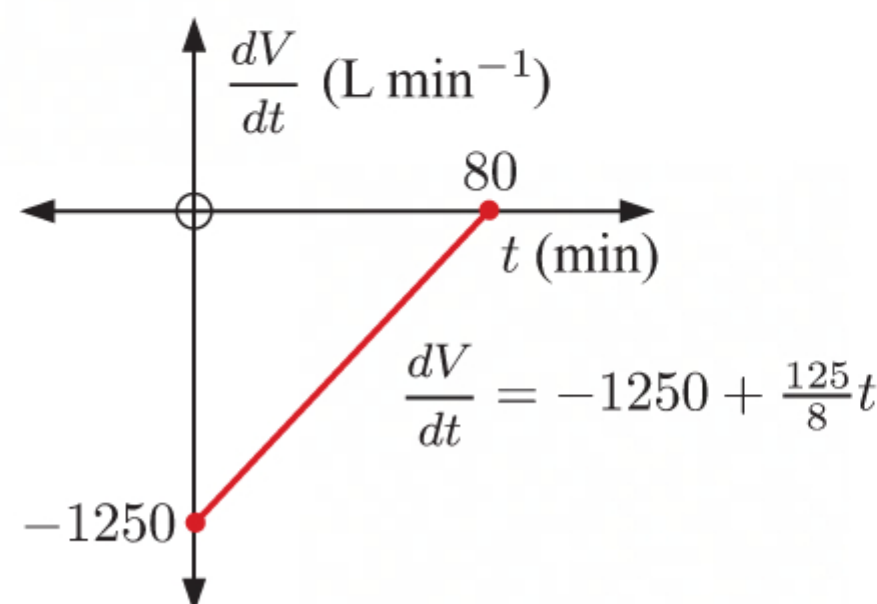
after 50 years, height = 18.576 m

The rate of growth is slowing as the years progress.



9 a

$$\begin{aligned} V &= 50\,000 \left(1 - \frac{t}{80}\right)^2 \text{ L}, \quad 0 \leq t \leq 80 \\ &= 50\,000 \left(1 - \frac{2t}{80} + \frac{t^2}{80^2}\right) \\ &= 50\,000 - 1250t + \frac{125}{16}t^2 \\ \therefore \frac{dV}{dt} &= -1250 + \frac{125}{8}t \text{ L min}^{-1} \end{aligned}$$



b When $t = 40$, $\frac{dV}{dt} = -1250 + \frac{125}{8}(40)$
 $= -625$

After 40 minutes, the tank is draining water at a rate of 625 L min⁻¹.

c The outflow is fastest when $\frac{dV}{dt} = -1250 + \frac{125}{8}t$ is at a minimum.

Looking at the graph in **a**, the minimum value of $\frac{dV}{dt}$ is -1250 L min⁻¹ which occurs when $t = 0$.

So, the outflow is fastest at $t = 0$, when the tap was first opened.

EXERCISE 12A.2

$$\begin{aligned}
 \mathbf{1} \quad \mathbf{a} \quad T &= r^2 - \frac{100}{r} \\
 &= r^2 - 100r^{-1} \\
 \therefore \frac{dT}{dr} &= 2r + 100r^{-2} \\
 &= 2r + \frac{100}{r^2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad A &= 2\pi h + \frac{1}{4}h^2 \\
 \therefore \frac{dA}{dh} &= 2\pi + \frac{1}{2}h
 \end{aligned}$$

2 Since C is in pounds, and x is the number of items, $\frac{dC}{dx}$ is in pounds per item produced.

3 $C(x) = -0.000\,007\,2x^3 + 0.0061x^2 + 18x + 14\,230$ dollars, $0 \leq x \leq 1200$

a $C(0) = \$14\,230$, which is the fixed operation cost without producing any items.

$$\mathbf{b} \quad C'(x) = -0.000\,021\,6x^2 + 0.0122x + 18$$

This is the rate at which the production cost (in dollars) is changing per item when x items are made. It gives an estimate of the cost of making the $(x+1)$ th item each day.

$$\begin{aligned}
 \mathbf{c} \quad C'(300) &= -0.000\,021\,6(300)^2 + 0.0122(300) + 18 \\
 &= 19.716
 \end{aligned}$$

The cost of producing the 301st item each day is about \$19.72.

$$\begin{aligned}
 \mathbf{d} \quad C(301) - C(300) &= -0.000\,007\,2(301)^3 + 0.0061(301)^2 + 18(301) + 14\,230 \\
 &\quad - [-0.000\,007\,2(300)^3 + 0.0061(300)^2 + 18(300) + 14\,230] \\
 &\approx \$20\,004.32 - \$19\,984.60 \\
 &\approx \$19.72
 \end{aligned}$$

The actual cost of producing the 301st item is about \$19.72.

4 $C(x) = 7800 + 7x - 0.0001x^2$ dollars

a The marginal cost function is $C'(x) = 7 - 0.0002x$ dollars per pair.

$$\begin{aligned}
 \mathbf{b} \quad C'(220) &= 7 - 0.0002(220) \\
 &= 6.956 \\
 &\approx \$6.96
 \end{aligned}$$

This estimates the cost of making the 221st pair of jeans if 220 pairs are currently being made.

$$\begin{aligned}
 \mathbf{c} \quad C(221) - C(220) &= 7800 + 7(221) - 0.0001(221)^2 - [7800 + 7(220) - 0.0001(220)^2] \\
 &= 6.9559 \\
 &\approx \$6.96
 \end{aligned}$$

This is the actual cost of making the 221st pair of jeans.

The answer in **b** is a very good estimate.

5 a $C(v) = \frac{1}{5}v^2 + \frac{200\,000}{v}$ euros

i $C(50) = \frac{1}{5}(50)^2 + \frac{200\,000}{50}$
 $= 500 + 4000$
 $= 4500$

\therefore if the average speed is 50 km h^{-1} , the total cost of the journey is 4500 euros.

ii $C(100) = \frac{1}{5}(100)^2 + \frac{200\,000}{100}$
 $= 2000 + 2000$
 $= 4000$

\therefore if the average speed is 100 km h^{-1} , the total cost of the journey is 4000 euros.

b $C(v) = \frac{1}{5}v^2 + \frac{200\,000}{v}$ euros, $v > 0$
 $= \frac{1}{5}v^2 + 200\,000v^{-1}$

$\therefore C'(v) = \frac{2}{5}v - 200\,000v^{-2}$
 $= \frac{2}{5}v - \frac{200\,000}{v^2}$ euros per km h^{-1}

i $C'(30) = \frac{2}{5}(30) - \frac{200\,000}{30^2}$
 $= 12 - \frac{2000}{9}$
 ≈ -210.22

\therefore if the average speed is 30 km h^{-1} , the rate of change in the cost of running the train is decreasing at about 210.22 euros per km h^{-1} .

ii $C'(90) = \frac{2}{5}(90) - \frac{200\,000}{90^2}$
 $= 36 - \frac{2000}{81}$
 ≈ 11.31

\therefore if the average speed is 90 km h^{-1} , the rate of change in the cost of running the train is increasing at about 11.31 euros per km h^{-1} .

c $C(v)$ is a minimum when $C'(v) = 0$

$$\therefore \frac{2}{5}v - \frac{200\,000}{v^2} = 0$$

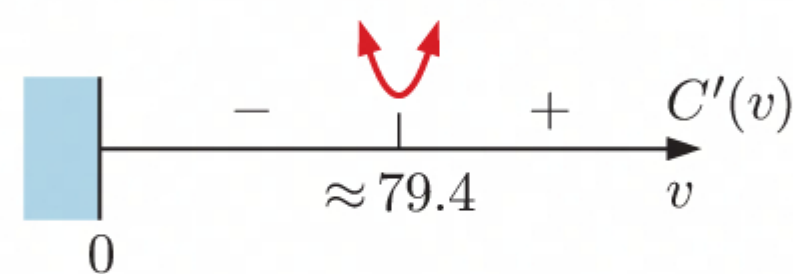
$$\therefore \frac{2}{5}v^3 - 200\,000 = 0$$

$$\therefore \frac{2}{5}v^3 = 200\,000$$

$$\therefore v^3 = 500\,000$$

$$\therefore v \approx 79.4 \text{ km h}^{-1}$$

$C'(v)$ has sign diagram:



\therefore the cost of running the train is a minimum when the average speed of the train is about 79.4 km h^{-1} .

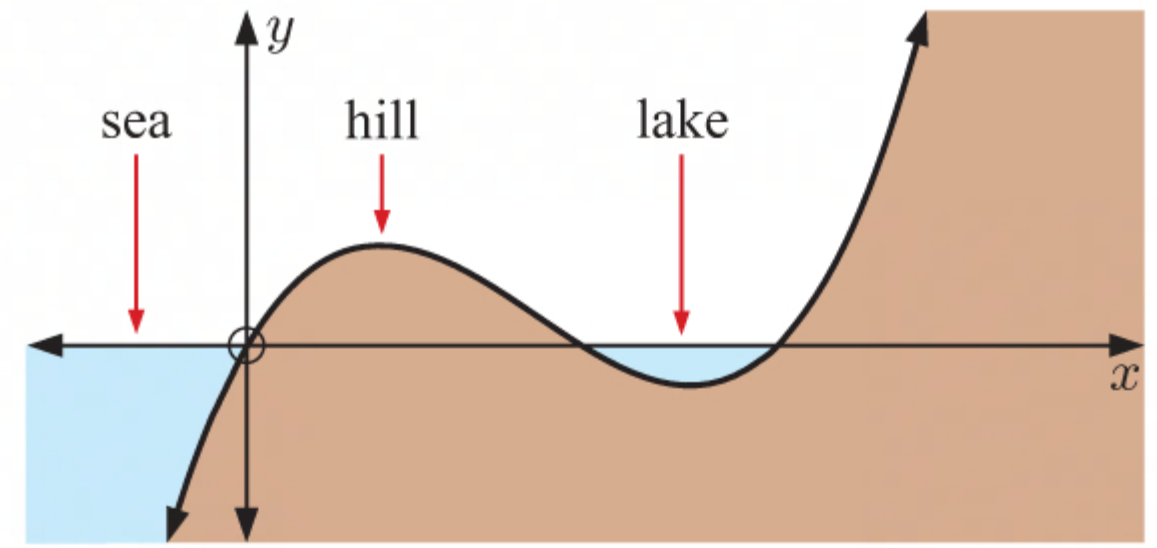
6 a $y = \frac{1}{10}x(x-2)(x-3)$

The edges of the lake correspond to values of x such that $y = 0$.

$$\therefore \frac{1}{10}x(x-2)(x-3) = 0$$

$$\therefore x = 0, 2, \text{ or } 3$$

From the graph, we can see that the nearest part of the lake is 2 km from the sea, and the furthest part is 3 km.



b

$$y = \frac{1}{10}x(x-2)(x-3)$$

$$= \frac{1}{10}x(x^2 - 5x + 6)$$

$$= \frac{1}{10}x^3 - \frac{1}{2}x^2 + \frac{3}{5}x$$

$$\therefore \frac{dy}{dx} = \frac{3}{10}x^2 - x + \frac{3}{5}$$

When $x = \frac{1}{2}$, $\frac{dy}{dx} = \frac{3}{10}\left(\frac{1}{2}\right)^2 - \frac{1}{2} + \frac{3}{5}$

$$= 0.175$$

\therefore the height of the hill is increasing when $x = \frac{1}{2}$ km, as the gradient is positive.
So the land is sloping upwards at this point.

When $x = 1\frac{1}{2} = \frac{3}{2}$, $\frac{dy}{dx} = \frac{3}{10}\left(\frac{3}{2}\right)^2 - \frac{3}{2} + \frac{3}{5}$

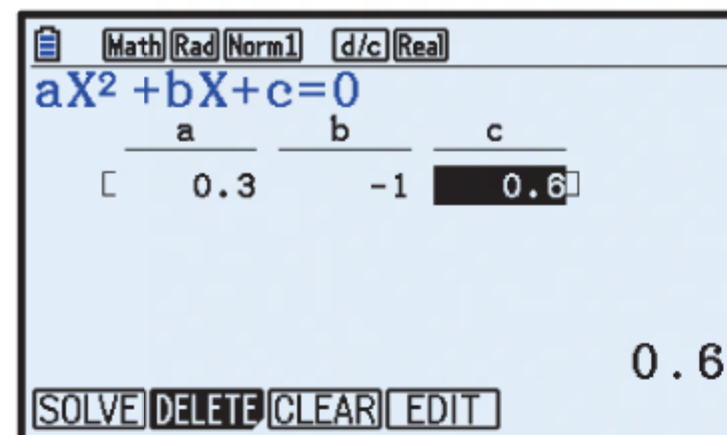
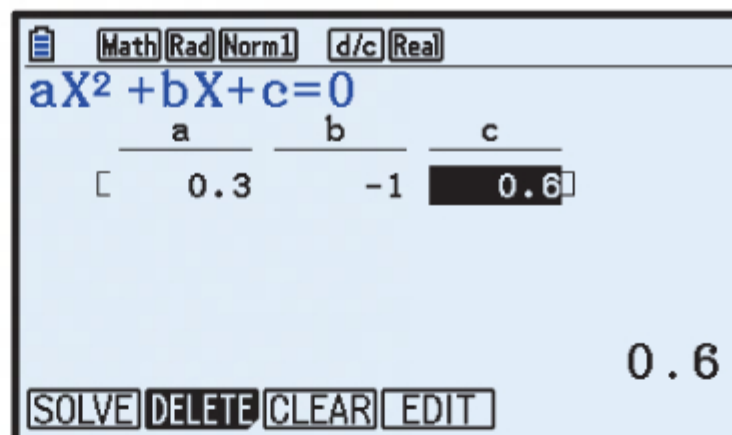
$$= -0.225$$

\therefore the height of the hill is decreasing when $x = 1\frac{1}{2}$ km, as the gradient is negative.
So the land is sloping downwards at this point.

This means the top of the hill is between $x = \frac{1}{2}$ km and $x = 1\frac{1}{2}$ km.

c The deepest point of the lake occurs when the slope of the land is 0, which is when $\frac{dy}{dx} = 0$.

$$\therefore \frac{3}{10}x^2 - x + \frac{3}{5} = 0$$



Using technology, $x \approx 0.785$ or 2.55 .

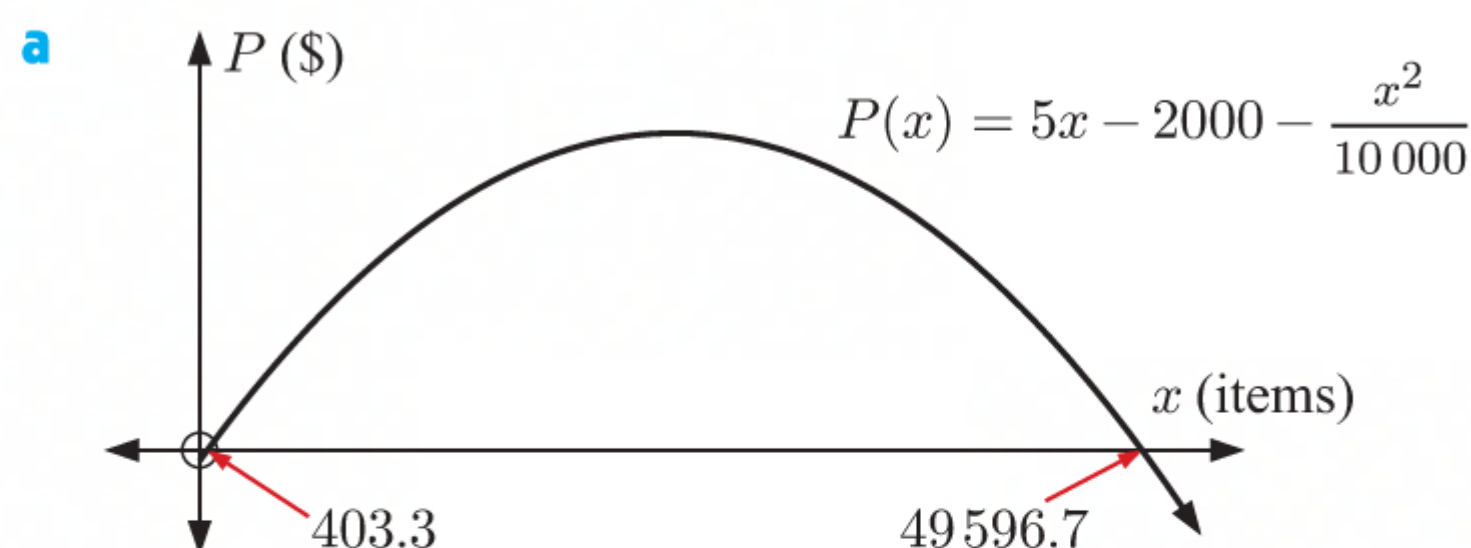
The deepest point of the lake is a turning point which lies between the edges of the lake at $x = 2$ and $x = 3$. So we only consider the value of x which lies between 2 and 3.

When $x = 2.55$, $y \approx -0.0631$ km

$$\approx -63.1 \text{ m}$$

So, the deepest point of the lake is about 2.55 km from the sea, and about 63.1 m deep.

7 $P(x) = 5x - 2000 - \frac{x^2}{10\,000}$ dollars



b A profit is made if $P(x) > 0$, so $404 \leq x \leq 49\,596$.

c

$$P(x) = 5x - 2000 - \frac{x^2}{10\,000}$$

$$= 5x - 2000 - \frac{1}{10\,000}x^2$$

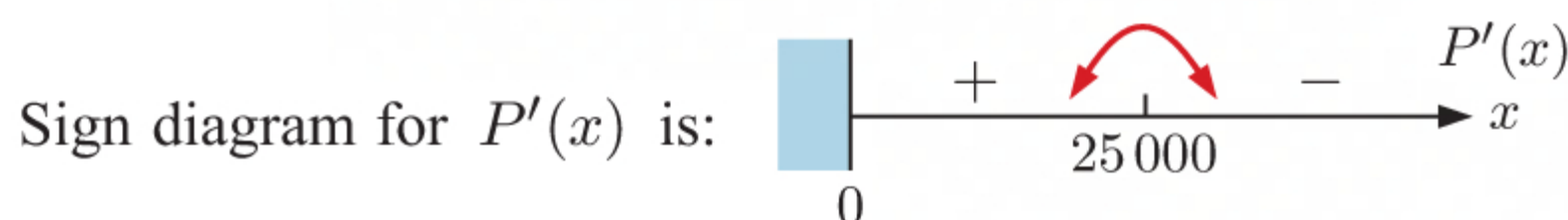
$$\therefore P'(x) = 5 - \frac{x}{5000}$$

This is the rate at which the profit earned is increasing or decreasing when producing x items per year.

d $P'(x) = 0$ when $5 - \frac{x}{5000} = 0$

$$\therefore 5 = \frac{x}{5000}$$

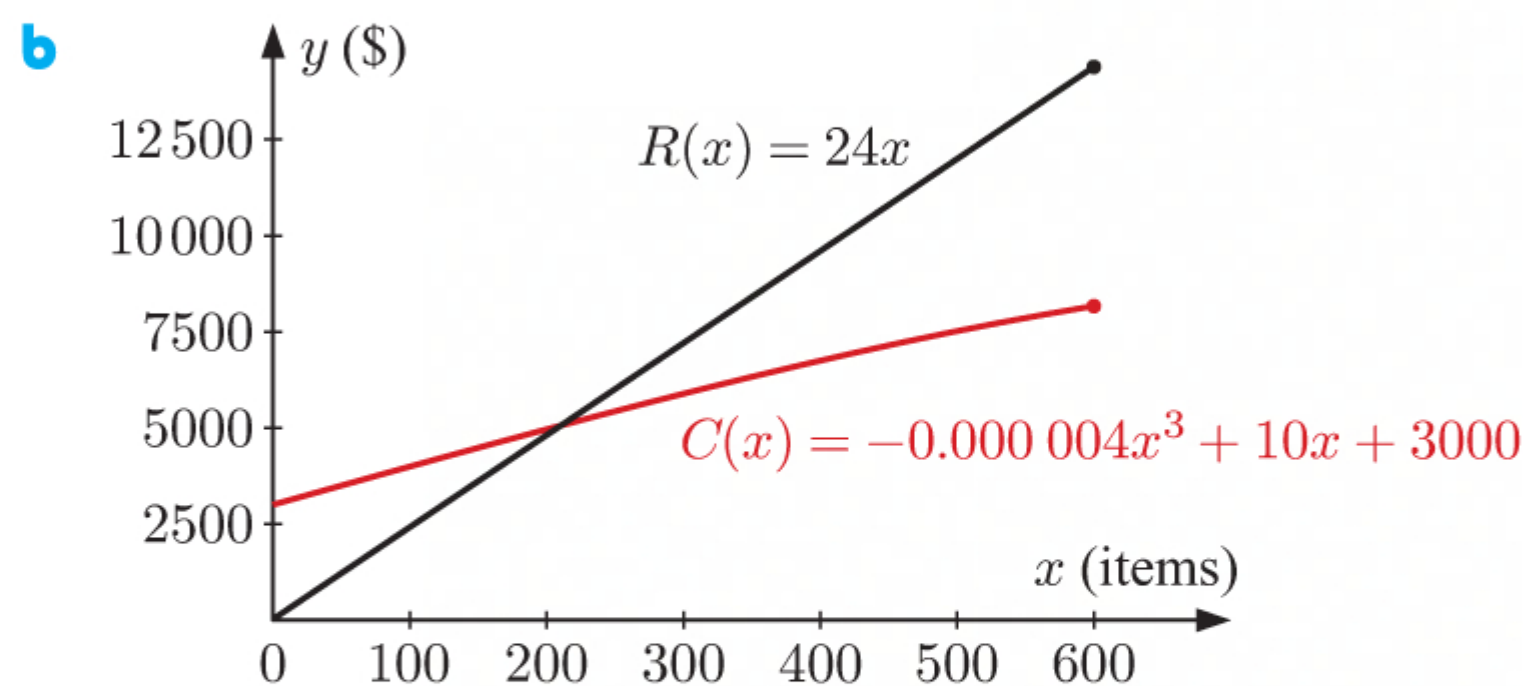
$$\therefore x = 25\,000$$



So, the profit is increasing when $0 \leq x \leq 25\,000$.

8 $C(x) = -0.000\,004x^3 + 10x + 3000$ dollars, $0 \leq x \leq 600$

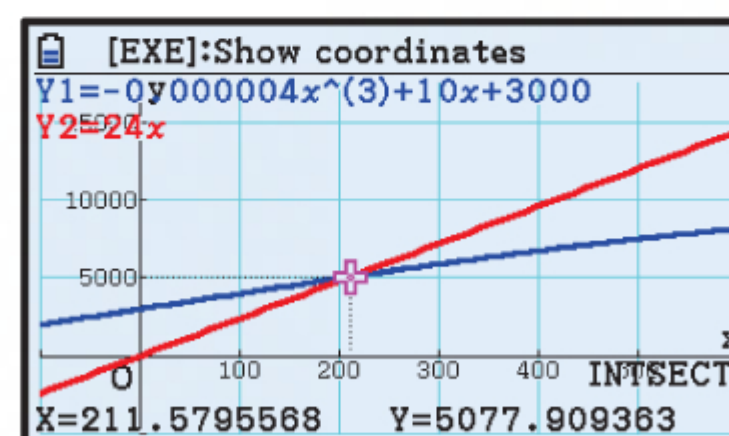
a The revenue is the amount of income generated, so $R(x) = 24x$.



c $C(x) = R(x)$ when $-0.000\,004x^3 + 10x + 3000 = 24x$.

Using technology, $x \approx 212$.

This is the breakeven point, when the revenue is equal to the cost of producing the items.



$$\begin{aligned}
 \text{d } P(x) &= R(x) - C(x) \\
 &= 24x - (-0.000\,004x^3 + 10x + 3000) \\
 &= 0.000\,004x^3 + 14x - 3000 \text{ dollars}
 \end{aligned}$$

$$\text{e } P'(x) = 0.000\,012x^2 + 14$$

$$\begin{aligned}
 \text{f } P'(120) &= 0.000\,012(120)^2 + 14 \\
 &= 14.1728 \\
 &\approx 14.17
 \end{aligned}$$

This means that the total profit is increasing at a rate of about \$14.17 per item when 120 items are produced.

EXERCISE 12B.1

$$1 \quad P(x) = -0.022x^2 + 11x - 720$$

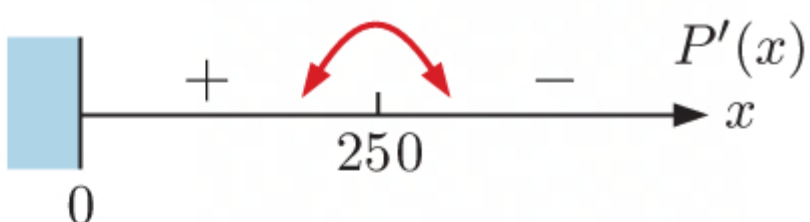
$$\therefore P'(x) = -0.044x + 11$$

$$\text{Now, } P'(x) = 0 \text{ when } -0.044x + 11 = 0$$

$$\therefore 11 = 0.044x$$

$$\begin{aligned}
 \therefore x &= \frac{11}{0.044} \\
 &= 250
 \end{aligned}$$

So, $P'(x)$ has sign diagram:



So, the profit is maximised when 250 items are made per day.

$$2 \quad h(t) = 1.6 + 32t - 4.9t^2$$

$$\therefore h'(t) = 32 - 9.8t$$

$$\text{Now } h'(t) = 0 \text{ when } 32 - 9.8t = 0$$

$$\therefore 32 = 9.8t$$

$$\therefore t = \frac{32}{9.8}$$

$$\therefore t \approx 3.27$$

$h'(t)$ has sign diagram:



\therefore the maximum height occurs when $t \approx 3.27$

$$\begin{aligned}
 h(3.27) &\approx 1.6 + 32(3.27) - 4.9(3.27)^2 \\
 &\approx 53.8
 \end{aligned}$$

So, the maximum height reached by the stone is about 53.8 m.

- 3 a** Let the remaining fence have length y m.

The total length of the fence is 60 m

$$\therefore 2x + y = 60$$

$$\therefore y = 60 - 2x$$

The area of the enclosure $A = \text{width} \times \text{length}$

$$= xy$$

$$= x(60 - 2x) \text{ m}^2$$

\therefore the area of the enclosure is given by $A(x) = x(60 - 2x) \text{ m}^2$.

b
$$A(x) = x(60 - 2x)$$

$$= 60x - 2x^2$$

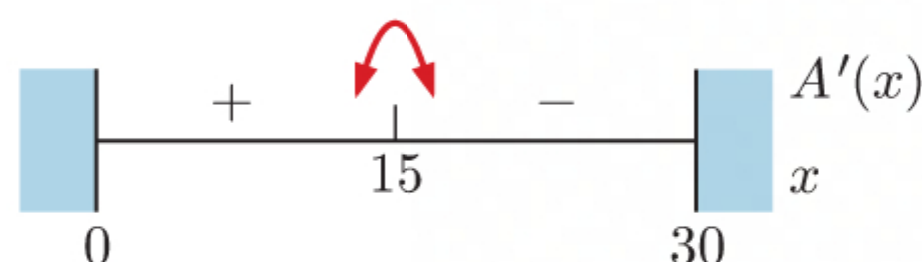
$$\therefore A'(x) = 60 - 4x$$

So, $A'(x) = 0$ when $60 - 4x = 0$

$$\therefore 4x = 60$$

$$\therefore x = 15$$

$A'(x)$ has sign diagram:

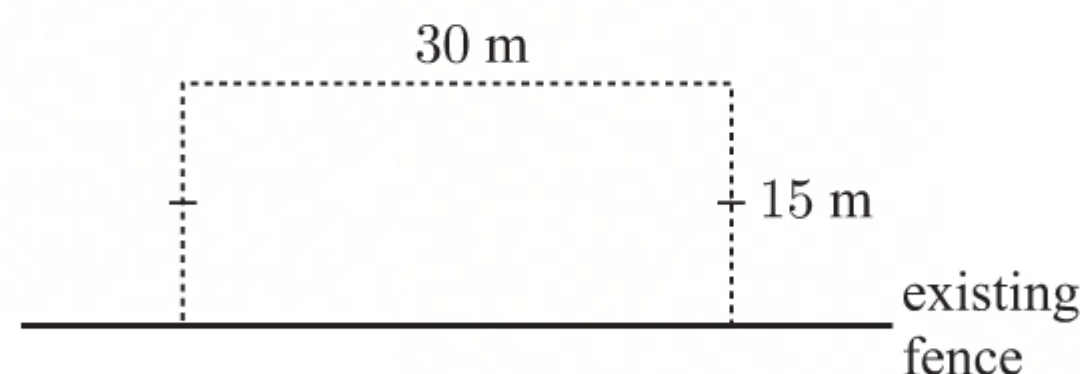
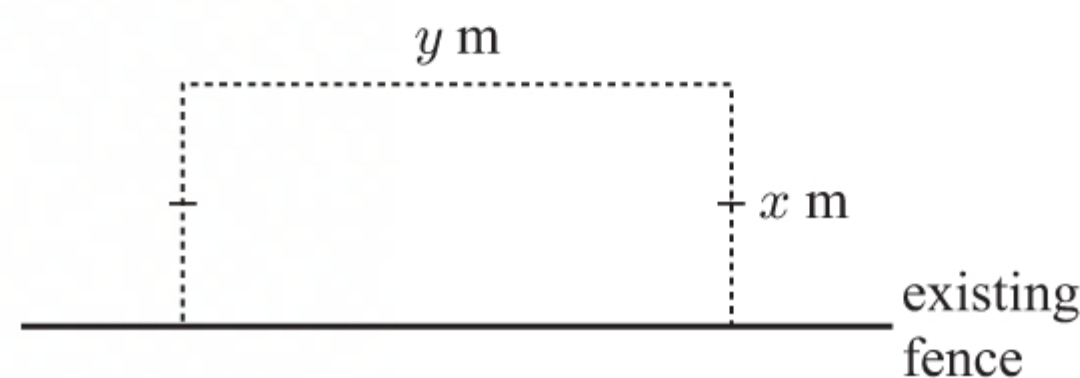


The area is maximised when $x = 15$

$$\text{and } y = 60 - 2(15)$$

$$= 30$$

The area of the enclosure is maximised by constructing a fence with two sides of length 15 m, and one side of length 30 m.



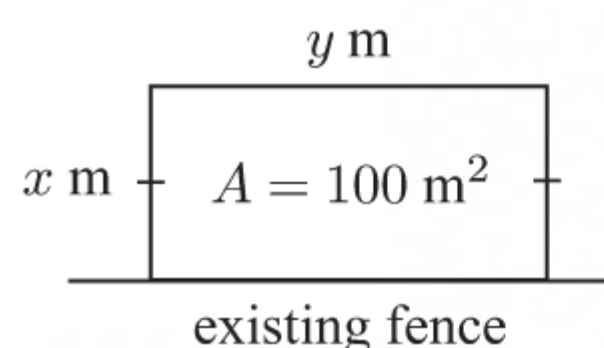
- 4 a** Now the area $A = 100 \text{ m}^2$

$$\therefore xy = 100$$

$$\therefore y = \frac{100}{x}$$

$$\text{So, } L = 2x + y$$

$$\therefore L = 2x + \frac{100}{x}$$



b
$$L = 2x + 100x^{-1}$$

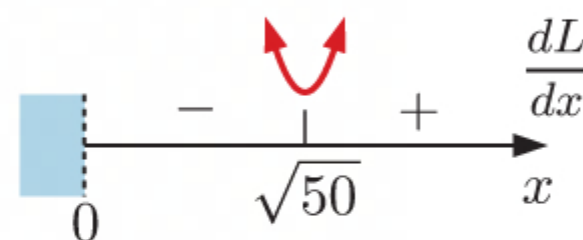
$$\therefore \frac{dL}{dx} = 2 - 100x^{-2} = 2 - \frac{100}{x^2}$$

which is 0 when $\frac{100}{x^2} = 2$

$$\therefore x^2 = 50$$

$$\therefore x = \sqrt{50} \quad \{x > 0\}$$

So, $\frac{dL}{dx}$ has sign diagram:



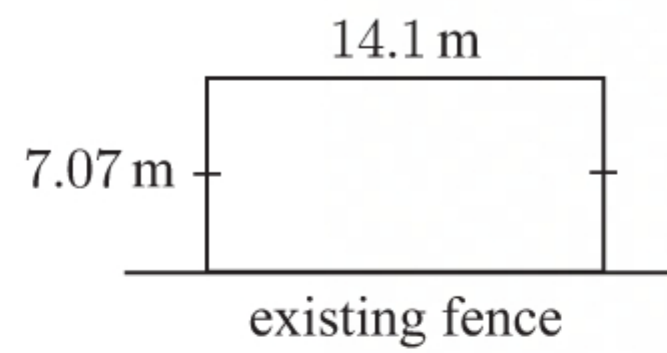
So, the length is a minimum when $x = \sqrt{50}$.

$$\begin{aligned}\text{When } x = \sqrt{50}, \quad L &= 2\sqrt{50} + \frac{100}{\sqrt{50}} \\ &\approx 28.3\end{aligned}$$

So, the minimum value of L is about 28.3 m which occurs when $x = \sqrt{50} \approx 7.07$.

c $x = \sqrt{50} \approx 7.07$ and $y = \frac{100}{\sqrt{50}} \approx 14.1$

So, the optimal situation is:



- 5 a** Inner surface area = area of square base + area of 4 rectangular sides

$$\begin{aligned}\therefore 108 &= x \times x + 4 \times x \times y \\ \therefore x^2 + 4xy &= 108\end{aligned}$$

b $x^2 + 4xy = 108$

$$\therefore 4xy = 108 - x^2$$

$$\therefore y = \frac{108 - x^2}{4x} \quad \{\text{as } x > 0\}$$

- c** Capacity of container $C = \text{area of base} \times \text{height}$

$$\begin{aligned}\therefore C &= x \times x \times y \\ &= x^2 \times \frac{108 - x^2}{4x} \\ &= \frac{108x^2 - x^4}{4x} \\ &= 27x - \frac{x^3}{4}\end{aligned}$$

d
$$\begin{aligned}C &= 27x - \frac{x^3}{4} \\ &= 27x - \frac{1}{4}x^3\end{aligned}$$

$$\therefore \frac{dC}{dx} = 27 - \frac{3}{4}x^2$$

e $\frac{dC}{dx} = 0$ when $27 - \frac{3}{4}x^2 = 0$

$$\therefore 27 = \frac{3}{4}x^2$$

$$\begin{aligned}\therefore x^2 &= 27 \times \frac{4}{3} \\ &= 36\end{aligned}$$

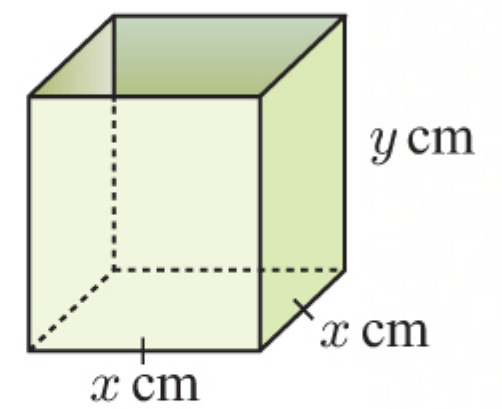
$$\therefore x = 6 \quad \{x > 0\}$$

f $\frac{dC}{dx}$ has sign diagram:

A sign diagram for the derivative $\frac{dC}{dx}$. It consists of a horizontal axis labeled x . A vertical dashed line is drawn at $x = 0$. A blue shaded region is to the left of $x = 0$. A red double-headed arrow is drawn above the axis between $x = 0$ and $x = 6$. A tick mark is at $x = 6$. The label $\frac{dC}{dx}$ is at the right end of the axis.

The capacity is a maximum when $x = 6$.

So, the base must be 6 cm by 6 cm.



- 6 a** The base has dimensions in the ratio 2 : 1.

\therefore if one side is x cm, then the other side must be $2x$ cm.

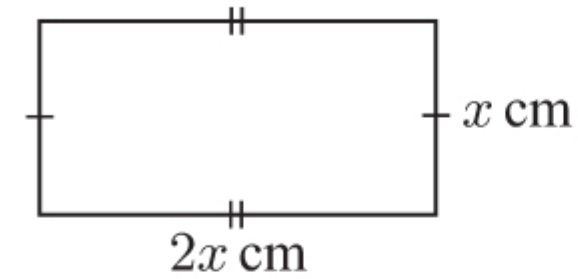
Inner volume $V = \text{area of base} \times \text{height}$

$$= 2x \times x \times h$$

$$= 2x^2h$$

but $V = 200 \text{ cm}^3$, $\therefore 2x^2h = 200$

$$\therefore x^2h = 100$$



- b** $x^2h = 100$ {from **a**}

$$\therefore h = \frac{100}{x^2}$$

Inner surface area

$$A(x) = 2(2x \times x) + 2\left(x \times \frac{100}{x^2}\right) + 2\left(2x \times \frac{100}{x^2}\right)$$

$$= 2\left(2x^2 + \frac{100}{x} + \frac{200}{x}\right)$$

$$= 2\left(2x^2 + \frac{300}{x}\right)$$

$$\therefore A(x) = 4x^2 + \frac{600}{x} \text{ cm}^2$$

c $A(x) = 4x^2 + 600x^{-1}$

$$\therefore A'(x) = 8x - 600x^{-2}$$

$$= 8x - \frac{600}{x^2}$$

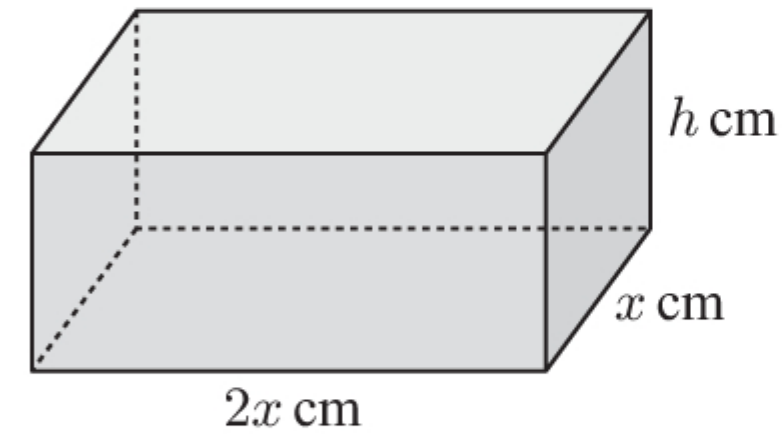
$$A'(x) = 0 \quad \text{when} \quad 8x - \frac{600}{x^2} = 0$$

$$\therefore 8x = \frac{600}{x^2}$$

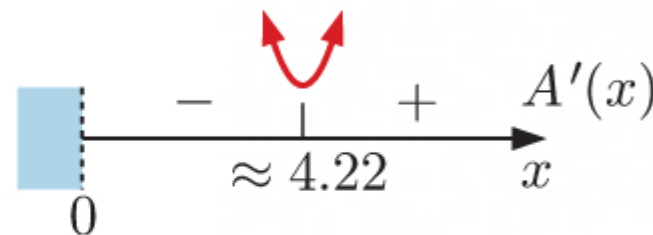
$$\therefore 8x^3 = 600$$

$$\therefore x^3 = 75$$

$$\therefore x = \sqrt[3]{75} \approx 4.22$$



$A'(x)$ has sign diagram:



So, the inner surface area of the box is a minimum when $x = \sqrt[3]{75} \approx 4.22$.

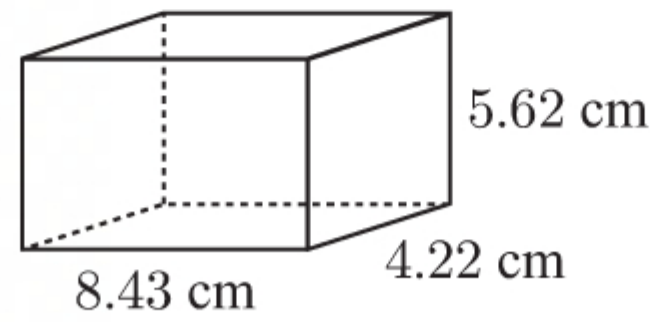
$$A(\sqrt[3]{75}) = 4(\sqrt[3]{75})^2 + \frac{600}{\sqrt[3]{75}}$$

$$\approx 213$$

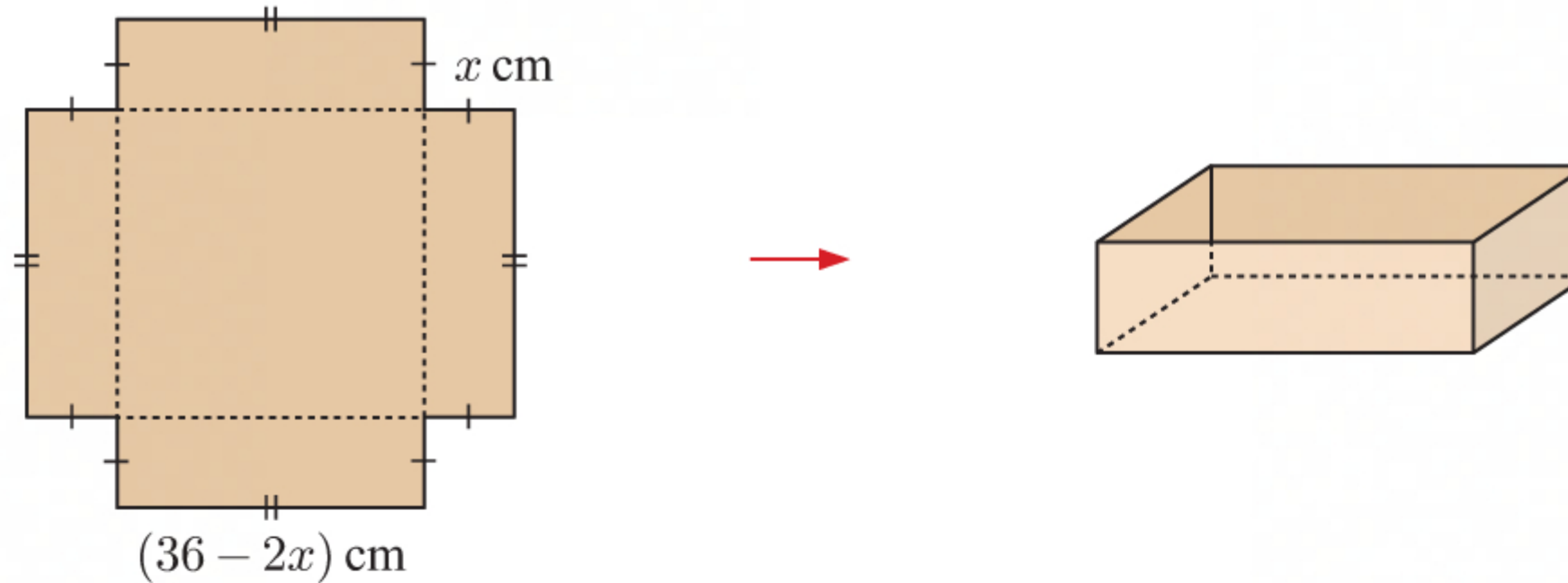
So, the minimum inner surface area is about 213 cm^2 , when $x = \sqrt[3]{75} \approx 4.22$.

$$\begin{aligned}
 \text{d} \quad x &= \sqrt[3]{75} \approx 4.22 \\
 \therefore 2x &= 2\sqrt[3]{75} \approx 8.43 \\
 h &= \frac{100}{x^2} = \frac{100}{(\sqrt[3]{75})^2} \approx 5.62
 \end{aligned}$$

So, the optimal box shape is:



7 a



The volume of the container is $V(x) = \text{area of base} \times \text{height}$

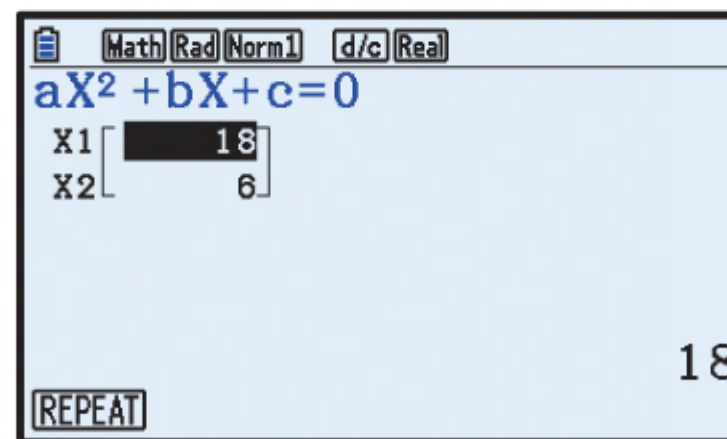
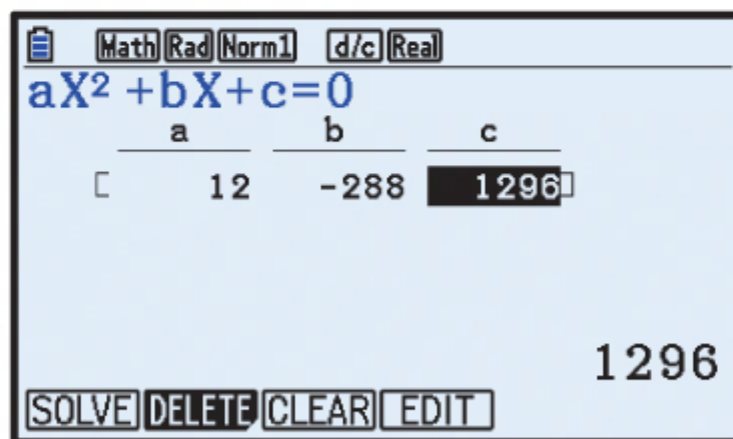
$$\therefore V(x) = (36 - 2x)^2 \times x$$

$$\therefore V(x) = x(36 - 2x)^2 \text{ cm}^3$$

$$\begin{aligned}
 \text{b} \quad V(x) &= x(36 - 2x)^2 \\
 &= x(1296 - 144x + 4x^2) \\
 &= 1296x - 144x^2 + 4x^3
 \end{aligned}$$

$$\therefore V'(x) = 1296 - 288x + 12x^2$$

$$V'(x) = 0 \text{ when } 1296 - 288x + 12x^2 = 0$$



Using technology, $x = 6$ or 18 .

$V'(x)$ has sign diagram:

A sign diagram for the derivative $V'(x)$. The horizontal axis is labeled x and has tick marks at 0, 6, and 18. Above the axis, there is a '+' sign between 0 and 6, and a '-' sign between 6 and 18. A red curved arrow points from the '+' sign to the '-' sign, indicating a change from positive to negative at $x = 6$.

The volume is a maximum when $x = 6$.

So, $6 \text{ cm} \times 6 \text{ cm}$ squares should be cut out to produce the container of greatest capacity.

8 Production cost $C(x) = \frac{1}{4}x^2 + 8x + 20$ pounds

Selling price $p(x) = 23 - \frac{1}{2}x$ pounds per blanket

Revenue $R(x) = xp(x) = 23x - \frac{1}{2}x^2$ pounds

Profit $P(x) = \text{revenue} - \text{cost}$

$$= (23x - \frac{1}{2}x^2) - (\frac{1}{4}x^2 + 8x + 20)$$

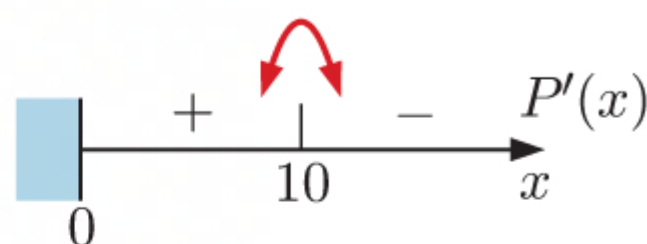
$$= -\frac{3}{4}x^2 + 15x - 20$$

$$\therefore P'(x) = -\frac{3}{2}x + 15$$

Now $P'(x) = 0$ when $-\frac{3}{2}x + 15 = 0$

$$\therefore x = \frac{15}{\frac{3}{2}} = 10$$

$P'(x)$ has sign diagram:



So, the profit is maximised when 10 blankets are produced per day.

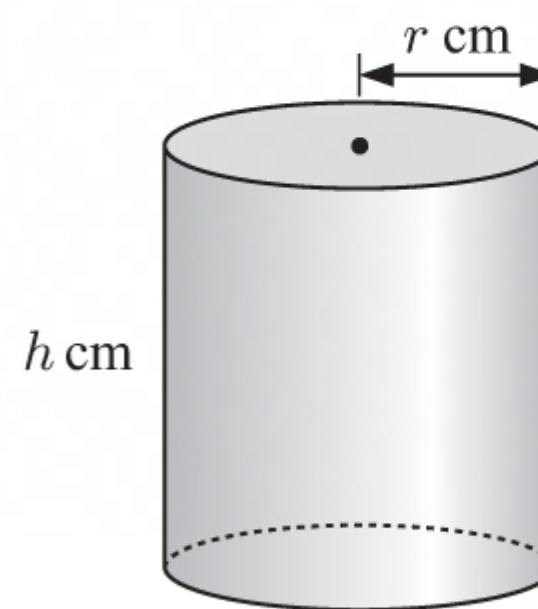
9 a volume of cylindrical can $= \pi r^2 h$

Now, capacity is 1 litre which is equivalent to 1000 cm^3 .

So, the volume $= 1000 \text{ cm}^3$

$$\therefore \pi r^2 h = 1000$$

$$\therefore h = \frac{1000}{\pi r^2} \text{ cm}$$



b Total surface area of cylindrical can $A = 2\pi r^2 + 2\pi r h$

$$\therefore A = 2\pi r^2 + 2\pi r \left(\frac{1000}{\pi r^2} \right) \quad \{\text{using a}\}$$

$$\therefore A = 2\pi r^2 + \frac{2000}{r} \text{ cm}^2$$

c $A = 2\pi r^2 + 2000r^{-1}$

$$\therefore \frac{dA}{dr} = 4\pi r - 2000r^{-2}$$

$$= 4\pi r - \frac{2000}{r^2}$$

$$\frac{dA}{dr} = 0 \text{ when } 4\pi r - \frac{2000}{r^2} = 0$$

$$\therefore 4\pi r = \frac{2000}{r^2}$$

$$\therefore 4\pi r^3 = 2000$$

$$\therefore r^3 = \frac{500}{\pi}$$

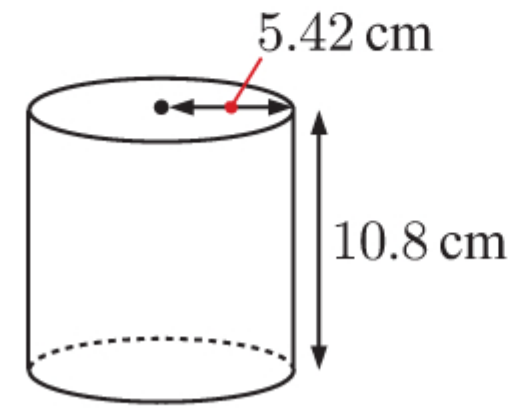
$$\therefore r = \sqrt[3]{\frac{500}{\pi}} \approx 5.42$$



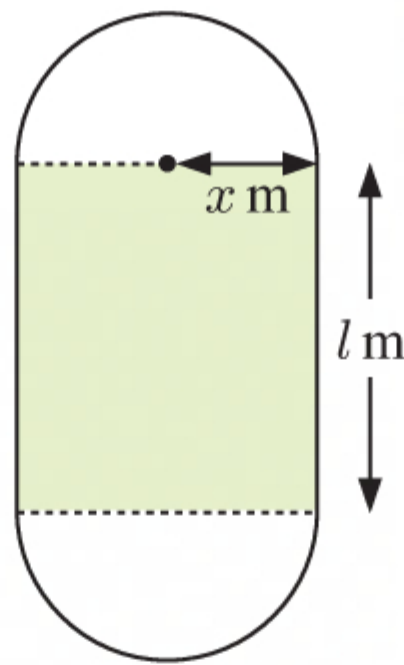
So, the total surface area is a minimum when $r = \sqrt[3]{\frac{500}{\pi}} \approx 5.42$.

$$\text{When } r = \sqrt[3]{\frac{500}{\pi}}, \quad h = \frac{1000}{\pi \left(\sqrt[3]{\frac{500}{\pi}} \right)^2} \approx 10.8$$

So, the can should have dimensions:



10 a



$$\text{Perimeter} = 2l + 2\pi x$$

$$\therefore 400 = 2l + 2\pi x$$

$$\therefore 2l = 400 - 2\pi x$$

$$\therefore l = 200 - \pi x$$

$x > 0$ and $l > 0$ for the track to exist

$$\therefore 200 - \pi x > 0$$

$$\therefore \pi x < 200$$

$$\therefore x < \frac{200}{\pi}$$

$$\text{So, } 0 < x < \frac{200}{\pi} \approx 63.7$$

b Area of rectangle $A = 2x \times l$

$$= 2x \times (200 - \pi x)$$

$$\therefore A = 400x - 2\pi x^2$$

$$\therefore \frac{dA}{dx} = 400 - 4\pi x$$

$$\frac{dA}{dx} = 0 \text{ when } 400 - 4\pi x = 0$$

$$\therefore 4\pi x = 400$$

$$\therefore x = \frac{100}{\pi}$$

$\frac{dA}{dx}$ has sign diagram:

The area is a maximum when $x = \frac{100}{\pi} \approx 31.8$

$$\begin{aligned} \therefore l &= 200 - \pi \left(\frac{100}{\pi} \right) \\ &= 100 \end{aligned}$$

$$\begin{aligned}
 \text{When } x = \frac{100}{\pi}, \quad A &= 400 \left(\frac{100}{\pi} \right) - 2\pi \left(\frac{100}{\pi} \right)^2 \\
 &= \frac{40\,000}{\pi} - \frac{20\,000}{\pi} \\
 &= \frac{20\,000}{\pi} \\
 &\approx 6370
 \end{aligned}$$

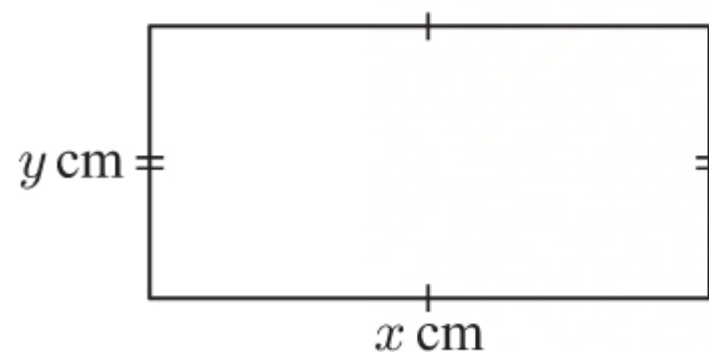
So, $l = 100$ and $x = \frac{100}{\pi} \approx 31.8$ give the maximum area $A = \frac{20\,000}{\pi} \approx 6370 \text{ m}^2$.

11 a Perimeter = 60 cm

$$\therefore 2x + 2y = 60$$

$$\therefore x + y = 30$$

$$\therefore y = 30 - x$$



b Area of rectangle $A = xy$

$$\therefore A(x) = x(30 - x) \text{ cm}^2 \quad \{\text{using a}\}$$

$$\text{c} \quad A(x) = 30x - x^2$$

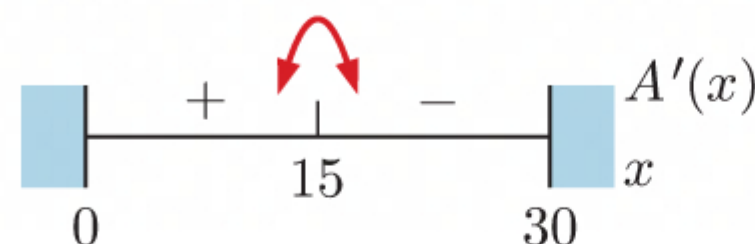
$$\therefore A'(x) = 30 - 2x$$

$$\text{d} \quad A'(x) = 0 \text{ when } 30 - 2x = 0$$

$$\therefore 2x = 30$$

$$\therefore x = 15$$

$A'(x)$ has sign diagram:



The area is a maximum when $x = 15$

$$\begin{aligned}
 \therefore y &= 30 - 15 \quad \{\text{using a}\} \\
 &= 15
 \end{aligned}$$

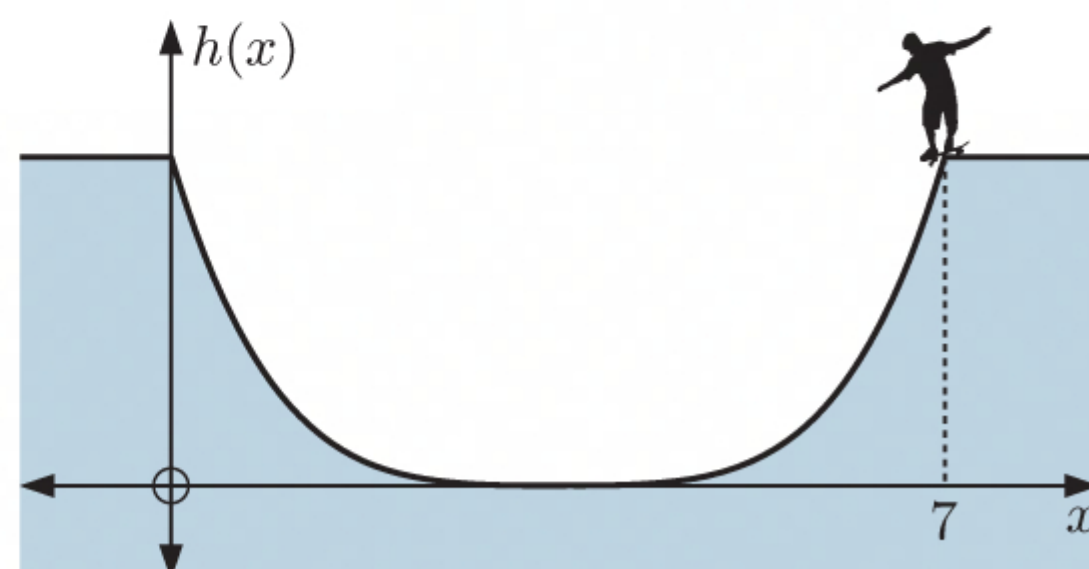
\therefore the dimensions of the rectangle with maximum area are $15 \text{ cm} \times 15 \text{ cm}$.

12 $h(x) = 0.014x^4 - 0.196x^3 + 1.039x^2 - 2.471x + 2.225$ metres, $0 \leq x \leq 7$

a The top of the half-pipe occurs where $x = 0$ and $x = 7$.

$$h(0) = 2.225 \quad \text{and} \quad h(7) = 2.225$$

So, the wall of the half-pipe is 2.225 m high.



b The lowest point occurs when $h'(x) = 0$ and the exact middle is where $x = 3.5$.

$$h'(x) = 0.056x^3 - 0.588x^2 + 2.078x - 2.471$$

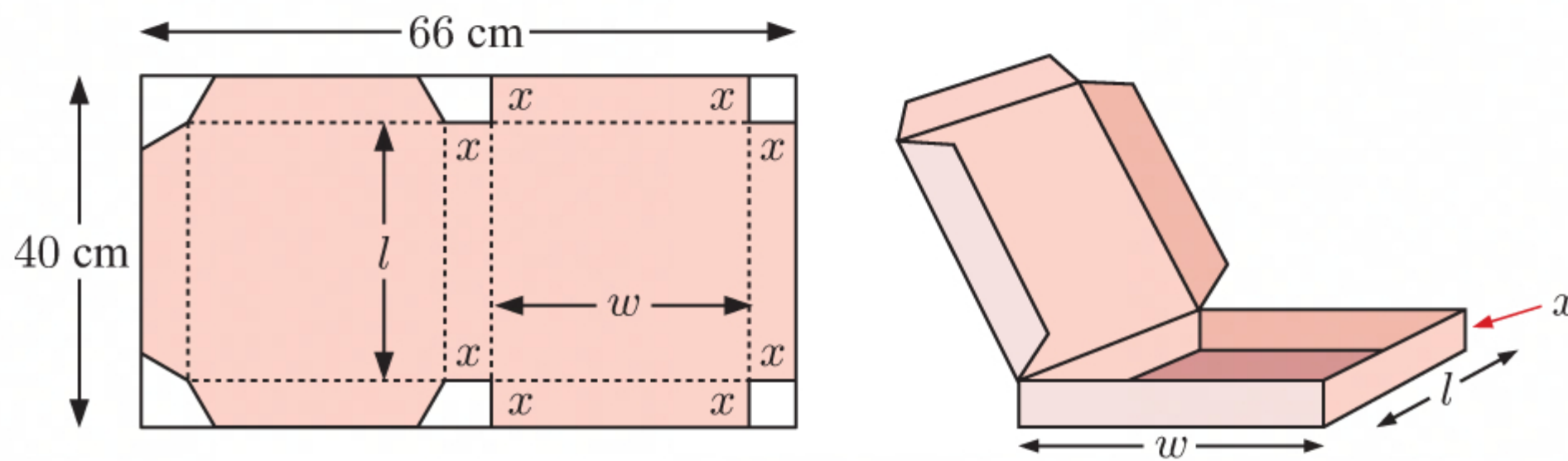
$$\begin{aligned}
 \therefore h'(3.5) &= 0.056(3.5)^3 - 0.588(3.5)^2 + 2.078(3.5) - 2.471 \\
 &= 0
 \end{aligned}$$

So the lowest point is in the exact middle of the half-pipe.

- c At the end points, $h'(0) = -2.471$ and $h'(7) = 2.471$.

The units are metres per metre.

13



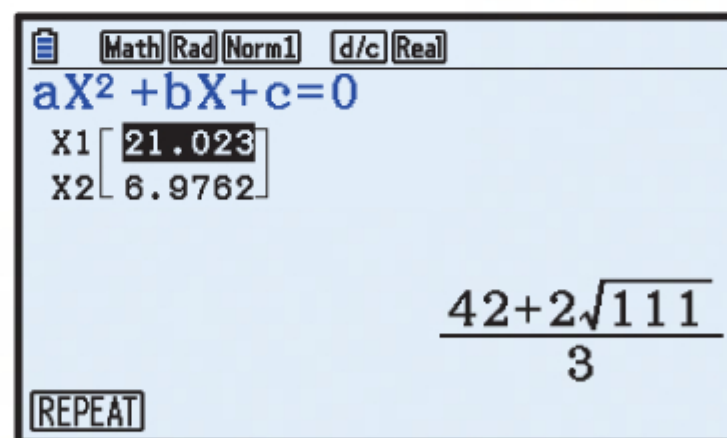
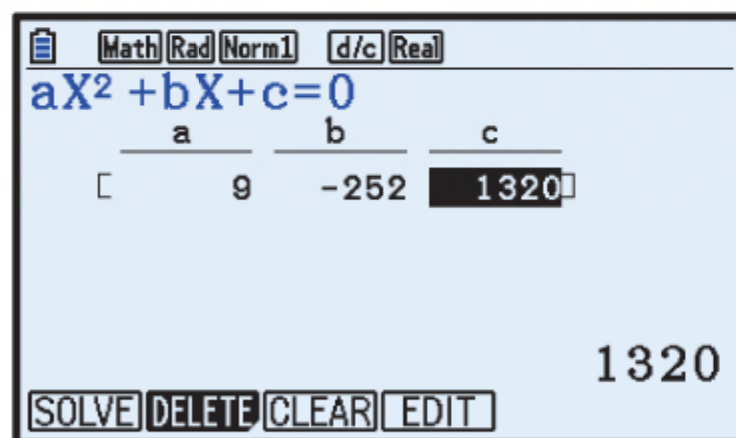
- a $x + l + x = 40$ cm and $x + w + x + w + x = 66$ cm
 $\therefore l = 40 - 2x$ cm $\therefore 2w + 3x = 66$ cm
 $\therefore w = 33 - \frac{3}{2}x$ cm

So, the dimensions of the closed box are $(40 - 2x)$ cm \times $(33 - \frac{3}{2}x)$ cm \times x cm.

- b Volume of the box $V = \text{area of base} \times \text{height}$
 $= (40 - 2x) \times (33 - \frac{3}{2}x) \times x$
 $= (1320 - 60x - 66x + 3x^2)x$
 $= (1320 - 126x + 3x^2)x$
 $= 3x^3 - 126x^2 + 1320x$ cm³

c $V'(x) = 9x^2 - 252x + 1320$

d $V'(x) = 0$ when $9x^2 - 252x + 1320 = 0$



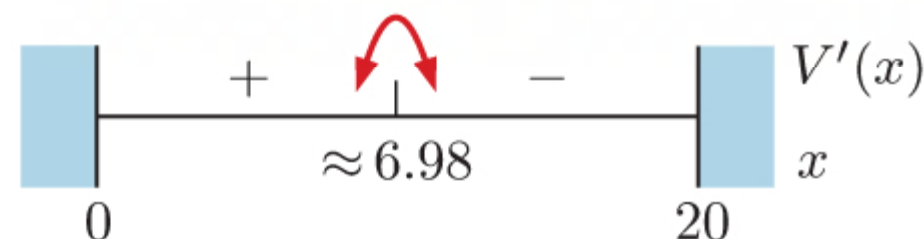
Using technology, $x \approx 21.0$ or $x \approx 6.98$.

Now $40 - 2x > 0$

$\therefore x < 20$

$\therefore x \approx 6.98$

$V'(x)$ has sign diagram:



\therefore the volume of the box is a maximum when $x \approx 6.98$.

$V(6.98) \approx 4100$

Also, when $x \approx 6.98$, $40 - 2x \approx 40 - 2(6.98) \approx 26.0$ and $33 - \frac{3}{2}(6.98) \approx 22.5$.

So, the maximum possible volume of the box is about 4100 cm³ when the dimensions are about 26.0 cm \times 22.5 cm \times 6.98 cm.

- 14 a** Consider triangle ABC which is right angled at B.

$$AB^2 + x^2 = 10^2 \quad \{\text{Pythagoras}\}$$

$$\therefore AB^2 = 100 - x^2$$

$$\therefore AB = \sqrt{100 - x^2} \text{ cm} \quad \{\text{as } AB > 0\}$$

The rectangle has area $A = \text{length} \times \text{width}$

$$= \sqrt{100 - x^2} \times x$$

$$\therefore A^2 = x^2(100 - x^2)$$

$$\therefore A^2 = 100x^2 - x^4$$

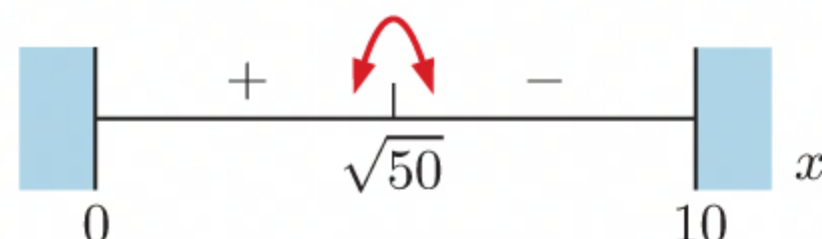
$$\begin{aligned} \mathbf{b} \quad \frac{d}{dx}(A^2) &= 200x - 4x^3 \\ &= 4x(50 - x^2) \end{aligned}$$

which is 0 when $x = 0$ or $50 - x^2 = 0$

$$\therefore x = 0 \quad \text{or} \quad x^2 = 50$$

$$\therefore x = 0 \quad \text{or} \quad x = \pm\sqrt{50}$$

$\frac{d}{dx}(A^2)$ has sign diagram:



$\therefore A^2$ is maximised when $x = \sqrt{50}$.

$$\begin{aligned} \mathbf{c} \quad \text{When } x = \sqrt{50}, \quad AB &= \sqrt{100 - (\sqrt{50})^2} \\ &= \sqrt{100 - 50} \\ &= \sqrt{50} \text{ cm} \end{aligned}$$

So, the largest rectangle which can be inscribed in the circle has dimensions

$$\sqrt{50} \text{ cm} \times \sqrt{50} \text{ cm}.$$

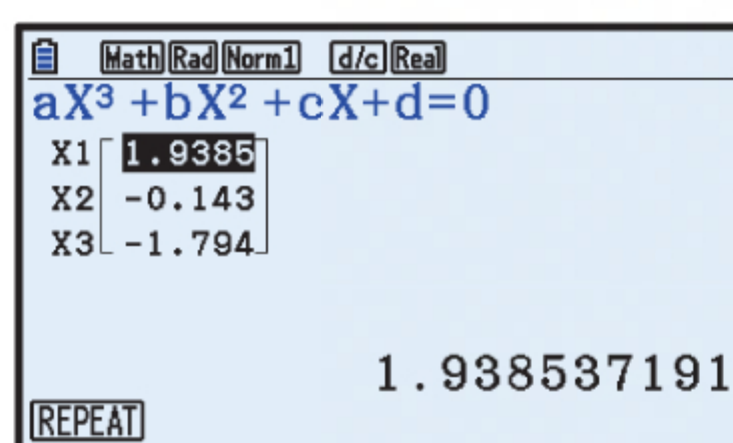
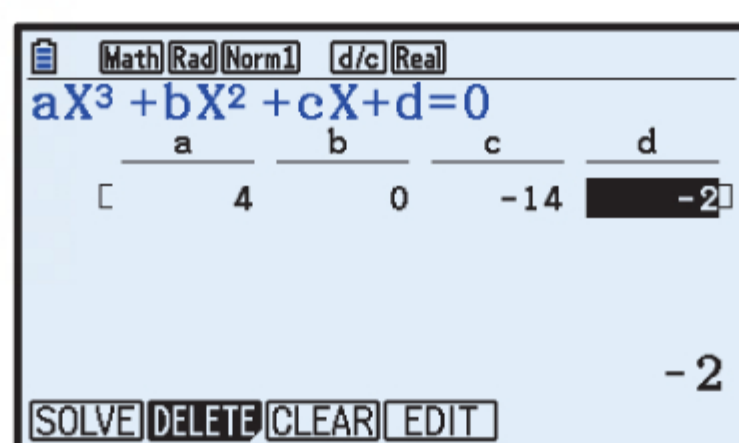
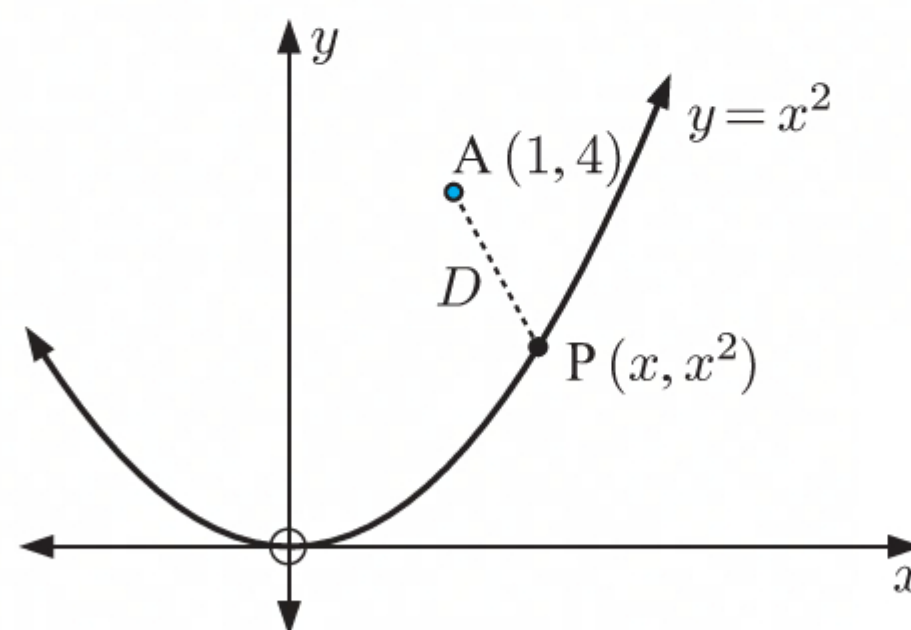
- 15 a** The distance from $A(1, 4)$ to $P(x, x^2)$ is

$$D = \sqrt{(x-1)^2 + (x^2-4)^2} \quad \{\text{distance formula}\}$$

$$\therefore D^2 = (x-1)^2 + (x^2-4)^2$$

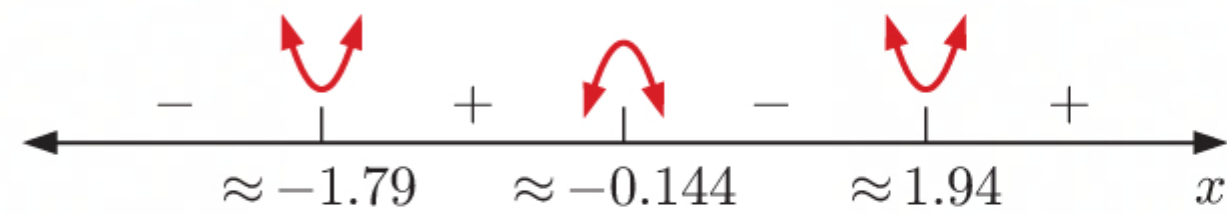
$$\begin{aligned} \mathbf{b} \quad D^2 &= (x-1)^2 + (x^2-4)^2 \\ &= x^2 - 2x + 1 + x^4 - 8x^2 + 16 \\ &= x^4 - 7x^2 - 2x + 17 \end{aligned}$$

$$\therefore \frac{d}{dx}(D^2) = 4x^3 - 14x - 2$$



Using technology, $\frac{d}{dx}(D^2) = 0$ when $x \approx 1.9385$, ≈ -0.1437 , or ≈ -1.7948

$\frac{d}{dx}(D^2)$ has sign diagram:



When $x \approx -1.7948$, $D^2 \approx 8.42$, and when $x \approx 1.9385$, $D^2 \approx 0.939$.

$\therefore D^2$ is minimised when $x \approx 1.94$.

- c** When $x \approx 1.94$, P has coordinates of about $(1.94, (1.94)^2)$ or $(1.94, 3.76)$.

When $x \approx 1.94$, $D^2 \approx 0.939$ {from **b**}

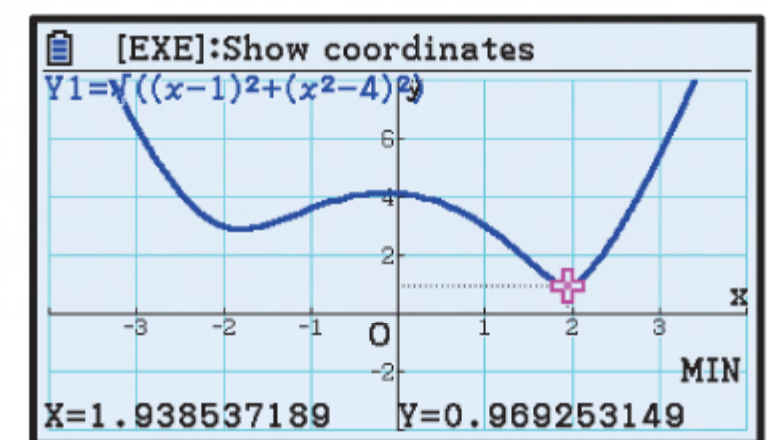
$$\therefore D \approx \sqrt{0.939} \quad \{D > 0\}$$

$$\approx 0.969$$

The closest point to A is $\approx (1.94, 3.76)$ which is about 0.969 units away.

- d** Using technology, the minimum value of D is about 0.969 which occurs when $x \approx 1.94$.

This agrees with the values we found in **c**.



- 16 a** Consider each boat's position t hours after 1:00 pm.

$$PA = 12t \quad \text{and} \quad QB = 8t$$

$$\therefore PB = 100 - 8t$$

Using the cosine rule in $\triangle PAB$,

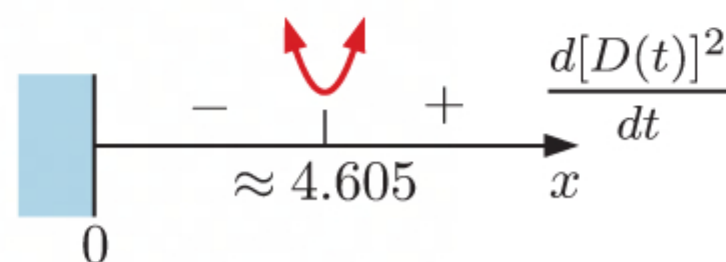
$$\begin{aligned} [D(t)]^2 &= PA^2 + PB^2 - 2 PA \times PB \cos 60^\circ \\ &= (12t)^2 + (100 - 8t)^2 - 2(12t)(100 - 8t)\left(\frac{1}{2}\right) \\ &= 144t^2 + 10\,000 - 1600t + 64t^2 - 12t(100 - 8t) \\ &= 144t^2 + 10\,000 - 1600t + 64t^2 - 1200t + 96t^2 \\ &= 304t^2 - 2800t + 10\,000 \end{aligned}$$

$$\therefore D(t) = \sqrt{304t^2 - 2800t + 10\,000} \quad \{D(t) > 0\}$$

- b** Now $\frac{d[D(t)]^2}{dt} = 608t - 2800$

$$\therefore \frac{d[D(t)]^2}{dt} = 0 \quad \text{when} \quad t = \frac{2800}{608} \approx 4.605$$

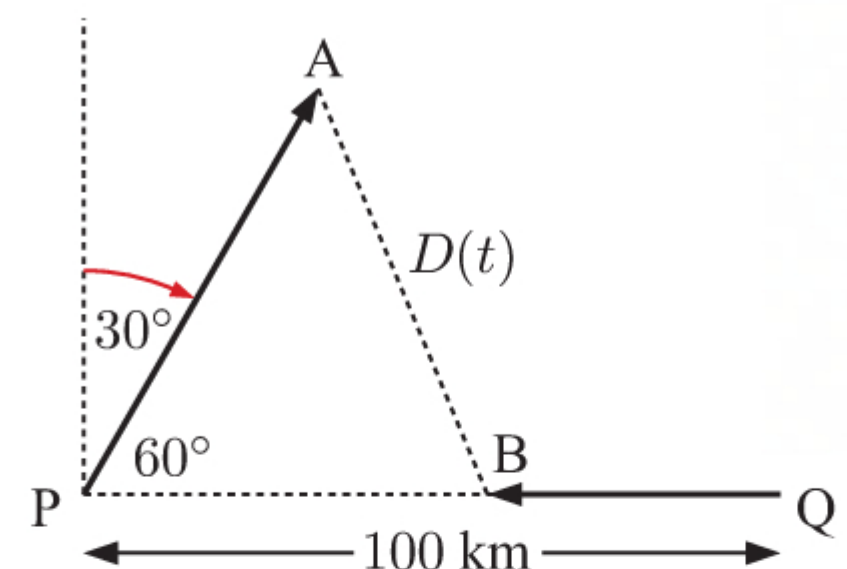
$\frac{d[D(t)]^2}{dt}$ has sign diagram:



$\therefore D(t)$ is a minimum when $t \approx 4.605$ hours after 1:00 pm

$$\begin{aligned} \text{When } t \approx 4.605, \quad [D(t)]^2 &\approx 304(4.605)^2 - 2800(4.605) + 10\,000 \\ &\approx 3550 \end{aligned}$$

\therefore the minimum value of D^2 is about 3550.



- c The ships are closest when $t \approx 4.605$ hours

$$0.605 \text{ hours} \approx 0.605 \times 60$$

$$\approx 36 \text{ minutes}$$

\therefore the ships are closest together about 4 hours and 36 minutes after 1:00 pm, which is about 5:36 pm.

- 17 a At time t , the mosquito is at $(3 - t^2, 2 + \sqrt{t}, 2 - \sqrt{t})$.

\therefore the mosquito's distance from the origin

$$\begin{aligned} D &= \sqrt{(3 - t^2)^2 + (2 + \sqrt{t})^2 + (2 - \sqrt{t})^2} \\ &= \sqrt{9 - 6t^2 + t^4 + 4 + 4\sqrt{t} + t + 4 - 4\sqrt{t} + t} \\ &= \sqrt{t^4 - 6t^2 + 2t + 17} \end{aligned}$$

$$\therefore D^2 = t^4 - 6t^2 + 2t + 17$$

b $\frac{d}{dt}(D^2) = 4t^3 - 12t + 2$

Using technology,

$$\frac{d}{dt}(D^2) = 0 \text{ when } t \approx 1.6418$$

$$\text{or } \approx 0.1683 \quad \{t \geq 0\}$$

$\frac{d}{dt}(D^2)$ has sign diagram:

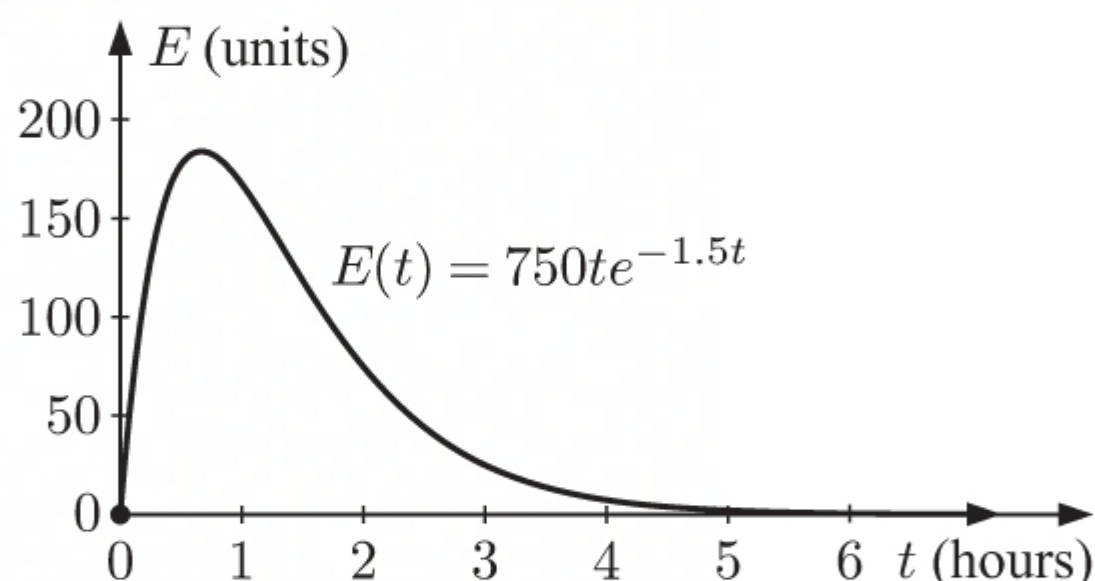
Now, D is minimised when D^2 is minimised, since $D > 0$.

$$D(1.6418) \approx 3.37$$

So, the closest the mosquito came to the source of the repellent was about 3.37 m.

EXERCISE 12B.2

- 1 a



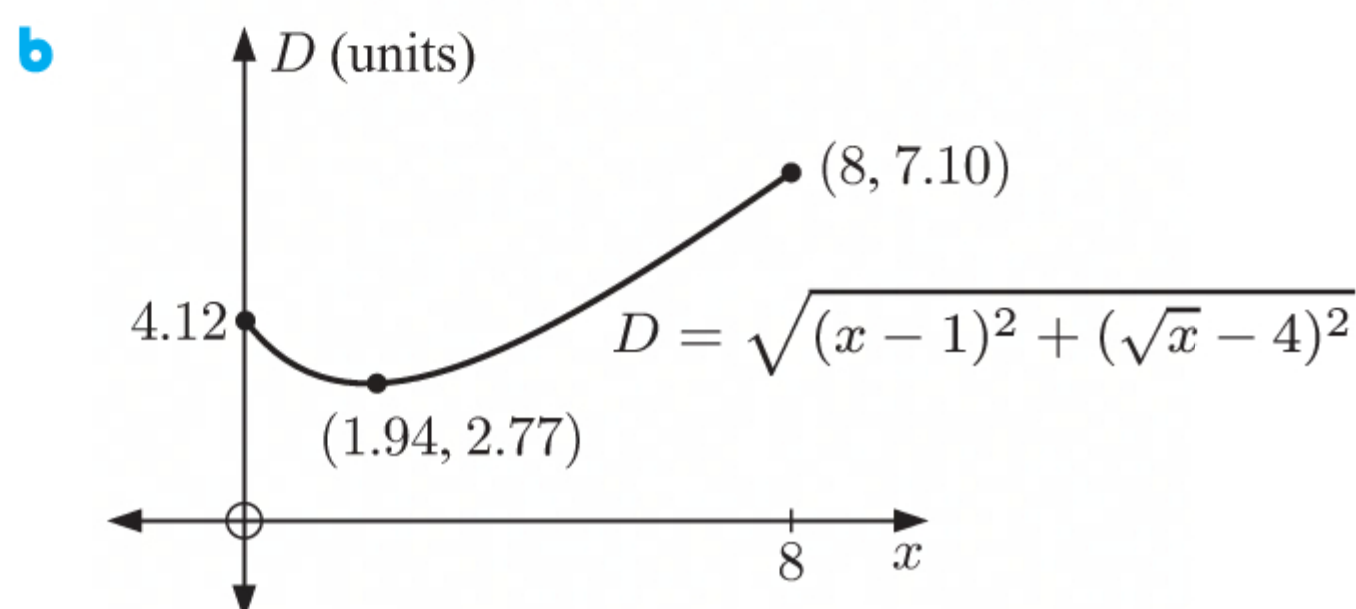
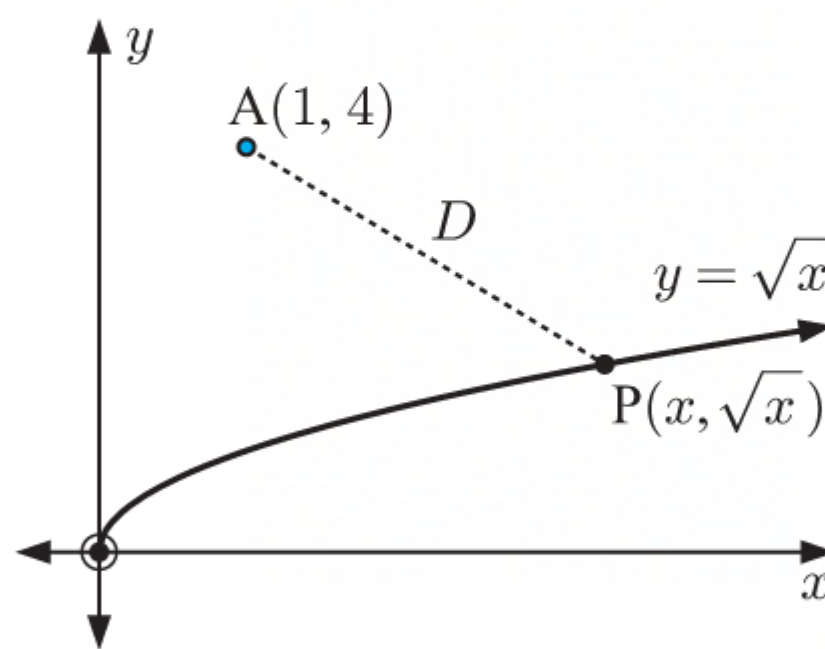
- b From a, we see that $E'(t) \geq 0$ for $0 \leq t \leq \frac{2}{3}$, and $E'(t) \leq 0$ for $t \geq \frac{2}{3}$.

$\therefore E'(t)$ has sign diagram:

- c $E(t)$ is at a maximum when $t = \frac{2}{3}$ hours, which is 40 minutes. So the anaesthetic is most effective 40 minutes after the injection.

- 2 a** Using the distance formula, the distance from $A(1, 4)$ to $P(x, \sqrt{x})$ is

$$D = \sqrt{(x-1)^2 + (\sqrt{x}-4)^2} \text{ units}$$

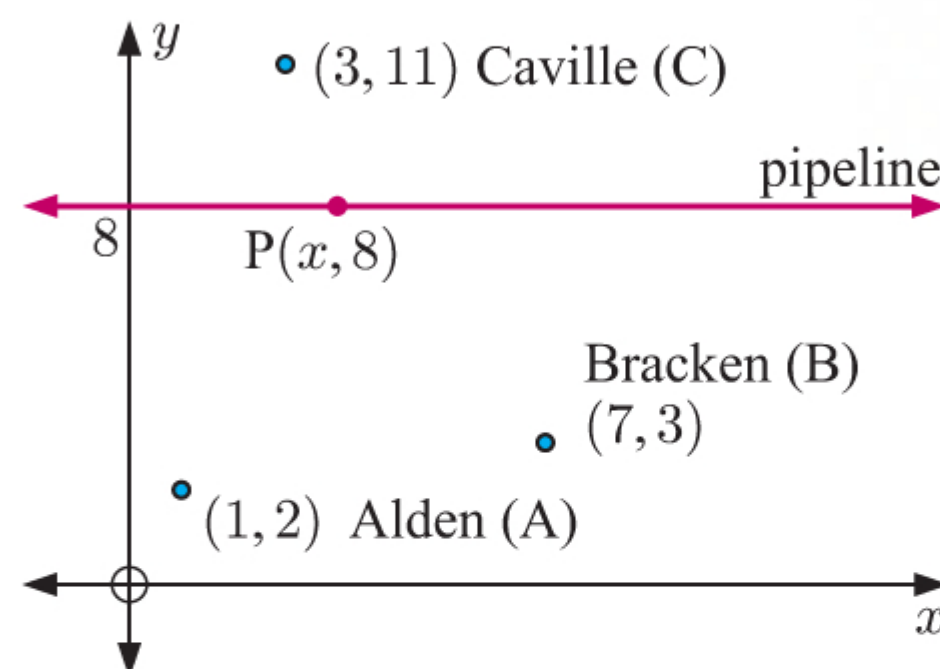


- c** From the graph in **b**, the minimum value of D occurs at about $(1.94, 2.77)$, so the smallest value of D is about 2.77 units, which occurs when $x \approx 1.94$.
- d** Since there is a local minimum at about $(1.94, 2.77)$, $\frac{dD}{dx} = 0$ when $x \approx 1.94$.

- 3 a** Using the distance formula, the distance from $C(3, 11)$ to $P(x, 8)$ is

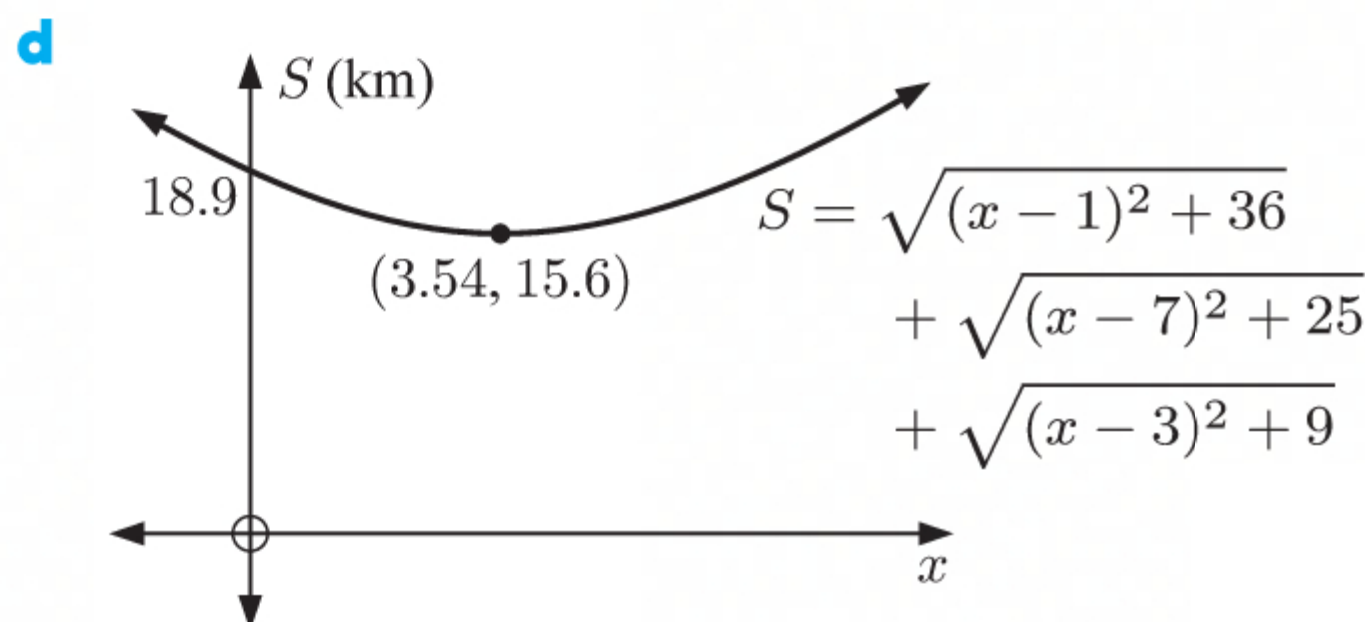
$$\begin{aligned} PC &= \sqrt{(x-3)^2 + (8-11)^2} \\ &= \sqrt{(x-3)^2 + 9} \end{aligned}$$

- b** Similarly, $PA = \sqrt{(x-1)^2 + (8-2)^2}$
 $= \sqrt{(x-1)^2 + 36}$
 and $PB = \sqrt{(x-7)^2 + (8-3)^2}$
 $= \sqrt{(x-7)^2 + 25}$



- c** $S = PA + PB + PC$

$$= \sqrt{(x-1)^2 + 36} + \sqrt{(x-7)^2 + 25} + \sqrt{(x-3)^2 + 9}$$



- e** From the graph in **d**, there is a local minimum at about $(3.54, 15.6)$.

$S'(x) \leq 0$ when $x \leq 3.54$, and $S'(x) \geq 0$ when $x \geq 3.54$.

$\therefore S'(x)$ has sign diagram:



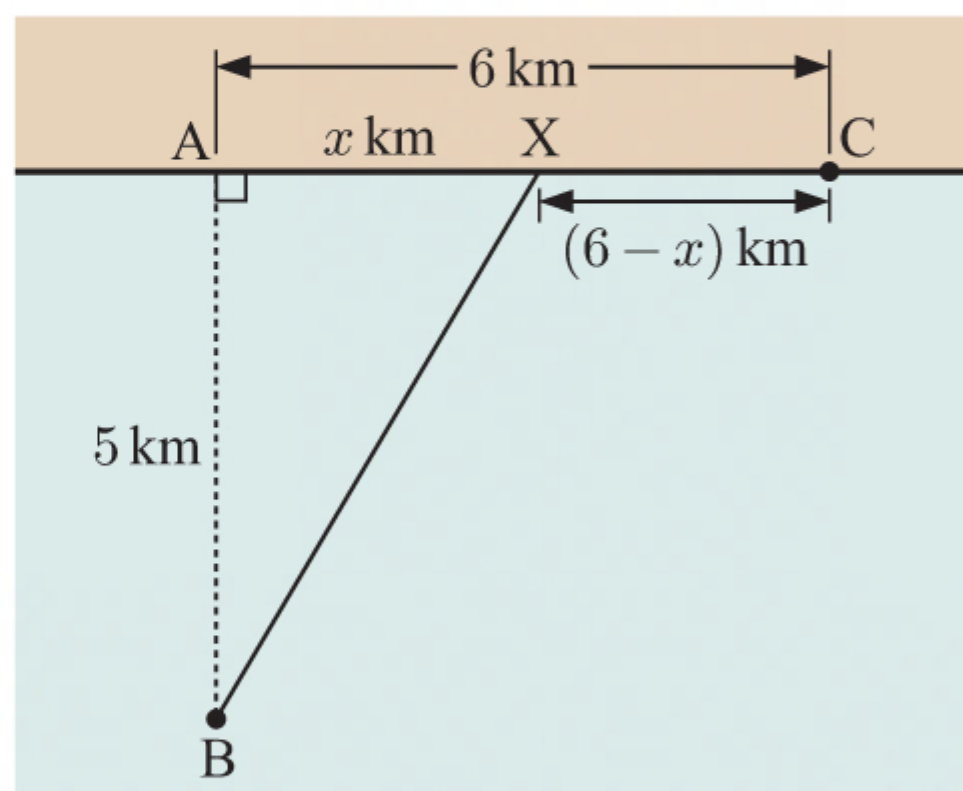
- f** The total length of connecting pipe needed is given by $S = PA + PB + PC$.

From **e**, the minimum value of S occurs when $x \approx 3.54$.

So, P should be located at about $(3.54, 8)$.

- 4 a** AC has length 6 km and X lies between A and C.

$\therefore 0 \leq x \leq 6$

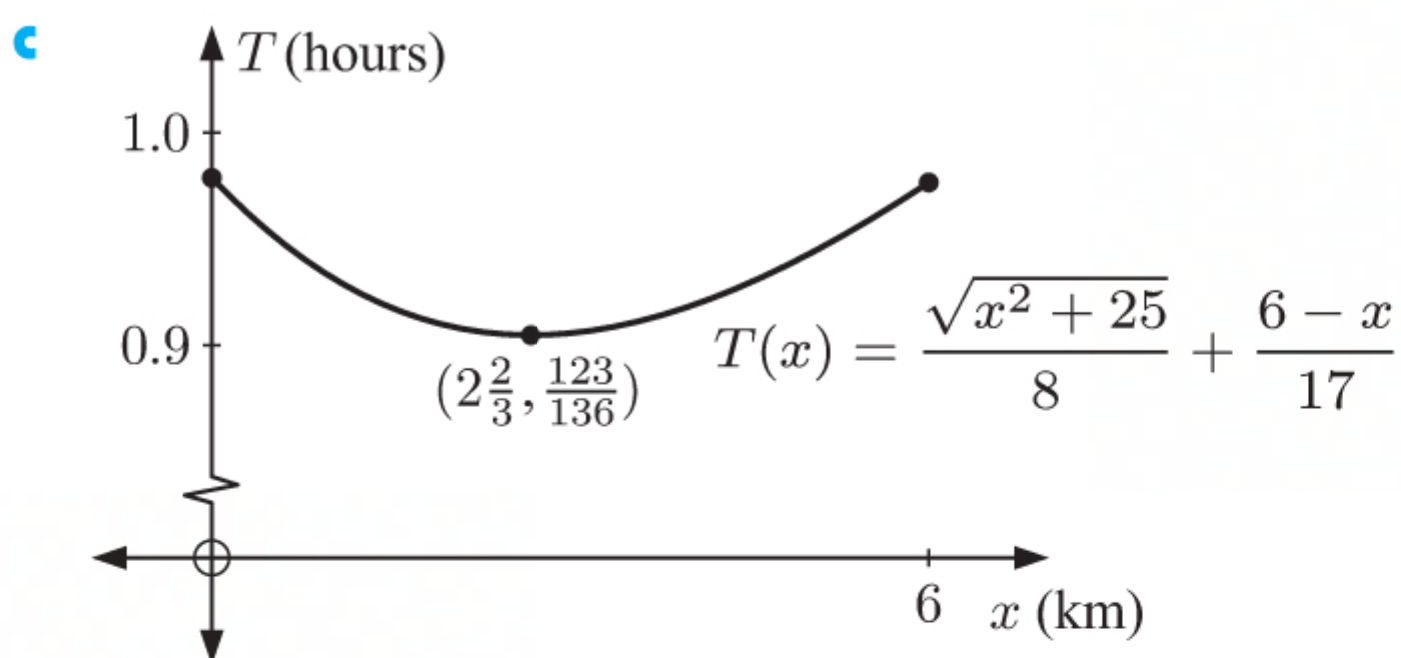


- b** Now $XC = 6 - x$ and $BX = \sqrt{x^2 + 5^2}$ {Pythagoras}

\therefore the time taken to row from B to X = $\frac{\text{distance}}{\text{speed}} = \frac{BX}{8} = \frac{\sqrt{x^2 + 5^2}}{8}$ hours

and the time taken to run from X to C = $\frac{\text{distance}}{\text{speed}} = \frac{XC}{17} = \frac{6 - x}{17}$ hours

\therefore the total time $T = \frac{\sqrt{x^2 + 25}}{8} + \frac{6 - x}{17}$ hours, $0 \leq x \leq 6$



- d** From the graph in **c**, there is a local minimum when $x = 2\frac{2}{3}$, so $\frac{dT}{dx} = 0$ when $x = 2\frac{2}{3}$.

This means that the time taken to travel to point C is minimised when Peter rows towards the point $2\frac{2}{3}$ km from A. So, Peter should row to the point which is $2\frac{2}{3}$ km from A, then run to C, to minimise his total time.

EXERCISE 12C

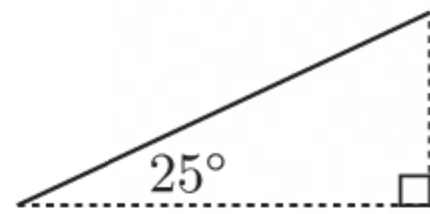
1 $h(x) = ax^2 + bx + c$ metres

- a The stone is thrown from 3 m above the water,
so $h(0) = 3$

$$\therefore c = 3$$

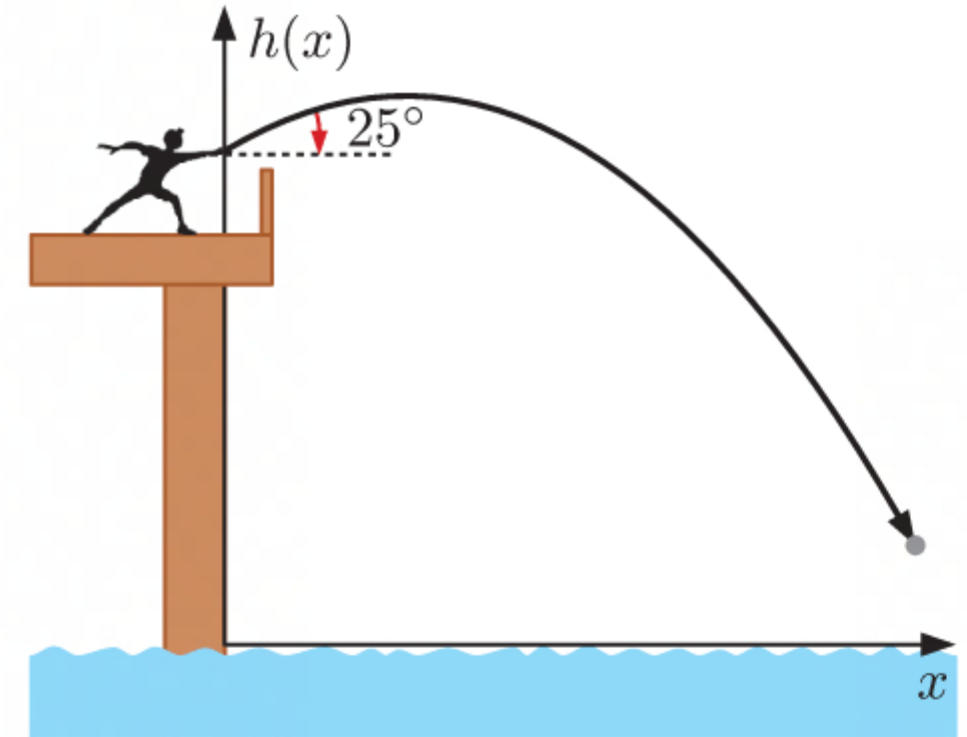
b $h'(x) = 2ax + b$

c



$$h'(0) = \text{gradient} = \tan 25^\circ$$

$$\therefore b = \tan 25^\circ$$



- d The stone reaches its maximum height when $x = 5$.

$$\therefore h'(5) = 0$$

$$\therefore 2a(5) + \tan 25^\circ = 0$$

$$\therefore 10a = -\tan 25^\circ$$

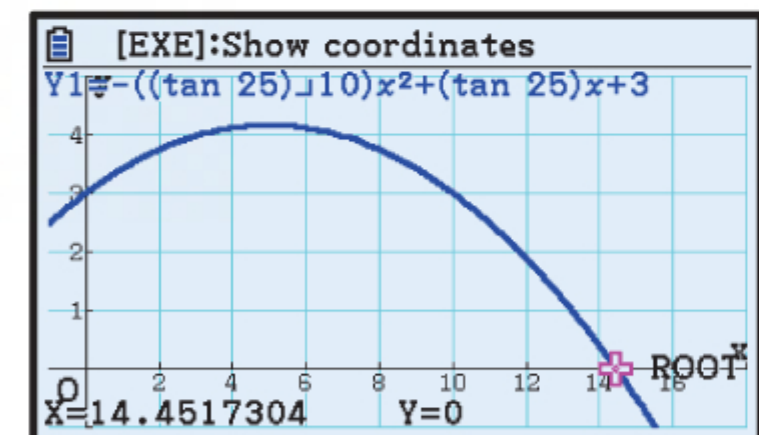
$$\therefore a = -\frac{\tan 25^\circ}{10}$$

$h(x) = 0$ when the stone lands in the water.

$$\therefore -\frac{\tan 25^\circ}{10}x^2 + (\tan 25^\circ)x + 3 = 0$$

Using technology, $x \approx 14.5$ $\{x > 0\}$

The stone lands in the water when $x \approx 14.5$.



2 $C(x) = ax^3 + bx^2 + cx + d$ euros

- a The maximum output is 140 cars, so the domain of $C(x)$ is $0 \leq x \leq 140$.

The fixed costs are $C(0) = \text{€}24\,500$, so $d = 24\,500$.

Differentiating with respect to x , $C'(x) = 3ax^2 + 2bx + c$.

Now $C'(0) \approx 3280$, so $c \approx 3280$

and $C'(80) \approx 2320$, so $3a(80)^2 + 2b(80) + 3280 \approx 2320$

$$\therefore 19\,200a + 160b \approx -960$$

$$\therefore 120a + b \approx -6 \quad \dots (1)$$

$C(80) = 294\,000$, so $a(80)^3 + b(80)^2 + c(80) + 24\,500 = 294\,000$

$$\therefore 512\,000a + 6400b + 262\,400 + 24\,500 \approx 294\,000 \quad \{c \approx 3280\}$$

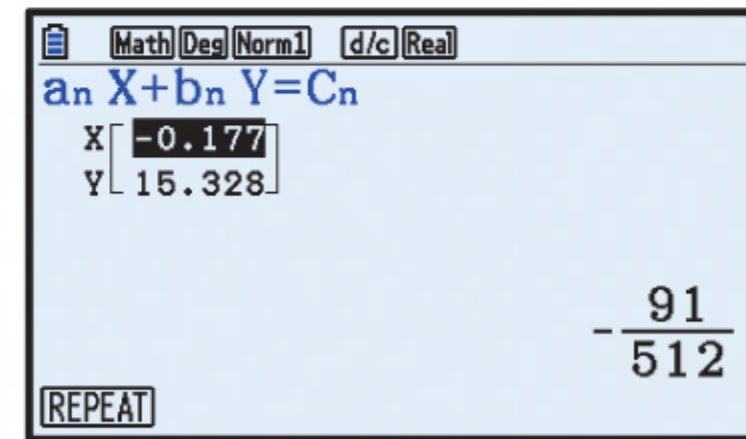
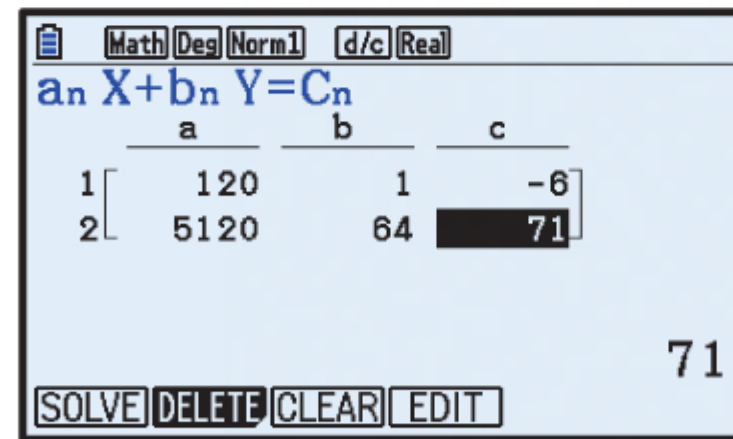
$$\therefore 512\,000a + 6400b \approx 7100$$

$$\therefore 5120a + 64b \approx 71 \quad \dots (2)$$

We solve (1) and (2) simultaneously using technology.

$$\therefore a \approx -0.178$$

and $b \approx 15.3$



So, $C(x) \approx -0.178x^3 + 15.3x^2 + 3280x + 24\,500$, $0 \leq x \leq 140$.

b $C(120) \approx -0.178(120)^3 + 15.3(120)^2 + 3280(120) + 24\,500$
 $\approx 331\,000$

\therefore it costs about €331 000 to produce the parts for 120 cars each day.

3 $P(t) = at^3 + bt^2 + ct + d$ million horses

a There were 21.5 million horses in 1900, so $P(0) = 21.5$
 $\therefore d = 21.5$

b $P'(t) = 3at^2 + 2bt + c$

c $P'(t) = 0$ when the population of horses is at a local maximum or a local minimum.
 The maximum number of horses occurred in 1915, when $t = 15$.
 So, $P'(t) = 0$ when $t = 15$.

d $P'(15) = 0$ {from **c**}
 $\therefore 3a(15)^2 + 2b(15) + c = 0$
 $\therefore 675a + 30b + c = 0$ (1)

The population was 26.5 million in 1915, when $t = 15$, so $P(15) = 26.5$
 $\therefore a(15)^3 + b(15)^2 + c(15) + 21.5 = 26.5$
 $\therefore 3375a + 225b + 15c = 5$ (2)

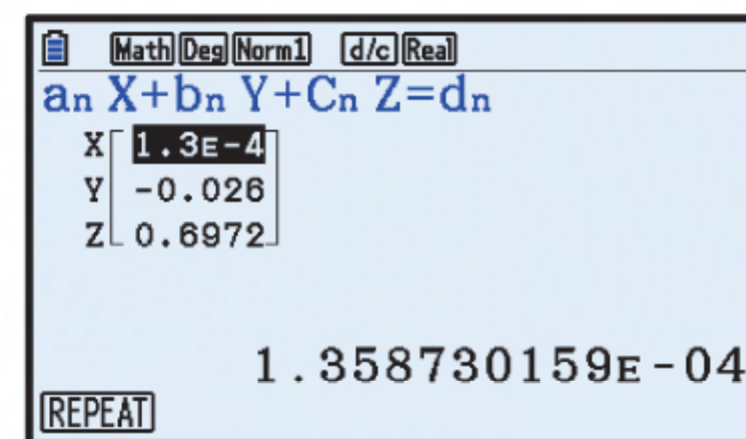
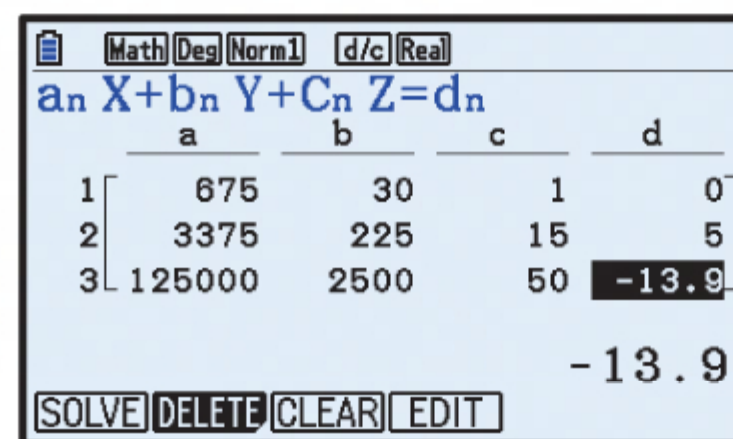
The population was 7.6 million in 1950, when $t = 50$, so $P(50) = 7.6$
 $\therefore a(50)^3 + b(50)^2 + c(50) + 21.5 = 7.6$
 $\therefore 125\,000a + 2500b + 50c = -13.9$ (3)

We solve (1), (2), and (3) simultaneously using technology.

$$\therefore a \approx 0.000\,136$$

$$b \approx -0.0263$$

and $c \approx 0.697$



So, $P(t) \approx 0.000\,136t^3 - 0.0263t^2 + 0.697t + 21.5$ million horses.

e i In 1930, $t = 30$.

$$P(30) \approx 0.000\,136(30)^3 - 0.0263(30)^2 + 0.697(30) + 21.5$$

$$\approx 22.4$$

The population of horses in 1930 was about 22.4 million.

ii In 1960, $t = 60$.

$$P(60) \approx 0.000136(60)^3 - 0.0263(60)^2 + 0.697(60) + 21.5 \\ \approx -1.98$$

The population of horses in 1960 was about -1.98 million.

f The model provided a reasonable estimate when interpolating between data points. The extrapolation in e ii however predicted a negative population, which is not possible.

4 $A(t) = at^3 + bt^2 + ct + d$

a There were no people at the fair outside of its opening hours.

So at 8 am, when $t = 0$, $A(0) = 0$, and at 6 pm, when $t = 10$, $A(10) = 0$.

At 10 am, when $t = 2$, the attendance was increasing at a rate of 1770 people per hour.

So $A'(2) = 1770$.

At 4 pm, when $t = 8$, the attendance was decreasing at a rate of 1800 people per hour.

So $A'(8) = -1800$.

b $A(0) = 0$

$$\therefore d = 0$$

$$A(10) = 0, \text{ so } a(10)^3 + b(10)^2 + c(10) = 0$$

$$\therefore 1000a + 100b + 10c = 0$$

$$\therefore 100a + 10b + c = 0 \quad \dots (1)$$

$$A'(t) = 3at^2 + 2bt + c$$

$$A'(2) = 1770, \text{ so } 3a(2)^2 + 2b(2) + c = 1770$$

$$\therefore 12a + 4b + c = 1770 \quad \dots (2)$$

$$A'(8) = -1800, \text{ so } 3a(8)^2 + 2b(8) + c = -1800$$

$$\therefore 192a + 16b + c = -1800 \quad \dots (3)$$

We solve (1), (2), and (3) simultaneously using technology.

$$\therefore a = -7.5$$

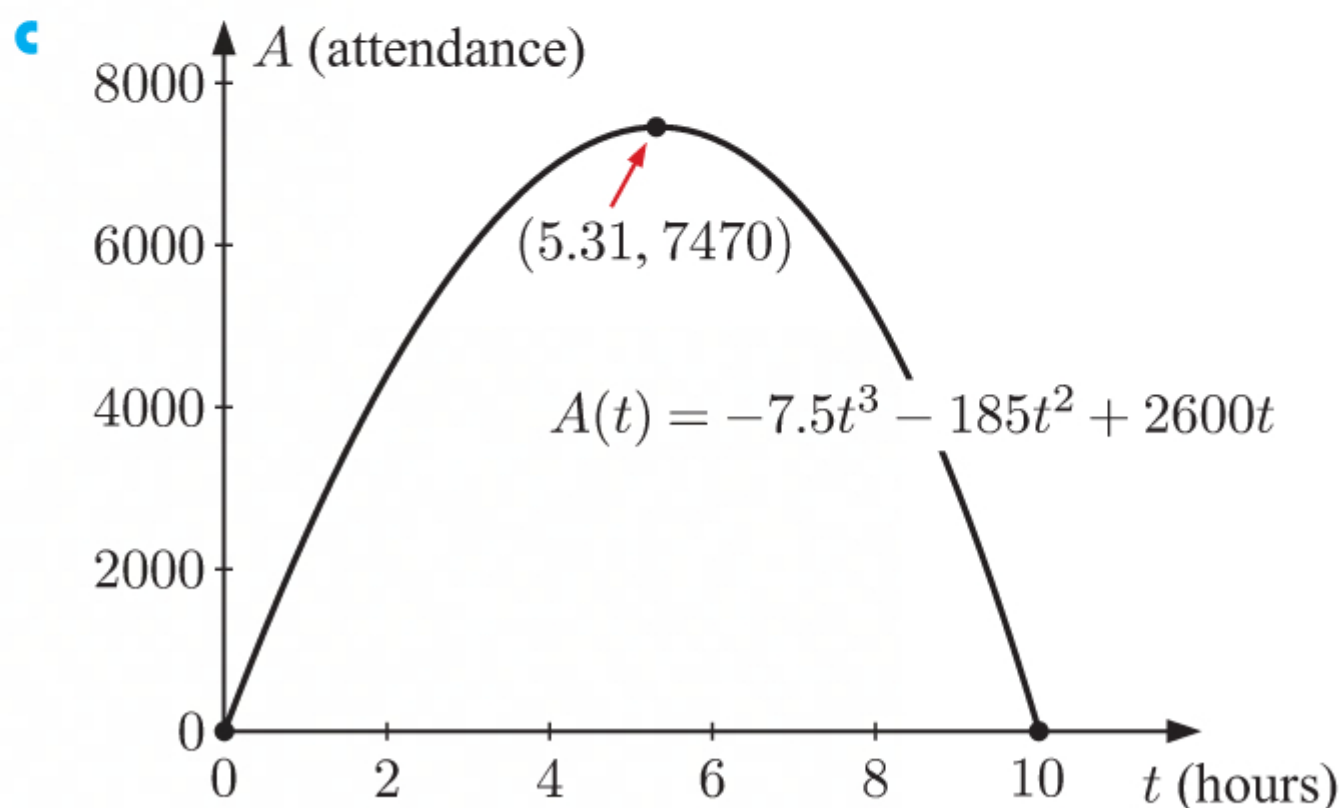
$$b = -185$$

and $c = 2600$

So, $A(t) = -7.5t^3 - 185t^2 + 2600t$, $0 \leq t \leq 10$.

	a	b	c	d
1	100	10	1	0
2	12	4	1	1770
3	192	16	1	-1800

	X	Y	Z
	-7.5	-185	2600



The mayor's model seems reasonable; the attendance was always increasing until shortly after 1 pm, then it began to decrease.

- d** From the graph in **c**, the maximum attendance was about 7470 people which occurred when $t \approx 5.31$, which was at about 1:19 pm.

5 a

Distance (r cm)	20	40	80
Illuminance (I lx)	450	112.3	28.1
Ir	9000	4492	2248
Ir^2	180 000	179 680	179 840
Ir^3	3 600 000	7 187 200	14 387 200

Model **B** $\left(I = \frac{k}{r^2}\right)$ best fits the data as Ir^2 is approximately constant.

b

	List 1	List 2	List 3	List 4
SUB				
1	20	450		
2	40	112.3		
3	80	28.1		
4				

	Deg	Norm1	d/c	Real
PowerReg				
a	180266.021			
b	-2.0006414			
r	-0.9999998			
r ²	0.99999969			
MSe	1.1876 × 10 ⁻⁶			
y = a · x ^b				

The correlation coefficient r is very close to 1, and the power is very close to -2 , so it is reasonable to conclude that I is inversely proportional to r^2 .

The model is $I \approx \frac{180\,266}{r^2}$, which agrees with our answer to **a**.

$$\begin{aligned}
 \text{c} \quad I &\approx \frac{180\,266}{r^2} \\
 &\approx 180\,266r^{-2} \\
 \therefore \frac{dI}{dr} &\approx -360\,532r^{-3} \\
 &\approx -\frac{360\,532}{r^3}
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad \text{When } r = 1 \text{ m} = 100 \text{ cm, } \frac{dI}{dr} &\approx -\frac{360\,532}{100^3} \\
 &\approx -0.361
 \end{aligned}$$

So, the illuminance is decreasing at a rate of about 0.361 lux per cm at the distance 1 m from the light.

- e** The surface area of a sphere is proportional to the square of its radius. The total illuminance from a point is a sphere.

Since $SA \propto r^2$ and $I \propto \frac{1}{r^2}$, the total illuminance is a constant.

REVIEW SET 12A

1 $C(x) = -0.0002x^3 + 24x + 5400$ dollars

a $C(0) = 5400$ dollars, which is the fixed operating cost without producing any items.

b $C'(x) = -0.0006x^2 + 24$

This is the rate at which the production cost (in dollars) is changing per lot of one thousand pairs of chopsticks when x thousand pairs of chopsticks are produced. It estimates the cost of producing the $(x + 1)$ th lot of one thousand pairs of chopsticks.

c $C'(100) = -0.0006(100)^2 + 24$
 $= 18$

The cost of producing the 101st lot of one thousand pairs of chopsticks is approximately \$18.

d $C(101) - C(100) = -0.0002(101)^3 + 24(101) + 5400 - [-0.0002(100)^3 + 24(100) + 5400]$
 $= 17.9398$
 ≈ 17.94

The actual cost of producing the 101st lot of one thousand pairs of chopsticks is approximately \$17.94.

2 $C(v) = 10v + \frac{90}{v}$ euros

a i $C(15) = 10(15) + \frac{90}{15}$
 $= 156$

So, the cost of running the barge for 2 hours at 15 km h^{-1} is
 $2 \times 156 = \text{€}312$.

b $C(v) = 10v + \frac{90}{v}$
 $= 10v + 90v^{-1}$

$\therefore C'(v) = 10 - 90v^{-2}$
 $= 10 - \frac{90}{v^2}$

i $C'(10) = 10 - \frac{90}{10^2}$
 $= 9.1$

So, the rate of change in the cost of running the barge at 10 km h^{-1} is
 $\text{€}9.10$ per hour per km h^{-1} .

ii $C(24) = 10(24) + \frac{90}{24}$
 $= 243.75$

So, the cost of running the barge for 5 hours at 24 km h^{-1} is
 $5 \times 243.75 = \text{€}1218.75$.

ii $C'(6) = 10 - \frac{90}{6^2}$
 $= 7.5$

So, the rate of change in the cost of running the barge at 6 km h^{-1} is
 $\text{€}7.50$ per hour per km h^{-1} .

c Now $C'(v) = 0$ when $10 - \frac{90}{v^2} = 0$

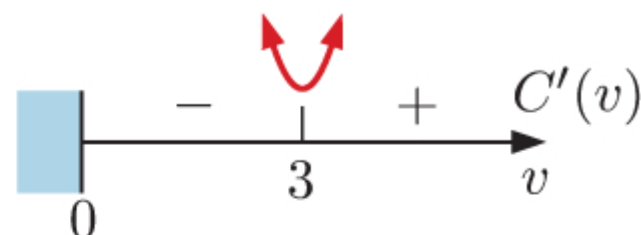
$$\therefore 10 = \frac{90}{v^2}$$

$$\therefore v^2 = \frac{90}{10}$$

$$\therefore v^2 = 9$$

$$\therefore v = \quad \{\text{as } v > 0\}$$

$C'(v)$ has sign diagram:



So, the minimum cost per hour occurs when $v = 3 \text{ km h}^{-1}$.

3 a Let $OD = x$, so C has coordinates $(x, 9 - x^2)$.

Area of rectangle ABCD = length \times width

$$\begin{aligned} \therefore A(x) &= 2x \times (9 - x^2) \\ &= 18x - 2x^3 \end{aligned}$$

b $A(x) = 18x - 2x^3$

$$\therefore A'(x) = 18 - 6x^2$$

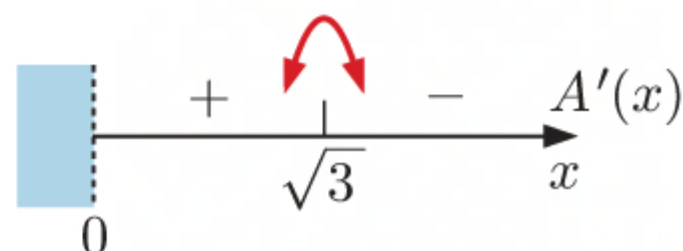
$$A'(x) = 0 \quad \text{when} \quad 18 - 6x^2 = 0$$

$$\therefore 6x^2 = 18$$

$$\therefore x^2 = 3$$

$$\therefore x = \sqrt{3} \quad \{x > 0\}$$

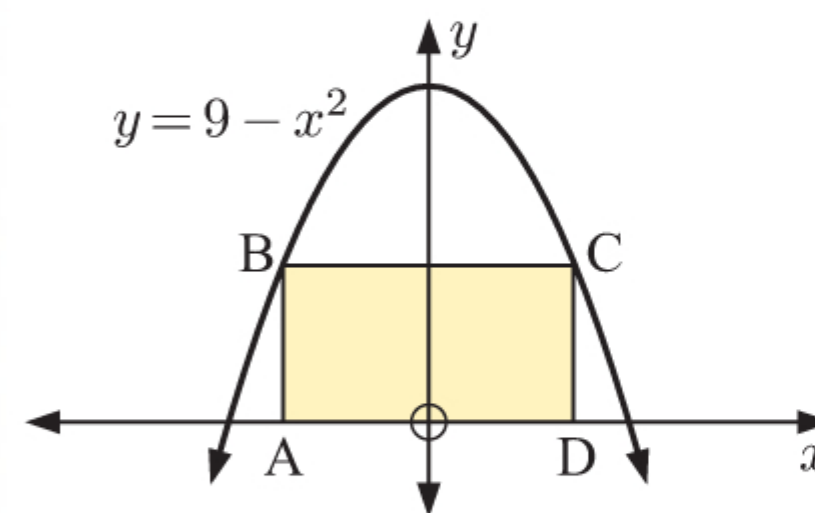
which has sign diagram:



So, the area is a maximum when $x = \sqrt{3}$.

$$\text{When } x = \sqrt{3}, y = 9 - (\sqrt{3})^2 = 6$$

So, C has coordinates $(\sqrt{3}, 6)$.

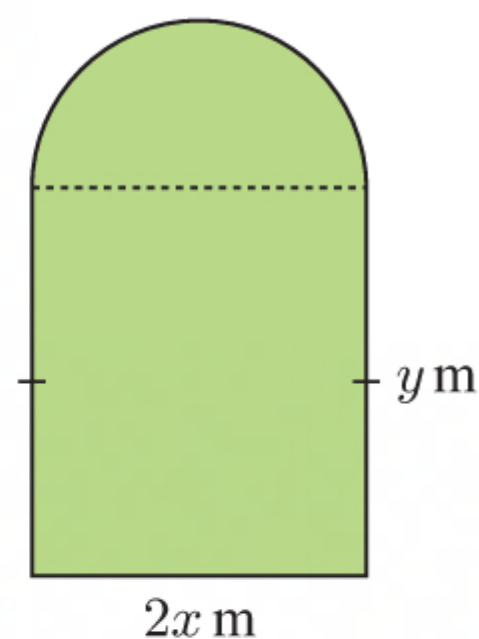


4 a perimeter = $2x + 2y + \pi x$

$$\therefore 200 = 2x + 2y + \pi x$$

$$\therefore 2y = 200 - 2x - \pi x$$

$$\therefore y = 100 - x - \frac{\pi}{2}x$$



b area of lawn $A = \text{area of rectangle} + \text{area of semi-circle}$

$$= 2x \times y + \frac{1}{2}\pi x^2$$

$$= 2x(100 - x - \frac{\pi}{2}x) + \frac{\pi}{2}x^2 \quad \{\text{using a}\}$$

$$= 200x - 2x^2 - \pi x^2 + \frac{\pi}{2}x^2$$

$$= 200x - 2x^2 - \frac{\pi}{2}x^2$$

$$\therefore A = 200x - x^2(2 + \frac{\pi}{2}) \text{ m}^2$$

c $\frac{dA}{dx} = 200 - 4x - \pi x$

Now $\frac{dA}{dx} = 0$ when $200 - 4x - \pi x = 0$

$$\therefore 4x + \pi x = 200$$

$$\therefore x(4 + \pi) = 200$$

$$\therefore x = \frac{200}{4 + \pi}$$

$$\therefore x \approx 28.0$$

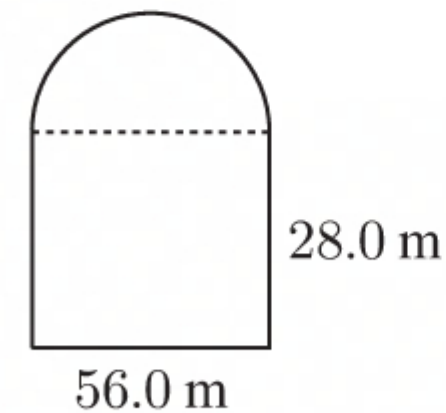


The area of the lawn is maximised when $x = \frac{200}{4 + \pi} \approx 28.0$

and $y = 100 - \frac{200}{4 + \pi} - \frac{\pi}{2} \left(\frac{200}{4 + \pi} \right)$

$$\approx 28.0$$

The dimensions of the lawn of maximum area are:

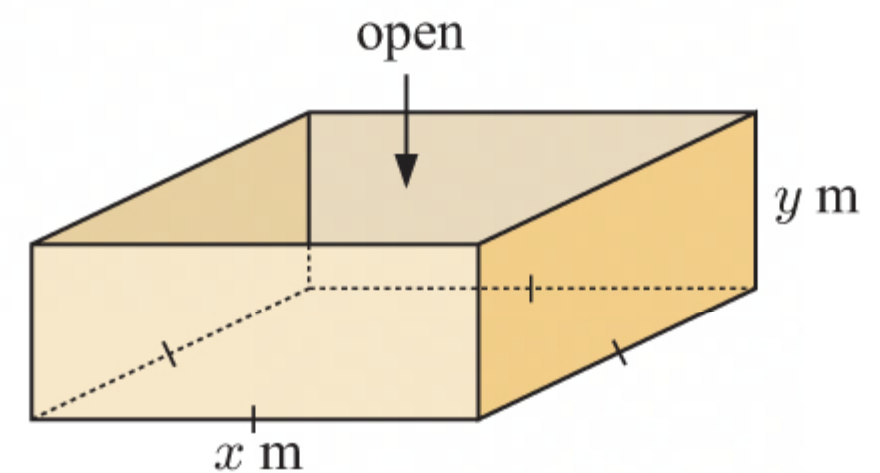


5 a capacity = 1 kL $\equiv 1 \text{ m}^3$

volume of box = area of base \times height

$$\therefore 1 = x^2 y$$

$$\therefore y = \frac{1}{x^2}, \quad x > 0$$



b area of steel needed = area of base + area of 4 sides

$$= x^2 + 4xy$$

$$= x^2 + 4x \left(\frac{1}{x^2} \right) \quad \{\text{from a}\}$$

$$= x^2 + \frac{4}{x}$$

Steel costs \$24 per m^2 , so total cost of steel = $\left(x^2 + \frac{4}{x} \right) \times 24$

$$\therefore C(x) = 24x^2 + \frac{96}{x} \text{ dollars}$$

$$c \quad C(x) = 24x^2 + \frac{96}{x} = 24x^2 + 96x^{-1}$$

$$\therefore C'(x) = 48x - 96x^{-2} = 48x - \frac{96}{x^2}$$

$$d \quad C'(x) = 0 \text{ when } 48x - \frac{96}{x^2} = 0$$

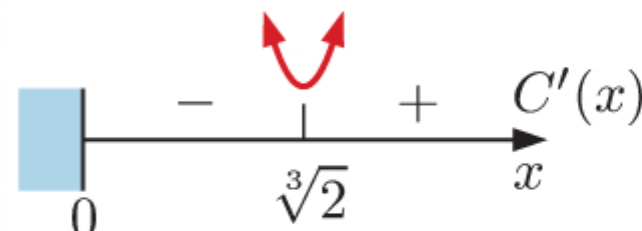
$$\therefore 48x = \frac{96}{x^2}$$

$$\therefore 48x^3 = 96$$

$$\therefore x^3 = 2$$

$$\therefore x = \sqrt[3]{2}$$

$C'(x)$ has sign diagram:



So, the cost is a minimum when $x = \sqrt[3]{2} \approx 1.26$

$$\text{When } x = \sqrt[3]{2}, \quad y = \frac{1}{(\sqrt[3]{2})^2} \approx 0.630$$

$$C(\sqrt[3]{2}) = 24(\sqrt[3]{2})^2 + \frac{96}{\sqrt[3]{2}} \approx 114.29$$

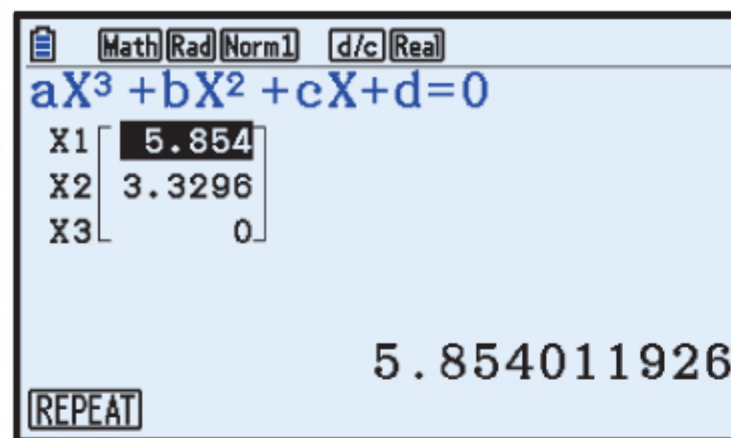
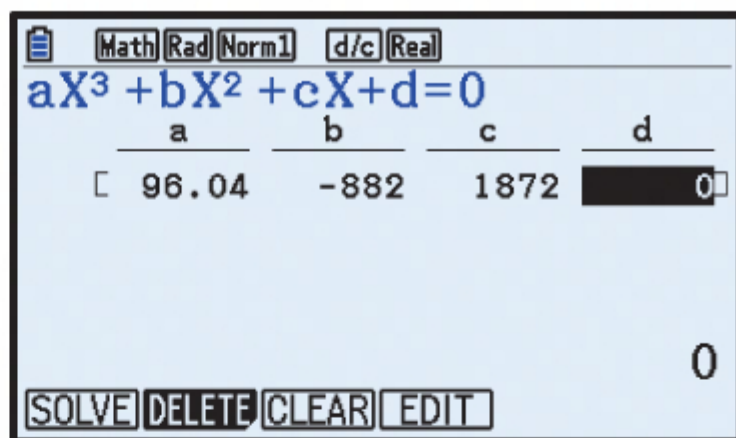
So, the dimensions which minimise the cost of the box are about $1.26 \text{ m} \times 1.26 \text{ m} \times 0.630 \text{ m}$.
The cost of the steel in this case would be about \$114.29.

$$6 \quad D(t) = \sqrt{24.01t^4 - 294t^3 + 936t^2} \text{ metres}$$

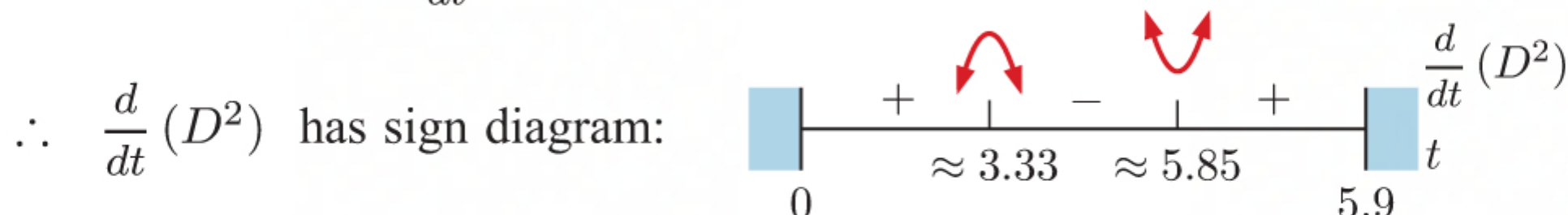
a The stone flies for 5.9 seconds before landing, so the domain of $D(t)$ is $\{t \mid 0 \leq t \leq 5.9\}$.

$$b \quad D^2 = 24.01t^4 - 294t^3 + 936t^2$$

$$\therefore \frac{d}{dt}(D^2) = 96.04t^3 - 882t^2 + 1872t$$



Using technology, $\frac{d}{dt}(D^2) = 0$ when $t = 0, 3.33$, or 5.85 .



So, D^2 is maximised when $t \approx 3.33$ seconds.

c D^2 is maximised when $t \approx 3.33$ seconds, so D is also maximised when $t \approx 3.33$ seconds.

$$D(3.33) \approx \sqrt{24.01(3.33)^4 - 294(3.33)^3 + 936(3.33)^2} \approx 49.8$$

So, the maximum distance between the stone and Max is about 49.8 m.

- d** The stone lands when $t = 5.9$ seconds.

$$D(5.9) = \sqrt{24.01(5.9)^4 - 294(5.9)^3 + 936(5.9)^2} \\ \approx 36.0$$

So, the stone lands about 36.0 m from Max.

7 $h(t) = at^2 + bt + c$ metres

- a** The ball is thrown from a height of 1.6 m above the ground, so $h(0) = 1.6$
 $\therefore c = 1.6$

- b** The ball initially gains height at 16.4 m s^{-1} , so $h'(0) = 16.4$
 Now $h'(t) = 2at + b$
 $\therefore b = 16.4$

- c** After 2 seconds the ball is falling at 3.2 m s^{-1} , so $h'(2) = -3.2$
 $\therefore 2a(2) + 16.4 = -3.2$
 $\therefore 4a = -19.6$
 $\therefore a = -4.9$

- d** $h'(t) = -9.8t + 16.4 = 0$ when $-9.8t = -16.4$
 $\therefore t = \frac{-16.4}{-9.8}$
 $\therefore t \approx 1.67$

$h'(t)$ has sign diagram:



$$h(t) = -4.9t^2 + 16.4t + 1.6$$

$$\therefore h(1.67) \approx -4.9(1.67)^2 + 16.4(1.67) + 1.6$$

$$\approx 15.3$$

So, the maximum height reached by the ball was about 15.3 m, which occurred about 1.67 seconds after the ball was thrown.

REVIEW SET 12B

1 $H(t) = 1.5 + 19t - 0.8t^2$ metres

a $H'(t) = 19 - 1.6t \text{ m s}^{-1}$

b $H'(0) = 19 \text{ m s}^{-1}$

$$H'(10) = 19 - 1.6(10) = 3 \text{ m s}^{-1}$$

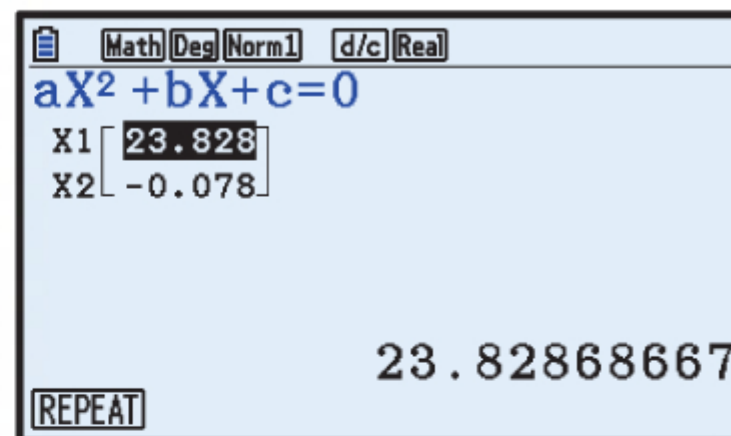
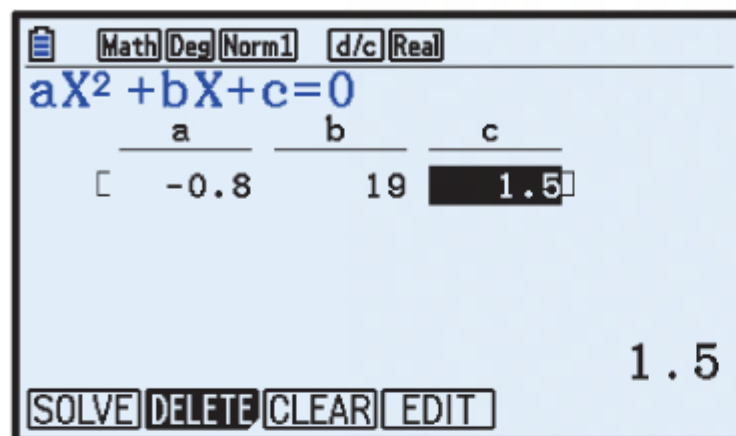
$$H'(20) = 19 - 1.6(20) = -13 \text{ m s}^{-1}$$

These are the instantaneous speeds at $t = 0, 10$, and 20 s.

A positive sign means the ball is travelling upwards.

A negative sign means the ball is travelling downwards.

c $H(t) = 0$ when $1.5 + 19t - 0.8t^2 = 0$



Using technology, $t \approx -0.0787$ or 23.8 s.

Since $t \geq 0$, the ball returns to the ground after about 23.8 seconds.

2 a $C(v) = \frac{v^2}{20} + \frac{50\,000}{v}$ dollars

$$\begin{aligned} C(64) &= \frac{64^2}{20} + \frac{50\,000}{64} \\ &= 204.8 + 781.25 \\ &= 986.05 \end{aligned}$$

\therefore the cost of running the train for 1 hour at 64 km h^{-1} is \$986.05

\therefore the cost of running the train for 5 hours at 64 km h^{-1} is \$4930.25.

b $C(v) = \frac{v^2}{20} + \frac{50\,000}{v}$ euros, $v > 0$

$$= \frac{1}{20}v^2 + 50\,000v^{-1}$$

$$\therefore C'(v) = \frac{1}{10}v - 50\,000v^{-2}$$

$$= \frac{1}{10}v - \frac{50\,000}{v^2}$$

i $C'(75) = \frac{1}{10}(75) - \frac{50\,000}{75^2}$

$$\begin{aligned} &= \frac{75}{10} - \frac{80}{9} \\ &= -\frac{25}{18} \\ &\approx -1.39 \end{aligned}$$

\therefore if the average speed is 75 km h^{-1} , the rate of change in the cost of running the train is decreasing at about \$1.39 per km h^{-1} .

ii $C'(90) = \frac{1}{10}(90) - \frac{50\,000}{90^2}$

$$\begin{aligned} &= 9 - \frac{500}{81} \\ &= \frac{229}{81} \\ &\approx 2.83 \end{aligned}$$

\therefore if the average speed is 90 km h^{-1} , the rate of change in the cost of running the train is increasing at about \$2.83 per km h^{-1} .

c $C(v)$ is a minimum when $C'(v) = 0$

$$\therefore \frac{1}{10}v - \frac{50\,000}{v^2} = 0$$

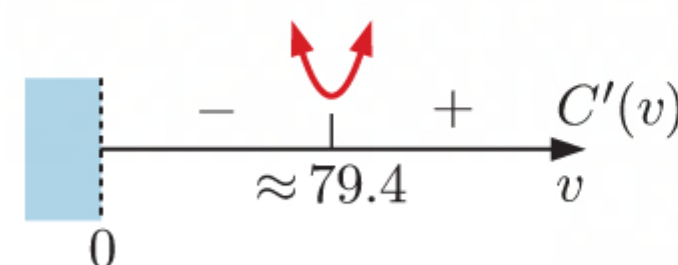
$$\therefore \frac{1}{10}v^3 - 50\,000 = 0$$

$$\therefore \frac{1}{10}v^3 = 50\,000$$

$$\therefore v^3 = 500\,000$$

$$\therefore v \approx 79.4 \text{ km h}^{-1}$$

$C'(v)$ has sign diagram:



\therefore the cost of running the train is a minimum when the average speed of the train is about 79.4 km h^{-1} .

- 3 Suppose the sheet is bent x cm from each end. To maximise the water carried we need to maximise the area of the cross-section.

$$A = x(24 - 2x), \quad 0 \leq x \leq 12$$

$$= 24x - 2x^2$$

$$\therefore \frac{dA}{dx} = 24 - 4x$$

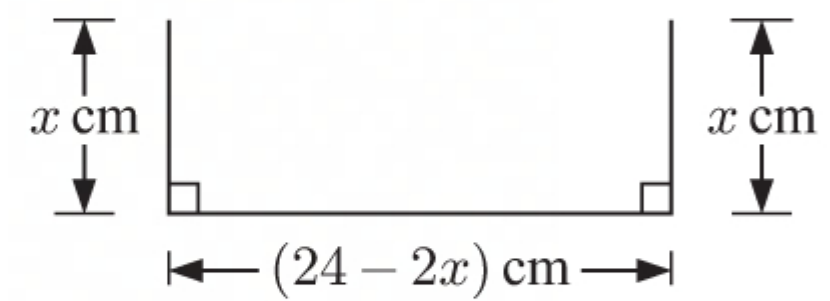
So, $\frac{dA}{dx} = 0$ when $24 - 4x = 0$

$$\therefore x = 6$$

$\frac{dA}{dx}$ has sign diagram:

The maximum water is held when $x = 6$

\therefore the bends must be made 6 cm from each end.



- 4 4 squares with sides x cm are cut from the corners.
- \therefore the remaining sides have length $(20 - 2x)$ cm and $(10 - 2x)$ cm.

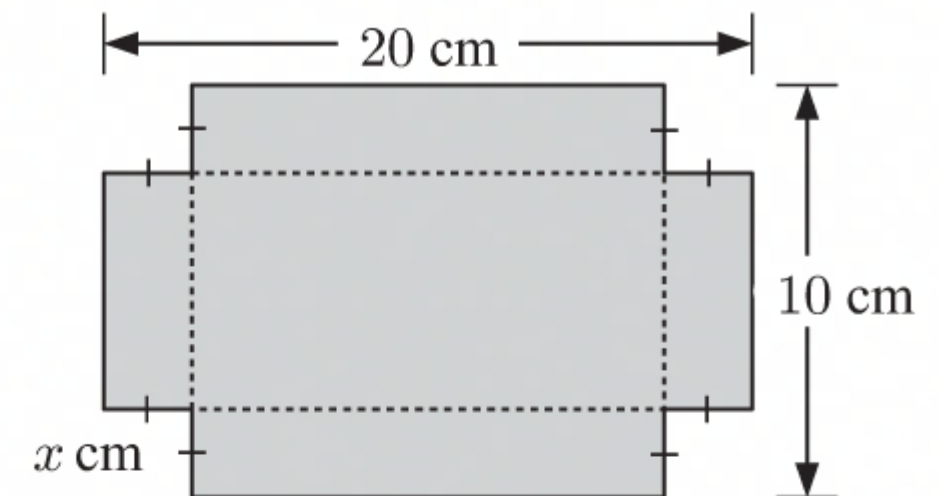
Now, volume $V = \text{length} \times \text{width} \times \text{depth}$

$$= (20 - 2x)(10 - 2x)x$$

$$= (200 - 40x - 20x + 4x^2)x$$

$$= (200 - 60x + 4x^2)x$$

$$= 200x - 60x^2 + 4x^3 \text{ cm}^3$$



Since the side lengths must be positive, $x > 0$ and $10 - 2x > 0$

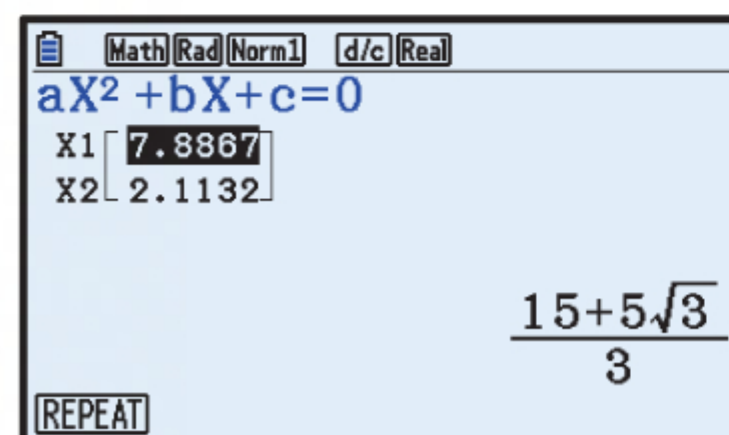
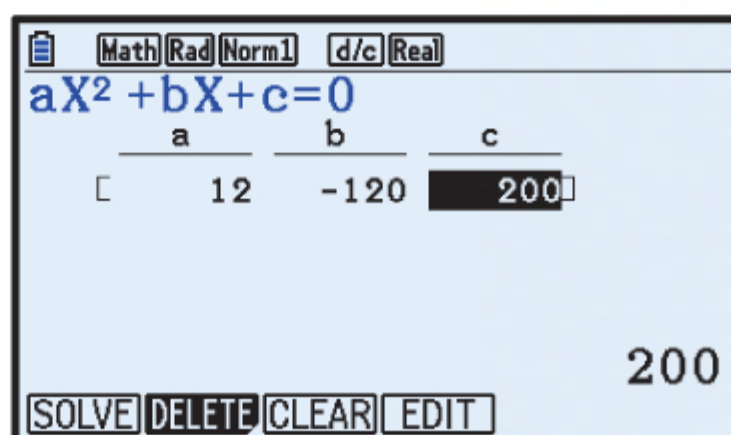
$$\therefore 2x < 10$$

$$\therefore 0 < x < 5$$

$$V = 4x^3 - 60x^2 + 200x$$

$$\therefore \frac{dV}{dx} = 12x^2 - 120x + 200$$

So, $\frac{dV}{dx} = 0$ when $12x^2 - 120x + 200 = 0$

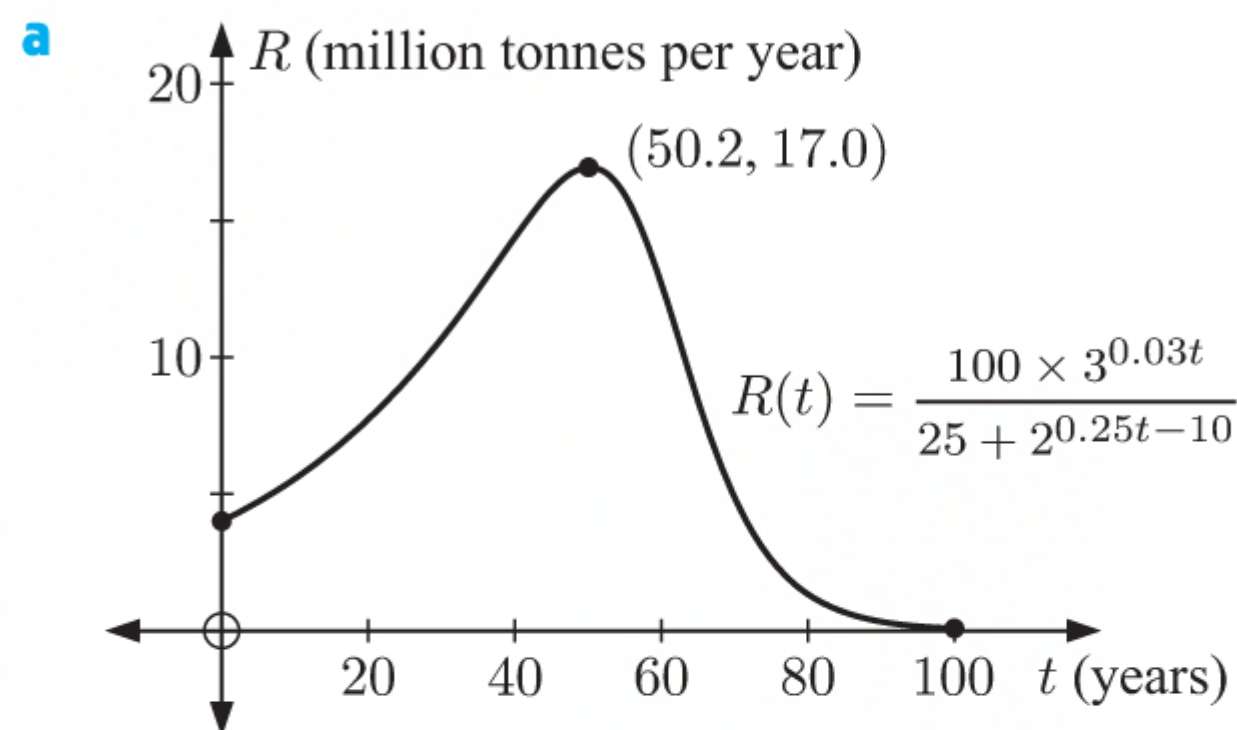


Using technology, $x \approx 2.11$ $\{0 < x < 5\}$

$\frac{dV}{dx}$ has sign diagram:

\therefore the capacity of the container is maximised when $x \approx 2.11$.

5 $R(t) = \frac{100 \times 3^{0.03t}}{25 + 2^{0.25t-10}}$ million tonnes per year, $t \geq 0$



b From a, we see that $R'(t) \geq 0$ for $0 \leq t \leq 50.2$, and $R'(t) \leq 0$ for $50.2 \leq t \leq 100$.



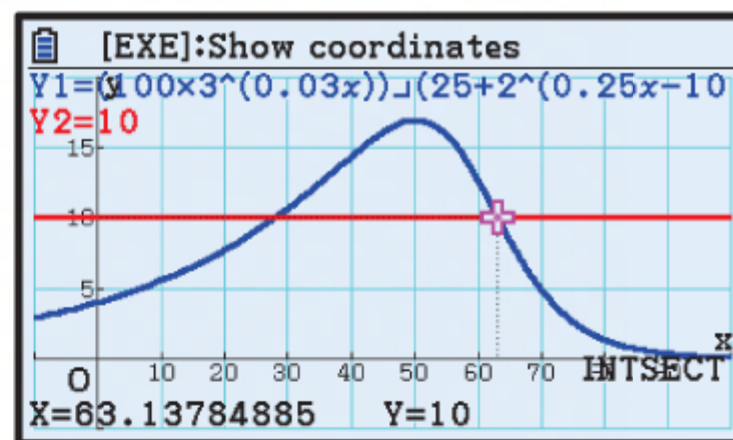
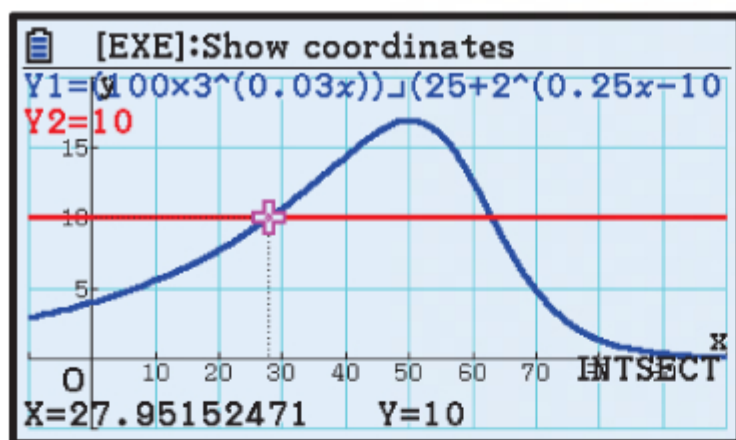
c
$$R(20) = \frac{100 \times 3^{0.03(20)}}{25 + 2^{0.25(20)-10}}$$

$$= \frac{100 \times 3^{0.6}}{25 + 2^{-5}}$$

$$\approx 7.72$$

So, the ore will be mined at a rate of about 7.72 million tonnes per year after 20 years.

d The rate of mining will be 10 million tonnes per year when $\frac{100 \times 3^{0.03t}}{25 + 2^{0.25t-10}} = 10$.



Using technology, $t \approx 28.0$ or 63.1

So, the rate of mining will be 10 million tonnes per year about 28.0 years and about 63.1 years after mining begins.

e From the graph in a, there is a local maximum at about $(50.2, 17.0)$.

So, the maximum rate of mining will be about 17.0 million tonnes per year, and will occur after about 50.2 years.

6 $C(x) = 16\,900 + 950x$ pounds

a The revenue after x days is $R(x) = \left(7500 - \frac{15\,000}{x}\right) \times 20$

$$= 150\,000 - \frac{300\,000}{x}$$

Profit $P(x) = R(x) - C(x)$

$$= 150\,000 - \frac{300\,000}{x} - (16\,900 + 950x)$$

$$= 133\,100 - \frac{300\,000}{x} - 950x \text{ pounds}$$

b $P(x) = 133\,100 - \frac{300\,000}{x} - 950x$

$$= 133\,100 - 300\,000x^{-1} - 950x$$

$$\therefore P'(x) = 300\,000x^{-2} - 950$$

$$= \frac{300\,000}{x^2} - 950$$

$P'(x) = 0$ when $\frac{300\,000}{x^2} - 950 = 0$

$$\therefore \frac{300\,000}{x^2} = 950$$

$$\therefore x^2 = \frac{300\,000}{950}$$

$$\therefore x = \sqrt{\frac{300\,000}{950}} \quad \{x > 0\}$$

$$\therefore x \approx 17.8$$

$P'(x)$ has sign diagram:



So, the campaign should last for about 18 days in order to maximise the profit.

7 $S(t) = at^3 + bt^2 + ct + d$ dollars, $0 \leq t \leq 6.5$

a The share price was initially \$22.81, so $S(0) = 22.81$
 $\therefore d = 22.81$

b The share price immediately began to fall at a rate of 16 cents per hour,
 so $S'(0) = -0.16$
 Now $S'(t) = 3at^2 + 2bt + c$
 $\therefore c = -0.16$

c At 12:30 pm, when $t = 3$, the share price was \$22.49, so $S(3) = 22.49$
 $\therefore a(3)^3 + b(3)^2 - 0.16(3) + 22.81 = 22.49$
 $\therefore 27a + 9b - 0.48 + 22.81 = 22.49$
 $\therefore 27a + 9b = 0.16 \quad \dots (1)$

At 2:30 pm, when $t = 5$, the share price was \$22.60, so $S(5) = 22.6$

$$\therefore a(5)^3 + b(5)^2 - 0.16(5) + 22.81 = 22.6$$

$$\therefore 125a + 25b - 0.8 + 22.81 = 22.6$$

$$\therefore 125a + 25b = 0.59 \quad \dots (2)$$

We solve (1) and (2) simultaneously using technology.

$$\therefore a \approx 0.00291$$

and $b \approx 0.00904$

Math Rad Norm1 d/c Real

$a_n X + b_n Y = C_n$

	a	b	c
1	27	9	0.16
2	125	25	0.59

0.59

SOLVE DELETE CLEAR EDIT

Math Rad Norm1 d/c Real

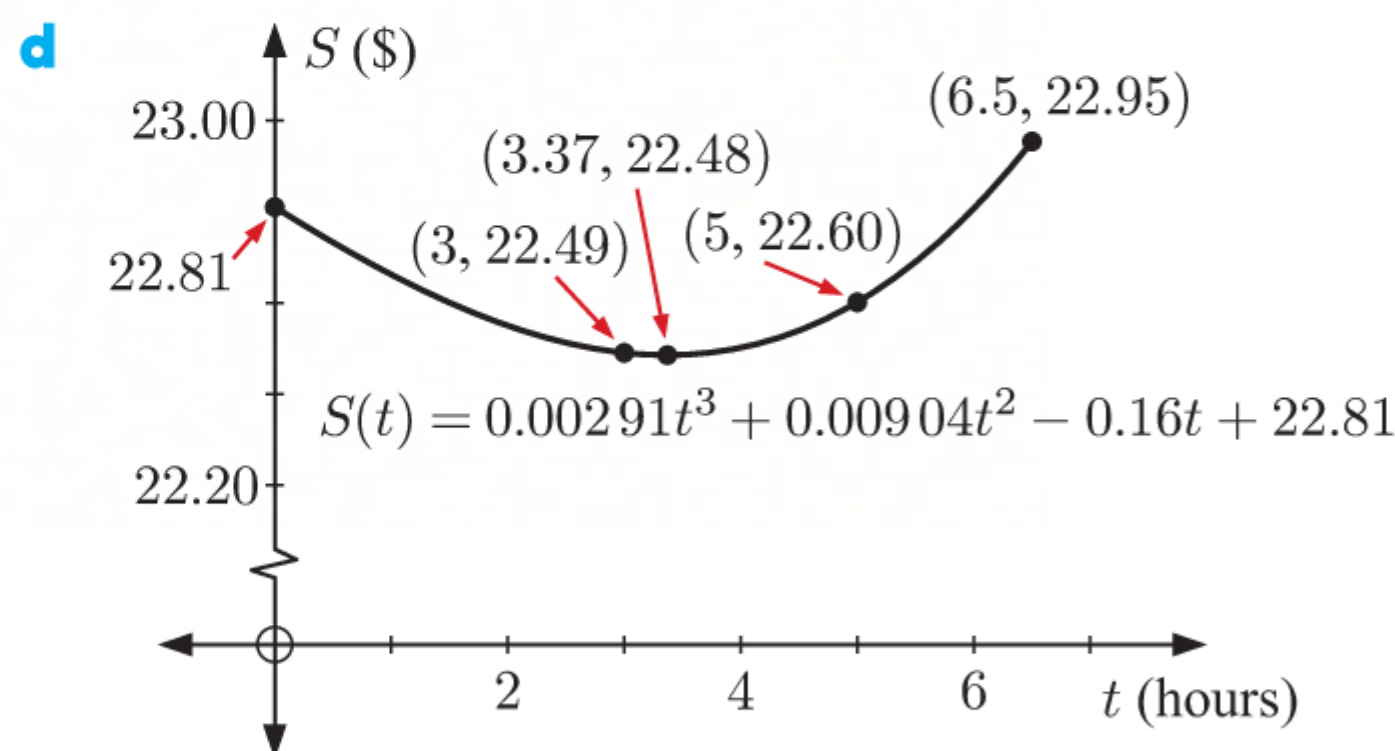
$a_n X + b_n Y = C_n$

X $2.9E-3$

Y $9E-3$

$2.911111111 \times 10^{-3}$

REPEAT



e At 4 pm, $t = 6.5$

$$S(6.5) \approx 0.00291(6.5)^3 + 0.00904(6.5)^2 - 0.16(6.5) + 22.81$$

$$\approx 22.95$$

So, the share price at 4 pm was about \$22.95.

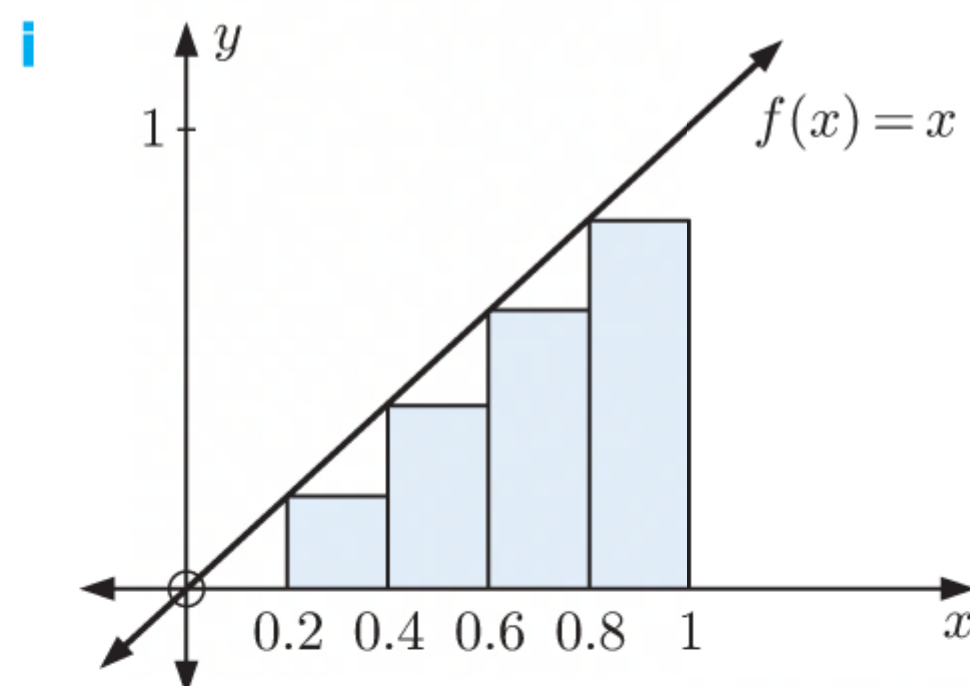
f Using the graph in **d**, the share price is at a minimum value of about \$22.48 after about 3.37 hours, or around 12:52 pm, and at a maximum value of about \$22.95 when the market closed at 4 pm.

Chapter 13

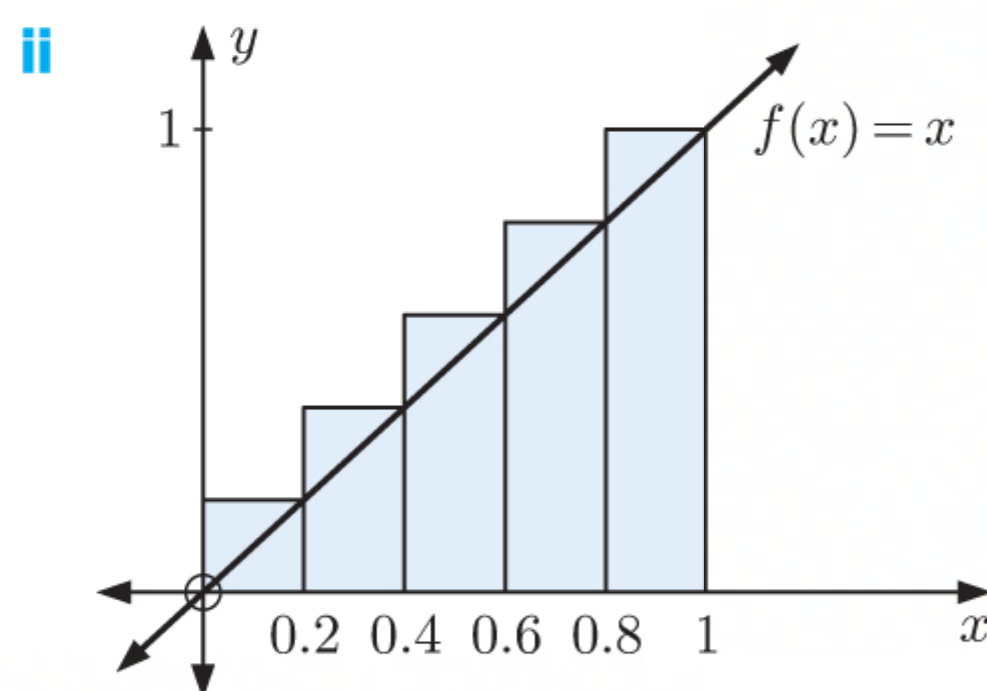
INTEGRATION

EXERCISE 13A.1

- 1 a The rectangles are $\frac{1}{5} = 0.2$ units wide.



$$\begin{aligned} A_L &= 0.2 \times f(0) + 0.2 \times f(0.2) + 0.2 \times f(0.4) \\ &\quad + 0.2 \times f(0.6) + 0.2 \times f(0.8) \\ &= (0.2 \times 0) + (0.2 \times 0.2) + (0.2 \times 0.4) \\ &\quad + (0.2 \times 0.6) + (0.2 \times 0.8) \\ &= 0.4 \text{ units}^2 \end{aligned}$$



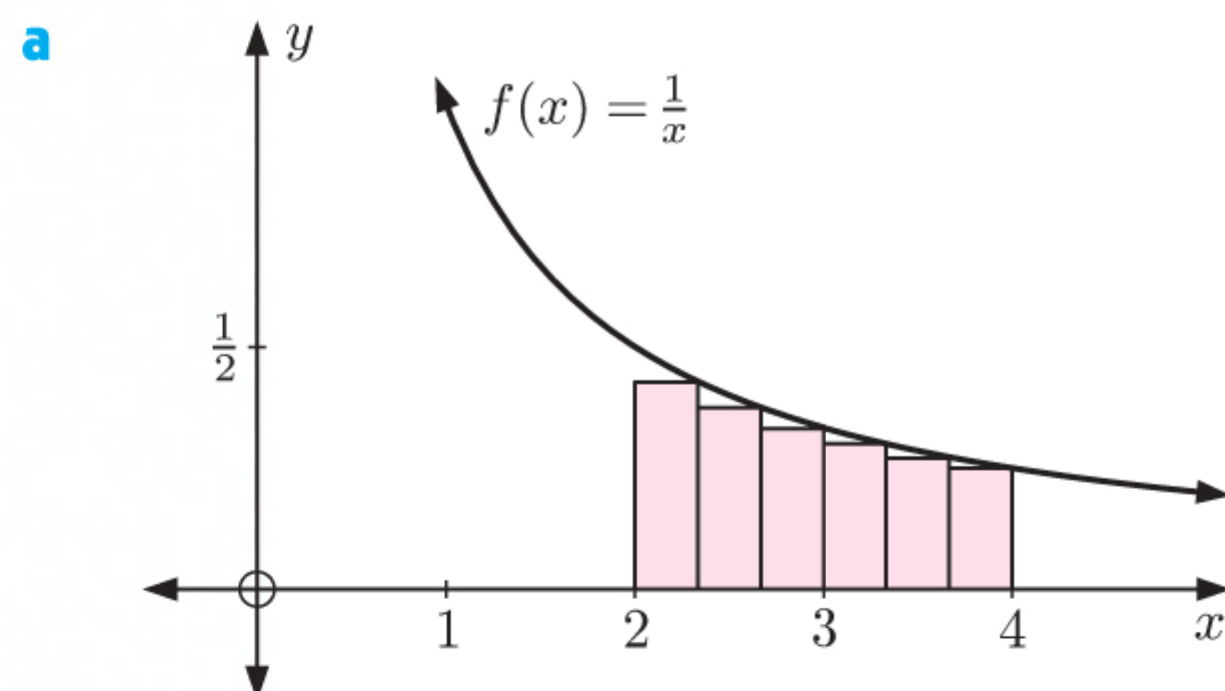
$$\begin{aligned} A_U &= 0.2 \times f(0.2) + 0.2 \times f(0.4) + 0.2 \times f(0.6) \\ &\quad + 0.2 \times f(0.8) + 0.2 \times f(1) \\ &= (0.2 \times 0.2) + (0.2 \times 0.4) + (0.2 \times 0.6) \\ &\quad + (0.2 \times 0.8) + (0.2 \times 1) \\ &= 0.6 \text{ units}^2 \end{aligned}$$

- b The area between $y = x$ and the x -axis from $x = 0$ to $x = 1$ is a triangle.

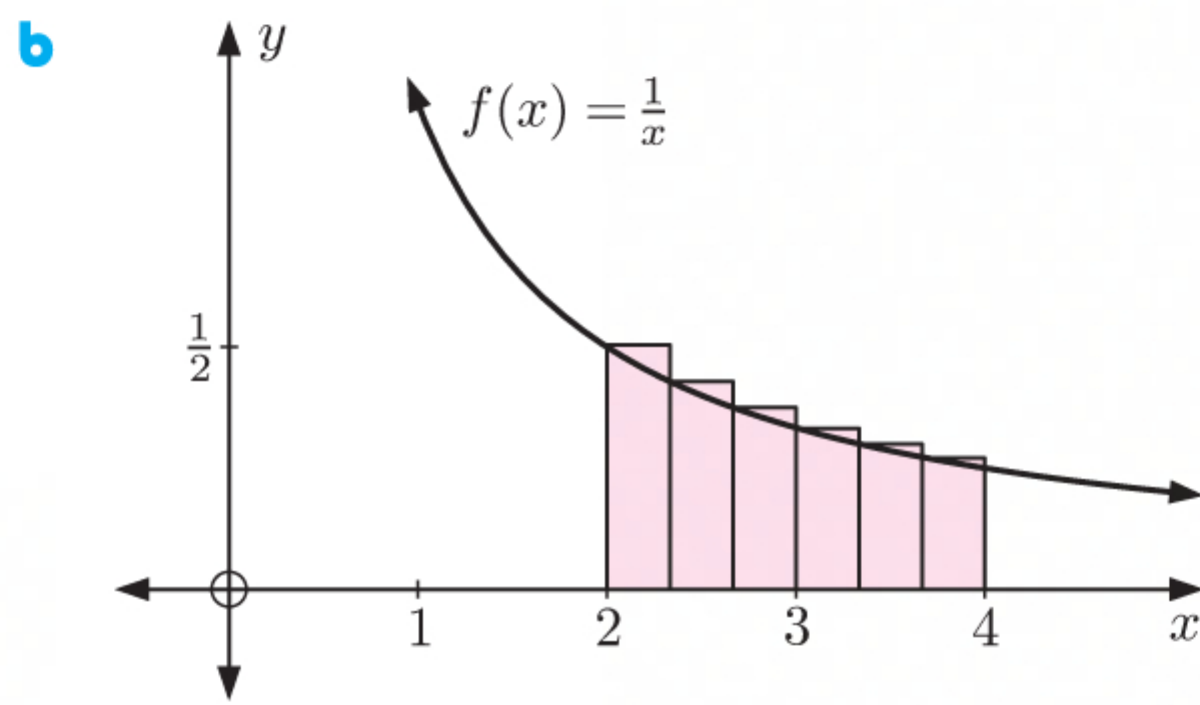
$$\begin{aligned} \therefore \text{area} &= \frac{1}{2} \times \text{base} \times \text{height} \\ &= \frac{1}{2} \times 1 \times 1 \\ &= 0.5 \text{ units}^2 \end{aligned}$$

$\therefore A_L < \text{area} < A_U$, and both A_L and A_U are within 0.1 units^2 , or 20%, of the actual area.

- 2 The rectangles are $\frac{2}{6} = \frac{1}{3}$ units wide.



$$\begin{aligned} A_L &= \frac{1}{3} \times f\left(\frac{7}{3}\right) + \frac{1}{3} \times f\left(\frac{8}{3}\right) + \frac{1}{3} \times f(3) + \frac{1}{3} \times f\left(\frac{10}{3}\right) + \frac{1}{3} \times f\left(\frac{11}{3}\right) + \frac{1}{3} \times f(4) \\ &= \left(\frac{1}{3} \times \frac{3}{7}\right) + \left(\frac{1}{3} \times \frac{3}{8}\right) + \left(\frac{1}{3} \times \frac{1}{3}\right) + \left(\frac{1}{3} \times \frac{3}{10}\right) + \left(\frac{1}{3} \times \frac{3}{11}\right) + \left(\frac{1}{3} \times \frac{1}{4}\right) \\ &\approx 0.653 \text{ units}^2 \end{aligned}$$



$$\begin{aligned}
 A_U &= \frac{1}{3} \times f(2) + \frac{1}{3} \times f\left(\frac{7}{3}\right) + \frac{1}{3} \times f\left(\frac{8}{3}\right) + \frac{1}{3} \times f(3) + \frac{1}{3} \times f\left(\frac{10}{3}\right) + \frac{1}{3} \times f\left(\frac{11}{3}\right) \\
 &= \left(\frac{1}{3} \times \frac{1}{2}\right) + \left(\frac{1}{3} \times \frac{3}{7}\right) + \left(\frac{1}{3} \times \frac{3}{8}\right) + \left(\frac{1}{3} \times \frac{1}{3}\right) + \left(\frac{1}{3} \times \frac{3}{10}\right) + \left(\frac{1}{3} \times \frac{3}{11}\right) \\
 &\approx 0.737 \text{ units}^2
 \end{aligned}$$

3 Using provided software,

n	A_L	A_U
10	2.1850	2.4850
25	2.2736	2.3936
50	2.3034	2.3634
100	2.3184	2.3484
500	2.3303	2.3363

A_L and A_U converge to $\frac{7}{3} = 2.\bar{3}$

4 a i

n	A_L	A_U
5	0.400 00	0.600 00
10	0.450 00	0.550 00
50	0.490 00	0.510 00
100	0.495 00	0.505 00
500	0.499 00	0.501 00
1000	0.499 50	0.500 50
10 000	0.499 95	0.500 05

ii

n	A_L	A_U
5	0.160 00	0.360 00
10	0.202 50	0.302 50
50	0.240 10	0.260 10
100	0.245 03	0.255 03
500	0.249 00	0.251 00
1000	0.249 50	0.250 50
10 000	0.249 95	0.250 05

iii

n	A_L	A_U
5	0.113 28	0.313 28
10	0.153 33	0.253 33
50	0.190 13	0.210 13
100	0.195 03	0.205 03
500	0.199 00	0.201 00
1000	0.199 50	0.200 50
10 000	0.199 95	0.200 05

iv

n	A_L	A_U
5	0.083 20	0.283 20
10	0.120 83	0.220 83
50	0.156 83	0.176 83
100	0.161 71	0.171 71
500	0.165 67	0.167 67
1000	0.166 17	0.167 17
10 000	0.166 62	0.166 72

b i A_L and A_U converge to $0.5 = \frac{1}{2} = \frac{1}{1+1}$

ii A_L and A_U converge to $0.25 = \frac{1}{4} = \frac{1}{3+1}$

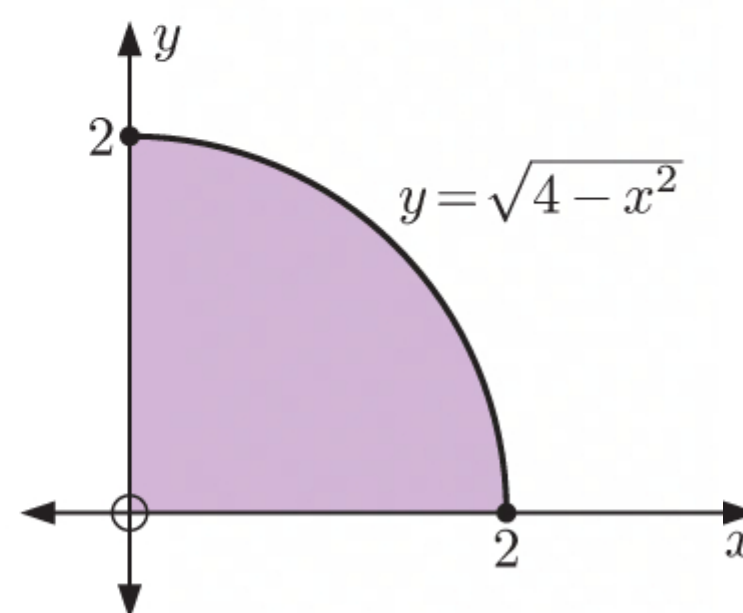
iii A_L and A_U converge to $0.2 = \frac{1}{5} = \frac{1}{4+1}$

iv A_L and A_U converge to $0.1\bar{6} = \frac{1}{6} = \frac{1}{5+1}$

- c** From **b**, it appears that the area between the graph of $y = x^a$ and the x -axis for $0 \leq x \leq 1$ and any integer $a > 0$ is $\frac{1}{a+1}$.

5 a

n	Rational bounds for π
10	$2.9045 < \pi < 3.3045$
50	$3.0983 < \pi < 3.1783$
100	$3.1204 < \pi < 3.1604$
200	$3.1312 < \pi < 3.1512$
1000	$3.1396 < \pi < 3.1436$
10 000	$3.1414 < \pi < 3.1418$



- b** $3\frac{10}{71} < \pi < 3\frac{1}{7}$ is approximately $3.1408 < \pi < 3.1429$

This is a better approximation than our estimates in **a** using $n = 10, 50, 100, 200$, or 1000 rectangles. Only $n = 10\,000$ gives us a better estimate than that of Archimedes.

EXERCISE 13A.2

- 1 a** $n = 4, a = 2, b = 4, f(x) = \frac{2}{\sqrt{x}}$
 $h = \frac{b-a}{n} = \frac{1}{2}$
 $x_i = 2 + \frac{1}{2}i$

i	x_i	$f(x_i)$
0	2	1.414 214
1	$2\frac{1}{2}$	1.264 911
2	3	1.154 701
3	$3\frac{1}{2}$	1.069 045
4	4	1

Using the trapezoidal rule, the area $\approx \frac{h}{2} (f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4))$
 $\approx 2.3479 \text{ units}^2$

- b** $n = 4, a = 1, b = 3, f(x) = -x^2 + 6x - 4$
 $h = \frac{b-a}{n} = \frac{1}{2}$
 $x_i = 1 + \frac{1}{2}i$

i	x_i	$f(x_i)$
0	1	1
1	$1\frac{1}{2}$	2.75
2	2	4
3	$2\frac{1}{2}$	4.75
4	3	5

Using the trapezoidal rule, the area $\approx \frac{h}{2} (f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4))$
 $\approx 7.25 \text{ units}^2$

2 a $n = 6, a = 0, b = 3, f(x) = 3 - x$

$$h = \frac{b-a}{n} = \frac{1}{2}$$

$$x_i = 0 + \frac{1}{2}i$$

i	x_i	$f(x_i)$
0	0	3
1	$\frac{1}{2}$	$2\frac{1}{2}$
2	1	2
3	$1\frac{1}{2}$	$1\frac{1}{2}$
4	2	1
5	$2\frac{1}{2}$	$\frac{1}{2}$
6	3	0

Using the trapezoidal rule, the area $= \frac{h}{2} (f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_5) + f(x_6))$
 $= 4\frac{1}{2} \text{ units}^2$

- b** Since $y = 3 - x$ is a straight line, the area of the i th trapezium is the same as the area between the x -axis and $y = f(x)$ from $x = x_{i-1}$ to $x = x_i$.
 So, the calculation in **a** is exact for this function.

3 a $n = 8, a = 0, b = 4, f(x) = \sqrt{x}$

$$h = \frac{b-a}{n} = \frac{1}{2}$$

$$x_i = 0 + \frac{1}{2}i$$

i	x_i	$f(x_i)$
0	0	0
1	$\frac{1}{2}$	0.707 107
2	1	1
3	$1\frac{1}{2}$	1.224 745
4	2	1.414 214
5	$2\frac{1}{2}$	1.581 139
6	3	1.732 051
7	$3\frac{1}{2}$	1.870 829
8	4	2

Using the trapezoidal rule, the area $\approx \frac{h}{2} (f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_7) + f(x_8))$
 $\approx 5.2650 \text{ units}^2$

b $n = 8, a = 0, b = 1, f(x) = \sqrt{x}e^{-\pi x}$

$$h = \frac{b-a}{n} = \frac{1}{8}$$

$$x_i = 0 + \frac{1}{8}i$$

i	x_i	$f(x_i)$
0	0	0
1	$\frac{1}{8}$	0.238 731
2	$\frac{1}{4}$	0.227 969
3	$\frac{3}{8}$	0.188 527
4	$\frac{1}{2}$	0.146 993
5	$\frac{5}{8}$	0.110 970
6	$\frac{3}{4}$	0.082 082
7	$\frac{7}{8}$	0.059 865
8	1	0.043 214

Using the trapezoidal rule, the area $\approx \frac{h}{2} (f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_7) + f(x_8))$
 $\approx 0.1346 \text{ units}^2$

• $n = 8, a = -0.6, b = 1, f(x) = x^3 - 2x^2 + 1$

$$h = \frac{b-a}{n} = 0.2$$

$$x_i = -0.6 + 0.2i$$

i	x_i	$f(x_i)$
0	-0.6	0.064
1	-0.4	0.616
2	-0.2	0.912
3	0	1
4	0.2	0.928
5	0.4	0.744
6	0.6	0.496
7	0.8	0.232
8	1	0

Using the trapezoidal rule, the area $\approx \frac{h}{2} (f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_7) + f(x_8))$
 $\approx 0.992 \text{ units}^2$

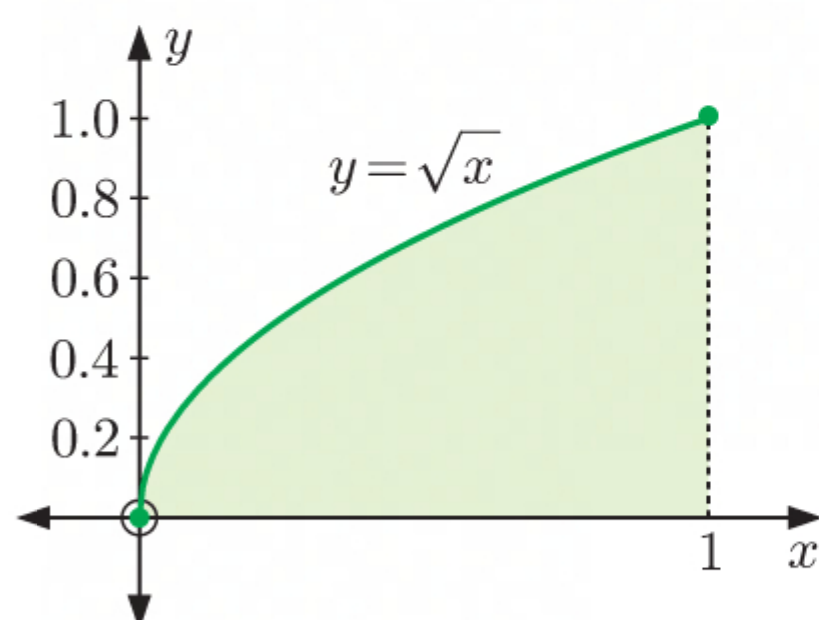
4 a Using the software provided:

n	Area estimate
8	3.0898
40	3.1369
100	3.1404
1000	3.1416

b From **Exercise 13A.1** question 5 b, Archimedes' approximation is about $3.1408 < \pi < 3.1428$.
 Only $n = 1000$ gives us a better estimate than that of Archimedes.

EXERCISE 13B

1 a



b

n	A_L	A_U
5	0.5497	0.7497
10	0.6105	0.7105
50	0.6561	0.6761
100	0.6615	0.6715
500	0.6656	0.6676

• $\int_0^1 \sqrt{x} \, dx \approx 0.67$

d $n = 8, a = 0, b = 1, f(x) = \sqrt{x}$

$$h = \frac{b-a}{n} = \frac{1}{8}$$

$$x_i = 0 + \frac{1}{8}i$$

i	x_i	$f(x_i)$
0	0	0
1	$\frac{1}{8}$	0.353 553
2	$\frac{1}{4}$	0.5
3	$\frac{3}{8}$	0.612 372
4	$\frac{1}{2}$	0.707 107
5	$\frac{5}{8}$	0.790 569
6	$\frac{3}{4}$	0.866 025
7	$\frac{7}{8}$	0.935 414
8	1	1

Using the trapezoidal rule, the area $\approx \frac{h}{2} (f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_7) + f(x_8))$
 $\approx 0.6581 \text{ units}^2$

$$\therefore \int_0^1 \sqrt{x} \, dx \approx 0.6581$$

With just 8 subintervals, the trapezoidal rule is more accurate than lower and upper rectangles were with 50 subintervals.

2 a The rectangles will have width $\frac{2-0}{n} = \frac{2}{n}$.

Let $x_i = \frac{2i}{n}$ for $i = 0, \dots, n$.

Since $y = \sqrt{1+x^3}$ is increasing on $0 \leq x \leq 2$, the i th lower rectangle has height

$$\sqrt{1+x_{i-1}^3} \text{ and the } i\text{th upper rectangle has height } \sqrt{1+x_i^3}.$$

The lower rectangle sum will be

$$\begin{aligned} A_L &= \frac{2}{n} \times \sqrt{1+x_0^3} + \frac{2}{n} \times \sqrt{1+x_1^3} + \dots + \frac{2}{n} \times \sqrt{1+x_{n-1}^3} \\ &= \frac{2}{n} \sum_{i=0}^{n-1} \sqrt{1+x_i^3} \end{aligned}$$

and the upper rectangle sum will be

$$\begin{aligned} A_U &= \frac{2}{n} \sqrt{1+x_1^3} + \frac{2}{n} \sqrt{1+x_2^3} + \dots + \frac{2}{n} \sqrt{1+x_n^3} \\ &= \frac{2}{n} \sum_{i=1}^n \sqrt{1+x_i^3} \end{aligned}$$

b

n	A_L	A_U
50	3.2016	3.2816
100	3.2214	3.2614
500	3.2373	3.2453

c $\int_0^2 \sqrt{1+x^3} \, dx \approx 3.24$

d $n = 10, a = 0, b = 2, f(x) = \sqrt{1+x^3}$

$$h = \frac{b-a}{n} = 0.2$$

$$x_i = 0 + 0.2i$$

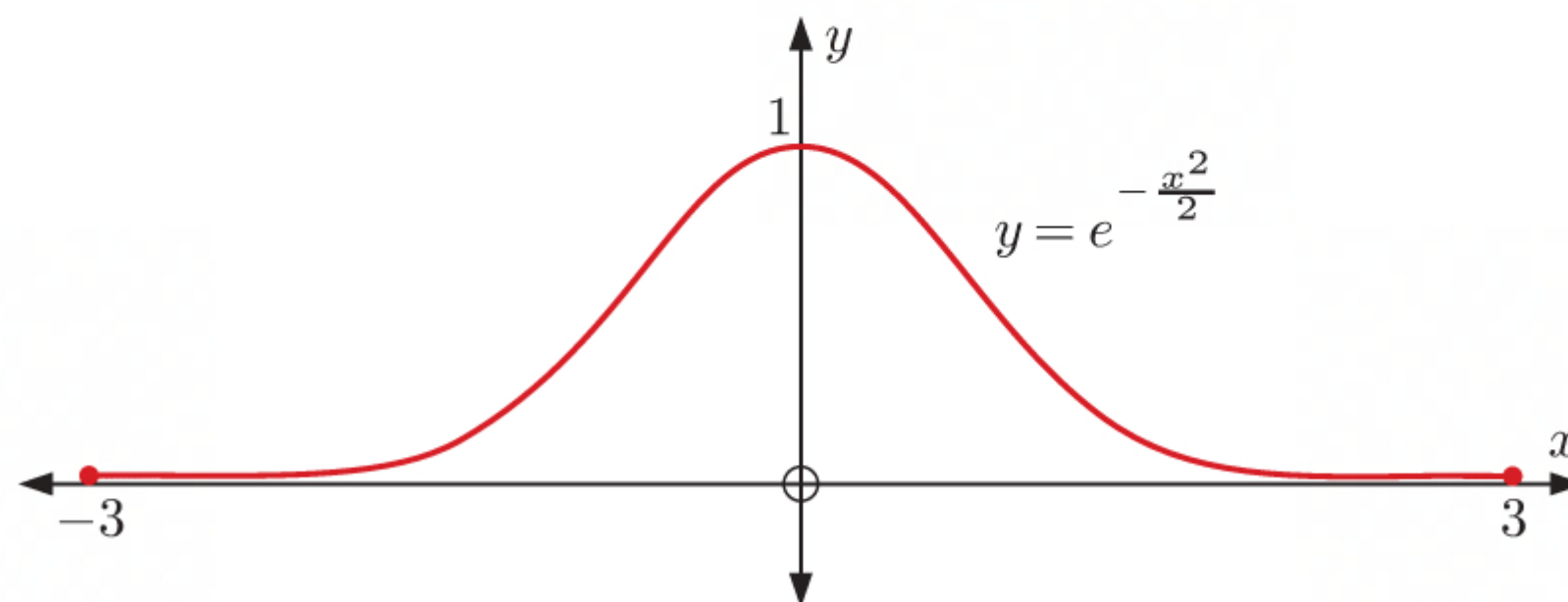
i	x_i	$f(x_i)$
0	0	1
1	0.2	1.003 992
2	0.4	1.031 504
3	0.6	1.102 724
4	0.8	1.229 634
5	1	1.414 214
6	1.2	1.651 666
7	1.4	1.934 942
8	1.6	2.257 432
9	1.8	2.613 809
10	2	3

Using the trapezoidal rule, the area $\approx \frac{h}{2} (f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_9) + f(x_{10}))$
 $\approx 3.2480 \text{ units}^2$

$$\therefore \int_0^2 \sqrt{1+x^3} dx \approx 3.2480$$

With just 10 subintervals, the trapezoidal rule is more accurate than lower and upper rectangles were with 100 subintervals.

3 a



b Using provided software with the following settings:

From: 0

To: 3

Method: Upper/lower rectangles

Partitions: 2250

we find that $A_L \approx 1.2493$ and $A_U \approx 1.2506$

c Since $y = e^{-\frac{x^2}{2}}$ is symmetrical about the y -axis, the lower and upper rectangle sums for $-3 \leq x \leq 0$ with $n = 2250$ is equivalent to the lower and upper rectangle sums for $0 \leq x \leq 3$ with $n = 2250$.

$$\therefore A_L \approx 1.2493 \text{ and } A_U \approx 1.2506$$

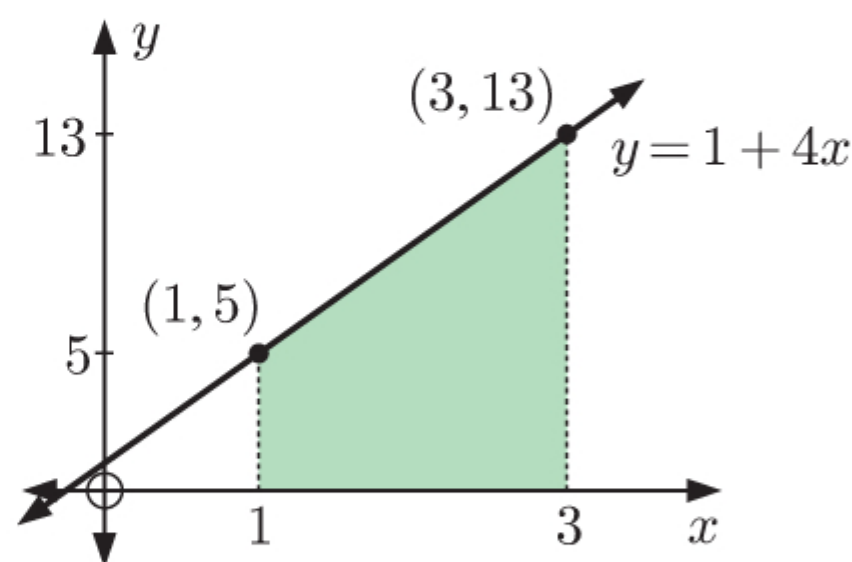
$$\text{So, } \int_{-3}^3 e^{-\frac{x^2}{2}} dx = 2 \int_0^3 e^{-\frac{x^2}{2}} dx \approx 2 \left(\frac{A_L + A_U}{2} \right) \approx 2.4999.$$

Using technology, $\sqrt{2\pi} \approx 2.5066$, which is close to our estimate.

- d** Using the trapezoidal rule with $n = 6$ subintervals, the area ≈ 1.2493 , which does not lie between the values for A_L and A_U calculated in **b**.

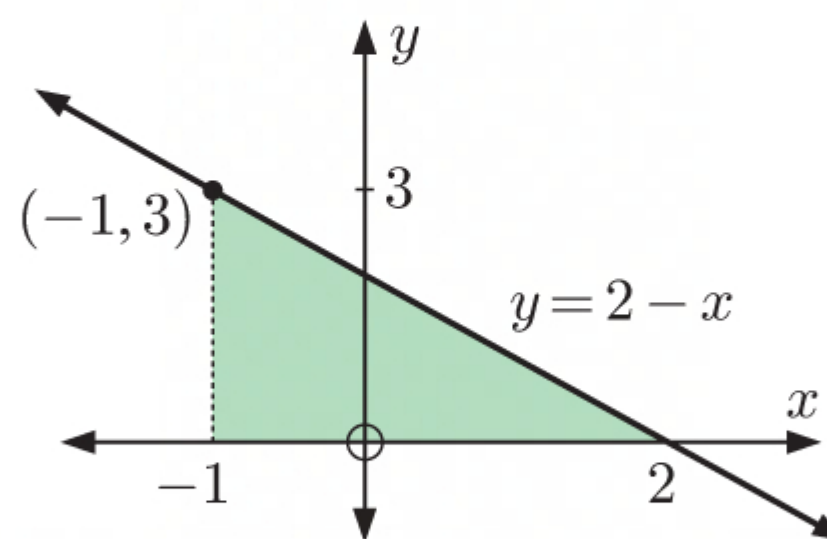
Using the trapezoidal rule with $n = 7$ subintervals, the area ≈ 1.2494 , which lies between $A_L \approx 1.2493$ and $A_U \approx 1.2506$, so 7 subintervals are necessary to get a more accurate estimate than lower and upper rectangles with 2250 subintervals.

4 a



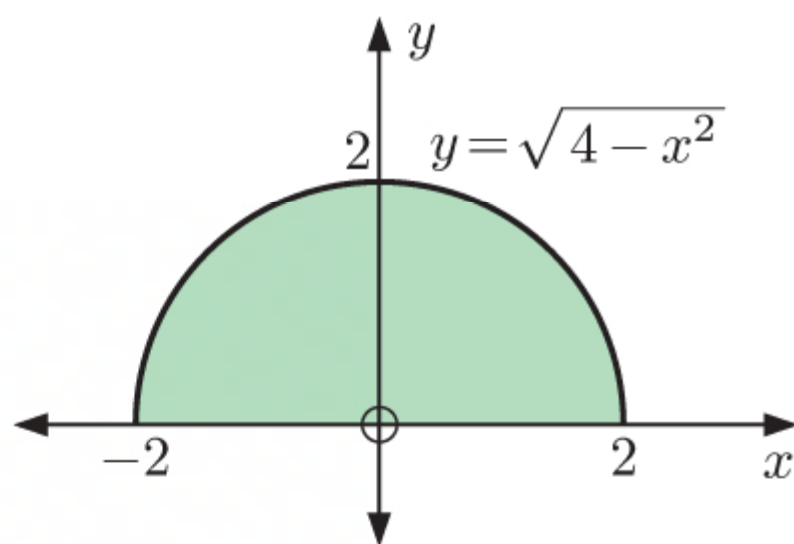
$$\begin{aligned} \int_1^3 (1 + 4x) dx &= \text{shaded area} \\ &= \left(\frac{5 + 13}{2} \right) \times 2 \\ &= 18 \end{aligned}$$

b



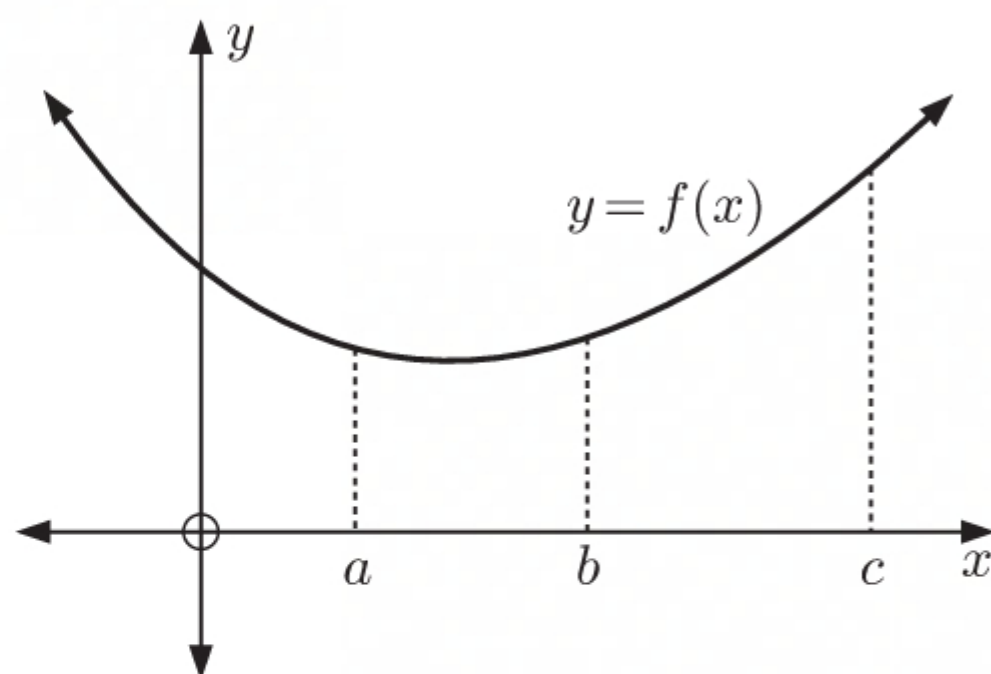
$$\begin{aligned} \int_{-1}^2 (2 - x) dx &= \text{shaded area} \\ &= \frac{1}{2}(3 \times 3) \\ &= 4.5 \end{aligned}$$

c



$$\begin{aligned} \int_{-2}^2 \sqrt{4 - x^2} dx &= \text{shaded area} \\ &= \frac{1}{2}(\pi \times 2^2) \\ &= 2\pi \end{aligned}$$

5 a



- i** From the diagram, we can see that there is no area under the curve for $a \leq x \leq a$.

$$\therefore \int_a^a f(x) dx = 0 \text{ for any positive function } f(x).$$

- ii** From the diagram, we can see that the area under the curve for $a \leq x \leq c$ is made up of the area under the curve for $a \leq x \leq b$ and $b \leq x \leq c$.

$$\begin{aligned} \therefore \int_a^b f(x) dx + \int_b^c f(x) dx &= \int_a^c f(x) dx \text{ for any positive function } f(x), \\ &\text{provided that } a \leq b \leq c. \end{aligned}$$

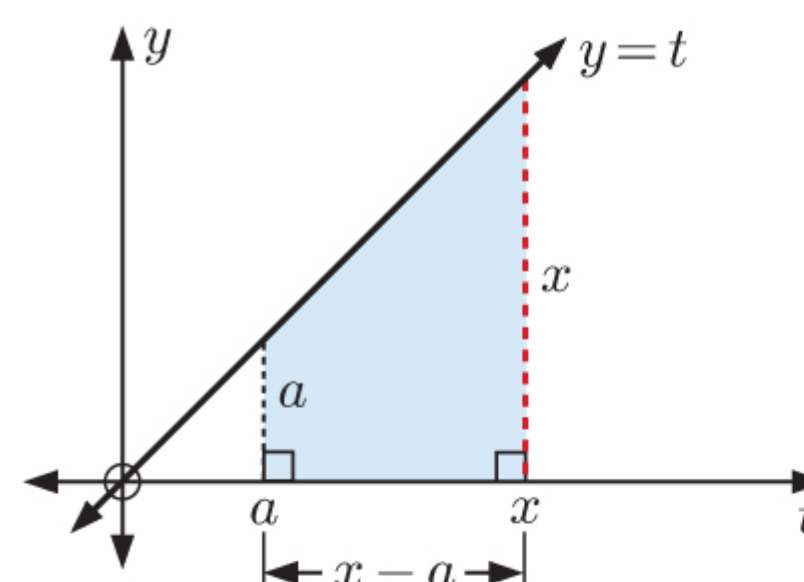
b i $\int_5^5 f(x) dx = 0 \quad \left\{ \int_a^a f(x) dx = 0 \right\}$

$$\begin{aligned}
 \text{ii} \quad & \int_2^9 f(x) \, dx \\
 &= \int_2^5 f(x) \, dx + \int_5^9 f(x) \, dx \quad \left\{ \int_a^b f(x) \, dx + \int_b^c f(x) \, dx = \int_a^c f(x) \, dx \right\} \\
 &= 10 + 12 \\
 &= 22
 \end{aligned}$$

INVESTIGATION 1

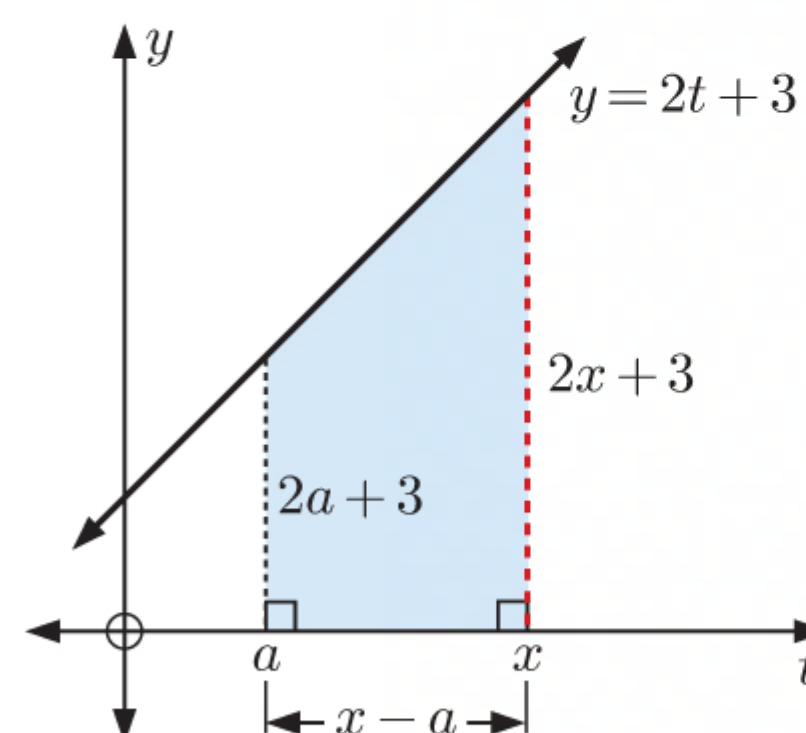
THE AREA FUNCTION

$$\begin{aligned}
 \text{1 a} \quad A(x) &= \left(\frac{x+a}{2} \right) (x-a) \\
 &= \frac{x^2 - a^2}{2} \\
 &= \frac{1}{2}x^2 - \frac{1}{2}a^2 \\
 &= F(x) - F(a) \quad \text{where } F(t) = \frac{1}{2}t^2
 \end{aligned}$$



$$\begin{aligned}
 \text{b} \quad F(t) &= \frac{1}{2}t^2 \\
 \therefore F'(t) &= t = f(t) \\
 \therefore F(t) &\text{ is the antiderivative of } f(t).
 \end{aligned}$$

$$\begin{aligned}
 \text{2 a} \quad A(x) &= \left(\frac{2x+3+2a+3}{2} \right) (x-a) \\
 &= \left(\frac{2(x+a)+6}{2} \right) (x-a) \\
 &= (x+a+3)(x-a) \\
 &= (x+a)(x-a) + 3(x-a) \\
 &= x^2 - a^2 + 3x - 3a \\
 &= x^2 + 3x - (a^2 + 3a) \\
 &= F(x) - F(a) \quad \text{where } F(t) = t^2 + 3t
 \end{aligned}$$

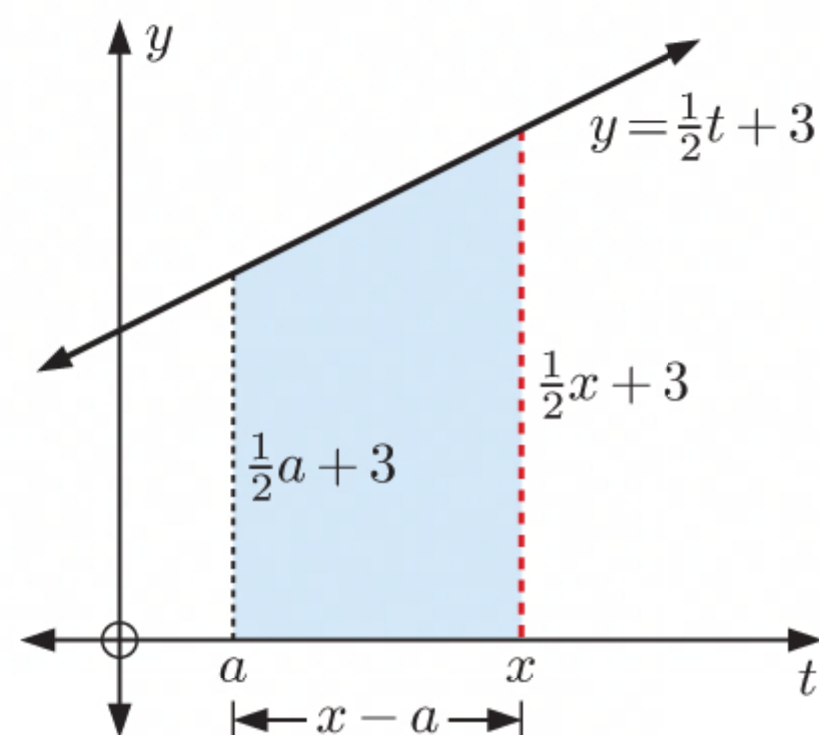


$$\begin{aligned}
 \text{b} \quad F(t) &= t^2 + 3t \\
 \therefore F'(t) &= 2t + 3 = f(t) \\
 \therefore F(t) &\text{ is the antiderivative of } f(t).
 \end{aligned}$$

3 a Consider $f(t) = \frac{1}{2}t + 3$.

The corresponding area function is

$$\begin{aligned}
 A(x) &= \int_a^x \left(\frac{1}{2}t + 3\right) dt \\
 &= \text{shaded area} \\
 &= \left(\frac{\frac{1}{2}x + 3 + \frac{1}{2}a + 3}{2}\right)(x - a) \\
 &= \left(\frac{1}{4}x + \frac{3}{2} + \frac{1}{4}a + \frac{3}{2}\right)(x - a) \\
 &= \frac{1}{4}x^2 - \cancel{\frac{1}{4}ax} + \frac{3}{2}x - \frac{3}{2}a + \cancel{\frac{1}{4}ax} - \frac{1}{4}a^2 + \frac{3}{2}x - \frac{3}{2}a \\
 &= \frac{1}{4}x^2 + 3x - \frac{1}{4}a^2 - 3a \\
 &= \frac{1}{4}x^2 + 3x - \left(\frac{1}{4}a^2 + 3a\right) \\
 &= F(x) - F(a) \quad \text{where } F(t) = \frac{1}{4}t^2 + 3t
 \end{aligned}$$



Now $F(t) = \frac{1}{4}t^2 + 3t$

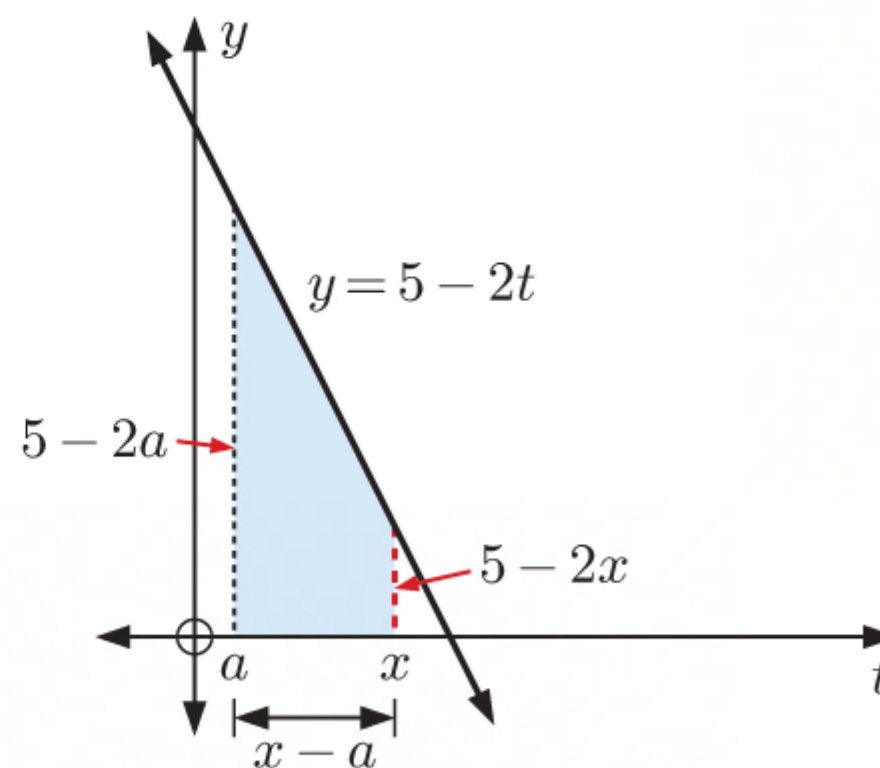
$$\therefore F'(t) = \frac{1}{2}t + 3 = f(t)$$

$\therefore F(t)$ is the antiderivative of $f(t)$.

b Consider $f(t) = 5 - 2t$.

The corresponding area function is

$$\begin{aligned}
 A(x) &= \int_a^x (5 - 2t) dt \\
 &= \text{shaded area} \\
 &= \left(\frac{5 - 2x + 5 - 2a}{2}\right)(x - a) \\
 &= \left(\frac{5}{2} - x + \frac{5}{2} - a\right)(x - a) \\
 &= \frac{5}{2}x - \frac{5}{2}a - x^2 + \cancel{ax} + \frac{5}{2}x - \frac{5}{2}a - \cancel{ax} + a^2 \\
 &= 5x - x^2 - 5a + a^2 \\
 &= 5x - x^2 - (5a - a^2) \\
 &= F(x) - F(a) \quad \text{where } F(t) = 5t - t^2
 \end{aligned}$$



Now $F(t) = 5t - t^2$

$$\therefore F'(t) = 5 - 2t = f(t)$$

$\therefore F(t)$ is the antiderivative of $f(t)$.

4 $f(t) = 3t^2 + 4t + 5$

We predict that $F(t)$ is the antiderivative of $f(t)$.

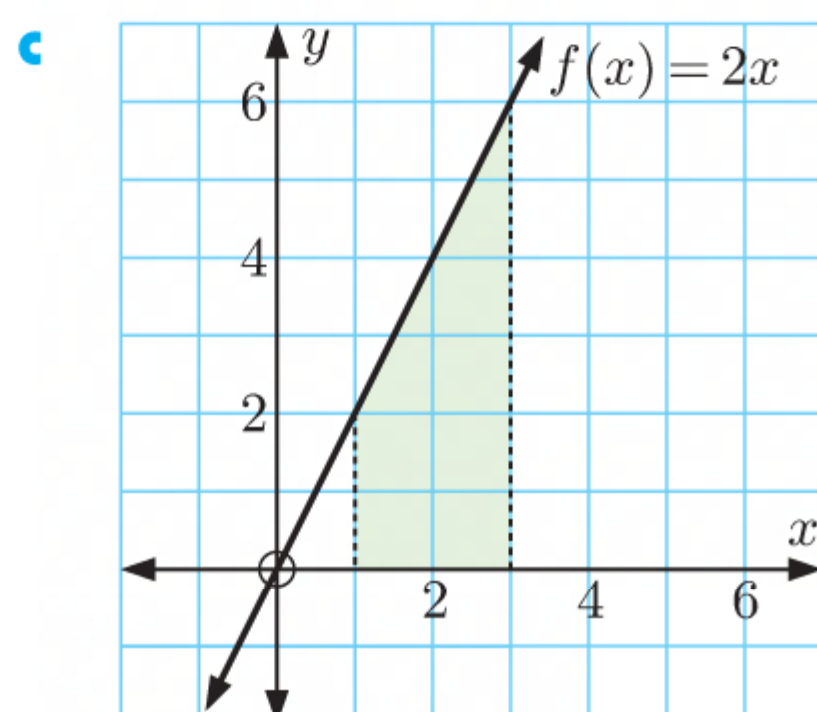
$$\frac{d}{dt}(t^3 + 2t^2 + 5t) = 3t^2 + 4t + 5 = f(t)$$

$$\therefore F(t) = t^3 + 2t^2 + 5t$$

EXERCISE 13C

- 1 a If $F(x) = x^2$ then $F'(x) = 2x$
 $\therefore F(x) = x^2$ is the antiderivative of $2x$.

b
$$\begin{aligned}\int_1^3 2x \, dx &= F(3) - F(1) \\ &= 3^2 - 1^2 \\ &= 8 \text{ units}^2\end{aligned}$$



$$\begin{aligned}\text{Shaded area} &= \int_1^3 2x \, dx \\ &= \left(\frac{2+6}{2} \right) \times 2 \quad \{\text{area of a trapezium}\} \\ &= 8 \text{ units}^2\end{aligned}$$

- 2 a If $F(x) = x^3$ then $F'(x) = 3x^2$
 $\therefore F(x) = x^3$ is the antiderivative of $3x^2$.

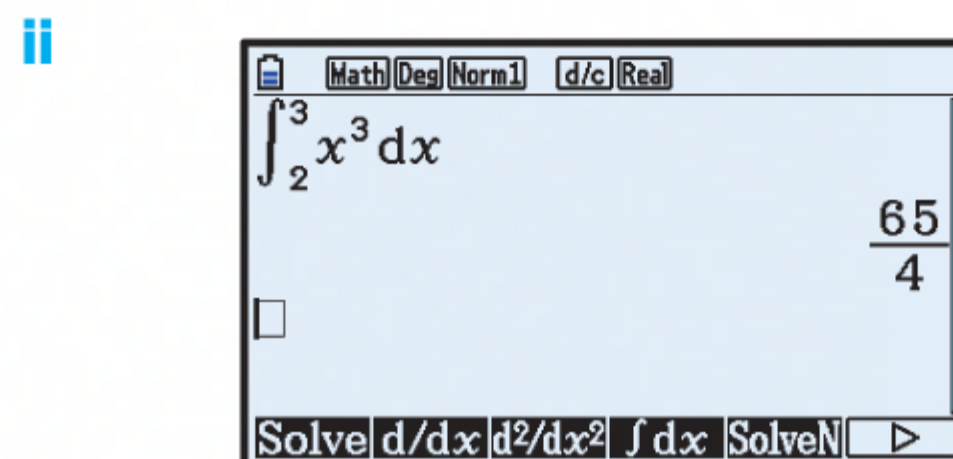
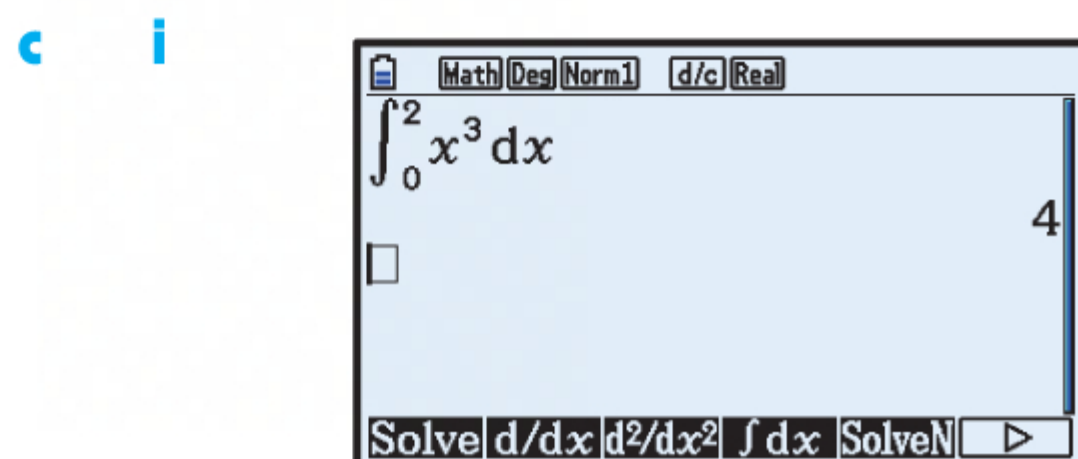
b
$$\begin{aligned}\int_0^1 3x^2 \, dx &= F(1) - F(0) \\ &= 1^3 - 0^3 \\ &= 1 \text{ unit}^2\end{aligned}$$

- 3 a If $F(x) = \frac{1}{4}x^4$ then $F'(x) = x^3$
 $\therefore F(x) = \frac{1}{4}x^4$ is the antiderivative of x^3 .

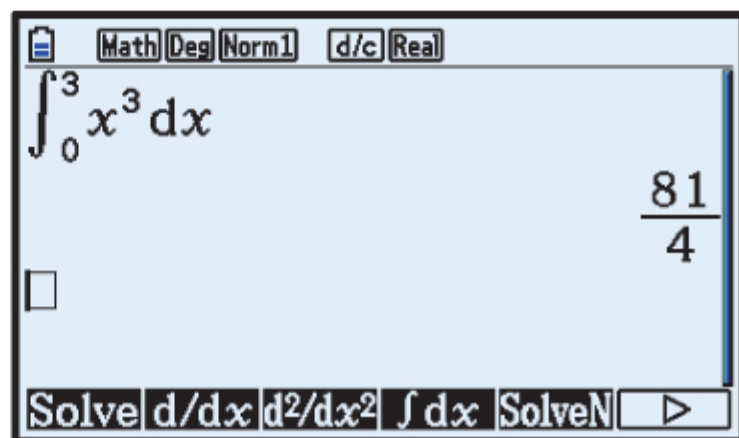
b i
$$\begin{aligned}\int_0^2 x^3 \, dx &= F(2) - F(0) \\ &= 4 - 0 \\ &= 4 \text{ units}^2\end{aligned}$$

ii
$$\begin{aligned}\int_2^3 x^3 \, dx &= F(3) - F(2) \\ &= \frac{81}{4} - 4 \\ &= 16\frac{1}{4} \text{ units}^2\end{aligned}$$

iii
$$\begin{aligned}\int_0^3 x^3 \, dx &= F(3) - F(0) \\ &= \frac{81}{4} - 0 \\ &= 20\frac{1}{4} \text{ units}^2\end{aligned}$$



iii



- d** We see that the property $\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$ holds true, as
- $$\int_0^2 x^3 dx + \int_2^3 x^3 dx = 4 + 16\frac{1}{4} = 20\frac{1}{4} = \int_0^3 x^3 dx.$$

4 Let $F(x)$ be the antiderivative of $f(x)$ and $G(x)$ be the antiderivative of $g(x)$.

- a** $\int_a^a f(x) dx = F(a) - F(a) = 0$
- $\int_a^a f(x) dx = \text{area of the region under the curve } y = f(x) \text{ between } x = a \text{ and } x = a.$

This region has 0 width, so its area = 0.

- c** $\int_b^a f(x) dx = F(a) - F(b)$
 $= -[F(b) - F(a)]$
 $= -\int_a^b f(x) dx$

- b** The antiderivative of $f(x) = k$ is $F(x) = kx$.

$$\begin{aligned} \therefore \int_a^b k dx &= F(b) - F(a) \\ &= kb - ka \\ &= k(b - a) \end{aligned}$$

- d** $\frac{d}{dx} F(x) = f(x)$

$$\therefore \frac{d}{dx} (k F(x)) = k f(x)$$

$\therefore k F(x)$ is the antiderivative of $k f(x)$.

$$\begin{aligned} \text{So, } \int_a^b k f(x) dx &= k F(b) - k F(a) \\ &= k[F(b) - F(a)] \\ &= k \int_a^b f(x) dx \end{aligned}$$

- e** $\frac{d}{dx} F(x) = f(x)$ and $\frac{d}{dx} G(x) = g(x)$

$$\therefore \frac{d}{dx} [F(x) + G(x)] = f(x) + g(x)$$

$\therefore F(x) + G(x)$ is the antiderivative of $f(x) + g(x)$.

$$\begin{aligned} \text{So, } \int_a^b [f(x) + g(x)] dx &= [F(b) + G(b)] - [F(a) + G(a)] \\ &= [F(b) - F(a)] + [G(b) - G(a)] \\ &= \int_a^b f(x) dx + \int_a^b g(x) dx \end{aligned}$$

$$\begin{aligned}
 \text{f } \int_a^b f(x) \, dx + \int_b^c f(x) \, dx &= \cancel{F(b)} - F(a) + F(c) - \cancel{F(b)} \\
 &= F(c) - F(a) \\
 &= \int_a^c f(x) \, dx
 \end{aligned}$$

EXERCISE 13D

$$1 \quad \frac{d}{dx} (x^2) = 2x$$

$$\therefore \frac{d}{dx} \left(\frac{1}{2} x^2 \right) = x$$

a The antiderivative of x is $\frac{1}{2}x^2$.

$$\text{b } \int x \, dx = \frac{1}{2}x^2 + c$$

$$2 \quad \frac{d}{dx} (x^3) = 3x^2$$

$$\therefore \frac{d}{dx} \left(\frac{1}{3} x^3 \right) = x^2$$

a The antiderivative of x^2 is $\frac{1}{3}x^3$.

$$\text{b } \int x^2 \, dx = \frac{1}{3}x^3 + c$$

$$3 \quad \frac{d}{dx} (x^{-1}) = -x^{-2}$$

$$\therefore \frac{d}{dx} (-x^{-1}) = x^{-2}$$

a The antiderivative of x^{-2} is $-x^{-1}$.

$$\begin{aligned}
 \text{b } \int x^{-2} \, dx &= -x^{-1} + c \\
 &= -\frac{1}{x} + c
 \end{aligned}$$

$$4 \quad \frac{d}{dx} (x^{-2}) = -2x^{-3}$$

$$\therefore \frac{d}{dx} \left(-\frac{1}{2} x^{-2} \right) = x^{-3}$$

a The antiderivative of x^{-3} is $-\frac{1}{2}x^{-2}$.

$$\begin{aligned}
 \text{b } \int x^{-3} \, dx &= -\frac{1}{2}x^{-2} + c \\
 &= -\frac{1}{2x^2} + c
 \end{aligned}$$

5 a In questions **1** to **4**, we saw that the antiderivative of x^n is $\frac{1}{n+1} x^{n+1}$.

This result does not apply when $n = -1$, as $\frac{1}{-1+1}$ is undefined.

So, we predict that the antiderivative of x^n is $\frac{1}{n+1} x^{n+1}$ for all $n \in \mathbb{Z}$ except $n = -1$.

b Using **a**, the antiderivative of x^5 is $\frac{1}{6}x^6$.

$$\frac{d}{dx} \left(\frac{1}{6} x^6 \right) = \frac{1}{6} (6x^5) = x^5 \quad \checkmark$$

$$6 \quad \frac{d}{dx}(x^3 + x^2) = 3x^2 + 2x$$

$$\therefore \int (3x^2 + 2x) dx = x^3 + x^2 + c$$

$$7 \quad \frac{d}{dx}(3x^4 - 2x^2) = 12x^3 - 4x$$

$$\therefore \int (12x^3 - 4x) dx = 3x^4 - 2x^2 + c$$

$$\therefore 4 \int (3x^3 - x) dx = 3x^4 - 2x^2 + c$$

$$\therefore \int (3x^3 - x) dx = \frac{3}{4}x^4 - \frac{1}{2}x^2 + c$$

$$8 \quad \frac{d}{dx}\left(\frac{1}{x} + 2x\right) = \frac{d}{dx}(x^{-1} + 2x)$$

$$= -x^{-2} + 2$$

$$= -\left(\frac{1}{x^2} - 2\right)$$

$$\therefore \int \left(\frac{1}{x^2} - 2\right) dx = -\left(\frac{1}{x} + 2x\right) + c$$

$$= -\frac{1}{x} - 2x + c$$

EXERCISE 13E

$$1 \quad \text{a} \quad \int 3 dx \\ = 3x + c$$

$$\text{b} \quad \int (-2) dx \\ = -2x + c$$

$$\text{c} \quad \int 2x dx \\ = \frac{2x^2}{2} + c \\ = x^2 + c$$

$$\text{d} \quad \int 3x^2 dx \\ = \frac{3x^3}{3} + c \\ = x^3 + c$$

$$\text{e} \quad \int 5x^4 dx \\ = \frac{5x^5}{5} + c \\ = x^5 + c$$

$$\text{f} \quad \int \left(-\frac{1}{2}x^3\right) dx \\ = \frac{-\frac{1}{2}x^4}{4} + c \\ = -\frac{1}{8}x^4 + c$$

$$\text{g} \quad \int \frac{2}{3}x^4 dx \\ = \frac{\frac{2}{3}x^5}{5} + c \\ = \frac{2}{15}x^5 + c$$

$$\text{h} \quad \int \frac{2}{x^2} dx \\ = \int 2x^{-2} dx \\ = \frac{2x^{-1}}{-1} + c \\ = -\frac{2}{x} + c$$

$$\text{i} \quad \int \frac{1}{2x^3} dx \\ = \int \frac{1}{2}x^{-3} dx \\ = \frac{\frac{1}{2}x^{-2}}{-2} + c \\ = -\frac{1}{4x^2} + c$$

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad & \int (2x - 1) \, dx \\
 &= \frac{2x^2}{2} - x + c \\
 &= x^2 - x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int (x + 3) \, dx \\
 &= \frac{x^2}{2} + 3x + c \\
 &= \frac{1}{2}x^2 + 3x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \int (4 - x) \, dx \\
 &= 4x - \frac{x^2}{2} + c \\
 &= 4x - \frac{1}{2}x^2 + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \int \frac{3x + 1}{2} \, dx \\
 &= \frac{1}{2} \int (3x + 1) \, dx \\
 &= \frac{1}{2} \left(\frac{3x^2}{2} + x \right) + c \\
 &= \frac{3}{4}x^2 + \frac{1}{2}x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & \int (x^2 - 2) \, dx \\
 &= \frac{x^3}{3} - 2x + c \\
 &= \frac{1}{3}x^3 - 2x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & \int (5 - x^2) \, dx \\
 &= 5x - \frac{x^3}{3} + c \\
 &= 5x - \frac{1}{3}x^3 + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad & \int \left(\frac{2}{3}x + 3x^2 \right) \, dx \\
 &= \frac{\frac{2}{3}x^2}{2} + \frac{3x^3}{3} + c \\
 &= \frac{1}{3}x^2 + x^3 + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad & \int \frac{2x^3 - 4}{3} \, dx \\
 &= \frac{1}{3} \int (2x^3 - 4) \, dx \\
 &= \frac{1}{3} \left(\frac{2x^4}{4} - 4x \right) + c \\
 &= \frac{1}{6}x^4 - \frac{4}{3}x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{i} \quad & \int \frac{x^3 + 1}{x^2} \, dx \\
 &= \int \left(\frac{x^3}{x^2} + \frac{1}{x^2} \right) \, dx \\
 &= \int (x + x^{-2}) \, dx \\
 &= \frac{x^2}{2} + \frac{x^{-1}}{-1} + c \\
 &= \frac{1}{2}x^2 - \frac{1}{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{3} \quad \mathbf{a} \quad & \frac{dy}{dx} = 6 \\
 \therefore y &= \int 6 \, dx \\
 &= 6x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \frac{dy}{dx} = 4x^2 \\
 \therefore y &= \int 4x^2 \, dx \\
 &= \frac{4x^3}{3} + c \\
 &= \frac{4}{3}x^3 + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \frac{dy}{dx} = \frac{1}{x^2} = x^{-2} \\
 \therefore y &= \int x^{-2} \, dx \\
 &= \frac{x^{-1}}{-1} + c \\
 &= -\frac{1}{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \frac{dy}{dx} = 2x^3 - 4 \\
 \therefore y &= \int (2x^3 - 4) \, dx \\
 &= \frac{2x^4}{4} - 4x + c \\
 &= \frac{1}{2}x^4 - 4x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & \frac{dy}{dx} = 4x^3 + 3x^2 \\
 \therefore y &= \int (4x^3 + 3x^2) \, dx \\
 &= \frac{4x^4}{4} + \frac{3x^3}{3} + c \\
 &= x^4 + x^3 + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & \frac{dy}{dx} = 2 - \frac{1}{x^2} = 2 - x^{-2} \\
 \therefore y &= \int (2 - x^{-2}) \, dx \\
 &= 2x - \frac{x^{-1}}{-1} + c \\
 &= 2x + \frac{1}{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{4 a} \quad & \int (x^2 + 3x - 2) \, dx \\
 &= \frac{x^3}{3} + \frac{3x^2}{2} - 2x + c \\
 &= \frac{1}{3}x^3 + \frac{3}{2}x^2 - 2x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \int (-x^3 + 4x^2 - 3) \, dx \\
 &= \frac{-x^4}{4} + \frac{4x^3}{3} - 3x + c \\
 &= -\frac{1}{4}x^4 + \frac{4}{3}x^3 - 3x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \int (x^4 - x^2 - x + 2) \, dx \\
 &= \frac{x^5}{5} - \frac{x^3}{3} - \frac{x^2}{2} + 2x + c \\
 &= \frac{1}{5}x^5 - \frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & \int \left(\frac{1}{2}x^3 - x^4 + x\right) \, dx \\
 &= \frac{\frac{1}{2}x^4}{4} - \frac{x^5}{5} + \frac{x^2}{2} + c \\
 &= \frac{1}{8}x^4 - \frac{1}{5}x^5 + \frac{1}{2}x^2 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{5 a} \quad & \int (2x + 1)^2 \, dx \\
 &= \int (4x^2 + 4x + 1) \, dx \\
 &= \frac{4x^3}{3} + \frac{4x^2}{2} + x + c \\
 &= \frac{4}{3}x^3 + 2x^2 + x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \int (2x^2 - 3x + 1) \, dx \\
 &= \frac{2x^3}{3} - \frac{3x^2}{2} + x + c \\
 &= \frac{2}{3}x^3 - \frac{3}{2}x^2 + x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \int \left(\frac{1}{2}x + x^2 + x^3\right) \, dx \\
 &= \frac{\frac{1}{2}x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + c \\
 &= \frac{1}{4}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \int \left(\frac{1}{3x^3} - \frac{2}{x^2}\right) \, dx \\
 &= \int \left(\frac{1}{3}x^{-3} - 2x^{-2}\right) \, dx \\
 &= \frac{\frac{1}{3}x^{-2}}{-2} - \frac{2x^{-1}}{-1} + c \\
 &= -\frac{1}{6x^2} + \frac{2}{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & \int \left(\frac{4}{x^2} + x^2 - \frac{1}{4}x^3\right) \, dx \\
 &= \int \left(4x^{-2} + x^2 - \frac{1}{4}x^3\right) \, dx \\
 &= \frac{4x^{-1}}{-1} + \frac{1}{3}x^3 - \frac{\frac{1}{4}x^4}{4} + c \\
 &= -\frac{4}{x} + \frac{1}{3}x^3 - \frac{1}{16}x^4 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \int \left(x + \frac{1}{x}\right)^2 \, dx \\
 &= \int \left(x^2 + 2 + \frac{1}{x^2}\right) \, dx \\
 &= \int (x^2 + 2 + x^{-2}) \, dx \\
 &= \frac{x^3}{3} + 2x + \frac{x^{-1}}{-1} + c \\
 &= \frac{1}{3}x^3 + 2x - \frac{1}{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \int (3 - x^2)^2 dx \\
 &= \int (9 - 6x^2 + x^4) dx \\
 &= 9x - \frac{6x^3}{3} + \frac{x^5}{5} + c \\
 &= 9x - 2x^3 + \frac{1}{5}x^5 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \int \left(\frac{2}{x^2} + 1 \right)^2 dx \\
 &= \int \left(\left(\frac{2}{x^2} \right)^2 + 2 \left(\frac{2}{x^2} \right) + 1 \right) dx \\
 &= \int (4x^{-4} + 4x^{-2} + 1) dx \\
 &= \frac{4x^{-3}}{-3} + \frac{4x^{-1}}{-1} + x + c \\
 &= -\frac{4}{3x^3} - \frac{4}{x} + x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \int (x + 1)^3 dx \\
 &= \int (x^2 + 2x + 1)(x + 1) dx \\
 &= \int (x^3 + x^2 + 2x^2 + 2x + x + 1) dx \\
 &= \int (x^3 + 3x^2 + 3x + 1) dx \\
 &= \frac{x^4}{4} + \frac{3x^3}{3} + \frac{3x^2}{2} + x + c \\
 &= \frac{1}{4}x^4 + x^3 + \frac{3}{2}x^2 + x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \int (x - 1)^4 dx \\
 &= \int (x^2 - 2x + 1)(x^2 - 2x + 1) dx \\
 &= \int (x^4 - 2x^3 + x^2 - 2x^3 + 4x^2 - 2x + x^2 - 2x + 1) dx \\
 &= \int (x^4 - 4x^3 + 6x^2 - 4x + 1) dx \\
 &= \frac{x^5}{5} - \frac{4x^4}{4} + \frac{6x^3}{3} - \frac{4x^2}{2} + x + c \\
 &= \frac{1}{5}x^5 - x^4 + 2x^3 - 2x^2 + x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{6 a} \quad & \frac{d}{dx} [F(x) + G(x)] = F'(x) + G'(x) \\
 &= f(x) + g(x)
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \text{Using a,} \quad \int [f(x) + g(x)] dx = F(x) + G(x) + c \\
 &= \int f(x) dx + \int g(x) dx
 \end{aligned}$$

EXERCISE 13F

1 a $f'(x) = 2x - 1$

$$\begin{aligned}\therefore f(x) &= \int (2x - 1) dx \\ &= \frac{2x^2}{2} - x + c \\ &= x^2 - x + c\end{aligned}$$

But $f(0) = 3$, so $0 - 0 + c = 3$
 $\therefore c = 3$

$$\therefore f(x) = x^2 - x + 3$$

c $f'(x) = 3 - 2x^2$

$$\begin{aligned}\therefore f(x) &= \int (3 - 2x^2) dx \\ &= 3x - \frac{2x^3}{3} + c \\ &= 3x - \frac{2}{3}x^3 + c\end{aligned}$$

But $f(1) = 1$, so $3(1) - \frac{2}{3}(1)^3 + c = 1$
 $\therefore \frac{7}{3} + c = 1$
 $\therefore c = -\frac{4}{3}$

$$\therefore f(x) = 3x - \frac{2}{3}x^3 - \frac{4}{3}$$

e $f'(x) = x^3 - 2$

$$\begin{aligned}\therefore f(x) &= \int (x^3 - 2) dx \\ &= \frac{x^4}{4} - 2x + c \\ &= \frac{1}{4}x^4 - 2x + c\end{aligned}$$

But $f(4) = 0$, so $\frac{1}{4}(4)^4 - 2(4) + c = 0$
 $\therefore 64 - 8 + c = 0$
 $\therefore c = -56$

$$\therefore f(x) = \frac{1}{4}x^4 - 2x - 56$$

b $f'(x) = 3x^2 + 2x$

$$\begin{aligned}\therefore f(x) &= \int (3x^2 + 2x) dx \\ &= \frac{3x^3}{3} + \frac{2x^2}{2} + c \\ &= x^3 + x^2 + c\end{aligned}$$

But $f(2) = 5$, so $8 + 4 + c = 5$
 $\therefore c = -7$

$$\therefore f(x) = x^3 + x^2 - 7$$

d $f'(x) = x - \frac{2}{x^2}$

$$\begin{aligned}\therefore f(x) &= \int \left(x - 2x^{-2}\right) dx \\ &= \frac{x^2}{2} - \frac{2x^{-1}}{-1} + c \\ &= \frac{1}{2}x^2 + \frac{2}{x} + c\end{aligned}$$

But $f(1) = 2$, so $\frac{1}{2}(1)^2 + \frac{2}{1} + c = 2$
 $\therefore \frac{5}{2} + c = 2$
 $\therefore c = -\frac{1}{2}$

$$\therefore f(x) = \frac{1}{2}x^2 + \frac{2}{x} - \frac{1}{2}$$

f $f'(x) = 3x^2 - 4x + 1$

$$\begin{aligned}\therefore f(x) &= \int (3x^2 - 4x + 1) dx \\ &= \frac{3x^3}{3} - \frac{4x^2}{2} + x + c \\ &= x^3 - 2x^2 + x + c\end{aligned}$$

But $f(0) = 12$, so $0 - 0 + 0 + c = 12$
 $\therefore c = 12$

$$\therefore f(x) = x^3 - 2x^2 + x + 12$$

$$2 \quad \frac{dy}{dx} = x - 2x^2$$

$$\begin{aligned}\therefore y &= \int (x - 2x^2) dx \\ &= \frac{1}{2}x^2 - \frac{2}{3}x^3 + c\end{aligned}$$

But the curve passes through (2, 4),
so when $x = 2$, $y = 4$.

$$\therefore 4 = \frac{1}{2}(2)^2 - \frac{2}{3}(2)^3 + c$$

$$\therefore 4 = 2 - \frac{16}{3} + c$$

$$\therefore c = \frac{22}{3}$$

$$\therefore y = \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{22}{3}$$

$$3 \quad \frac{dy}{dx} = 1 - \frac{3}{x^2} = 1 - 3x^{-2}$$

$$\begin{aligned}\therefore y &= \int (1 - 3x^{-2}) dx \\ &= x - \frac{3x^{-1}}{-1} + c \\ &= x + \frac{3}{x} + c\end{aligned}$$

But the curve has x -intercept 3,
so when $x = 3$, $y = 0$.

$$\therefore 3 + \frac{3}{3} + c = 0$$

$$\therefore c = -4$$

$$\therefore y = x + \frac{3}{x} - 4$$

$$4 \quad f'(x) = ax + 1$$

$$\begin{aligned}\therefore f(x) &= \int (ax + 1) dx \\ &= \frac{ax^2}{2} + x + c\end{aligned}$$

Now $f(0) = 3$, so $c = 3$

$$\therefore f(x) = \frac{1}{2}ax^2 + x + 3$$

and $f(3) = -3$

$$\therefore \frac{1}{2}a(3)^2 + 3 + 3 = -3$$

$$\therefore \frac{9}{2}a + 6 = -3$$

$$\therefore \frac{9}{2}a = -9$$

$$\therefore a = -2$$

$$\begin{aligned}\therefore f(x) &= \frac{1}{2}(-2)x^2 + x + 3 \\ &= -x^2 + x + 3\end{aligned}$$

$$5 \quad f'(x) = ax^2 + bx$$

$$\begin{aligned}\therefore f(x) &= \int (ax^2 + bx) dx \\ &= \frac{ax^3}{3} + \frac{bx^2}{2} + c\end{aligned}$$

Now $f(0) = 1$, so $c = 1$

$$\therefore f(x) = \frac{1}{3}ax^3 + \frac{1}{2}bx^2 + 1,$$

$$f(-1) = -2$$

$$\therefore \frac{1}{3}a(-1)^3 + \frac{1}{2}b(-1)^2 + 1 = -2$$

$$\therefore -\frac{1}{3}a + \frac{1}{2}b + 1 = -2$$

$$\therefore -\frac{1}{3}a + \frac{1}{2}b = -3$$

$$\text{and } f(1) = 4$$

$$\therefore \frac{1}{3}a + \frac{1}{2}b + 1 = 4$$

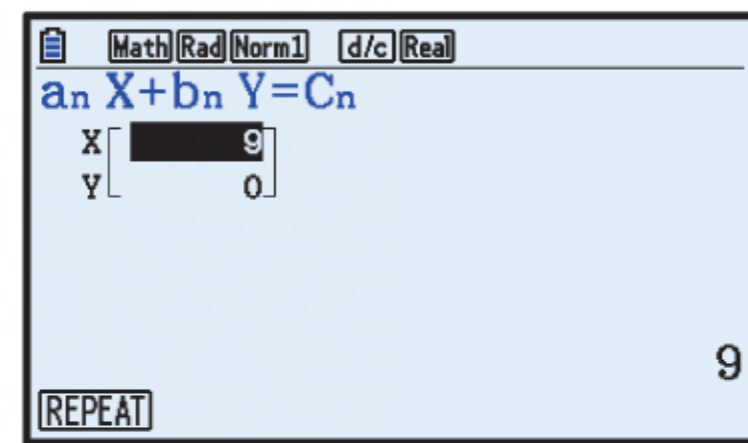
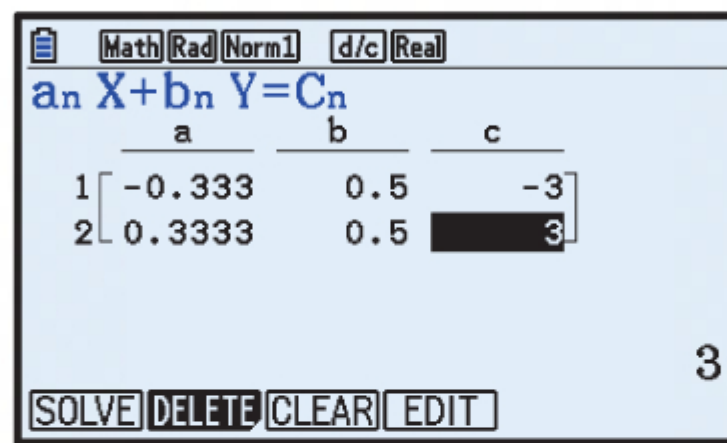
$$\therefore \frac{1}{3}a + \frac{1}{2}b = 3$$

Solving the system of equations

$$\begin{cases} -\frac{1}{3}a + \frac{1}{2}b = -3 \\ \frac{1}{3}a + \frac{1}{2}b = 3 \end{cases}$$

gives $a = 9$ and $b = 0$.

$$\begin{aligned} \therefore f(x) &= \frac{1}{3}(9)x^3 + \frac{1}{2}(0)x^2 + 1 \\ &= 3x^3 + 1 \end{aligned}$$



EXERCISE 13G

$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad \int_1^2 x^3 dx &= \left[\frac{x^4}{4} \right]_1^2 \\ &= \frac{2^4}{4} - \frac{1^4}{4} \\ &= 4 - \frac{1}{4} \\ &= \frac{15}{4} \end{aligned} \qquad \begin{aligned} \int_1^2 (-x^3) dx &= \left[-\frac{x^4}{4} \right]_1^2 \\ &= -\frac{2^4}{4} - \left(-\frac{1^4}{4} \right) \\ &= -4 + \frac{1}{4} \\ &= -\frac{15}{4} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \int_0^1 x^7 dx &= \left[\frac{x^8}{8} \right]_0^1 \\ &= \frac{1}{8} - 0 \\ &= \frac{1}{8} \end{aligned} \qquad \begin{aligned} \int_0^1 (-x^7) dx &= \left[-\frac{x^8}{8} \right]_0^1 \\ &= -\frac{1}{8} - 0 \\ &= -\frac{1}{8} \end{aligned}$$

Property: $\int_a^b [-f(x)] dx = -\int_a^b f(x) dx$

$$\begin{aligned} \mathbf{2} \quad \mathbf{a} \quad \int_0^1 x^2 dx &= \left[\frac{x^3}{3} \right]_0^1 \\ &= \frac{1}{3} - 0 \\ &= \frac{1}{3} \end{aligned} \qquad \begin{aligned} \mathbf{b} \quad \int_1^2 x^2 dx &= \left[\frac{x^3}{3} \right]_1^2 \\ &= \frac{8}{3} - \frac{1}{3} \\ &= \frac{7}{3} \end{aligned} \qquad \begin{aligned} \mathbf{c} \quad \int_0^2 x^2 dx &= \left[\frac{x^3}{3} \right]_0^2 \\ &= \frac{8}{3} - 0 \\ &= \frac{8}{3} \end{aligned} \qquad \begin{aligned} \mathbf{d} \quad \int_0^1 3x^2 dx &= [x^3]_0^1 \\ &= 1 - 0 \\ &= 1 \end{aligned}$$

Properties: $\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$

$$\int_a^b c f(x) dx = c \int_a^b f(x) dx, \text{ where } c \text{ is a constant}$$

$$\mathbf{3} \quad \mathbf{a} \quad \int_0^2 (x^3 - 4x) \, dx$$

$$= \left[\frac{x^4}{4} - 2x^2 \right]_0^2$$

$$= \left(\frac{16}{4} - 2(4) \right) - (0 - 0)$$

$$= -4$$

$$\mathbf{b} \quad \int_2^3 (x^3 - 4x) \, dx$$

$$= \left[\frac{x^4}{4} - 2x^2 \right]_2^3$$

$$= \left(\frac{81}{4} - 2(9) \right) - \left(\frac{16}{4} - 2(4) \right)$$

$$= \frac{25}{4}$$

$$= 6\frac{1}{4}$$

$$\mathbf{c} \quad \int_0^3 (x^3 - 4x) \, dx = \left[\frac{x^4}{4} - 2x^2 \right]_0^3$$

$$= \left(\frac{81}{4} - 2(9) \right) - (0 - 0)$$

$$= \frac{9}{4}$$

$$= 2\frac{1}{4}$$

Property: $\int_a^b f(x) \, dx + \int_b^c f(x) \, dx = \int_a^c f(x) \, dx$

$$\mathbf{4} \quad \mathbf{a} \quad \int_0^1 x^2 \, dx$$

$$= \left[\frac{x^3}{3} \right]_0^1$$

$$= \frac{1}{3}(1) - 0$$

$$= \frac{1}{3}$$

$$\mathbf{b} \quad \int_0^1 (-3x) \, dx$$

$$= \left[-\frac{3x^2}{2} \right]_0^1$$

$$= -\frac{3}{2} - 0$$

$$= -\frac{3}{2}$$

$$\mathbf{c} \quad \int_0^1 (x^2 - 3x) \, dx$$

$$= \left[\frac{x^3}{3} - \frac{3x^2}{2} \right]_0^1$$

$$= \left(\frac{1}{3} - \frac{3}{2} \right) - (0 - 0)$$

$$= \frac{2}{6} - \frac{9}{6}$$

$$= -\frac{7}{6}$$

Property: $\int_a^b f(x) \, dx + \int_a^b g(x) \, dx = \int_a^b [f(x) + g(x)] \, dx$

$$\mathbf{5} \quad \mathbf{a} \quad \int_0^1 x^3 \, dx = \left[\frac{x^4}{4} \right]_0^1$$

$$= \frac{1}{4} - 0$$

$$= \frac{1}{4}$$

$$\mathbf{b} \quad \int_0^2 (x^2 - x) \, dx = \left[\frac{x^3}{3} - \frac{x^2}{2} \right]_0^2$$

$$= \left(\frac{8}{3} - 2 \right) - (0 - 0)$$

$$= \frac{2}{3}$$

$$\begin{aligned}
 \text{c} \quad & \int_0^2 (3x^2 - x + 6) \, dx \\
 &= \left[\frac{3x^3}{3} - \frac{x^2}{2} + 6x \right]_0^2 \\
 &= \left[x^3 - \frac{x^2}{2} + 6x \right]_0^2 \\
 &= (8 - 2 + 12) - 0 \\
 &= 18
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & \int_1^3 \frac{1}{x^2} \, dx = \int_1^3 x^{-2} \, dx \\
 &= \left[\frac{x^{-1}}{-1} \right]_1^3 \\
 &= \left[-\frac{1}{x} \right]_1^3 \\
 &= -\frac{1}{3} - (-1) \\
 &= \frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{g} \quad & \int_1^2 \left(x^2 + \frac{1}{x^2} \right) \, dx = \int_1^2 (x^2 + x^{-2}) \, dx \\
 &= \left[\frac{x^3}{3} + \frac{x^{-1}}{-1} \right]_1^2 \\
 &= \left[\frac{x^3}{3} - \frac{1}{x} \right]_1^2 \\
 &= \left(\frac{8}{3} - \frac{1}{2} \right) - \left(\frac{1}{3} - 1 \right) \\
 &= \frac{17}{6}
 \end{aligned}$$

$$\begin{aligned}
 \text{i} \quad & \int_2^3 \left(\frac{4}{x^3} - \frac{6}{x^4} \right) \, dx = \int_2^3 (4x^{-3} - 6x^{-4}) \, dx \\
 &= \left[\frac{4x^{-2}}{-2} - \frac{6x^{-3}}{-3} \right]_2^3 \\
 &= \left[-\frac{2}{x^2} + \frac{2}{x^3} \right]_2^3 \\
 &= \left(-\frac{2}{9} + \frac{2}{27} \right) - \left(-\frac{2}{4} + \frac{2}{8} \right) \\
 &= \frac{11}{108}
 \end{aligned}$$

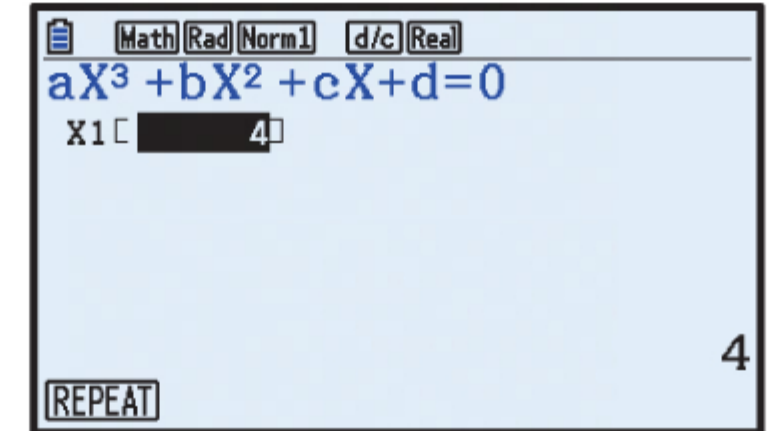
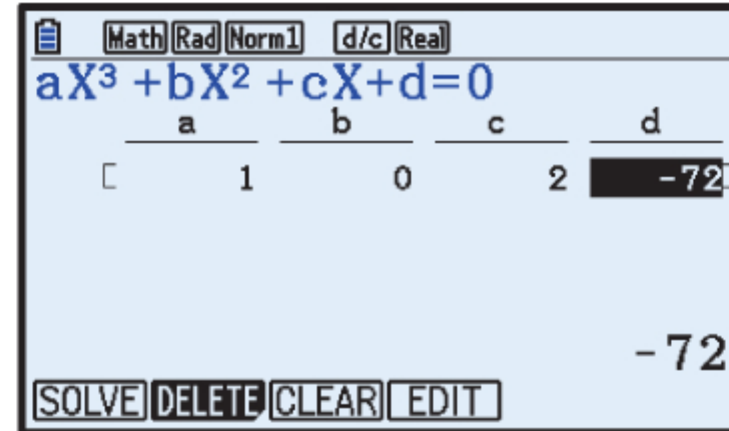
$$\begin{aligned}
 \text{d} \quad & \int_1^4 (3 + 2x - x^2) \, dx \\
 &= \left[3x + \frac{2x^2}{2} - \frac{x^3}{3} \right]_1^4 \\
 &= \left[3x + x^2 - \frac{x^3}{3} \right]_1^4 \\
 &= \left(3(4) + 4^2 - \frac{4^3}{3} \right) - \left(3 + 1 - \frac{1}{3} \right) \\
 &= 12 + 16 - \frac{64}{3} - 3 - 1 + \frac{1}{3} \\
 &= 3
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & \int_1^2 (x + 3)^2 \, dx \\
 &= \int_1^2 (x^2 + 6x + 9) \, dx \\
 &= \left[\frac{x^3}{3} + \frac{6x^2}{2} + 9x \right]_1^2 \\
 &= \left[\frac{x^3}{3} + 3x^2 + 9x \right]_1^2 \\
 &= \left(\frac{8}{3} + 12 + 18 \right) - \left(\frac{1}{3} + 3 + 9 \right) \\
 &= \frac{61}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{h} \quad & \int_1^4 \frac{x^2 + 5}{x^2} \, dx = \int_1^4 (1 + 5x^{-2}) \, dx \\
 &= \left[x + \frac{5x^{-1}}{-1} \right]_1^4 \\
 &= \left[x - \frac{5}{x} \right]_1^4 \\
 &= \left(4 - \frac{5}{4} \right) - \left(1 - 5 \right) \\
 &= \frac{27}{4}
 \end{aligned}$$

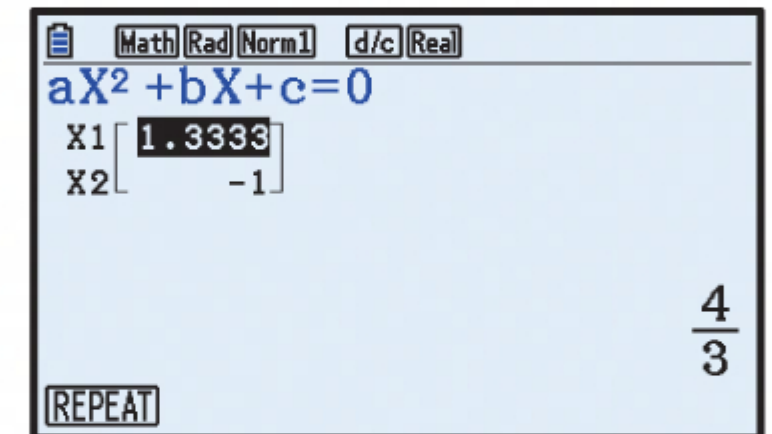
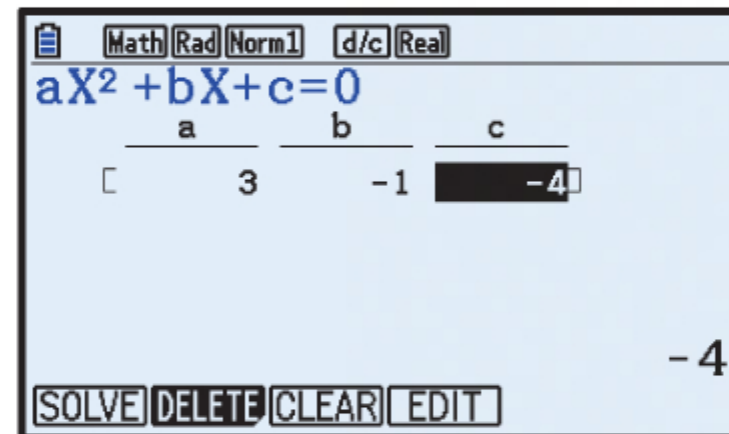
6 a $\int_0^m (3x^2 + 2) dx = 72$
 $\therefore [x^3 + 2x]_0^m = 72$
 $\therefore m^3 + 2m - 0 = 72$
 $\therefore m^3 + 2m - 72 = 0$

Using technology,
 $m = 4$



b $\int_m^{2m} (2x - 1) dx = 4$
 $\therefore [x^2 - x]_m^{2m} = 4$
 $\therefore (4m^2 - 2m) - (m^2 - m) = 4$
 $\therefore 3m^2 - m - 4 = 0$

Using technology,
 $m = -1$ or $\frac{4}{3}$



7 a $\int_2^6 \frac{1}{\sqrt{2x-3}} dx$

Using technology,

$$\int_2^6 \frac{1}{\sqrt{2x-3}} dx = 2$$

b $\int_{-3}^0 \sqrt{1-x} dx$

Using technology,

$$\int_{-3}^0 \sqrt{1-x} dx \approx 4.667$$

c $\int_0^1 e^x dx$

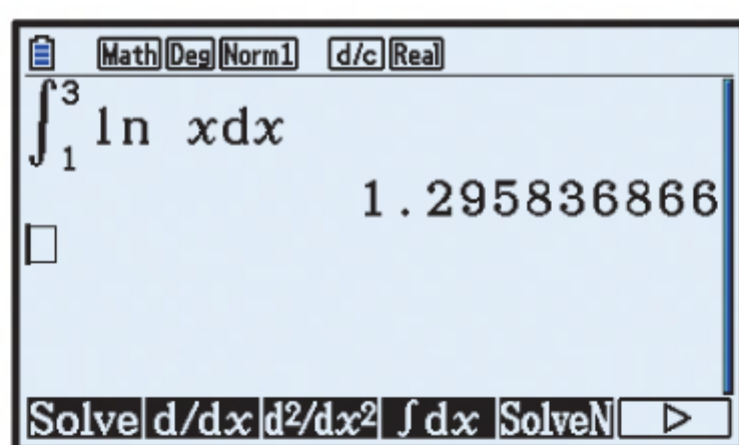
Using technology,

$$\int_0^1 e^x dx \approx 1.718$$

d $\int_0^1 x e^{1-x} dx$

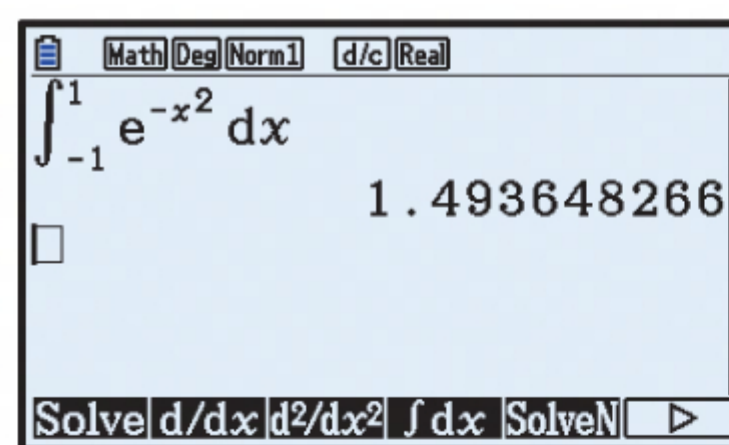
Using technology,

$$\int_0^1 x e^{1-x} dx \approx 0.7183$$

e

Using technology,

$$\int_1^3 \ln x \, dx \approx 1.296$$

f

Using technology,

$$\int_{-1}^1 e^{-x^2} \, dx \approx 1.494$$

$$8 \quad \mathbf{a} \quad \int_2^4 f(x) \, dx + \int_4^7 f(x) \, dx = \int_2^7 f(x) \, dx$$

$$\begin{aligned} \mathbf{b} \quad \int_4^5 f(x) \, dx - \int_6^5 f(x) \, dx &= \int_4^5 f(x) \, dx + \int_5^6 f(x) \, dx \\ &= \int_4^6 f(x) \, dx \end{aligned}$$

$$\mathbf{c} \quad \int_1^3 g(x) \, dx + \int_3^8 g(x) \, dx + \int_8^9 g(x) \, dx = \int_1^9 g(x) \, dx$$

$$\begin{aligned} 9 \quad \mathbf{a} \quad \int_1^3 f(x) \, dx + \int_3^6 f(x) \, dx &= \int_1^6 f(x) \, dx \\ \therefore \int_3^6 f(x) \, dx &= \int_1^6 f(x) \, dx - \int_1^3 f(x) \, dx \\ &= -3 - 2 \\ &= -5 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \int_0^2 f(x) \, dx + \int_2^4 f(x) \, dx + \int_4^6 f(x) \, dx &= \int_0^6 f(x) \, dx \\ \therefore \int_2^4 f(x) \, dx &= \int_0^6 f(x) \, dx - \int_4^6 f(x) \, dx - \int_0^2 f(x) \, dx \\ &= 7 - (-2) - 5 \\ &= 4 \end{aligned}$$

$$\begin{aligned} 10 \quad \mathbf{a} \quad \int_1^{-1} f(x) \, dx &= - \int_{-1}^1 f(x) \, dx \\ &= -(-4) \\ &= 4 \end{aligned}$$

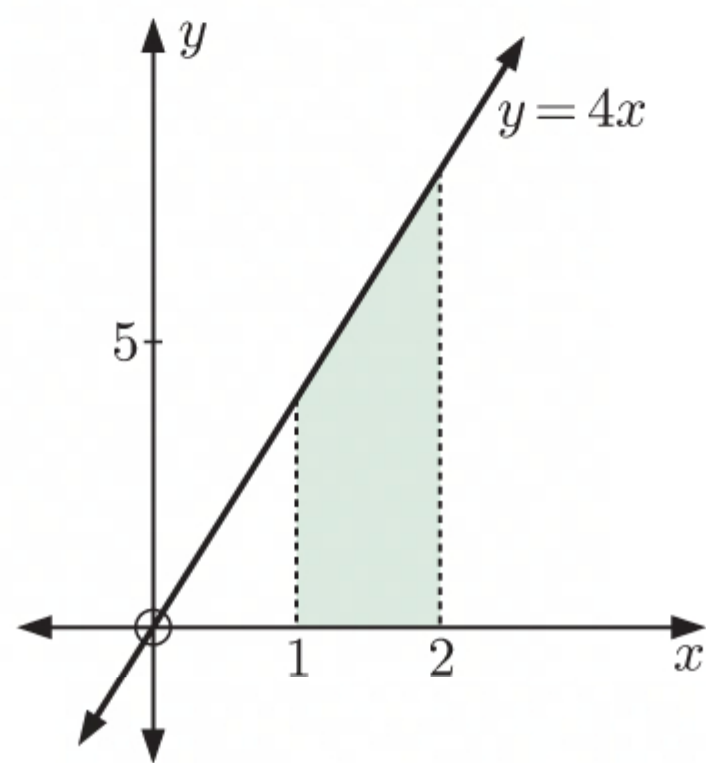
$$\begin{aligned} \mathbf{b} \quad \int_{-1}^1 (2 + f(x)) \, dx &= \int_{-1}^1 2 \, dx + \int_{-1}^1 f(x) \, dx \\ &= [2x]_{-1}^1 + (-4) \\ &= (2 - (-2)) - 4 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{c } \int_{-1}^1 2f(x) \, dx &= 2 \int_{-1}^1 f(x) \, dx \\ &= 2(-4) \\ &= -8 \end{aligned}$$

$$\begin{aligned} \text{d } \int_{-1}^1 kf(x) \, dx &= 7 \\ \therefore k \int_{-1}^1 f(x) \, dx &= 7 \\ \therefore k(-4) &= 7 \\ \therefore k &= -\frac{7}{4} \end{aligned}$$

EXERCISE 13H

1 a

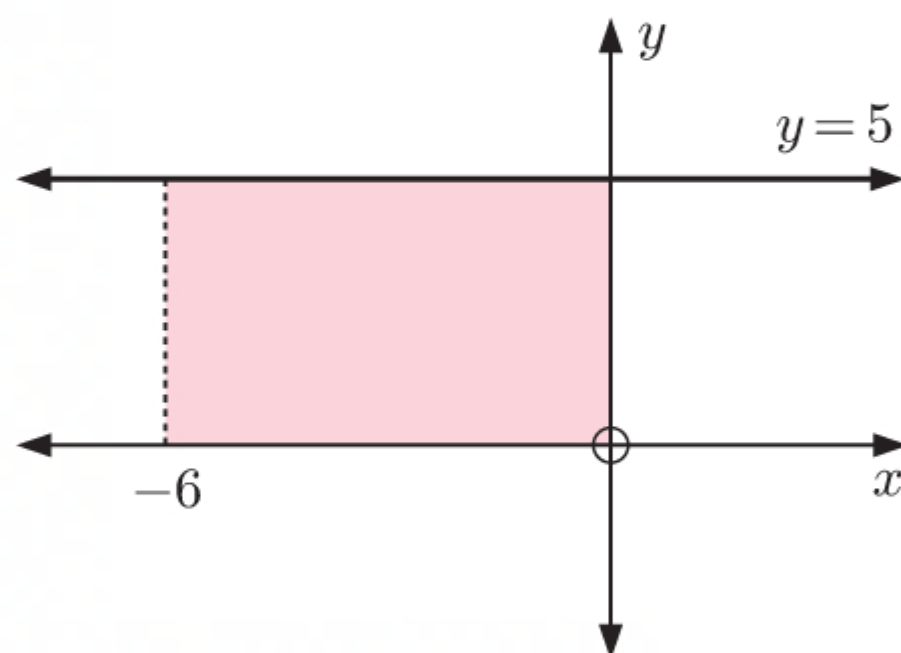
When $x = 1$, $y = 4(1) = 4$ When $x = 2$, $y = 4(2) = 8$

Area = area of trapezium

$$\begin{aligned} &= \left(\frac{4+8}{2} \right) \times 1 \\ &= 6 \text{ units}^2 \end{aligned}$$

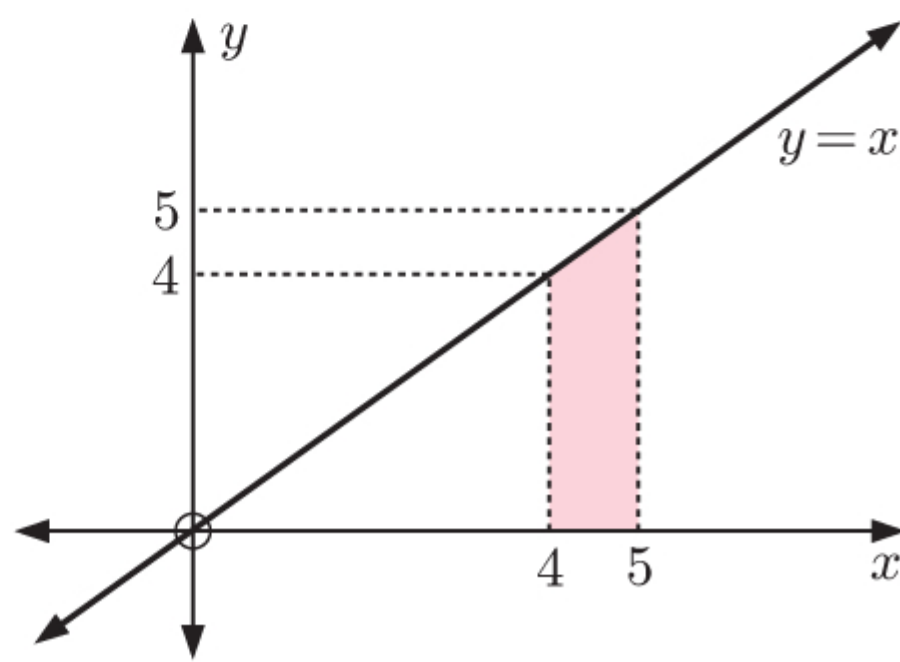
$$\begin{aligned} \text{b } \text{Area} &= \int_1^2 4x \, dx \\ &= [2x^2]_1^2 \\ &= 2(2)^2 - 2(1)^2 \\ &= 6 \text{ units}^2 \end{aligned}$$

2 a



$$\begin{aligned} \text{i } \text{Area} &= 6 \times 5 \\ &= 30 \text{ units}^2 \end{aligned}$$

$$\begin{aligned} \text{ii } \text{Area} &= \int_{-6}^0 5 \, dx \\ &= [5x]_{-6}^0 \\ &= 5(0) - 5(-6) \\ &= 30 \text{ units}^2 \end{aligned}$$

b**i** Area = area of trapezium

$$= \left(\frac{4+5}{2} \right) \times 1$$

$$= \frac{9}{2}$$

$$= 4\frac{1}{2} \text{ units}^2$$

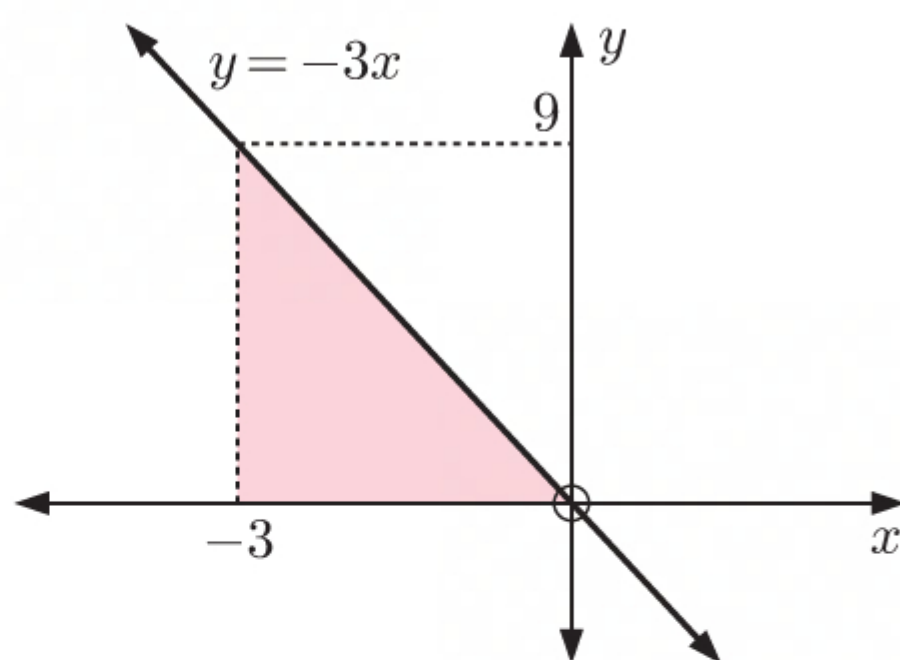
$$\text{ii Area} = \int_4^5 x \, dx$$

$$= \left[\frac{1}{2}x^2 \right]_4^5$$

$$= \frac{1}{2}(5)^2 - \frac{1}{2}(4)^2$$

$$= \frac{25}{2} - \frac{16}{2}$$

$$= 4\frac{1}{2} \text{ units}^2$$

c

$$\text{i Area} = \frac{1}{2} \times 3 \times 9$$

$$= 13\frac{1}{2} \text{ units}^2$$

$$\text{ii Area} = \int_{-3}^0 -3x \, dx$$

$$= \left[-\frac{3}{2}x^2 \right]_{-3}^0$$

$$= 0 - \left(-\frac{3}{2}(9) \right)$$

$$= 13\frac{1}{2} \text{ units}^2$$

3 a Area of blue shaded region

$$= \int_0^2 2x^2 \, dx$$

$$= \left[\frac{2}{3}x^3 \right]_0^2$$

$$= \frac{16}{3} - 0$$

$$= 5\frac{1}{3} \text{ units}^2$$

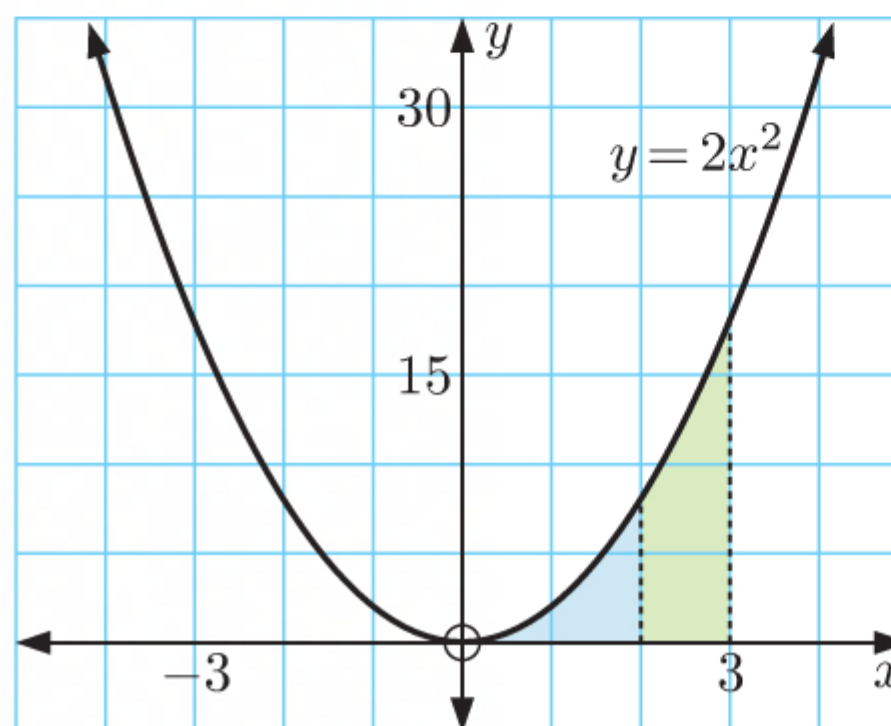
b Area of green shaded region

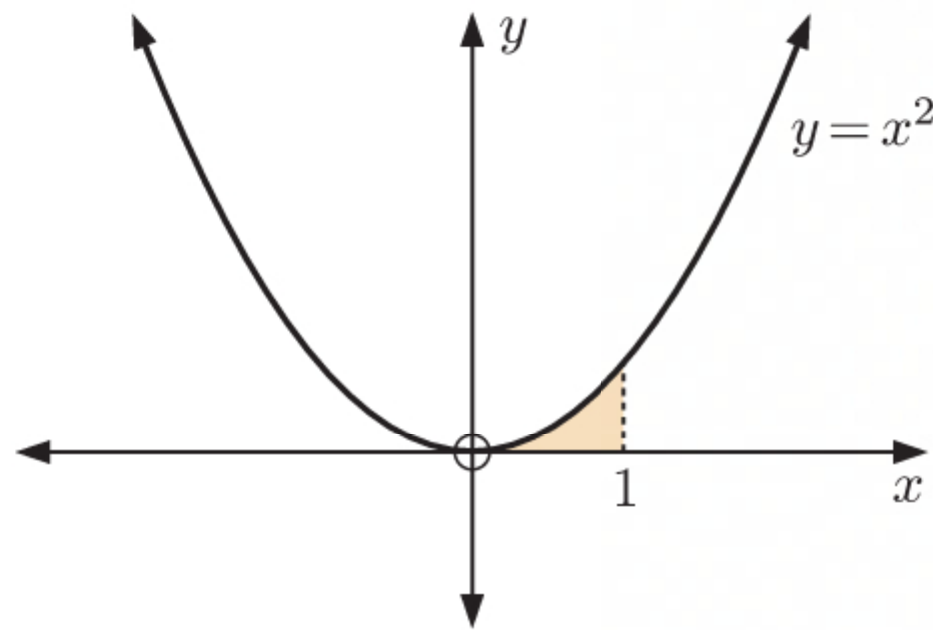
$$= \int_2^3 2x^2 \, dx$$

$$= \left[\frac{2}{3}x^3 \right]_2^3$$

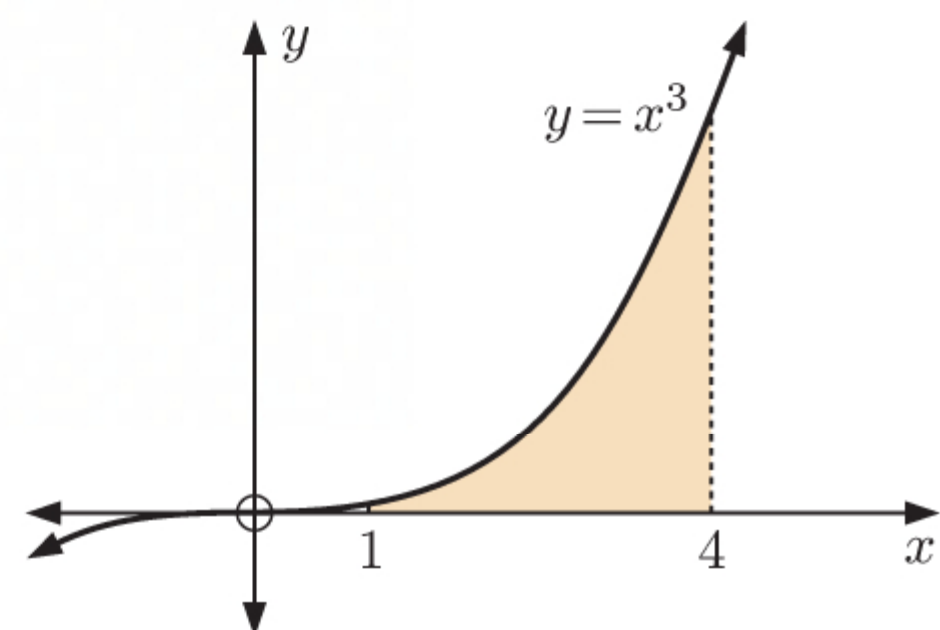
$$= \frac{54}{3} - \frac{16}{3}$$

$$= 12\frac{2}{3} \text{ units}^2$$

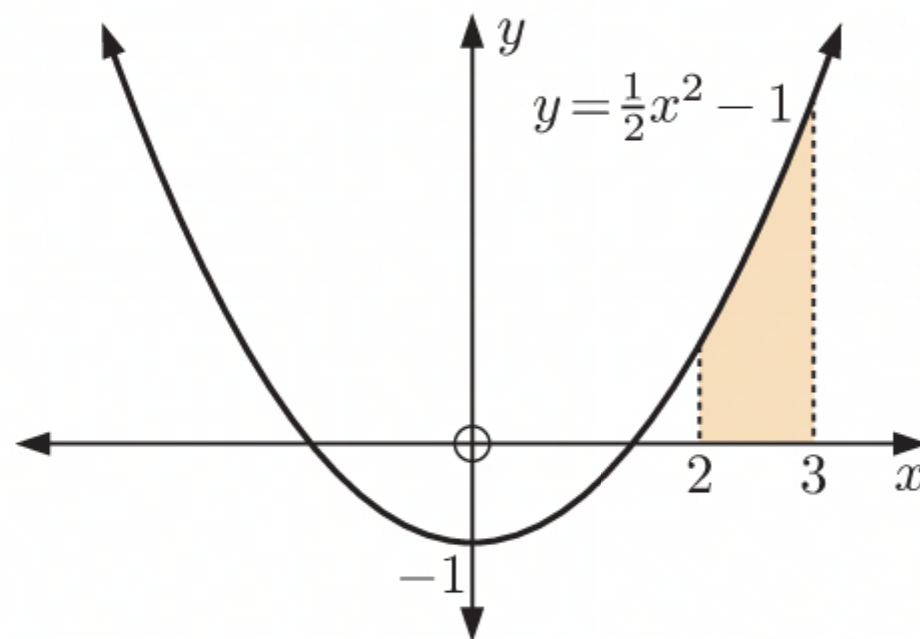


4 a

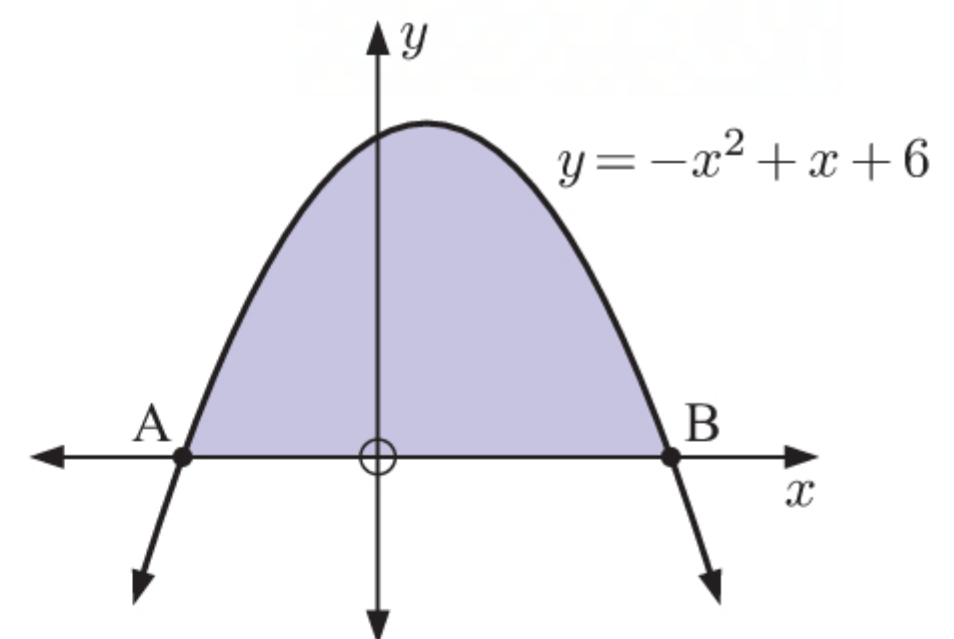
$$\begin{aligned}\text{Area} &= \int_0^1 x^2 dx \\ &= \left[\frac{1}{3}x^3 \right]_0^1 \\ &= \frac{1}{3} - 0 \\ &= \frac{1}{3} \text{ units}^2\end{aligned}$$

b

$$\begin{aligned}\text{Area} &= \int_1^4 x^3 dx \\ &= \left[\frac{1}{4}x^4 \right]_1^4 \\ &= 64 - \frac{1}{4} \\ &= 63\frac{3}{4} \text{ units}^2\end{aligned}$$

c

$$\begin{aligned}\text{Area} &= \int_2^3 \left(\frac{1}{2}x^2 - 1 \right) dx \\ &= \left[\frac{1}{6}x^3 - x \right]_2^3 \\ &= \left(\frac{27}{6} - 3 \right) - \left(\frac{8}{6} - 2 \right) \\ &= 2\frac{1}{6} \text{ units}^2\end{aligned}$$

5 a A and B are the x -intercepts of $y = -x^2 + x + 6$.When $y = 0$, $-x^2 + x + 6 = 0$ 

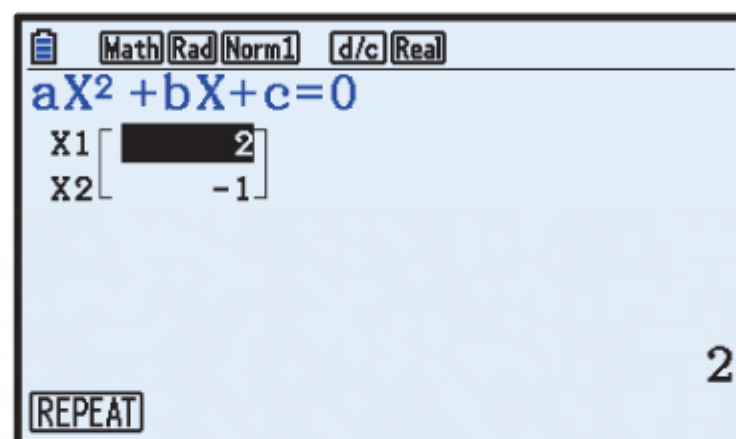
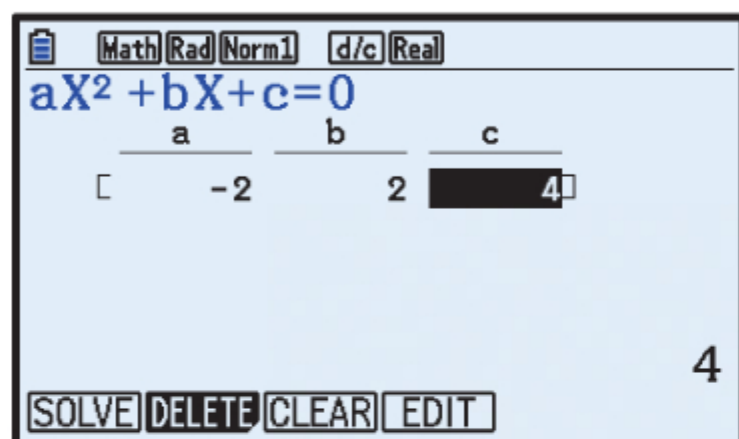
Calculator screen showing the quadratic equation $aX^2 + bX + c = 0$ with coefficients $a = -1$, $b = 1$, and $c = 6$. The screen also displays the number 6.

Calculator screen showing the solutions $X1 = 3$ and $X2 = -2$. The screen also displays the number 3.

Using technology, $x = -2$ or 3
 \therefore A is $(-2, 0)$ and B is $(3, 0)$.

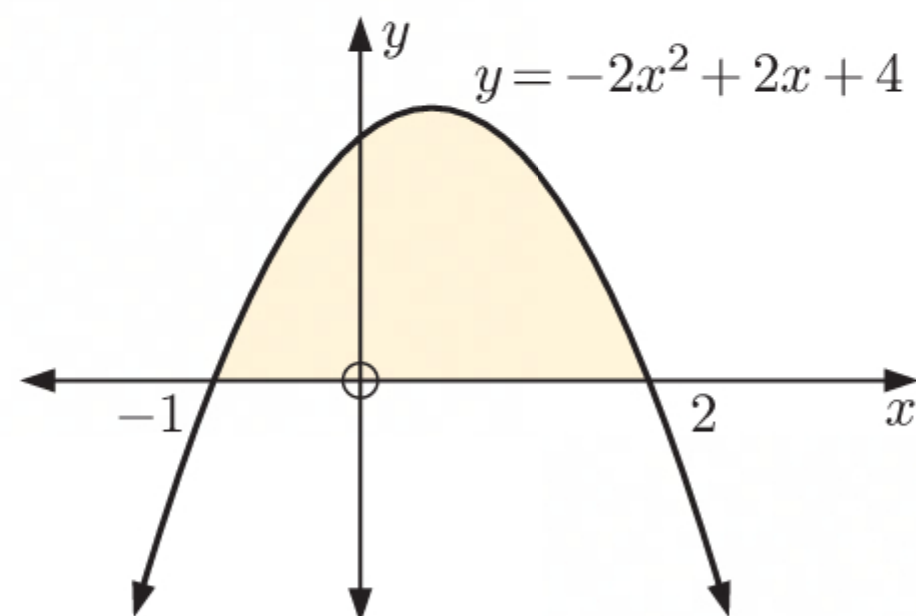
$$\begin{aligned}
 \text{b Area} &= \int_{-2}^3 (-x^2 + x + 6) dx \\
 &= \left[-\frac{1}{3}x^3 + \frac{1}{2}x^2 + 6x \right]_{-2}^3 \\
 &= \left(-9 + \frac{9}{2} + 18 \right) - \left(\frac{8}{3} + 2 - 12 \right) \\
 &= 13\frac{1}{2} - \left(-7\frac{1}{3} \right) \\
 &= 20\frac{5}{6} \text{ units}^2
 \end{aligned}$$

6 When $y = 0$, $-2x^2 + 2x + 4 = 0$



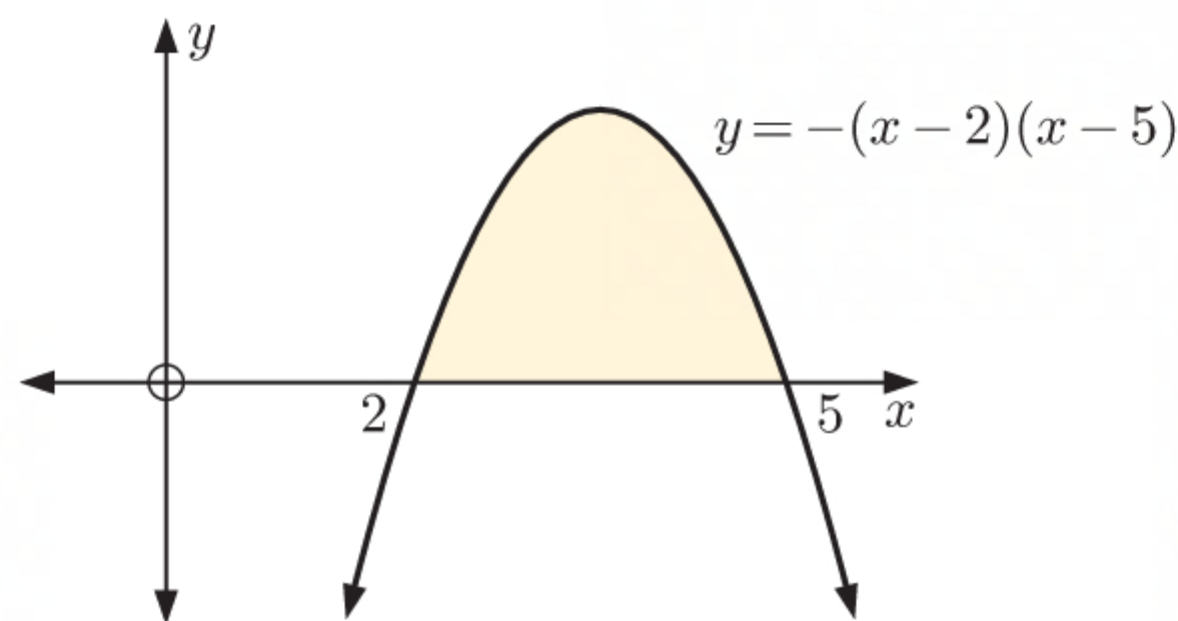
Using technology, $x = -1$ or 2

$$\begin{aligned}
 \therefore \text{area} &= \int_{-1}^2 (-2x^2 + 2x + 4) dx \\
 &= \left[-\frac{2}{3}x^3 + x^2 + 4x \right]_{-1}^2 \\
 &= \left(-\frac{2}{3}(8) + 4 + 8 \right) - \left(-\frac{2}{3}(-1) + 1 - 4 \right) \\
 &= 9 \text{ units}^2
 \end{aligned}$$



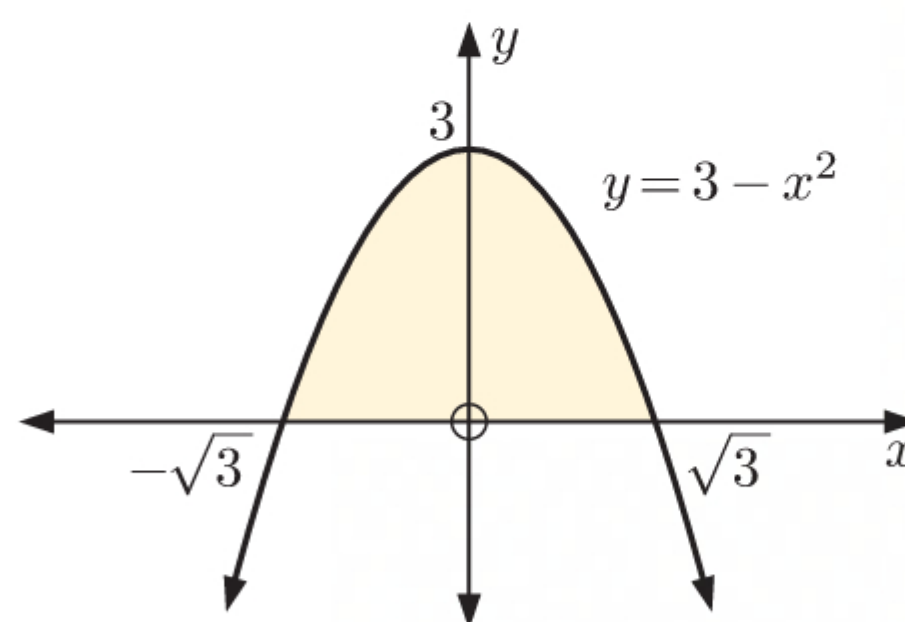
7 a When $y = 0$, $-(x - 2)(x - 5) = 0$
 $\therefore x = 2$ or 5

$$\begin{aligned}
 \therefore \text{area} &= \int_2^5 -(x - 2)(x - 5) dx \\
 &= \int_2^5 -(x^2 - 7x + 10) dx \\
 &= \int_2^5 (-x^2 + 7x - 10) dx \\
 &= \left[-\frac{1}{3}x^3 + \frac{7}{2}x^2 - 10x \right]_2^5 \\
 &= \left(-\frac{125}{3} + \frac{175}{2} - 50 \right) - \left(-\frac{8}{3} + 14 - 20 \right) \\
 &= 4\frac{1}{2} \text{ units}^2
 \end{aligned}$$

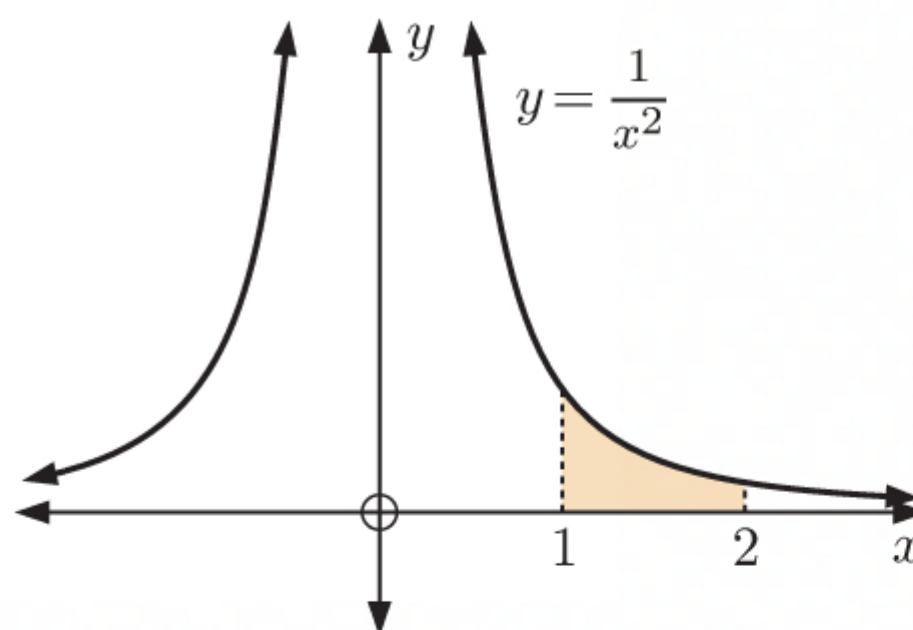


b When $y = 0$, $3 - x^2 = 0$
 $\therefore x^2 = 3$
 $\therefore x = \pm\sqrt{3}$

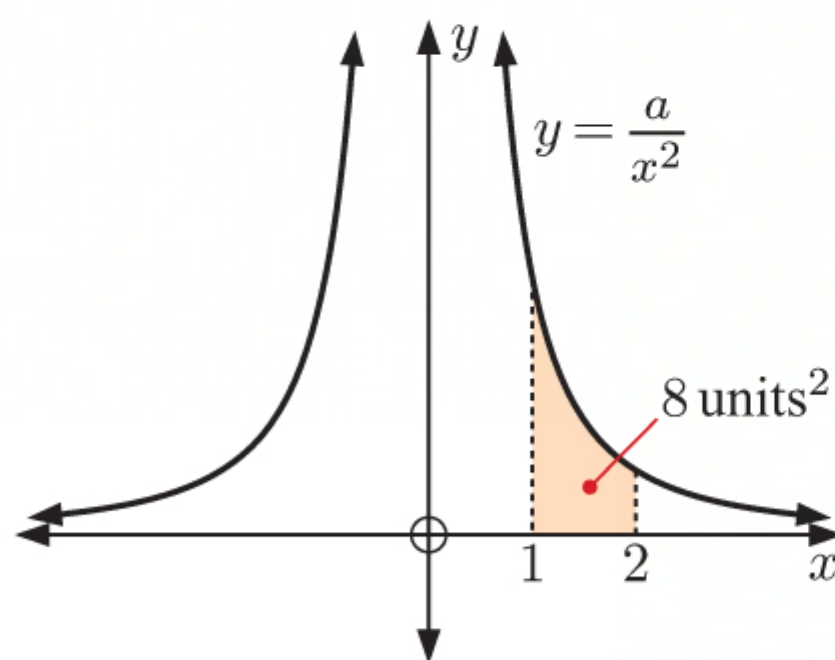
$$\begin{aligned}\therefore \text{area} &= \int_{-\sqrt{3}}^{\sqrt{3}} (3 - x^2) dx \\ &= \left[3x - \frac{1}{3}x^3 \right]_{-\sqrt{3}}^{\sqrt{3}} \\ &= \left(3\sqrt{3} - \frac{1}{3}(\sqrt{3})^3 \right) - \left(-3\sqrt{3} - \frac{1}{3}(-\sqrt{3})^3 \right) \\ &= 2\sqrt{3} - (-2\sqrt{3}) \\ &= 4\sqrt{3} \text{ units}^2\end{aligned}$$



8 Area $= \int_1^2 \frac{1}{x^2} dx$
 $= \int_1^2 x^{-2} dx$
 $= \left[-\frac{1}{x} \right]_1^2$
 $= -\frac{1}{2} - (-1)$
 $= \frac{1}{2} \text{ units}^2$



9 a Area $= \int_1^2 \frac{a}{x^2} dx$
 $\therefore 8 = \int_1^2 ax^{-2} dx$
 $\therefore 8 = \left[-ax^{-1} \right]_1^2$
 $\therefore 8 = -\frac{a}{2} - \left(-\frac{a}{1} \right)$
 $\therefore 8 = \frac{a}{2}$
 $\therefore a = 16$



b

$$\text{Area} = \int_{-a}^a (x^2 + 2) dx$$

$$\therefore 6a = \left[\frac{1}{3}x^3 + 2x \right]_{-a}^a$$

$$\therefore 6a = \left(\frac{1}{3}a^3 + 2a \right) - \left(-\frac{1}{3}a^3 - 2a \right)$$

$$\therefore 6a = \frac{2}{3}a^3 + 4a$$

$$\therefore \frac{2}{3}a^3 - 2a = 0$$

$$\therefore a^3 - 3a = 0$$

$$\therefore a(a^2 - 3) = 0$$

$$\therefore a = 0, \pm\sqrt{3}$$

$$\text{but } a > 0, \therefore a = \sqrt{3}$$

c

$$\text{When } y = 0, \quad a^2 - x^2 = 0$$

$$\therefore x^2 = a^2$$

$$\therefore x = a \quad \{\text{as } x > 0\}$$

$$\therefore \text{area} = \int_0^a (a^2 - x^2) dx$$

$$\therefore 18 = \left[a^2x - \frac{1}{3}x^3 \right]_0^a$$

$$\therefore 18 = (a^3 - \frac{1}{3}a^3) - 0$$

$$\therefore \frac{2}{3}a^3 = 18$$

$$\therefore a^3 = 27$$

$$\therefore a = 3$$

d

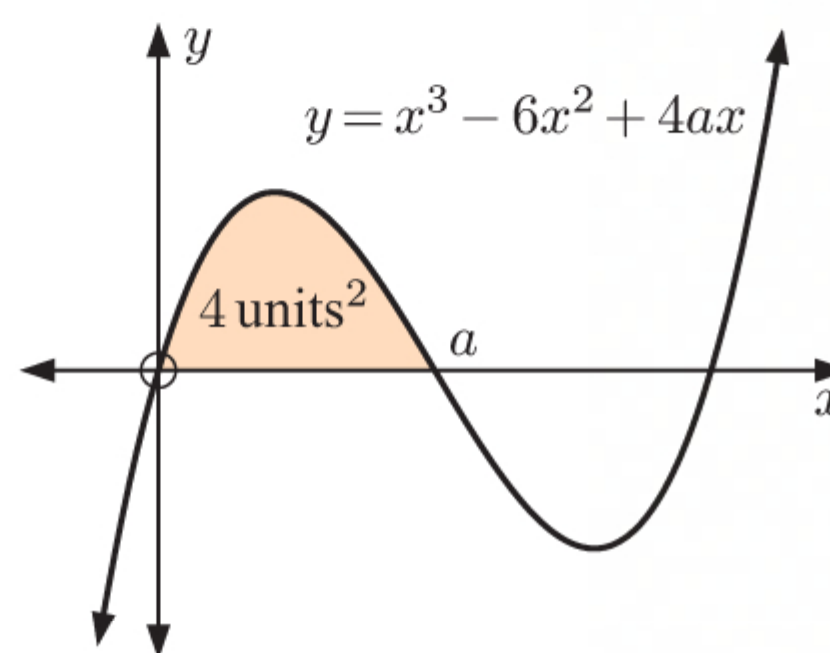
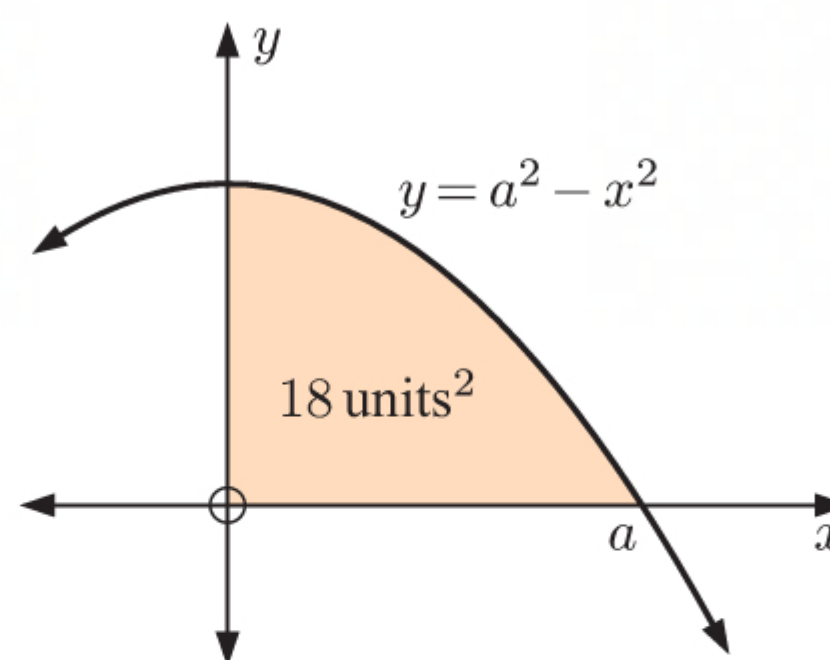
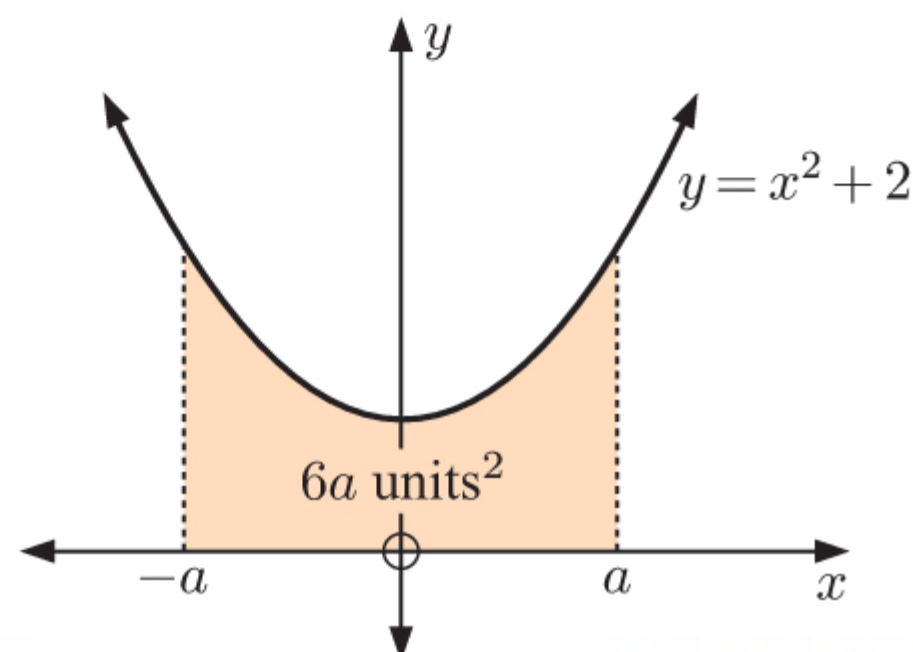
$$\text{Area} = \int_0^a (x^3 - 6x^2 + 4ax) dx$$

$$\therefore 4 = \left[\frac{1}{4}x^4 - 2x^3 + 2ax^2 \right]_0^a$$

$$\therefore 4 = \frac{1}{4}a^4 - 2a^3 + 2a^3 - 0$$

$$\therefore a^4 = 16$$

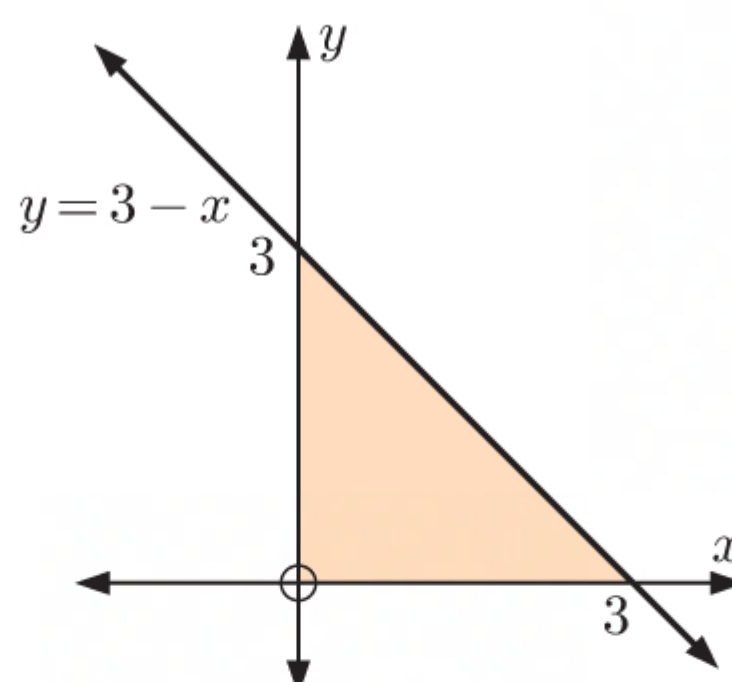
$$\therefore a = 2 \quad \{\text{as } a > 0\}$$



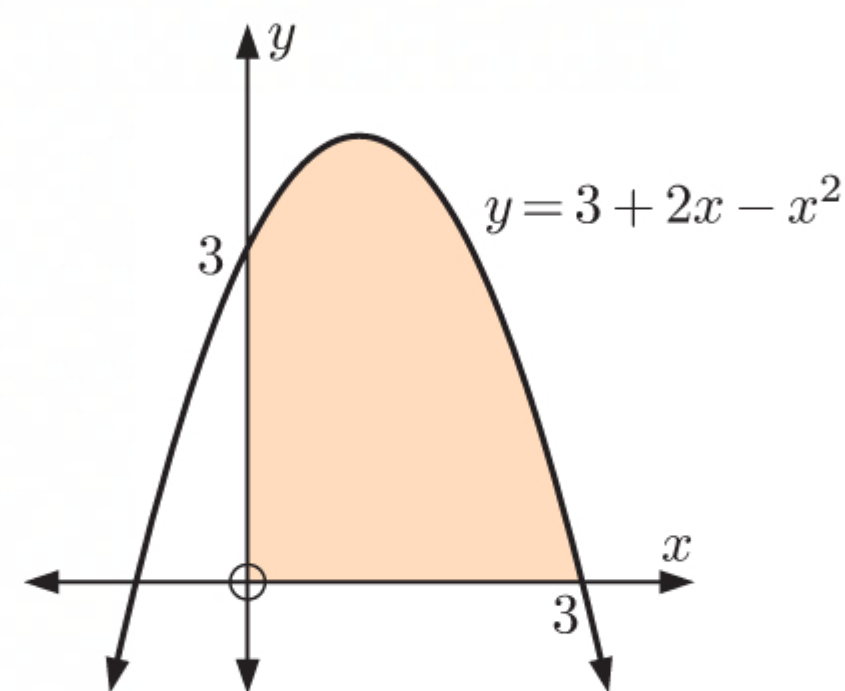
10 a i When $y = 0$, $3 - x = 0$
 $\therefore x = 3$

$$\therefore \text{area} = \frac{1}{2} \times 3 \times 3$$

$$= 4\frac{1}{2} \text{ units}^2$$

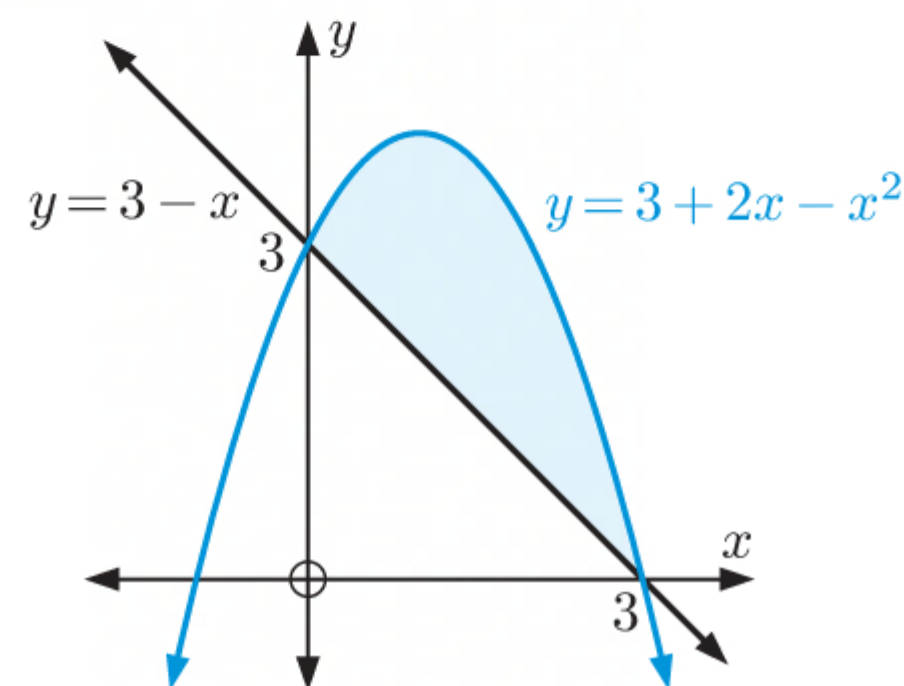


$$\begin{aligned}
 \text{ii Area} &= \int_0^3 (3 + 2x - x^2) dx \\
 &= \left[3x + x^2 - \frac{1}{3}x^3 \right]_0^3 \\
 &= \left(9 + 9 - \frac{27}{3} \right) - 0 \\
 &= 9 \text{ units}^2
 \end{aligned}$$



- b** When $x = 0$, $3 - x = 3$ and $3 + 2x - x^2 = 3$,
and when $x = 3$, $3 - x = 0$ and $3 + 2x - x^2 = 0$.

$$\begin{aligned}
 \therefore \text{shaded area} &= \int_0^3 (3 + 2x - x^2) dx - \int_0^3 (3 - x) dx \\
 &= 9 - 4\frac{1}{2} \\
 &= 4\frac{1}{2} \text{ units}^2
 \end{aligned}$$



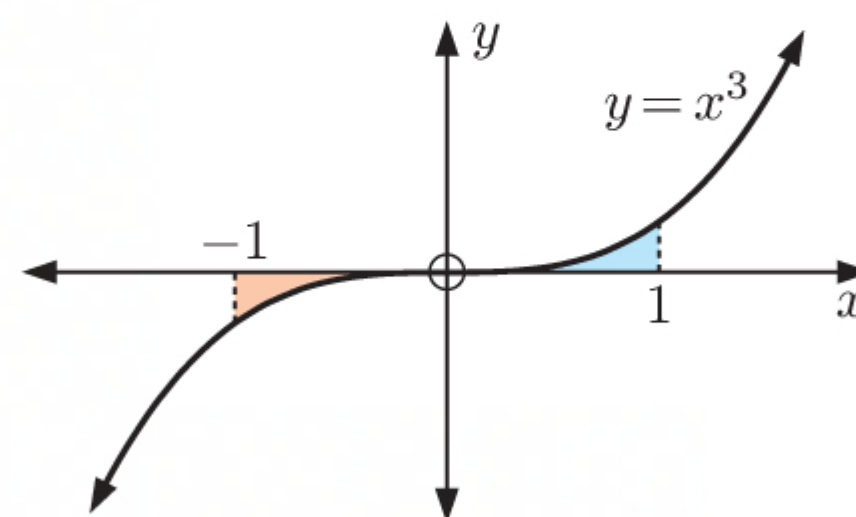
INVESTIGATION 2

$\int_a^b f(x) dx$ AND AREAS

$$\begin{aligned}
 \text{1 a } \int_0^1 x^3 dx &= \left[\frac{1}{4}x^4 \right]_0^1 & \text{and} & \int_{-1}^1 x^3 dx = \left[\frac{1}{4}x^4 \right]_{-1}^1 \\
 &= \frac{1}{4} - 0 & & = \frac{1}{4} - \frac{1}{4}(-1)^4 \\
 &= \frac{1}{4} & & = \frac{1}{4} - \frac{1}{4} \\
 & & & = 0
 \end{aligned}$$

- b** Since the curve lies on or above the x -axis for $0 \leq x \leq 1$, the first integral in **a** is the area bounded by $y = x^3$, the x -axis, and the vertical lines $x = 0$ and $x = 1$.

The second integral in **a** does *not* give an area as the curve lies on or below the x -axis for $-1 \leq x \leq 0$.



$$\begin{aligned}
 \text{c } \int_{-1}^0 x^3 dx &= \left[\frac{1}{4}x^4 \right]_{-1}^0 \\
 &= 0 - \frac{1}{4}(-1)^4 \\
 &= -\frac{1}{4}
 \end{aligned}$$

The answer is negative since the curve lies on or below the x -axis for $-1 \leq x \leq 0$.

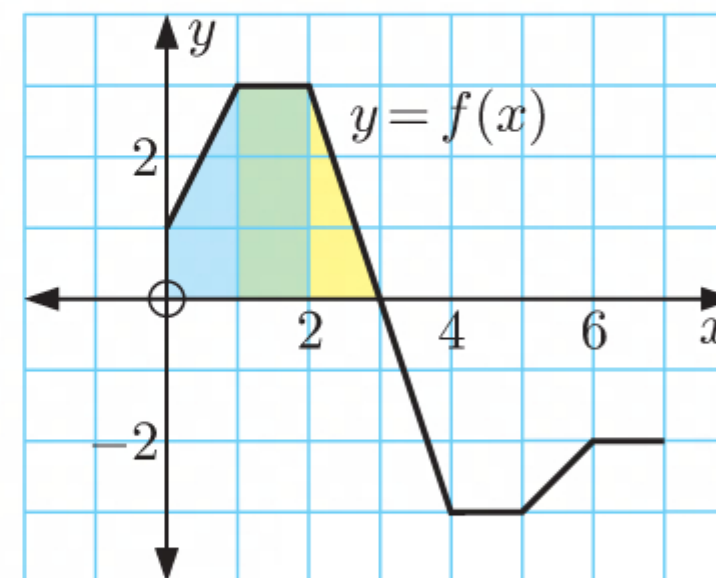
$$\begin{aligned} \text{d} \quad \int_{-1}^0 x^3 dx + \int_0^1 x^3 dx &= -\frac{1}{4} + \frac{1}{4} \quad \{\text{using a and c}\} \\ &= 0 \\ &= \int_{-1}^1 x^3 dx \end{aligned}$$

$$\begin{aligned} \text{e} \quad \int_0^{-1} x^3 dx &= -\int_{-1}^0 x^3 dx \\ &= -\left(-\frac{1}{4}\right) \quad \{\text{from c}\} \\ &= \frac{1}{4} \end{aligned}$$

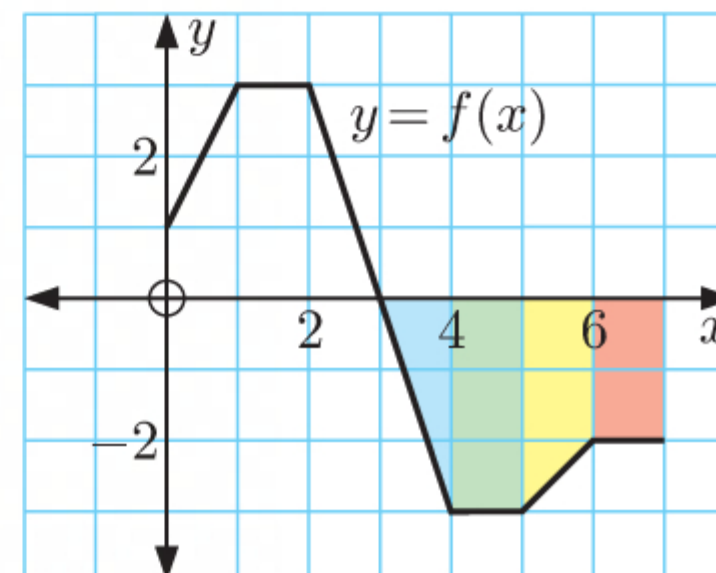
The area between the curve and the x -axis between $x = -1$ and $x = 0$ is $\frac{1}{4}$ units².

$$\text{2} \quad \text{Area} = -\int_a^b f(x) dx = \int_b^a f(x) dx$$

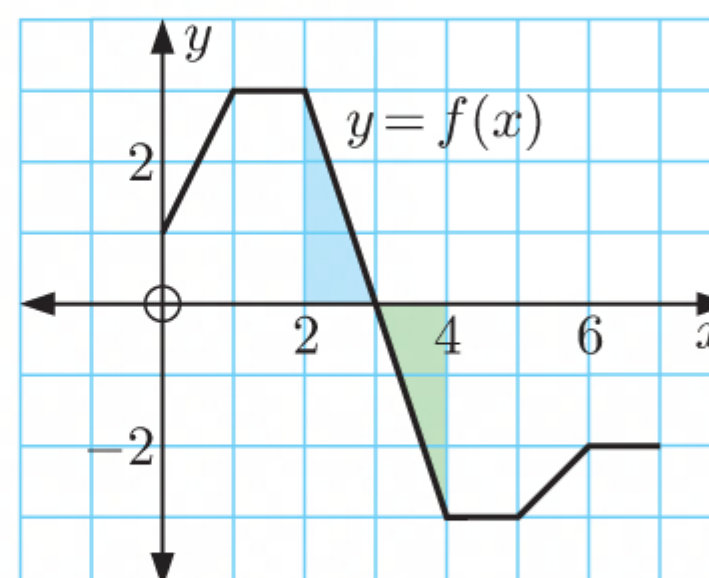
$$\begin{aligned} \text{3 a} \quad \int_0^3 f(x) dx &= \text{area of blue trapezium} \\ &\quad + \text{area of green rectangle} \\ &\quad + \text{area of yellow triangle} \\ &= \left(\frac{1+3}{2}\right) \times 1 + (1 \times 3) + \left(\frac{1}{2} \times 1 \times 3\right) \\ &= 2 + 3 + \frac{3}{2} \\ &= \frac{13}{2} \end{aligned}$$



$$\begin{aligned} \text{b} \quad \int_3^7 f(x) dx &= -(\text{area of blue triangle} \\ &\quad + \text{area of green rectangle} \\ &\quad + \text{area of yellow trapezium} \\ &\quad + \text{area of red rectangle}) \\ &= -\left[\left(\frac{1}{2} \times 1 \times 3\right) + (1 \times 3) + \left(\frac{2+3}{2}\right) \times 1 + (1 \times 2)\right] \\ &= -\left(\frac{3}{2} + 3 + \frac{5}{2} + 2\right) \\ &= -9 \end{aligned}$$



$$\begin{aligned} \text{c} \quad \int_2^4 f(x) dx &= \text{area of blue triangle} \\ &\quad - \text{area of green triangle} \\ &= \left(\frac{1}{2} \times 1 \times 3\right) - \left(\frac{1}{2} \times 1 \times 3\right) \\ &= 0 \end{aligned}$$



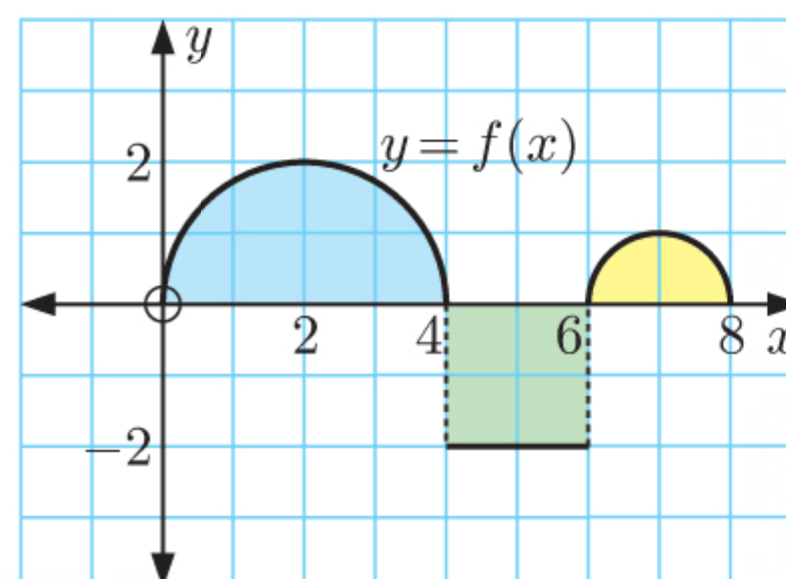
$$\begin{aligned}
 \text{d } \int_0^7 f(x) \, dx &= \int_0^3 f(x) \, dx + \int_3^7 f(x) \, dx \\
 &= \frac{13}{2} + (-9) \quad \{\text{using a and b}\} \\
 &= -\frac{5}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{4 a } \int_0^4 f(x) \, dx &= \text{area of blue semi-circle} \\
 &= \frac{1}{2} \times \pi \times 2^2 \\
 &= 2\pi
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \int_4^6 f(x) \, dx &= -\text{area of green square} \\
 &= -(2 \times 2) \\
 &= -4
 \end{aligned}$$

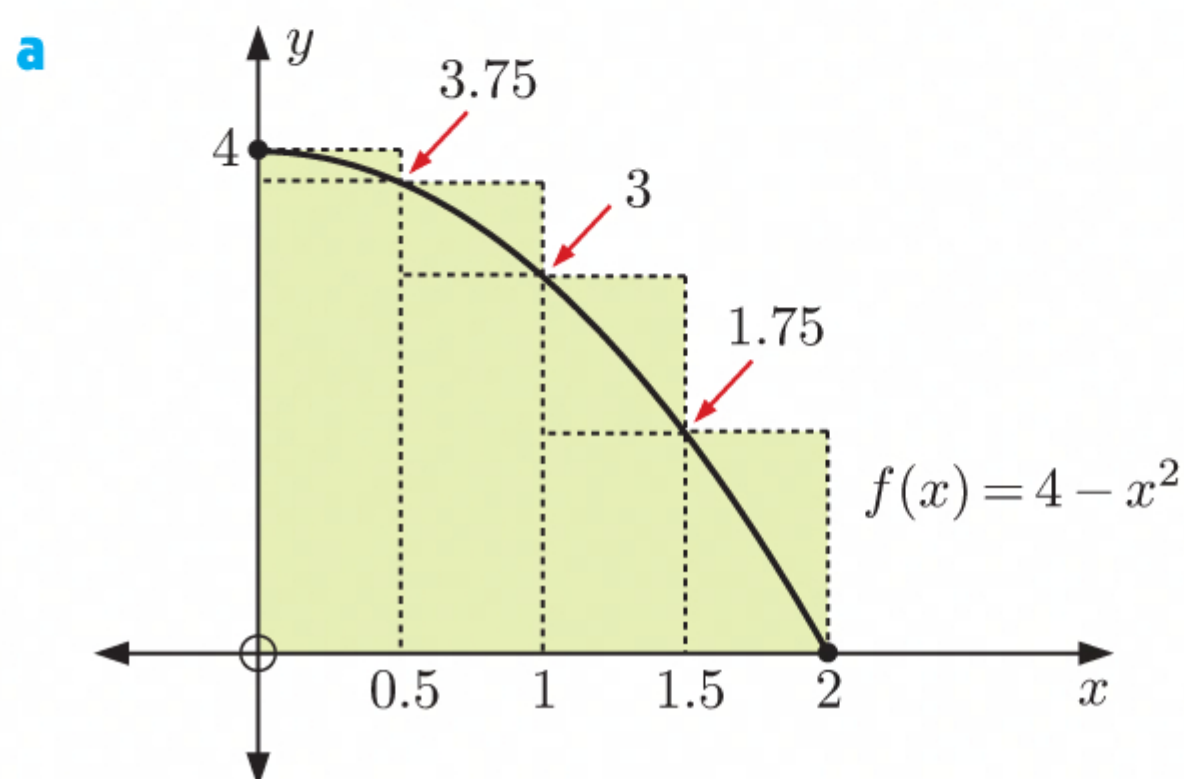
$$\begin{aligned}
 \text{c } \int_6^8 f(x) \, dx &= \text{area of yellow semi-circle} \\
 &= \frac{1}{2} \times \pi \times 1^2 \\
 &= \frac{\pi}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{d } \int_0^8 f(x) \, dx &= \int_0^4 f(x) \, dx + \int_4^6 f(x) \, dx + \int_6^8 f(x) \, dx \\
 &= 2\pi + (-4) + \frac{\pi}{2} \quad \{\text{using a, b, and c}\} \\
 &= \frac{5\pi}{2} - 4
 \end{aligned}$$



REVIEW SET 13A

- 1 The rectangles are $\frac{2}{4} = \frac{1}{2}$ units wide.



$$\begin{aligned}
 A_L &= \frac{1}{2} \left[f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right) + f(2) \right] \\
 &= \frac{1}{2} \left(\frac{15}{4} + 3 + \frac{7}{4} + 0 \right) \\
 &= \frac{17}{4}
 \end{aligned}$$

$$\begin{aligned}
 A_U &= \frac{1}{2} \left[f(0) + f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right) \right] \\
 &= \frac{1}{2} \left(4 + \frac{15}{4} + 3 + \frac{7}{4} \right) \\
 &= \frac{25}{4}
 \end{aligned}$$

$$\therefore \frac{17}{4} < \int_0^2 (4 - x^2) \, dx < \frac{25}{4}$$

$$\therefore A = \frac{17}{4}, \quad B = \frac{25}{4}$$

b $n = 8, a = 0, b = 2, f(x) = 4 - x^2$

$$h = \frac{b-a}{n} = \frac{1}{4}$$

$$x_i = 0 + \frac{1}{4}i$$

i	x_i	$f(x_i)$
0	0	4
1	$\frac{1}{4}$	3.9375
2	$\frac{1}{2}$	3.75
3	$\frac{3}{4}$	3.4375
4	1	3
5	$1\frac{1}{4}$	2.4375
6	$1\frac{1}{2}$	1.75
7	$1\frac{3}{4}$	0.9375
8	2	0

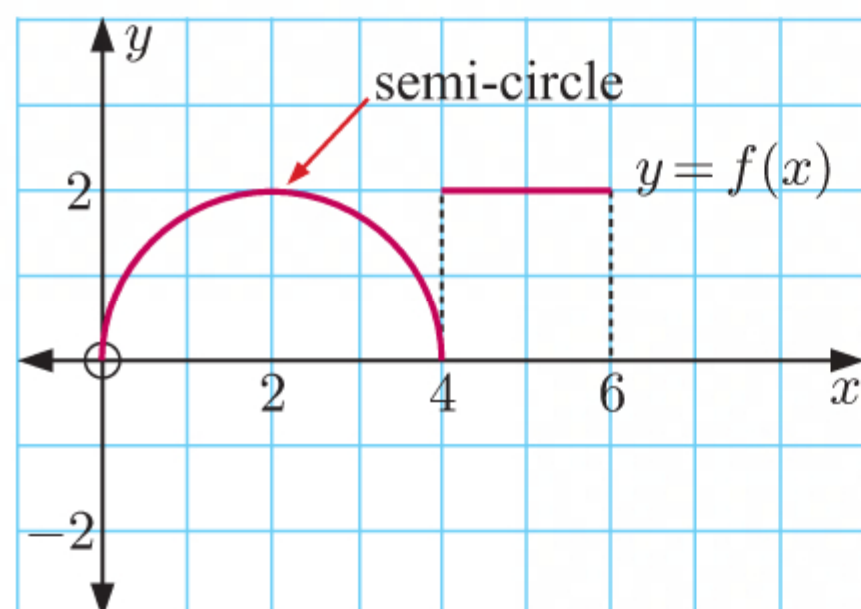
Using the trapezoidal rule, the area $\approx \frac{h}{2} (f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_7) + f(x_8))$
 $\approx 5.3125 \text{ units}^2$

$$\therefore \int_0^2 (4 - x^2) dx \approx 5.3125$$

c
$$\begin{aligned} \int_0^2 (4 - x^2) dx &= \left[4x - \frac{1}{3}x^3 \right]_0^2 \\ &= \left(8 - \frac{8}{3} \right) - 0 \\ &= \frac{16}{3} \\ &= 5\frac{1}{3} \end{aligned}$$

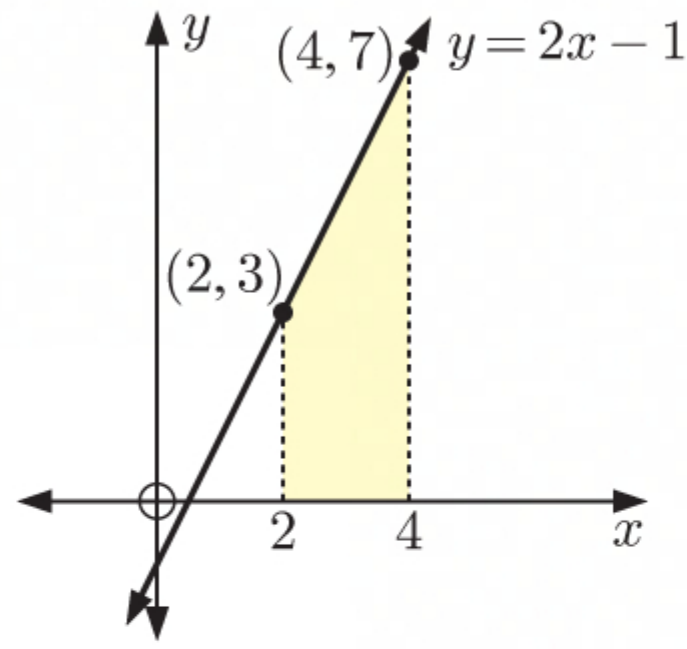
The estimate from **a**, $\approx \frac{A+B}{2} \approx 5.25$, and the estimate from **b**, ≈ 5.3125 , both provide good approximations of the integral.

2

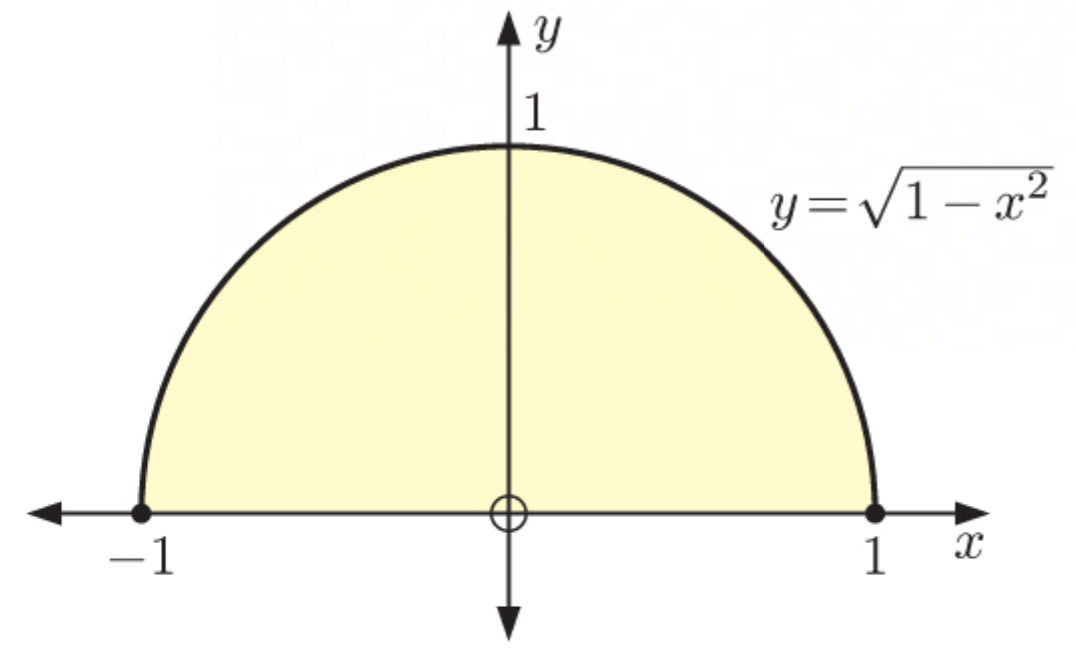


a
$$\begin{aligned} \int_0^4 f(x) dx &= \text{area of semi-circle} \\ &\quad \text{with radius 2} \\ &= \frac{1}{2} \times \pi(2)^2 \\ &= 2\pi \end{aligned}$$

b
$$\begin{aligned} \int_4^6 f(x) dx &= \text{area of square} \\ &= 2^2 \\ &= 4 \end{aligned}$$

3 a

$$\begin{aligned}\int_2^4 (2x - 1) dx &= \text{shaded area} \\ &= \left(\frac{3+7}{2}\right) \times 2 \\ &= 10\end{aligned}$$

b

$$\begin{aligned}\int_{-1}^1 \sqrt{1 - x^2} dx &= \text{shaded area} \\ &= \frac{1}{2}\pi r^2 \\ &= \frac{1}{2}(\pi \times 1^2) \\ &= \frac{\pi}{2}\end{aligned}$$

4

$$\begin{aligned}\frac{d}{dx}(x^4 - x^2) &= 4x^3 - 2x \\ \therefore \int (4x^3 - 2x) dx &= x^4 - x^2 + c \\ \therefore 2 \int (2x^3 - x) dx &= x^4 - x^2 + c \\ \therefore \int (2x^3 - x) dx &= \frac{1}{2}x^4 - \frac{1}{2}x^2 + c\end{aligned}$$

5 a

$$\begin{aligned}\int 5 dx \\ &= 5x + c\end{aligned}$$

b

$$\begin{aligned}\int 6x^2 dx \\ &= \frac{6x^3}{3} + c \\ &= 2x^3 + c\end{aligned}$$

c

$$\begin{aligned}\int (3 - 2x) dx \\ &= 3x - \frac{2x^2}{2} + c \\ &= 3x - x^2 + c\end{aligned}$$

6 a

$$\begin{aligned}\int \left(4x - \frac{2}{x^2}\right) dx \\ &= \int (4x - 2x^{-2}) dx \\ &= \frac{4x^2}{2} - \frac{2x^{-1}}{-1} + c \\ &= 2x^2 + \frac{2}{x} + c\end{aligned}$$

b

$$\begin{aligned}\int \left(\frac{1}{3}x^3 + 2x\right) dx \\ &= \frac{\frac{1}{3}x^4}{4} + x^2 + c \\ &= \frac{1}{12}x^4 + x^2 + c\end{aligned}$$

c

$$\begin{aligned}\int \frac{1 - 2x}{x^3} dx \\ &= \int \left(\frac{1}{x^3} - \frac{2}{x^2}\right) dx \\ &= \int (x^{-3} - 2x^{-2}) dx \\ &= \frac{x^{-2}}{-2} - \frac{2x^{-1}}{-1} + c \\ &= -\frac{1}{2}x^{-2} + 2x^{-1} + c \\ &= -\frac{1}{2x^2} + \frac{2}{x} + c\end{aligned}$$

$$\begin{aligned}
 \text{7 a } \int (-3x^4 + 6x^2) dx \\
 &= -\frac{3x^5}{5} + \frac{6x^3}{3} + c \\
 &= -\frac{3}{5}x^5 + 2x^3 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \int \frac{3x^3 - x^2 - 1}{x^2} dx \\
 &= \int \left(3x - 1 - \frac{1}{x^2} \right) dx \\
 &= \int (3x - 1 - x^{-2}) dx \\
 &= \frac{3x^2}{2} - x - \frac{x^{-1}}{-1} + c \\
 &= \frac{3}{2}x^2 - x + \frac{1}{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \int (2x - 3)^2 dx \\
 &= \int (4x^2 - 12x + 9) dx \\
 &= \frac{4x^3}{3} - \frac{12x^2}{2} + 9x + c \\
 &= \frac{4}{3}x^3 - 6x^2 + 9x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{8 } f'(x) &= 3x^2 - 4x + 1 \\
 \therefore f(x) &= \int (3x^2 - 4x + 1) dx \\
 &= \frac{3x^3}{3} - \frac{4x^2}{2} + x + c \\
 &= x^3 - 2x^2 + x + c
 \end{aligned}$$

But $f(0) = 2$, so $0 - 0 + 0 + c = 2$
 $\therefore c = 2$

$$\therefore f(x) = x^3 - 2x^2 + x + 2$$

$$\begin{aligned}
 \text{9 } f'(x) &= ax + 3 \\
 \therefore f(x) &= \int (ax + 3) dx \\
 &= \frac{ax^2}{2} + 3x + c
 \end{aligned}$$

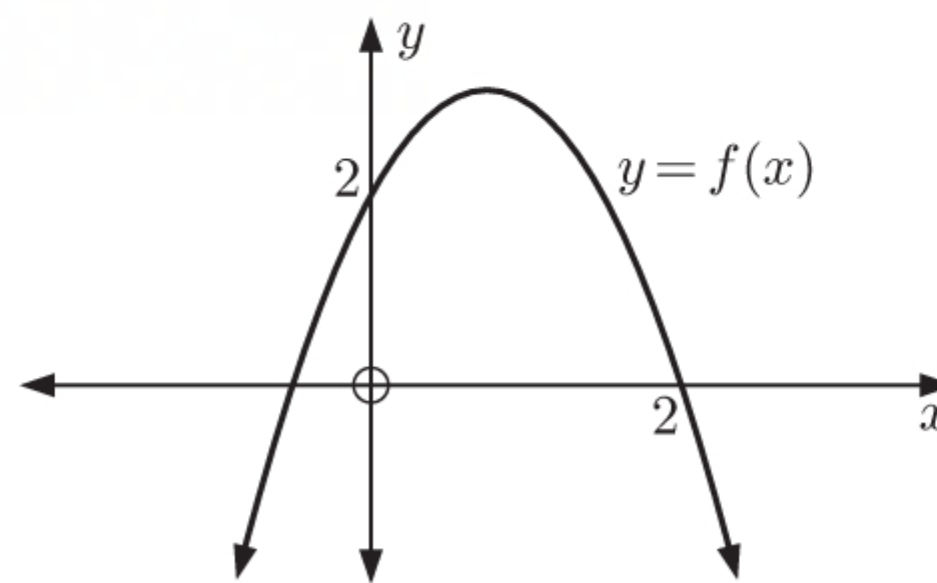
But the y -intercept is 2, so $f(0) = 2$
 $\therefore c = 2$

$$\therefore f(x) = \frac{ax^2}{2} + 3x + 2$$

and the x -intercept is 2, so $f(2) = 0$

$$\begin{aligned}
 \therefore \frac{a(2)^2}{2} + 3(2) + 2 &= 0 \\
 \therefore 2a + 6 + 2 &= 0 \\
 \therefore 2a &= -8 \\
 \therefore a &= -4
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{the equation of the curve is } y = f(x) &= \frac{(-4)x^2}{2} + 3x + 2 \\
 &= -2x^2 + 3x + 2
 \end{aligned}$$



$$\begin{aligned}
 10 \quad a \quad \int_{-2}^0 (1 - 3x) \, dx &= \left[x - \frac{3}{2}x^2 \right]_{-2}^0 \\
 &= 0 - \left(-2 - \frac{3}{2}(4) \right) \\
 &= 0 - (-8) \\
 &= 8
 \end{aligned}$$

$$\begin{aligned}
 b \quad \int_0^{\frac{1}{2}} (x^2 - x) \, dx &= \left[\frac{x^3}{3} - \frac{x^2}{2} \right]_0^{\frac{1}{2}} \\
 &= \frac{\frac{1}{8}}{3} - \frac{\frac{1}{4}}{2} - 0 \\
 &= \frac{1}{24} - \frac{1}{8} \\
 &= -\frac{1}{12}
 \end{aligned}$$

$$\begin{aligned}
 c \quad \int_1^2 (x^2 + 1)^2 \, dx \\
 &= \int_1^2 (x^4 + 2x^2 + 1) \, dx \\
 &= \left[\frac{1}{5}x^5 + \frac{2}{3}x^3 + x \right]_1^2 \\
 &= \left(\frac{32}{5} + \frac{16}{3} + 2 \right) - \left(\frac{1}{5} + \frac{2}{3} + 1 \right) \\
 &= 11\frac{13}{15}
 \end{aligned}$$

$$\begin{aligned}
 11 \quad a \quad \int_0^b (x - b)^2 \, dx &= 9 \\
 \therefore \int_0^b (x^2 - 2bx + b^2) \, dx &= 9 \\
 \therefore \left[\frac{1}{3}x^3 - bx^2 + b^2x \right]_0^b &= 9 \\
 \therefore \left(\frac{1}{3}b^3 - \cancel{b^3} + \cancel{b^3} \right) - 0 &= 9 \\
 \therefore \frac{1}{3}b^3 &= 9 \\
 \therefore b^3 &= 27 \\
 \therefore b &= 3
 \end{aligned}$$

$$\begin{aligned}
 b \quad \int_0^b \left(x^2 + \frac{1}{2}x \right) \, dx &= 3 \\
 \therefore \left[\frac{1}{3}x^3 + \frac{1}{4}x^2 \right]_0^b &= 3 \\
 \therefore \left(\frac{1}{3}b^3 + \frac{1}{4}b^2 \right) - 0 &= 3 \\
 \therefore \frac{1}{3}b^3 + \frac{1}{4}b^2 &= 3 \\
 \therefore \frac{1}{3}b^3 + \frac{1}{4}b^2 - 3 &= 0 \\
 \therefore b &\approx 1.86 \\
 &\text{\{using technology\}}
 \end{aligned}$$

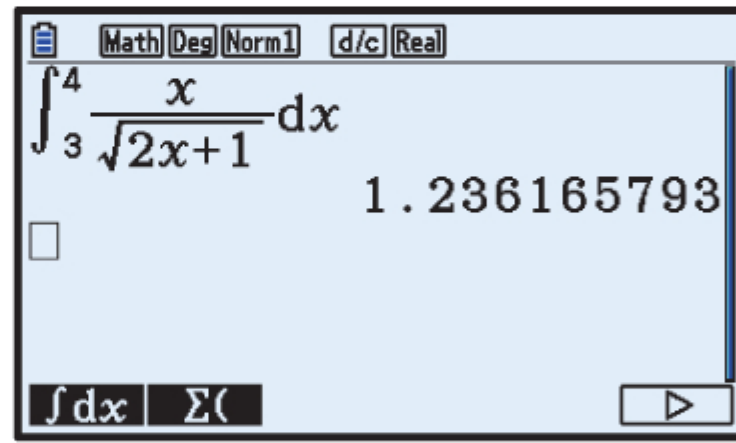
Math Deg Norm1 d/c Real
 $aX^3 + bX^2 + cX + d = 0$

a	b	c	d
0.3333	0.25	0	-3

 SOLVE DELETE CLEAR EDIT

Math Deg Norm1 d/c Real
 $aX^3 + bX^2 + cX + d = 0$
 X1 1.8577
 1.857753178
 REPEAT

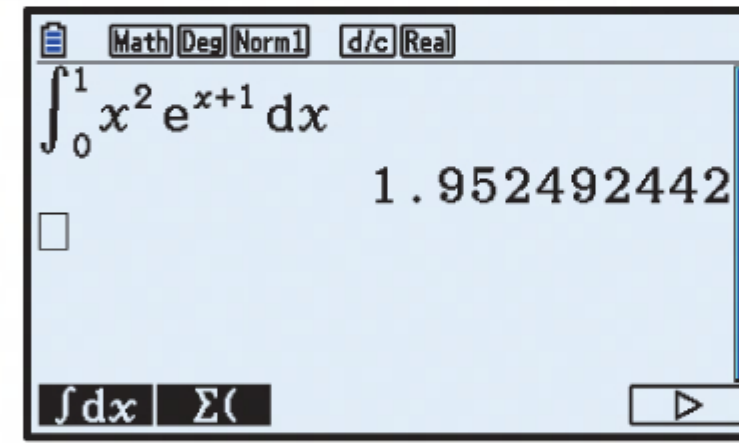
12 a



Using technology,

$$\int_3^4 \frac{x}{\sqrt{2x+1}} dx \approx 1.23617$$

b

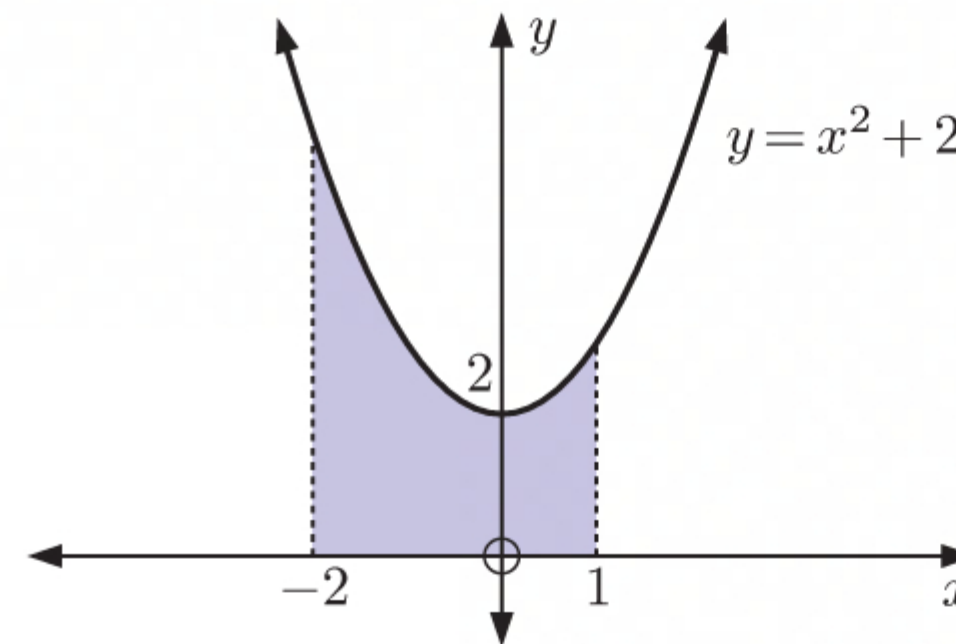


Using technology,

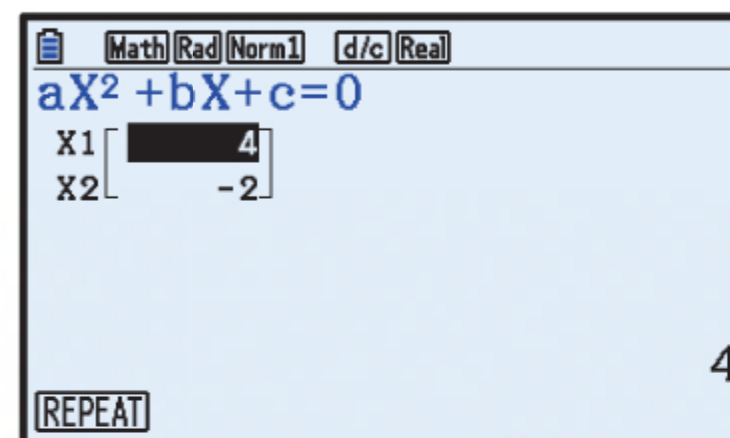
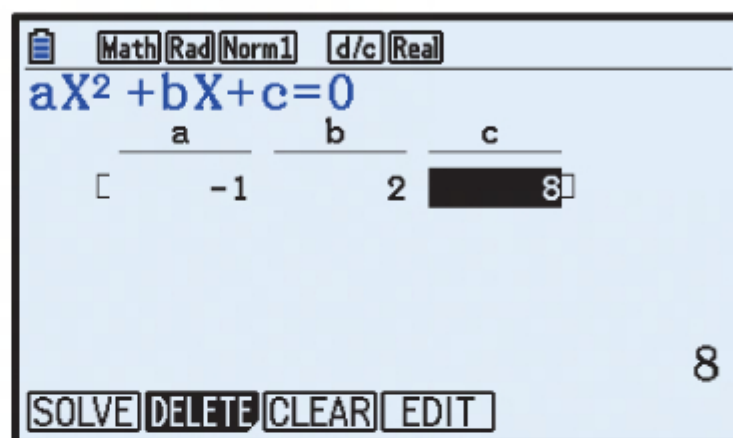
$$\int_0^1 x^2 e^{x+1} dx \approx 1.95249$$

13 a

$$\begin{aligned} \text{Area} &= \int_{-2}^1 (x^2 + 2) dx \\ &= \left[\frac{1}{3}x^3 + 2x \right]_{-2}^1 \\ &= \left(\frac{1}{3} + 2 \right) - \left(-\frac{8}{3} - 4 \right) \\ &= 9 \text{ units}^2 \end{aligned}$$

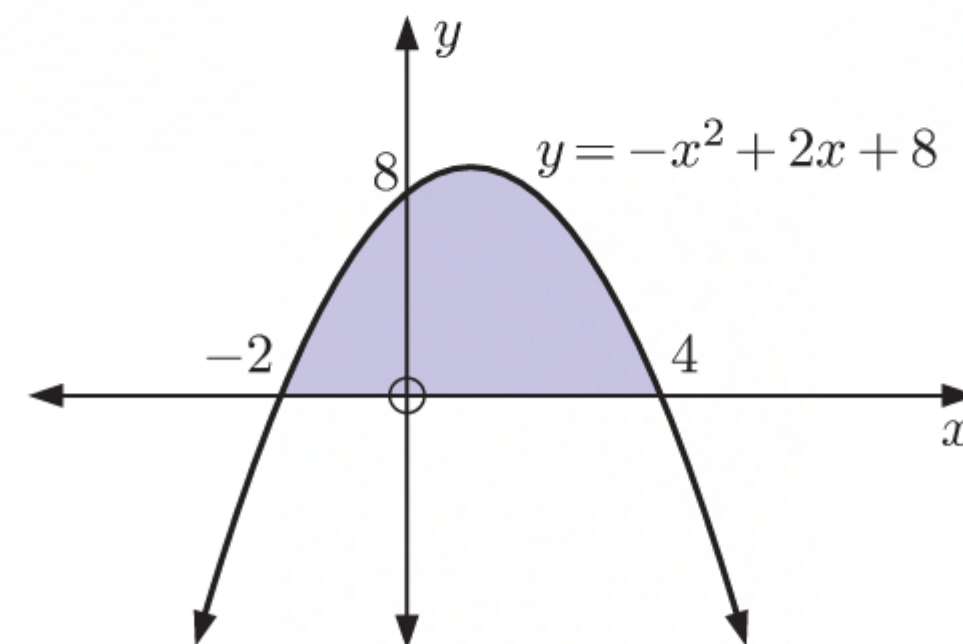


b The curve cuts the x -axis when $y = 0$
 $\therefore -x^2 + 2x + 8 = 0$



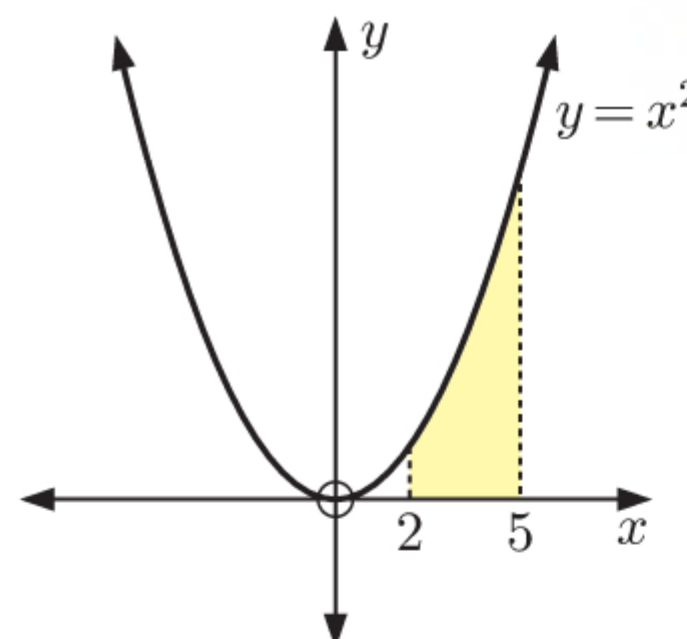
Using technology, $x = -2$ or 4
 \therefore the x -intercepts are -2 and 4 .

$$\begin{aligned} \therefore \text{area} &= \int_{-2}^4 (-x^2 + 2x + 8) dx \\ &= \left[-\frac{1}{3}x^3 + x^2 + 8x \right]_{-2}^4 \\ &= \left(-\frac{64}{3} + 16 + 32 \right) - \left(\frac{8}{3} + 4 - 16 \right) \\ &= \frac{80}{3} - \left(-\frac{28}{3} \right) \\ &= 36 \text{ units}^2 \end{aligned}$$



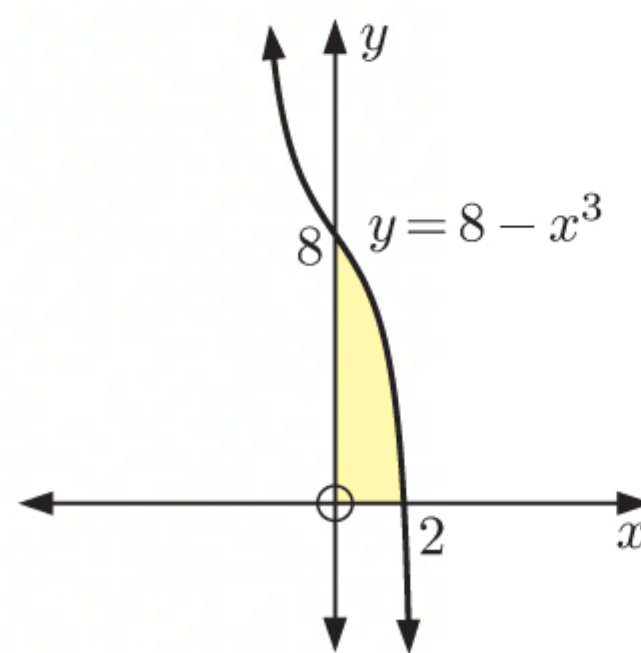
14 a

$$\begin{aligned} \text{Area} &= \int_2^5 x^2 dx \\ &= \left[\frac{1}{3}x^3 \right]_2^5 \\ &= \frac{125}{3} - \frac{8}{3} \\ &= 39 \text{ units}^2 \end{aligned}$$



b When $y = 0$, $8 - x^3 = 0$
 $\therefore x^3 = 8$
 $\therefore x = 2$

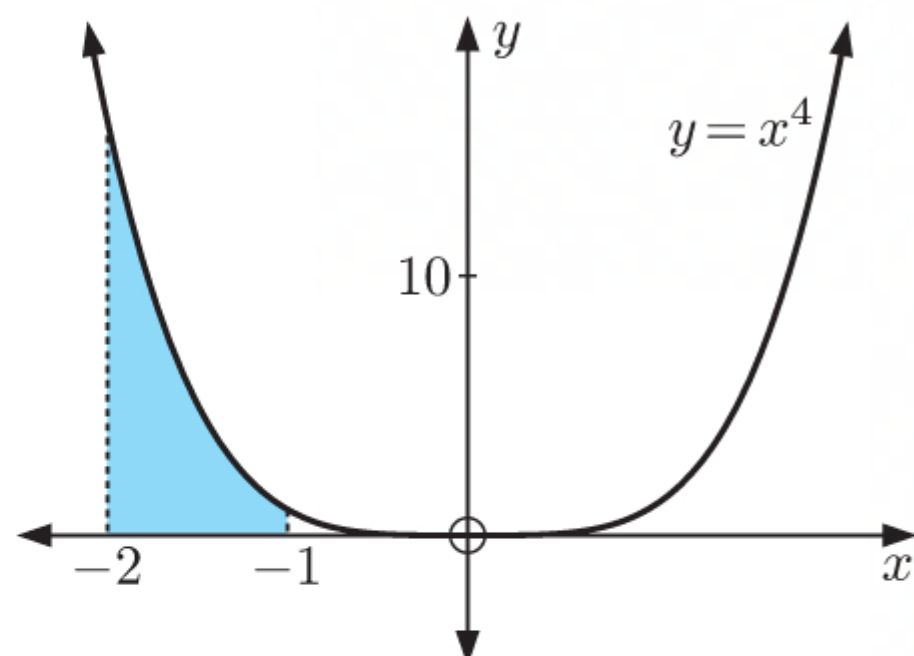
$$\begin{aligned}\therefore \text{area} &= \int_0^2 (8 - x^3) dx \\ &= \left[8x - \frac{1}{4}x^4 \right]_0^2 \\ &= 16 - \frac{16}{4} - 0 \\ &= 12 \text{ units}^2\end{aligned}$$



15 a $n = 8$, $a = -2$, $b = -1$, $f(x) = x^4$

$$h = \frac{b - a}{n} = \frac{1}{8}$$

$$x_i = -2 + \frac{1}{8}i$$



i	x_i	$f(x_i)$
0	-2	16
1	$-1\frac{7}{8}$	12.359 62
2	$-1\frac{3}{4}$	9.378 91
3	$-1\frac{5}{8}$	6.972 90
4	$-1\frac{1}{2}$	5.0625
5	$-1\frac{3}{8}$	3.574 46
6	$-1\frac{1}{4}$	2.441 41
7	$-1\frac{1}{8}$	1.601 81
8	-1	1

Using the trapezoidal rule, the area $\approx \frac{h}{2} (f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_7) + f(x_8))$
 $\approx 6.2365 \text{ units}^2$

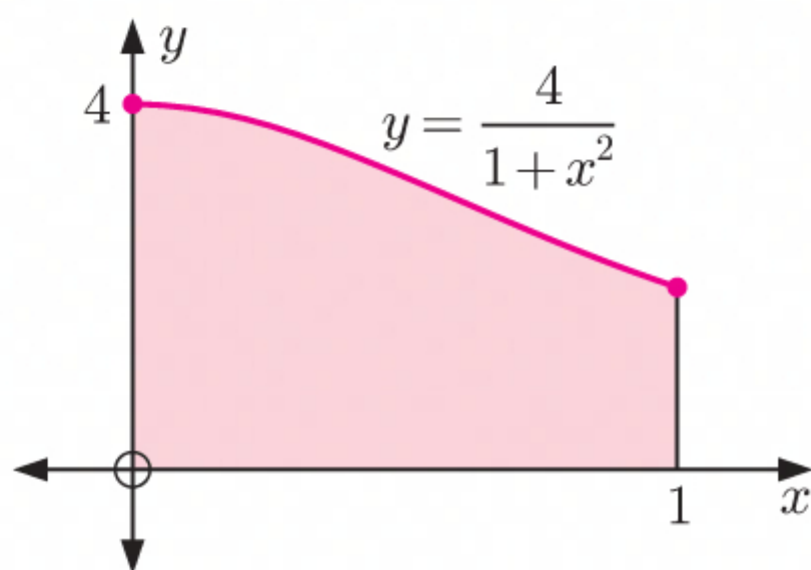
$$\therefore \int_{-2}^{-1} x^4 dx \approx 6.2365$$

b The estimate in **a** is an over estimate. Joining the interval endpoints with straight line segments would give a larger area than the shaded area.

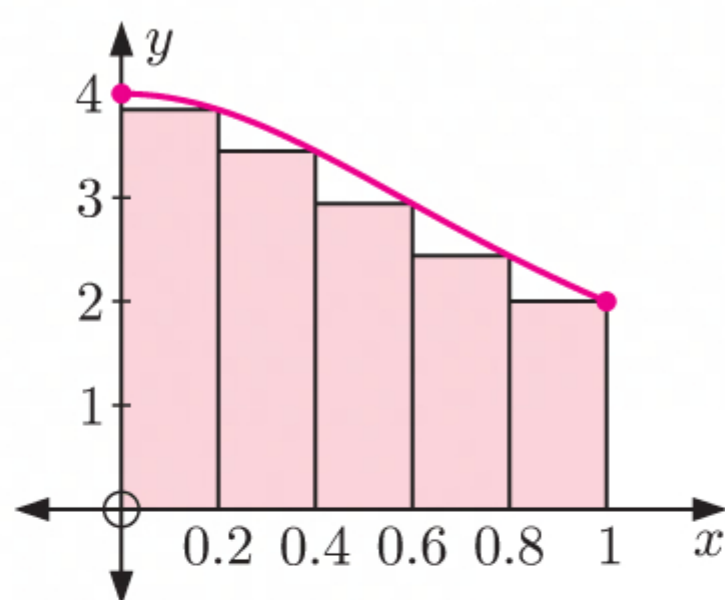
c
$$\begin{aligned}\int_{-2}^{-1} x^4 dx &= \left[\frac{1}{5}x^5 \right]_{-2}^{-1} \\ &= \frac{1}{5}(-1)^5 - \frac{1}{5}(-2)^5 \\ &= -\frac{1}{5} + \frac{32}{5} \\ &= \frac{31}{5} \\ &= 6.2 < 6.2365 \quad \checkmark\end{aligned}$$

REVIEW SET 13B

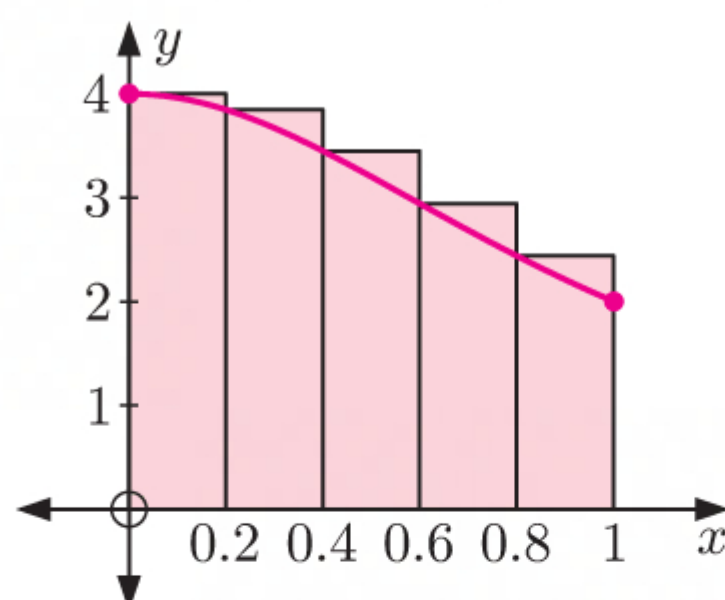
1 a



lower rectangles



upper rectangles



b

n	A_L	A_U
5	2.9349	3.3349
50	3.1215	3.1615
100	3.1316	3.1516
500	3.1396	3.1436

c Using $n = 500$ rectangles,

$$\begin{aligned} \int_0^1 \frac{4}{1+x^2} dx &\approx \frac{A_L + A_U}{2} \\ &\approx \frac{3.1396 + 3.1436}{2} \\ &\approx 3.1416 \end{aligned}$$

d $n = 10$, $a = 0$, $b = 1$, $f(x) = \frac{4}{1+x^2}$

$$h = \frac{b-a}{n} = 0.1$$

$$x_i = 0 + 0.1i$$

i	x_i	$f(x_i)$
0	0	4
1	0.1	3.9604
2	0.2	3.8462
3	0.3	3.6697
4	0.4	3.4483
5	0.5	3.2
6	0.6	2.9412
7	0.7	2.6846
8	0.8	2.4390
9	0.9	2.2099
10	1	2

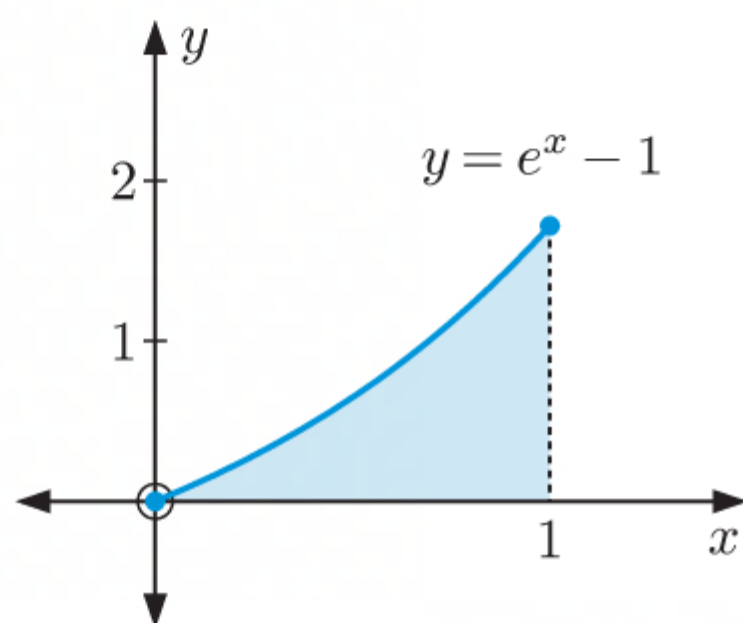
Using the trapezoidal rule, the area $\approx \frac{h}{2} (f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_9) + f(x_{10}))$
 $\approx 3.1399 \text{ units}^2$

$$\therefore \int_0^1 \frac{4}{1+x^2} dx \approx 3.1399$$

e $\int_0^1 \frac{4}{1+x^2} dx = \pi \approx 3.1416$

Both answers in **c** and **d** are good approximations of the integral, but the trapezoidal method required fewer subintervals and is hence more efficient.

2 a



b $n = 10, a = 0, b = 1, f(x) = e^x - 1$

$$h = \frac{b-a}{n} = 0.1$$

$$x_i = a + hi = 0.1i$$

i $A_L = h(f(x_0) + f(x_1) + \dots + f(x_9))$
 $\approx 0.6338 \text{ units}^2$

$$\therefore \int_0^1 (e^x - 1) dx \approx 0.6338$$

ii $A_U = h(f(x_1) + f(x_2) + \dots + f(x_{10}))$
 $\approx 0.8056 \text{ units}^2$

$$\therefore \int_0^1 (e^x - 1) dx \approx 0.8056$$

iii Using the trapezoidal rule, the area

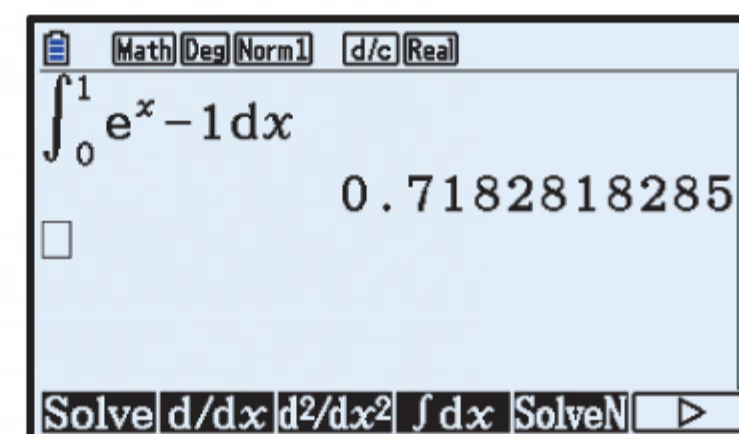
$$\approx \frac{h}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_9) + f(x_{10}))$$

$$\approx 0.7197 \text{ units}^2$$

$$\therefore \int_0^1 (e^x - 1) dx \approx 0.7197$$

c Using technology, $\int_0^1 (e^x - 1) dx \approx 0.7183.$

i	x_i	$f(x_i)$
0	0	0
1	0.1	0.105 17
2	0.2	0.221 40
3	0.3	0.349 86
4	0.4	0.491 83
5	0.5	0.648 72
6	0.6	0.822 12
7	0.7	1.013 75
8	0.8	1.225 54
9	0.9	1.459 60
10	1	1.718 28



Using the upper and lower rectangle sums in **b i** and **b ii**:

$$\int_0^1 (e^x - 1) dx \approx \frac{0.6338 + 0.8056}{2}$$

$$\approx 0.7197$$

Our approximations of the integral using the rectangle and trapezoidal methods with 10 subintervals are almost equal.

$$\begin{aligned}
 3 \quad 2x - \frac{3}{x} &= 2x - 3x^{-1} \\
 \therefore \frac{d}{dx} \left(2x - \frac{3}{x} \right) &= 2 + 3x^{-2} \\
 &= 2 + \frac{3}{x^2} \\
 &= - \left(-\frac{3}{x^2} - 2 \right) \\
 \therefore \int \left(-\frac{3}{x^2} - 2 \right) dx &= - \left(2x - \frac{3}{x} \right) + c \\
 &= \frac{3}{x} - 2x + c
 \end{aligned}$$

$$4 \quad \text{a The antiderivative of } -5x^2 \text{ is } -\frac{5x^3}{3} = -\frac{5}{3}x^3.$$

$$\text{b The antiderivative of } 6x^{-2} \text{ is } \frac{6x^{-1}}{-1} = -\frac{6}{x}.$$

$$\text{c The antiderivative of } 1 + \frac{x^2}{6} \text{ is } x + \frac{\frac{x^3}{6}}{3} = x + \frac{x^3}{18}.$$

$$\begin{aligned}
 5 \quad \text{a} \quad \int \frac{x^2 - 2}{x^2} dx \\
 &= \int \left(1 - \frac{2}{x^2} \right) dx \\
 &= \int (1 - 2x^{-2}) dx \\
 &= x - \frac{2x^{-1}}{-1} + c \\
 &= x + \frac{2}{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad \int (3x - 4)^2 dx \\
 &= \int (9x^2 - 24x + 16) dx \\
 &= \frac{9x^3}{3} - \frac{24x^2}{2} + 16x + c \\
 &= 3x^3 - 12x^2 + 16x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad \int (4 - 2x^2) dx \\
 &= 4x - \frac{2}{3}x^3 + c
 \end{aligned}$$

$$\begin{aligned}
 6 \quad \text{a} \quad \int \left(\frac{x+3}{3} \right) dx \\
 &= \int \left(\frac{1}{3}x + 1 \right) dx \\
 &= \frac{\frac{1}{3}x^2}{2} + x + c \\
 &= \frac{1}{6}x^2 + x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad \int (3x^2 - 2) dx \\
 &= \frac{3x^3}{3} - 2x + c \\
 &= x^3 - 2x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad \int (3 + 2x)^2 dx \\
 &= \int (9 + 12x + 4x^2) dx \\
 &= 9x + \frac{12x^2}{2} + \frac{4x^3}{3} + c \\
 &= 9x + 6x^2 + \frac{4}{3}x^3 + c
 \end{aligned}$$

7 $f'(x) = x^2 - 3x + 2$

$$\begin{aligned}\therefore f(x) &= \int (x^2 - 3x + 2) dx \\ &= \frac{x^3}{3} - \frac{3x^2}{2} + 2x + c \\ &= \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + c\end{aligned}$$

But $f(1) = 3$, so $\frac{1}{3} - \frac{3}{2} + 2 + c = 3$
 $\therefore c = \frac{13}{6}$

$$\therefore f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + \frac{13}{6}$$

8 The indefinite integral of a function $f(x)$ is every function that, when differentiated, gives us $f(x)$.

If $F(x)$ is the antiderivative of $f(x)$, then $\frac{d}{dx}(F(x)) = F'(x) = f(x)$.

However, $\frac{d}{dx}(F(x) + c) = F'(x) + 0 = f(x)$, so we must add a constant of integration c to the antiderivative to give us the indefinite integral.

9 a
$$\begin{aligned}\int_2^4 \frac{4}{x^2} dx &= \int_2^4 4x^{-2} dx \\ &= \left[\frac{4x^{-1}}{-1} \right]_2^4 \\ &= \left[-\frac{4}{x} \right]_2^4 \\ &= -1 - (-2) \\ &= 1\end{aligned}$$

b
$$\begin{aligned}\int_1^4 \left(x - \frac{1}{2}x^2\right) dx &= \left[\frac{1}{2}x^2 - \frac{1}{6}x^3\right]_1^4 \\ &= \left(\frac{1}{2}(16) - \frac{1}{6}(64)\right) - \left(\frac{1}{2} - \frac{1}{6}\right) \\ &= -\frac{8}{3} - \frac{1}{3} \\ &= -3\end{aligned}$$

c
$$\begin{aligned}\int_0^1 \left(x^2 + \frac{1}{3}\right)^2 dx &= \int_0^1 \left(x^4 + \frac{2}{3}x^2 + \frac{1}{9}\right) dx \\ &= \left[\frac{1}{5}x^5 + \frac{2}{9}x^3 + \frac{1}{9}x\right]_0^1 \\ &= \left(\frac{1}{5} + \frac{2}{9} + \frac{1}{9}\right) - 0 \\ &= \frac{8}{15}\end{aligned}$$

10
$$\begin{aligned}\int_0^a \left(x^2 - \frac{1}{2}x\right) dx &= \frac{9}{16} \\ \therefore \left[\frac{1}{3}x^3 - \frac{1}{4}x^2\right]_0^a &= \frac{9}{16} \\ \therefore \frac{1}{3}a^3 - \frac{1}{4}a^2 &= \frac{9}{16} \\ \therefore 16a^3 - 12a^2 &= 27 \\ \therefore 16a^3 - 12a^2 - 27 &= 0 \\ \therefore a &= \frac{3}{2} \\ \text{\{using technology\}}\end{aligned}$$

Math Rad Norm1 d/c Real
 $aX^3 + bX^2 + cX + d = 0$

a	b	c	d
16	-12	0	-27

 SOLVE DELETE CLEAR EDIT

Math Rad Norm1 d/c Real
 $aX^3 + bX^2 + cX + d = 0$
 X1 $\frac{3}{2}$
 REPEAT

11 a

A calculator screen showing the integral $\int_{-2}^0 4e^{-x^2} dx$ and the result 3.528325563. The screen also shows the mode settings: Math, Rad, Norm1, d/c, Real. At the bottom, there are buttons for $\int dx$, $\Sigma($, and a right arrow.

Using technology,

$$\int_{-2}^0 4e^{-x^2} dx \approx 3.528$$

b

A calculator screen showing the integral $\int_0^1 \frac{10x}{\sqrt{3x+1}} dx$ and the result 2.962962963. The screen also shows the mode settings: Math, Rad, Norm1, d/c, Real. At the bottom, there are buttons for $\int dx$, $\Sigma($, and a right arrow.

Using technology,

$$\int_0^1 \frac{10x}{\sqrt{3x+1}} dx \approx 2.963$$

12 $\int_1^4 f(x) dx = 3$

a

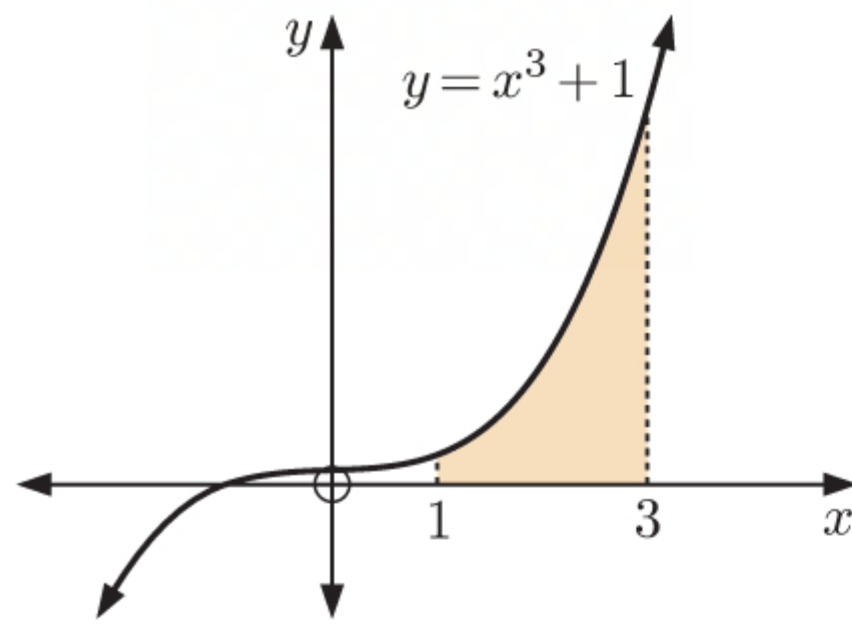
$$\begin{aligned} & \int_1^4 (f(x) + 1) dx \\ &= \int_1^4 f(x) dx + \int_1^4 1 dx \\ &= 3 + [x]_1^4 \\ &= 3 + (4 - 1) \\ &= 3 + 3 \\ &= 6 \end{aligned}$$

c

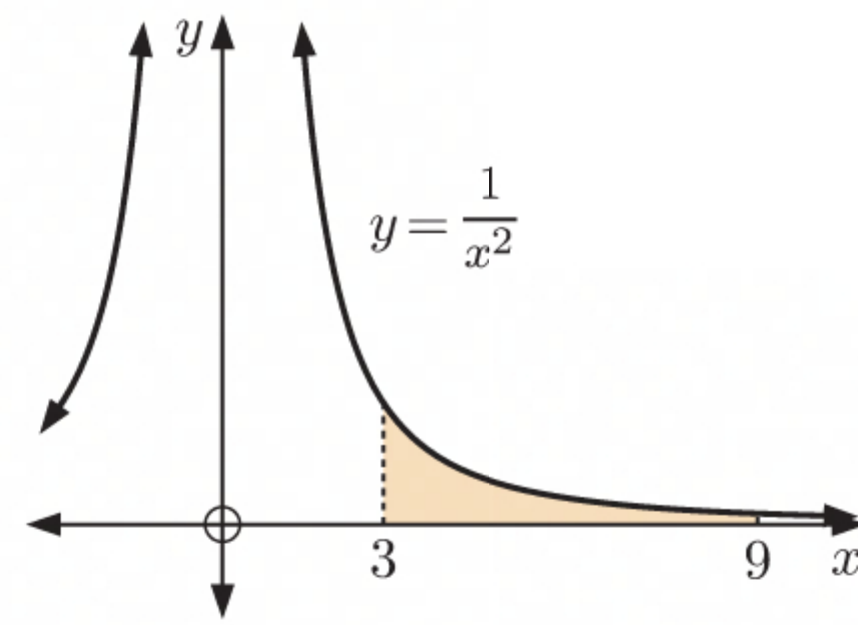
$$\begin{aligned} & \int_4^1 k f(x) dx = 5 \\ \therefore -k \int_1^4 f(x) dx &= 5 \\ \therefore -3k &= 5 \\ \therefore k &= -\frac{5}{3} \end{aligned}$$

b

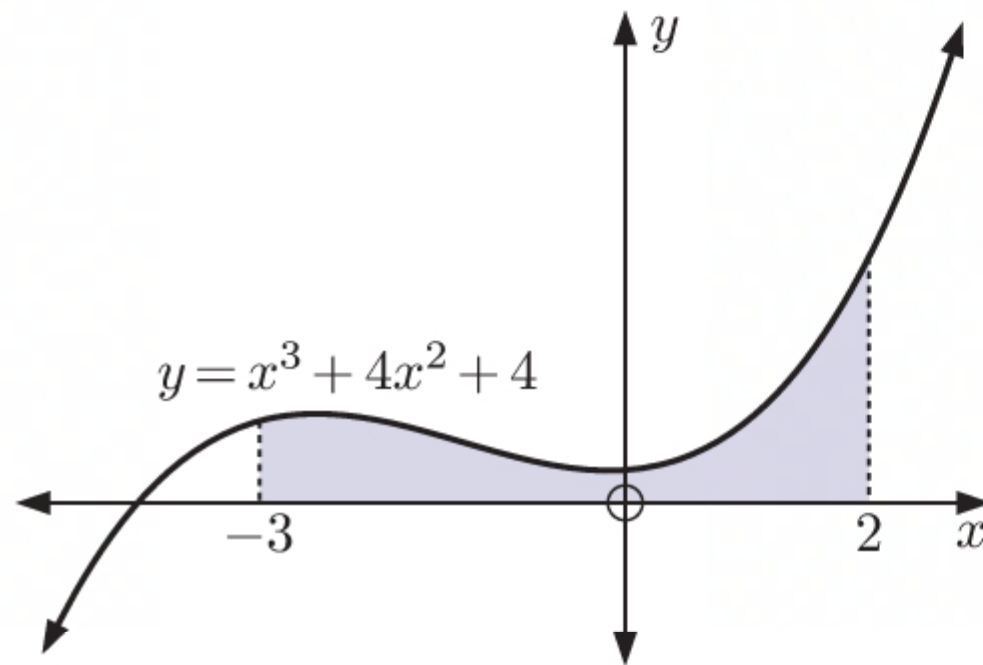
$$\begin{aligned} & \int_1^2 f(x) dx - \int_4^2 f(x) dx \\ &= \int_1^2 f(x) dx + \int_2^4 f(x) dx \\ &= \int_1^4 f(x) dx \\ &= 3 \end{aligned}$$

13 a

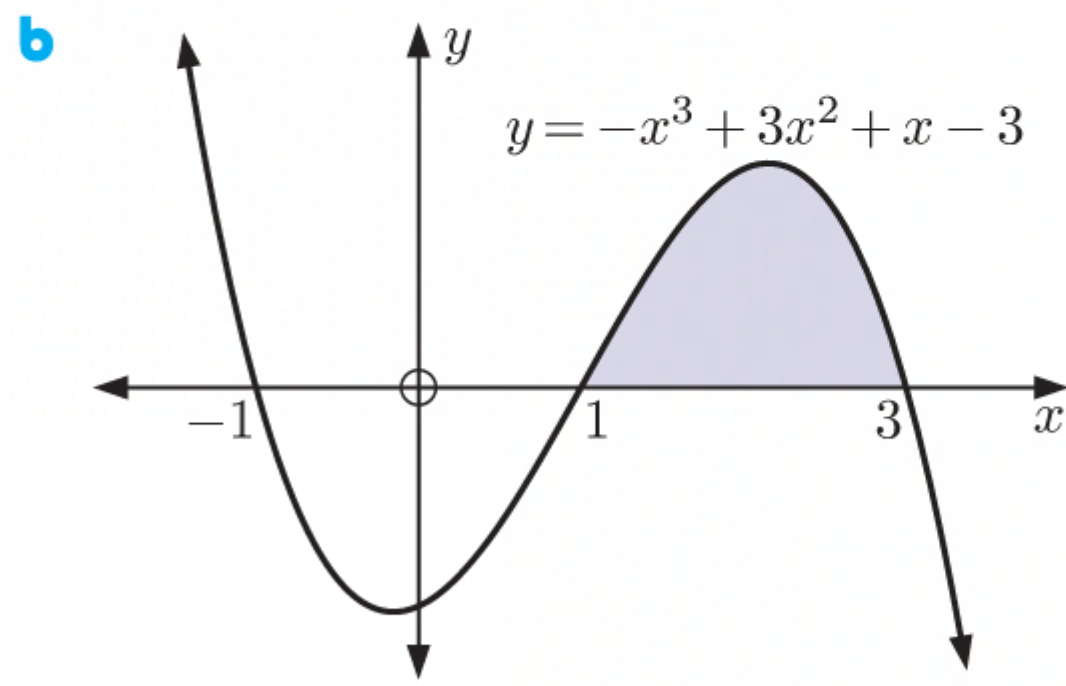
$$\begin{aligned}
 \text{Area} &= \int_1^3 (x^3 + 1) \, dx \\
 &= \left[\frac{1}{4}x^4 + x \right]_1^3 \\
 &= \left(\frac{81}{4} + 3 \right) - \left(\frac{1}{4} + 1 \right) \\
 &= 22 \text{ units}^2
 \end{aligned}$$

b

$$\begin{aligned}
 \text{Area} &= \int_3^9 \frac{1}{x^2} \, dx \\
 &= \int_3^9 x^{-2} \, dx \\
 &= \left[\frac{x^{-1}}{-1} \right]_3^9 \\
 &= \left[-\frac{1}{x} \right]_3^9 \\
 &= -\frac{1}{9} - \left(-\frac{1}{3} \right) \\
 &= \frac{2}{9} \text{ units}^2
 \end{aligned}$$

14 a

$$\begin{aligned}
 \text{Area} &= \int_{-3}^2 (x^3 + 4x^2 + 4) \, dx \\
 &= \left[\frac{1}{4}x^4 + \frac{4}{3}x^3 + 4x \right]_{-3}^2 \\
 &= \left(\frac{1}{4}(16) + \frac{4}{3}(8) + 4(2) \right) - \left(\frac{1}{4}(81) + \frac{4}{3}(-27) + 4(-3) \right) \\
 &= \frac{605}{12} \\
 &= 50\frac{5}{12} \text{ units}^2
 \end{aligned}$$



$$\begin{aligned}
 \text{Area} &= \int_1^3 (-x^3 + 3x^2 + x - 3) \, dx \\
 &= \left[-\frac{1}{4}x^4 + x^3 + \frac{1}{2}x^2 - 3x \right]_1^3 \\
 &= \left(-\frac{1}{4}(81) + 27 + \frac{1}{2}(9) - 9 \right) - \left(-\frac{1}{4} + 1 + \frac{1}{2} - 3 \right) \\
 &= 4 \text{ units}^2
 \end{aligned}$$

- 15** B has coordinates $(2, 2^2 + k)$, which is $(2, 4 + k)$, so the rectangle OABC has area $2(4 + k)$ units².

If the upper area U and the lower area L are equal, and the total area $U + L = 2(4 + k)$,

$$\text{then } 2L = 2(4 + k)$$

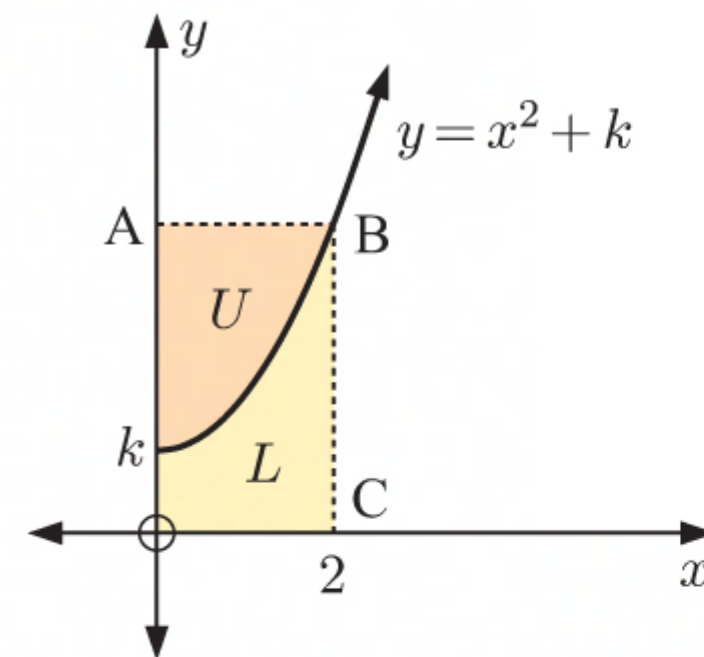
$$\therefore L = 4 + k$$

$$\therefore \int_0^2 (x^2 + k) \, dx = 4 + k$$

$$\therefore \left[\frac{1}{3}x^3 + kx \right]_0^2 = 4 + k$$

$$\therefore \frac{1}{3}(8) + 2k - 0 = 4 + k$$

$$\therefore k = \frac{4}{3}$$



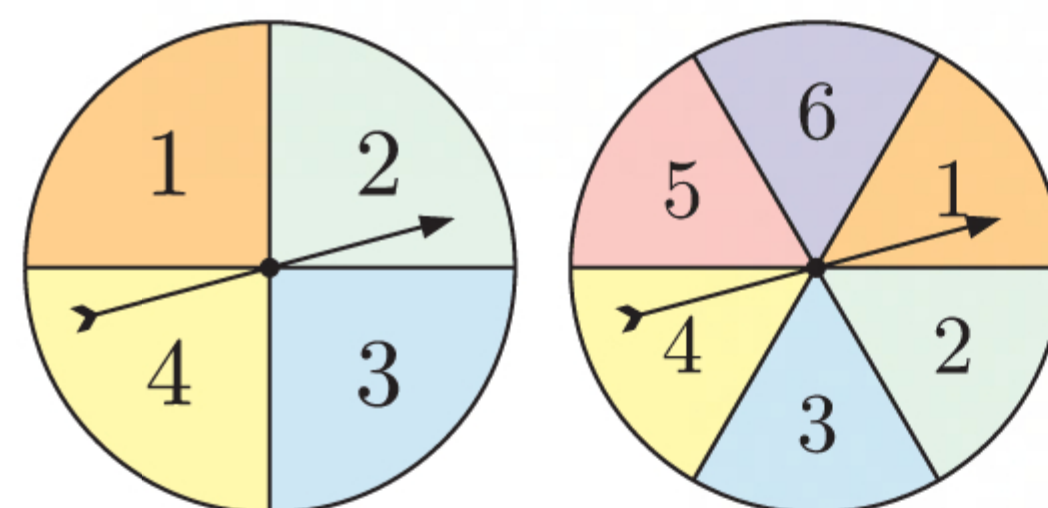
DISCRETE RANDOM VARIABLES

EXERCISE 14A

- 1**
 - a** The quantity of fat in a sausage is a continuous random variable.
 - b** The mark out of 50 for a geography test is a discrete random variable.
 - c** The weight of a Year 12 student is a continuous random variable.
 - d** The volume of water in a cup of coffee is a continuous random variable.
 - e** The number of trout in a lake is a discrete random variable.
 - f** The number of the hairs on a cat is a discrete random variable.
 - g** The length of a horse's mane is a continuous random variable.
 - h** The height of a skyscraper is a continuous random variable.
- 2**
 - a**
 - i** The random variable X is the height of water in the rain gauge.
 - ii** The variable is a continuous random variable. **iii** $0 \leq X \leq 400$ mm
 - b**
 - i** The random variable X is the stopping distance.
 - ii** The variable is a continuous random variable. **iii** $0 \leq X \leq 50$ m
 - c**
 - i** The random variable X is the number of times that the switch is turned off and on before it fails.
 - ii** The variable is a discrete random variable. **iii** X can be any integer ≥ 1

- 3 a** X is the sum of a number from one spinner and a number from the other spinner. So X is a discrete random variable because X has a set of distinct possible values.

b $X = 2, 3, 4, 5, 6, 7, 8, 9, \text{ or } 10$



- 4 a** The two teams play against each other until one team wins 4 games (best out of 7).
 $\therefore X = 4, 5, 6, \text{ or } 7$

- b** **i** $X = 5$ **ii** $X = 6$ or 7

- 5 a** There are four weighing devices and X is the number which are accurate.
 $\therefore X = 0, 1, 2, 3, \text{ or } 4$

- [illegible]

- c** **i** If exactly two devices are accurate, then $X = 2$.
ii If at least two devices are accurate, then 2, 3, or 4 are accurate $\therefore X = 2, 3, \text{ or } 4$.

6 a If 3 coins are tossed then the number of heads X can be 0, 1, 2, or 3.

b Let H represent heads, and T represent tails.

HHH	HHT	TTH	TTT
↓	HTH	THT	↓
	THH	HTT	
$(X = 3)$	$(X = 2)$	$(X = 1)$	$(X = 0)$

c $P(X = 0) = \frac{1}{8}, \quad P(X = 1) = \frac{3}{8}, \quad P(X = 2) = \frac{3}{8}, \quad P(X = 3) = \frac{1}{8}$

Since $P(X = 2) \neq P(X = 3)$, the possible values of X are not equally likely to occur.

EXERCISE 14B

1 a i

x	1	2	3	4
$P(X = x)$	0.2	0.4	0.15	0.25

$$\sum_{x=1}^4 P(X = x) = 0.2 + 0.4 + 0.15 + 0.25 = 1$$

Since $\sum_{x=1}^4 P(X = x) = 1$ and $0 \leq P(X = x) \leq 1$ for all x , it is a valid probability distribution.

ii

x	0	1	2	3
$P(X = x)$	0.2	0.3	0.4	0.2

$$\sum_{x=0}^3 P(X = x) = 0.2 + 0.3 + 0.4 + 0.2 = 1.1$$

Since $\sum_{x=0}^3 P(X = x) > 1$, it is not a valid probability distribution.

iii

x	0	1	2	3	4
$P(X = x)$	0.2	0.2	0.2	0.2	0.2

$$\sum_{x=0}^4 P(X = x) = 0.2 + 0.2 + 0.2 + 0.2 + 0.2 = 1$$

Since $\sum_{x=0}^4 P(X = x) = 1$ and $0 \leq P(X = x) \leq 1$ for all x , it is a valid probability distribution.

iv

x	2	3	4	5
$P(X = x)$	0.3	0.4	0.5	-0.2

Since $P(X = 5) = -0.2 < 0$, it is not a valid probability distribution.

- b** X is a uniform random variable for the probability distribution in **a iii**, since $p_i = 0.2$ for each value of i .

2 a

x	0	1	2
$P(X = x)$	0.3	k	0.5

$$\sum_{x=0}^2 P(X = x) = 1$$

$$\therefore 0.3 + k + 0.5 = 1$$

$$\therefore k = 0.2$$

b

x	0	1	2	3
$P(X = x)$	k	$2k$	$3k$	k

$$\sum_{x=0}^3 P(X = x) = 1$$

$$\therefore k + 2k + 3k + k = 1$$

$$\therefore 7k = 1$$

$$\therefore k = \frac{1}{7}$$

3 a

$$\sum_{x=0}^3 P(X = x) = 1$$

$$\therefore 0.1 + 0.25 + 0.45 + a = 1$$

$$\therefore a = 0.2$$

x	0	1	2	3
$P(X = x)$	0.1	0.25	0.45	a

b Since $P(X = 0) \neq P(X = 1)$, the probabilities of each outcome are not all equal, so X is not a uniform discrete random variable.

c Since $P(X = 2)$ is the greatest probability, 2 is the mode of the distribution.

d $P(X \geq 2) = P(X = 2) + P(X = 3)$
 $= 0.45 + 0.2$
 $= 0.65$

4

x	0	1	2	3	4	5
$P(x)$	a	0.3333	0.1088	0.0084	0.0007	0.0000

a From the table, $P(2) = 0.1088$.

b Since this is a probability distribution, $\sum_{i=1}^n P(x_i) = 1$

$$\therefore a + 0.3333 + 0.1088 + 0.0084 + 0.0007 + 0.0000 = 1$$

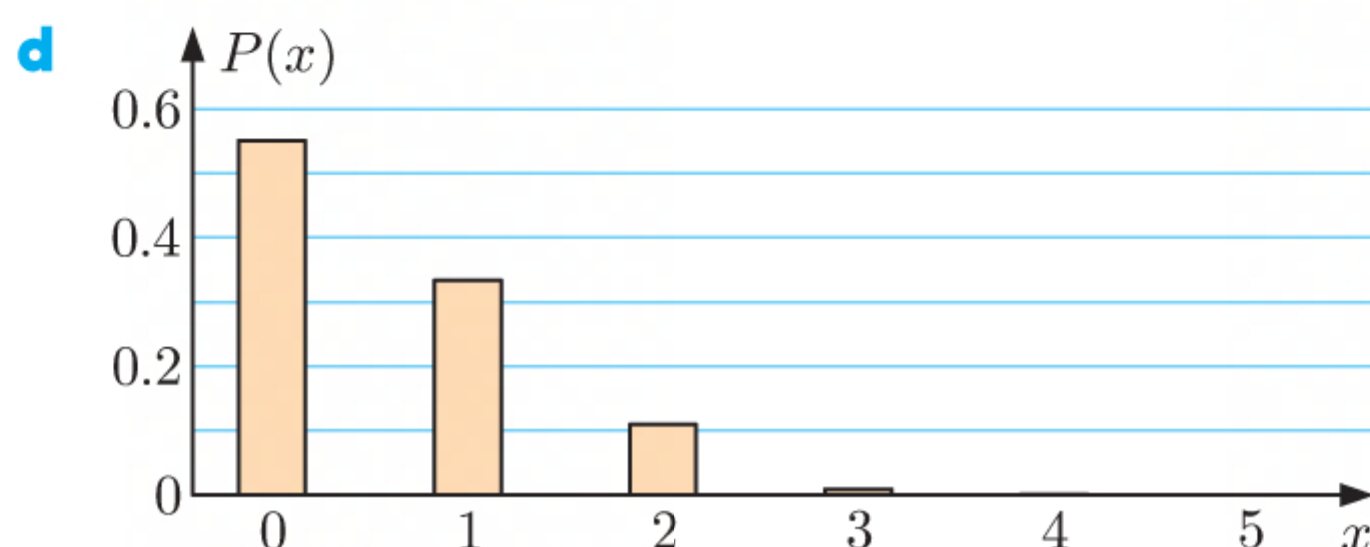
$$\therefore a + 0.4512 = 1$$

$$\therefore a = 0.5488$$

This is the probability that Jason does not hit a home run in a game.

c $P(1) + P(2) + P(3) + P(4) + P(5) = 0.3333 + 0.1088 + 0.0084 + 0.0007 + 0.0000$
 $= 0.4512$

This is the probability that Jason will hit one or more home runs in a game.



e Jason is most likely to score 0 home runs, so this is the mode of the distribution. Using **b**, $P(0) = 0.5488 \geq 0.5$, so the median is 0 home runs.

5	x	0	1	2	3	4
	$P(X = x)$	0.68	0.2	0.06	k	0.02

a
$$\sum_{x=0}^4 P(X = x) = 1$$

$$\therefore 0.68 + 0.2 + 0.06 + k + 0.02 = 1$$

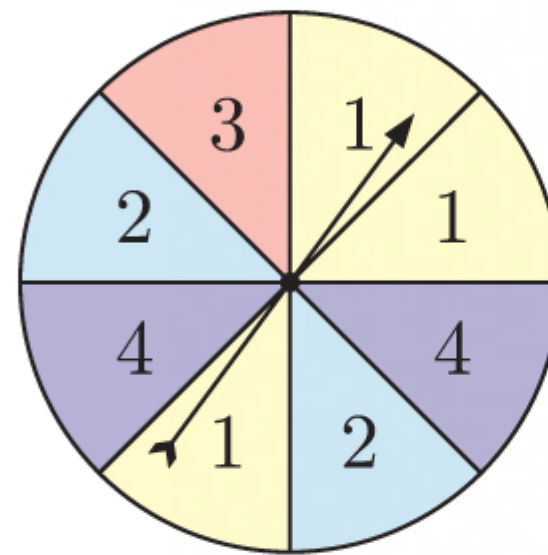
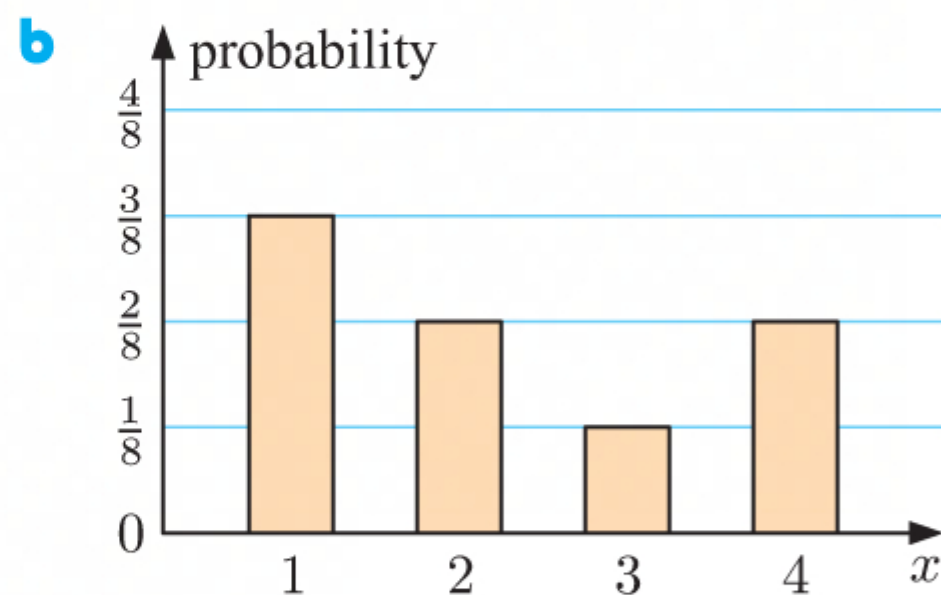
$$\therefore k = 0.04$$

b It is most likely that the number of tyres which needed replacing is 0, so the mode of the distribution is 0 tyres.

c
$$\begin{aligned} P(X > 1) &= P(X = 2) + P(X = 3) + P(X = 4) \\ &= 0.06 + 0.04 + 0.02 \\ &= 0.12 \end{aligned}$$

This is the probability that more than 1 tyre will need replacing on a car being inspected.

6	a	x	1	2	3	4
		$P(X = x)$	$\frac{3}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{2}{8}$



c The spinner is most likely to land on 1, so this is the mode of the distribution.

$$p_1 = \frac{3}{8} = 0.375$$

$$p_1 + p_2 = \frac{3}{8} + \frac{2}{8} = 0.625$$

Since $p_1 + p_2 \geq 0.5$, the median is 2.

d
$$\begin{aligned} P(X \leq 3) &= P(X = 1) + P(X = 2) + P(X = 3) \\ &= \frac{3}{8} + \frac{2}{8} + \frac{1}{8} \\ &= \frac{6}{8} \\ &= \frac{3}{4} \end{aligned}$$

7 **a** $X = 1, 2, 3$, or 4

b $P(X = 1) = \frac{24}{100} = 0.24$

$$P(X = 2) = \frac{35}{100} = 0.35$$

$$P(X = 3) = \frac{27}{100} = 0.27$$

$$P(X = 4) = \frac{14}{100} = 0.14$$

\therefore the probability table for X is

x	1	2	3	4
$P(X = x)$	0.24	0.35	0.27	0.14

- c** It is most likely for a randomly selected person to have 2 bedrooms in their house, so this is the mode of the distribution.

$$p_1 = 0.24$$

$$p_1 + p_2 = 0.24 + 0.35 = 0.59$$

Since $p_1 + p_2 \geq 0.5$, the median is 2 bedrooms.

- 8 a** $X = 1, 2, 3$, or 4

b $P(X = 1) = \frac{12}{25} = 0.48$

$$P(X = 2) = \frac{7}{25} = 0.28$$

$$P(X = 3) = \frac{2}{25} = 0.08$$

$$P(X = 4) = \frac{25 - (12 + 7 + 2)}{25} = 0.16$$

\therefore the probability table for X is

x	1	2	3	4
$P(X = x)$	0.48	0.28	0.08	0.16

- c** It is most likely for a randomly selected player to only need 1 shot to score a goal, so this is the mode of the distribution.

$$p_1 = 0.48$$

$$p_1 + p_2 = 0.48 + 0.28 = 0.76$$

Since $p_1 + p_2 \geq 0.5$, the median is 2 shots.

- 9 a** $P(x) = \frac{x+1}{10}$, $x = 0, 1, 2, 3$

$$\therefore P(0) = \frac{1}{10}, \quad P(1) = \frac{2}{10}, \quad P(2) = \frac{3}{10}, \quad P(3) = \frac{4}{10}$$

$$0 \leq P(x_i) \leq 1 \text{ in each case, and } \sum_{i=1}^n P(x_i) = \frac{1}{10} + \frac{2}{10} + \frac{3}{10} + \frac{4}{10} = 1$$

$\therefore P(x)$ is a valid probability mass function.

b $P(x) = \frac{6}{11x}$, $x = 1, 2, 3$

$$\therefore P(1) = \frac{6}{11}, \quad P(2) = \frac{6}{22} = \frac{3}{11}, \quad P(3) = \frac{6}{33} = \frac{2}{11}$$

$$0 \leq P(x_i) \leq 1 \text{ in each case, and } \sum_{i=1}^n P(x_i) = \frac{6}{11} + \frac{3}{11} + \frac{2}{11} = 1$$

$\therefore P(x)$ is a valid probability mass function.

- 10 a** $P(x) = k(x+2)$, $x = 1, 2, 3$

$$\therefore P(1) = 3k, \quad P(2) = 4k, \quad P(3) = 5k$$

$$\text{Since this is a probability distribution, } \sum_{i=1}^n P(x_i) = 1$$

$$\therefore 3k + 4k + 5k = 1$$

$$\therefore 12k = 1$$

$$\therefore k = \frac{1}{12}$$

b $P(x) = \frac{k}{x+1}, \quad x = 0, 1, 2, 3$

$$\therefore P(0) = k, \quad P(1) = \frac{k}{2}, \quad P(2) = \frac{k}{3}, \quad P(3) = \frac{k}{4}$$

Since this is a probability distribution, $\sum_{i=1}^n P(x_i) = 1$

$$\therefore k + \frac{k}{2} + \frac{k}{3} + \frac{k}{4} = 1$$

$$\therefore \frac{12k + 6k + 4k + 3k}{12} = 1$$

$$\therefore \frac{25k}{12} = 1$$

$$\therefore k = \frac{12}{25}$$

11 a $P(x) = \frac{4x - x^2}{a}, \quad x = 0, 1, 2, 3$

$$\therefore P(0) = 0, \quad P(1) = \frac{3}{a}, \quad P(2) = \frac{4}{a}, \quad P(3) = \frac{3}{a}$$

Since this is a probability distribution, $\sum_{i=1}^n P(x_i) = 1$

$$\therefore 0 + \frac{3}{a} + \frac{4}{a} + \frac{3}{a} = 1$$

$$\therefore \frac{10}{a} = 1$$

$$\therefore a = 10$$

b $P(X = 1) = P(1) = \frac{3}{a} = \frac{3}{10}$

c Since $P(X = 2) = P(2) = \frac{4}{10}$ is the greatest probability, the mode of the distribution is 2.

EXERCISE 14C.1

1 a

x_i	1	2	3
p_i	0.4	0.5	0.1

$$\begin{aligned} E(X) &= \sum_{i=1}^n x_i p_i \\ &= 1(0.4) + 2(0.5) + 3(0.1) \\ &= 1.7 \end{aligned}$$

b

x_i	0	1	2	3	4
p_i	0.1	0.2	0.15	0.2	0.35

$$\begin{aligned} E(X) &= \sum_{i=1}^n x_i p_i \\ &= 0(0.1) + 1(0.2) + 2(0.15) + 3(0.2) + 4(0.35) \\ &= 2.5 \end{aligned}$$

c

x_i	0	2	5	10
p_i	0.2	0.35	0.27	0.18

$$\begin{aligned}
 E(X) &= \sum_{i=1}^n x_i p_i \\
 &= 0(0.2) + 2(0.35) + 5(0.27) + 10(0.18) \\
 &= 3.85
 \end{aligned}$$

d

x_i	10	15	30	60
p_i	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{1}{3}$

$$\begin{aligned}
 E(X) &= \sum_{i=1}^n x_i p_i \\
 &= 10\left(\frac{1}{4}\right) + 15\left(\frac{1}{3}\right) + 30\left(\frac{1}{12}\right) + 60\left(\frac{1}{3}\right) \\
 &= 30
 \end{aligned}$$

- 2 a** Since this is a probability distribution, $\sum_{i=1}^n P(x_i) = 1$
- $$\therefore \frac{2}{5} + a + \frac{1}{10} = 1$$
- $$\therefore a = \frac{1}{2}$$

x	1	3	5
$P(X = x)$	$\frac{2}{5}$	a	$\frac{1}{10}$

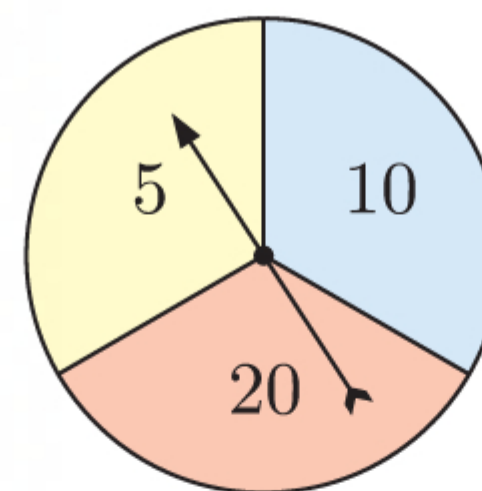
- b** Since $P(X = 3)$ is the greatest probability, 3 is the mode of the distribution.

c

$$\begin{aligned}
 \mu = E(X) &= 1\left(\frac{2}{5}\right) + 3\left(\frac{1}{2}\right) + 5\left(\frac{1}{10}\right) \\
 &= \frac{2}{5} + \frac{3}{2} + \frac{5}{10} \\
 &= \frac{4}{10} + \frac{15}{10} + \frac{5}{10} \\
 &= 2\frac{2}{5}
 \end{aligned}$$

- 3** Each coloured region on the spinner has the same area.
The probability table is:

<i>Number of points</i>	5	10	20
<i>Probability</i>	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$



$$\begin{aligned}
 E(X) &= \sum_{i=1}^n x_i p_i \\
 &= \left(5 \times \frac{1}{3}\right) + \left(10 \times \frac{1}{3}\right) + \left(20 \times \frac{1}{3}\right) \\
 &= \frac{35}{3} \\
 &\approx 11.7 \text{ points}
 \end{aligned}$$

In the long term, we can expect to be awarded an average of about 11.7 points per spin.

4

<i>Number of fish</i>	0	1	2	3
<i>Probability</i>	0.17	0.28	0.36	0.19

$$\begin{aligned}
 E(X) &= \sum_{i=1}^n x_i p_i \\
 &= (0 \times 0.17) + (1 \times 0.28) + (2 \times 0.36) + (3 \times 0.19) \\
 &= 0.28 + 0.72 + 0.57 \\
 &= 1.57 \text{ fish}
 \end{aligned}$$

On average, you would expect Ernie to catch 1.57 fish per trip.

5

<i>Number of books</i>	1	2	3	4	5
<i>Probability</i>	0.16	0.15	a	0.28	0.16

a Since this is a probability distribution, $\sum_{i=1}^n P(x_i) = 1$

$$\therefore 0.16 + 0.15 + a + 0.28 + 0.16 = 1$$

$$\therefore a = 0.25$$

b Pam is most likely to borrow 4 books when she visits the library, so this is the mode of the distribution.

c $E(X) = \sum_{i=1}^n x_i p_i$

$$\begin{aligned}
 &= (1 \times 0.16) + (2 \times 0.15) + (3 \times 0.25) + (4 \times 0.28) + (5 \times 0.16) \\
 &= 0.16 + 0.30 + 0.75 + 1.12 + 0.80 \\
 &= 3.13 \text{ books}
 \end{aligned}$$

On average, Pam borrows 3.13 books per visit.

6

<i>Colour</i>	<i>Number of lollies</i>
Red	4
Green	6
White	10

There are 5 red balls, 2 green balls, and 1 white ball, so in total there are $5 + 2 + 1 = 8$ balls.

<i>Number of lollies</i>	4	6	10
<i>Probability</i>	$\frac{5}{8} = 0.625$	$\frac{2}{8} = 0.25$	$\frac{1}{8} = 0.125$

$$\begin{aligned}
 E(X) &= \sum_{i=1}^n x_i p_i \\
 &= (4 \times 0.625) + (6 \times 0.25) + (10 \times 0.125) \\
 &= 2.5 + 1.5 + 1.25 \\
 &= 5.25 \text{ lollies}
 \end{aligned}$$

On average, Lachlan can expect to receive 5.25 lollies.

$$\begin{aligned}
 7 \quad a \quad P(\text{all ten pins}) &= 1 - \frac{1}{3} - \frac{2}{5} \\
 &= \frac{15}{15} - \frac{5}{15} - \frac{6}{15} \\
 &= \frac{4}{15}
 \end{aligned}$$

<i>Number of pins knocked down</i>	8	9	10
<i>Probability</i>	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{4}{15}$

$$\begin{aligned}
 b \quad E(X) &= \sum_{i=1}^n x_i p_i \\
 &= \left(8 \times \frac{1}{3}\right) + \left(9 \times \frac{2}{5}\right) + \left(10 \times \frac{4}{15}\right) \\
 &= \frac{40}{15} + \frac{54}{15} + \frac{40}{15} \\
 &= \frac{134}{15} \\
 &\approx 8.93 \text{ pins}
 \end{aligned}$$

On average, Jenna knocks down about 8.93 pins with her first bowl.

8 Since this is a probability distribution,

$$\sum_{i=1}^n P(x_i) = 1$$

$$\therefore 0.3 + a + b + 0.2 = 1$$

$$\therefore b = 0.5 - a \quad \dots (*)$$

$$\text{Now, } E(X) = 2.5$$

$$\therefore (1 \times 0.3) + (2 \times a) + (3 \times b) + (4 \times 0.2) = 2.5$$

$$\therefore 0.3 + 2a + 3(0.5 - a) + 0.8 = 2.5 \quad \{\text{using } (*)\}$$

$$\therefore 2a + 1.5 - 3a = 1.4$$

$$\therefore a = 0.1 \text{ and } b = 0.4$$

<i>x</i>	1	2	3	4
$P(X = x)$	0.3	<i>a</i>	<i>b</i>	0.2

9 a When Brad's soccer team plays an offensive strategy, $P(\text{draw}) = 1 - 0.3 - 0.55 = 0.15$

When Brad's soccer team plays a defensive strategy, $P(\text{draw}) = 1 - 0.2 - 0.3 = 0.5$

b Let X be the number of points awarded per game when Brad's soccer team plays an offensive strategy.

$$\begin{aligned}
 E(X) &= (3 \times 0.3) + (1 \times 0.15) + (0 \times 0.55) \\
 &= 0.9 + 0.15 \\
 &= 1.05 \text{ points per game}
 \end{aligned}$$

<i>Result</i>	W	D	L
<i>Points</i>	3	1	0
<i>Probability</i>	0.3	0.15	0.55

Let Y be the number of points awarded per game when Brad's soccer team plays a defensive strategy.

$$\begin{aligned}
 E(Y) &= (3 \times 0.2) + (1 \times 0.5) + (0 \times 0.3) \\
 &= 0.6 + 0.5 \\
 &= 1.1 \text{ points per game}
 \end{aligned}$$

<i>Result</i>	W	D	L
<i>Points</i>	3	1	0
<i>Probability</i>	0.2	0.5	0.3

c It is better for the team to play a defensive strategy in the long run as the team is expected to gain more points per game.

- d** If 4 points are awarded instead of 3 points for a win:

$$\begin{aligned} E(X) &= (4 \times 0.3) + (1 \times 0.15) + (0 \times 0.55) \\ &= 1.2 + 0.15 \\ &= 1.35 \text{ points per game} \end{aligned}$$

Result	W	D	L
Points	4	1	0
Probability	0.3	0.15	0.55

and $E(Y) = (4 \times 0.2) + (1 \times 0.5) + (0 \times 0.3)$
 $= 0.8 + 0.5$
 $= 1.3 \text{ points per game}$

Result	W	D	L
Points	4	1	0
Probability	0.2	0.5	0.3

The team is expected to gain more points per game when they play an offensive strategy. The team should change their strategy.

- 10 a**
- i** car park B
 - ii** car park A
 - iii** car park B

Car park A		Car park B	
Time	Cost	Time	Cost
0 - 1 hour	\$7	0 - 1 hour	\$6.50
1 - 2 hours	\$12	1 - 2 hours	\$11
2 - 3 hours	\$15	2 - 3 hours	\$16
3 - 4 hours	\$19	3 - 4 hours	\$18.50

- b** Let X be the amount Zoe pays for parking.

When Zoe parks her car at car park A:

$$\begin{aligned} E(X) &= (7 \times 0) + (12 \times 0.2) + (15 \times 0.7) + (19 \times 0.1) \\ &= 2.4 + 10.5 + 1.9 \\ &= \$14.80 \end{aligned}$$

When Zoe parks her car at car park B:

$$\begin{aligned} E(X) &= (6.5 \times 0) + (11 \times 0.2) + (16 \times 0.7) + (18.5 \times 0.1) \\ &= 2.2 + 11.2 + 1.85 \\ &= \$15.25 \end{aligned}$$

\therefore Zoe should choose car park A as it has the lower expected cost.

- 11** The probability of the ring not being stolen or lost is $P(\text{ring is safe}) = 1 - 0.0025 - 0.03$
 $= 0.9675$

Let X be the amount the insurance company pays the policy owner.

$$\begin{aligned} E(X) &= (0 \times 0.9675) + (20\,000 \times 0.0025) + (8000 \times 0.03) \\ &= 50 + 240 \\ &= \$290 \text{ per policy} \end{aligned}$$

\therefore the insurance company should charge \$390 per policy to have an expected return of \$100.

EXERCISE 14C.2

- 1 Let X denote the return from one game.

<i>Number</i>	1	2	3	4	5	6
<i>Winnings</i>	\$3	\$1	\$3	\$1	\$3	\$1
<i>Probability</i>	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$\begin{aligned}
 E(X) &= 3\left(\frac{1}{6} \times 3\right) + 3\left(\frac{1}{6} \times 1\right) \\
 &= \frac{9}{6} + \frac{3}{6} \\
 &= \frac{12}{6} \\
 &= 2
 \end{aligned}$$

So, \$2 is the expected return.

Since the game costs \$2 to play, the expected gain = expected return – \$2
 $= \$2 - \$2 = \$0$

Since the expected gain is zero, the game is fair.

- 2 a Let X denote the return from each roll.

$$\begin{aligned}
 E(X) &= \left(\frac{1}{6} \times 1\right) + \left(\frac{1}{6} \times 2\right) + \left(\frac{1}{6} \times 3\right) + \left(\frac{1}{6} \times 4\right) + \left(\frac{1}{6} \times 5\right) + \left(\frac{1}{6} \times 6\right) \\
 &= \frac{1}{6} \times 21 \\
 &= \$3.50
 \end{aligned}$$

- b The expected gain is $\$3.50 - \$4 = -\$0.50$

- c The player should not play many games, as on average he would expect to lose \$0.50 with each roll.

- 3 a Let X denote the return from each bet.

$$\begin{aligned}
 E(X) &= \left(\frac{18}{37} \times 2\right) + \left(\frac{19}{37} \times -2\right) \\
 &= \frac{36}{37} - \frac{38}{37} \\
 &= -\frac{2}{37} \\
 &\approx -\$0.05
 \end{aligned}$$

- b $100 \times -\frac{2}{37} \approx -\5.41

From 100 bets, I would expect to lose about \$5.41.

4

<i>Result</i>	<i>Win</i>
HH	\$10
HT or TH	\$3
TT	\$1

Let X be the gain from each game, and
 Y be the return from each game.

$$\begin{aligned}
 E(Y) &= \left(\frac{1}{4} \times 10\right) + \left(\frac{2}{4} \times 3\right) + \left(\frac{1}{4} \times 1\right) \\
 &= \frac{10}{4} + \frac{6}{4} + \frac{1}{4} \\
 &= \$4.25
 \end{aligned}$$

The expected return per game is \$4.25. It costs \$5.00 to play the game.

So, the expected gain $E(X) = E(Y) - \$5$
 $= \$4.25 - \5.00
 $= -\$0.75$

So we expect a loss of \$0.75 per game on average.

- 5 a i $P(\text{win 5 tokens}) = \frac{6}{20}$ {there are 6 multiples of 3 between 1 and 20}
 $= \frac{3}{10}$
 $= 0.3$

$$\begin{aligned} \text{ii } P(\text{win 10 tokens}) &= \frac{2}{20} \quad \{\text{there are 2 multiples of 10 between 1 and 20}\} \\ &= \frac{1}{10} \\ &= 0.1 \end{aligned}$$

$$\begin{aligned} \text{b } E(X) &= (0 \times 0.6) + (5 \times 0.3) + (10 \times 0.1) \\ &= 1.5 + 1 \\ &= 2.5 \text{ tokens} \end{aligned}$$

- c It costs 3 tokens to play the game. So, the expected gain = $2.5 - 3 = -0.5$ tokens.
We do not recommend playing the game many times as the player can expect to lose half a token on average per game.

6 a

<i>Disc colour</i>	Black	Blue	Gold
<i>Winnings</i>	\$1	\$5	\$20
<i>Probability</i>	$\frac{10}{15}$	$\frac{4}{15}$	$\frac{1}{15}$

Let X be the return from each game.

$$\begin{aligned} E(X) &= \left(1 \times \frac{10}{15}\right) + \left(5 \times \frac{4}{15}\right) + \left(20 \times \frac{1}{15}\right) \\ &= \frac{10 + 20 + 20}{15} \\ &= \frac{50}{15} \\ &\approx \$3.33 \end{aligned}$$

The expected return per game is \$3.33. It costs \$4.00 to play the game.

So, the expected gain $\approx \$3.33 - \4.00

$\approx -\$0.67 \neq \0 , so the game is not fair.

- b Let the new prize money for selecting the gold disc be \$ x .
Now, for the game to be fair, the expected return must be equal to the cost of each game.

$$\therefore E(X) = \left(1 \times \frac{10}{15}\right) + \left(5 \times \frac{4}{15}\right) + \left(x \times \frac{1}{15}\right) = 4 \quad \{\text{the cost of the game is \$4}\}$$

$$\therefore \frac{10}{15} + \frac{20}{15} + \frac{x}{15} = 4$$

$$\therefore \frac{30 + x}{15} = 4$$

$$\therefore 30 + x = 60$$

$$\therefore x = 30$$

So, the new prize money for selecting the gold disc is \$30.

$$\text{7 } P(\text{RRR}) = \frac{5}{12} \times \frac{4}{11} \times \frac{3}{10} = \frac{60}{1320}$$

$$P(\text{BBB}) = \frac{4}{12} \times \frac{3}{11} \times \frac{2}{10} = \frac{24}{1320}$$

$$P(\text{GGG}) = \frac{3}{12} \times \frac{2}{11} \times \frac{1}{10} = \frac{6}{1320}$$

$$P(\text{RBG}) = P(\text{RGB}) = P(\text{BRG}) = P(\text{BGR}) = P(\text{GRB}) = P(\text{GBR}) = \frac{5}{12} \times \frac{4}{11} \times \frac{3}{10} = \frac{60}{1320}$$

$$P(\text{winning}) = P(\text{all the same colour or one of each})$$

$$= P(\text{RRR}) + P(\text{BBB}) + P(\text{GGG}) + P(\text{RBG}) + P(\text{RGB})$$

$$+ P(\text{BRG}) + P(\text{BGR}) + P(\text{GRB}) + P(\text{GBR})$$

$$= \frac{60}{1320} + \frac{24}{1320} + \frac{6}{1320} + \frac{60}{1320} \times 6$$

$$= \frac{60+24+6+360}{1320}$$

$$= \frac{450}{1320} = \frac{15}{44}$$

The player expects to win $11 \times \frac{15}{44} = \3.75

The organiser makes \$1 when the player loses \$1.

Now, the expected gain for the player = expected win – cost to play

$$\therefore -\$1.00 = \$3.75 - \text{cost to play}$$

$$\therefore \text{cost to play} = \$4.75$$

EXERCISE 14D

- 1
 - a The binomial distribution applies, as tossing a coin has two possible outcomes (a head or a tail) and each toss is independent of every other toss.
 - b The binomial distribution applies, as this is equivalent to tossing one coin 100 times.
 - c The binomial distribution applies as we can draw out a red or a blue marble with the same chances each time.
 - d The binomial distribution does not apply as the result of each draw is dependent upon the results of previous draws.
 - e The binomial distribution does not apply, assuming that ten bolts are drawn without replacement, as we do not have a repetition of independent trials. However, since there is such a large number of bolts in the bin, the trials are approximately independent, so the distribution is approximately binomial.

2	1	1							$n = 1$
	1	2	1						$n = 2$
	1	3	3	1					$n = 3$
	1	4	6	4	1				$n = 4$
	1	5	10	10	5	1			$n = 5$
	1	6	15	20	15	6	1		$n = 6$

- 3 The number of trials is $n = 4$.
The probability of success with each toss is $p = \frac{1}{2}$.
Let X be the number of heads tossed.

$$\therefore X \sim B(4, \frac{1}{2})$$

$$\begin{aligned} \therefore P(X = x) &= \binom{4}{x} \left(\frac{1}{2}\right)^x \left(1 - \frac{1}{2}\right)^{4-x} \\ &= \binom{4}{x} \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x} \end{aligned}$$

a $P(4 \text{ heads})$

$$= P(X = 4)$$

$$= \binom{4}{4} \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^{4-4}$$

$$= \left(\frac{1}{2}\right)^4$$

$$= \frac{1}{16}$$

b $P(3 \text{ heads})$

$$= P(X = 3)$$

$$= \binom{4}{3} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{4-3}$$

$$= 4 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^1$$

$$= \frac{4}{16}$$

$$= \frac{1}{4}$$

c $P(2 \text{ heads})$

$$= P(X = 2)$$

$$= \binom{4}{2} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{4-2}$$

$$= 6 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2$$

$$= \frac{6}{16}$$

$$= \frac{3}{8}$$

4 The number of trials is $n = 5$.

The probability of success with each toss is $p = \frac{1}{2}$.

Let X be the number of heads from each toss.

$$\therefore X \sim B(5, \frac{1}{2})$$

$$\begin{aligned}\therefore P(X = x) &= \binom{5}{x} \left(\frac{1}{2}\right)^x \left(1 - \frac{1}{2}\right)^{5-x} \\ &= \binom{5}{x} \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{5-x}\end{aligned}$$

a $P(4 \text{ heads})$

$$\begin{aligned}&= P(X = 4) \\ &= \binom{5}{4} \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^{5-4} \\ &= 5 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right) \\ &= \frac{5}{32}\end{aligned}$$

b $P(2 \text{ heads})$

$$\begin{aligned}&= P(X = 2) \\ &= \binom{5}{2} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{5-2} \\ &= 10 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^3 \\ &= \frac{10}{32} \\ &= \frac{5}{16}\end{aligned}$$

c $P(4 \text{ heads then 1 tail})$

$$\begin{aligned}&= P(\text{HHHHT}) \\ &= \left(\frac{1}{2}\right)^4 \times \left(\frac{1}{2}\right) \\ &= \frac{1}{32}\end{aligned}$$

5 The number of trials is $n = 4$.

The probability of success (getting a strawberry cream) is $p = \frac{2}{2+1} = \frac{2}{3}$.

Let X be the number of strawberry creams selected.

$$\therefore X \sim B(4, \frac{2}{3})$$

$$\begin{aligned}\therefore P(X = x) &= \binom{4}{x} \left(\frac{2}{3}\right)^x \left(1 - \frac{2}{3}\right)^{4-x} \\ &= \binom{4}{x} \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{4-x}\end{aligned}$$

a $P(\text{all strawberry creams})$

$$\begin{aligned}&= P(X = 4) \\ &= \binom{4}{4} \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^{4-4} \\ &= \left(\frac{2}{3}\right)^4 \\ &= \frac{16}{81}\end{aligned}$$

b $P(\text{two of each type})$

$$\begin{aligned}&= P(X = 2) \\ &= \binom{4}{2} \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right)^{4-2} \\ &= 6 \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right)^2 \\ &= \frac{8}{27}\end{aligned}$$

c $P(\text{at least 2 strawberry creams})$

$$\begin{aligned}&= P(X \geq 2) \\ &= P(X = 2) + P(X = 3) + P(X = 4) \\ &= \frac{8}{27} + \binom{4}{3} \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^1 + \frac{16}{81} \quad \{\text{using a and b}\} \\ &= \frac{24}{81} + 4 \left(\frac{8}{27}\right) \left(\frac{1}{3}\right) + \frac{16}{81} \\ &= \frac{72}{81} \\ &= \frac{8}{9}\end{aligned}$$

- 6 The number of trials is $n = 6$.

The probability of success (selecting a “flat back”) is $p = \frac{1}{3+1} = \frac{1}{4}$

Let X be the number of “flat backs” selected.

$$\therefore X \sim B(6, \frac{1}{4})$$

$$\begin{aligned}\therefore P(X = x) &= \binom{6}{x} \left(\frac{1}{4}\right)^x \left(1 - \frac{1}{4}\right)^{6-x} \\ &= \binom{6}{x} \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{6-x}\end{aligned}$$

a $P(\text{two “flat backs”})$

$$\begin{aligned}&= P(X = 2) \\ &= \binom{6}{2} \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^{6-2} \\ &= 15 \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^4 \\ &= \frac{1215}{4096}\end{aligned}$$

b $P(\text{at least 3 “flat backs”})$

$$\begin{aligned}&= P(X \geq 3) \\ &= P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6) \\ &= \binom{6}{3} \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^3 + \binom{6}{4} \left(\frac{1}{4}\right)^4 \left(\frac{3}{4}\right)^2 + \binom{6}{5} \left(\frac{1}{4}\right)^5 \left(\frac{3}{4}\right) + \binom{6}{6} \left(\frac{1}{4}\right)^6 \\ &= \frac{20 \times 27}{4096} + \frac{15 \times 9}{4096} + \frac{6 \times 3}{4096} + \frac{1 \times 1}{4096} \\ &= \frac{694}{4096} = \frac{347}{2048}\end{aligned}$$

c $P(\text{at most 3 “normal” kiwis}) = P(\text{at least 3 “flat backs”})$

$$= \frac{347}{2048} \quad \{\text{from b}\}$$

INVESTIGATION 1

THE GRAPH OF A BINOMIAL DISTRIBUTION

1 **a** $X \sim B(n, p)$

When $n = 25$, $p = 0.1$, the mode of X is 2.

b The distribution is positively skewed.

2 When $p = 0.5$, the distribution is symmetric.

When $p < 0.5$, the distribution is positively skewed.

When $p > 0.5$, the distribution is negatively skewed.

3 $p = 0.1$, and the value of n is free to change.

As n increases, the distribution becomes approximately symmetrical.

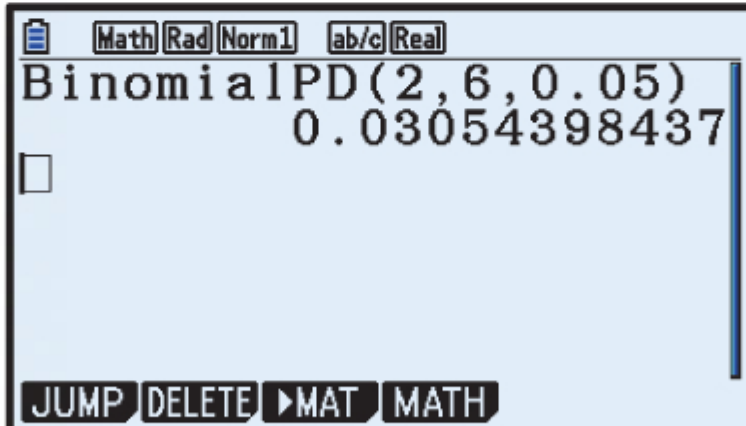
EXERCISE 14E

1 Let X be the number of defective light bulbs.

$n = 6$, so $X = 0, 1, 2, 3, 4, 5$, or 6 , and $p = 5\% = 0.05$

$$\therefore X \sim B(6, 0.05)$$

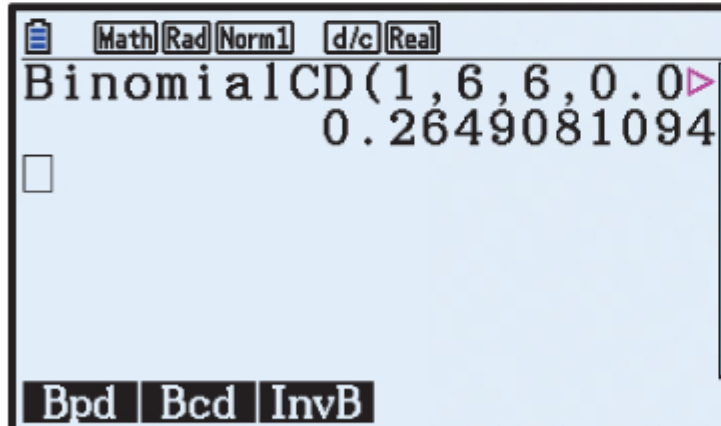
a



BinomialPD(2,6,0.05)
0.03054398437

$$P(X = 2) \approx 0.0305$$

b



BinomialCD(1,6,6,0.05)
0.2649081094

$$P(X \geq 1) \approx 0.265$$

- 2** Let X be the number of faulty items.

$n = 12$, so $X = 0, 1, 2, 3, \dots$, or 12 , and $p = 6\% = 0.06$

$$\therefore X \sim B(12, 0.06)$$

- a** $P(\text{none will be faulty})$

$$= P(X = 0)$$

$$= \binom{12}{0} (0.06)^0 (0.94)^{12}$$

$$\approx 0.476$$

- b** $P(\text{at most one is faulty})$

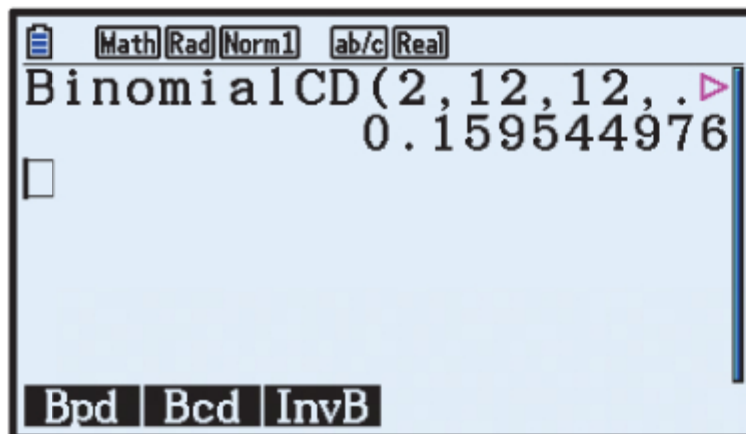
$$= P(X \leq 1)$$

$$= P(X = 0) + P(X = 1)$$

$$\approx 0.476 + \binom{12}{1} (0.06)^1 (0.94)^{11}$$

$$\approx 0.840$$

c



$$P(\text{at least two are faulty}) = P(X \geq 2)$$

$$\approx 0.160$$

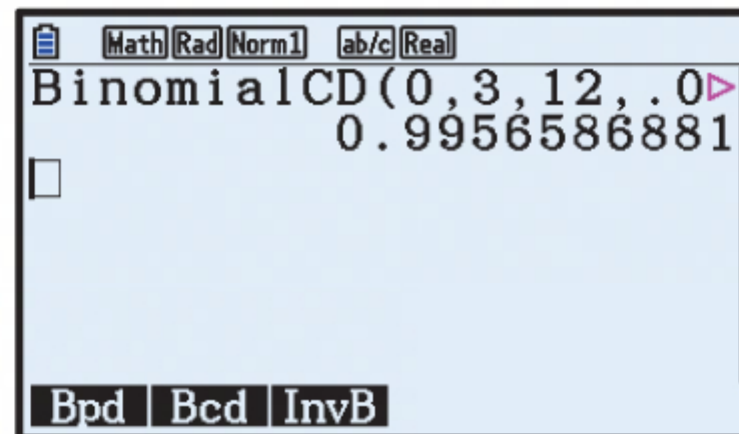
or $P(\text{at least two are faulty})$

$$= 1 - P(\text{at most one is faulty})$$

$$\approx 1 - 0.840 \quad \{\text{from b}\}$$

$$\approx 0.160$$

d



$$P(\text{less than four are faulty}) = P(X < 4)$$

$$= P(X \leq 3)$$

$$\approx 0.996$$

- 3** Let X be the number of times in a week that the bus is on time.

Since it is late 2 in every 5 days, then it is on time 3 in every 5 days, so $p = \frac{3}{5} = 0.6$.

$n = 7$, so $X = 0, 1, 2, 3, 4, 5, 6$, or 7 , and $X \sim B(7, 0.6)$.

a $P(X = 7) = \binom{7}{7} (0.6)^7 (0.4)^0$

$$\approx 0.0280$$

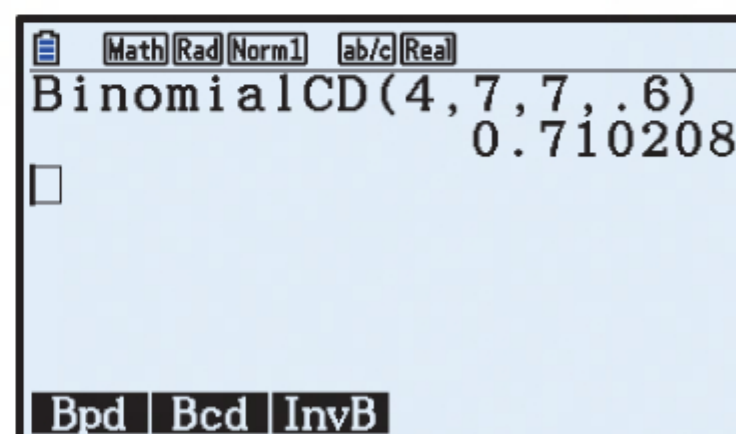
b $P(\text{on time only on Monday}) = 0.6 \times (0.4)^6$

$$\approx 0.00246$$

c $P(X = 6) = \binom{7}{6} (0.6)^6 (0.4)$

$$\approx 0.131$$

d



$$P(X \geq 4) \approx 0.710$$

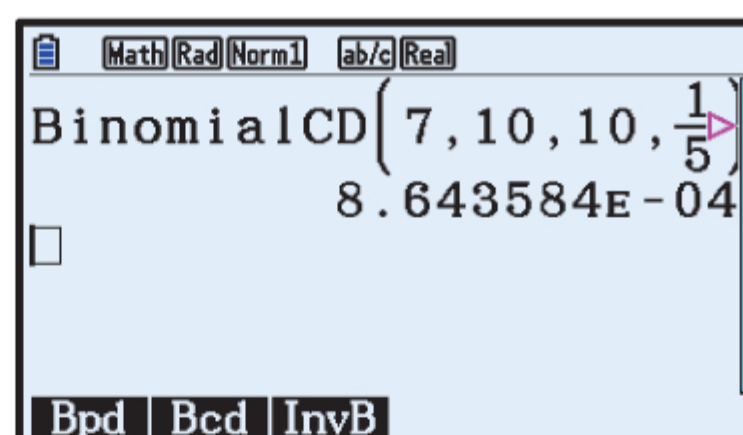
- 4** Let X denote the number of questions Raj answers correctly.

$n = 10$, so $X = 0, 1, 2, \dots$, or 10 , and $p = \frac{1}{5}$

$$\therefore X \sim B(10, \frac{1}{5})$$

$$P(\text{Raj passes}) = P(X \geq 7)$$

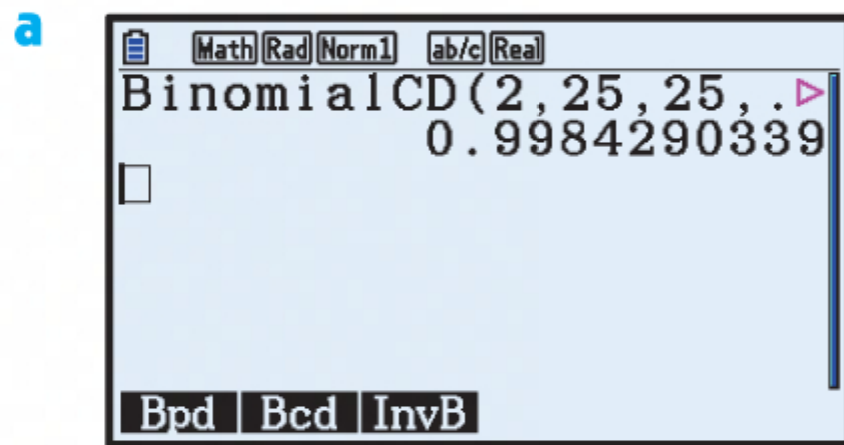
$$\approx 0.000864$$



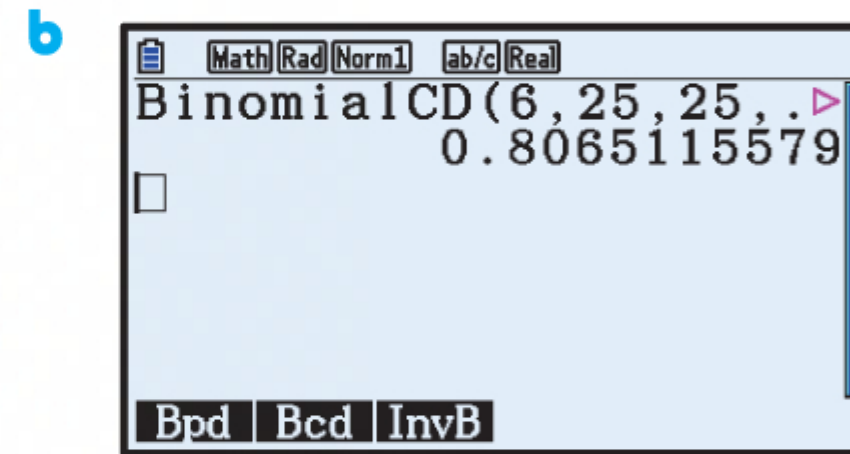
- 5 Let X be the number of students with the flu.

$n = 25$, so $X = 0, 1, 2, 3, \dots$, or 25 , and $p = 0.3$

$$\therefore X \sim B(25, 0.3)$$



$$P(X \geq 2) \approx 0.998$$



$$20\% \text{ of } 25 = 5$$

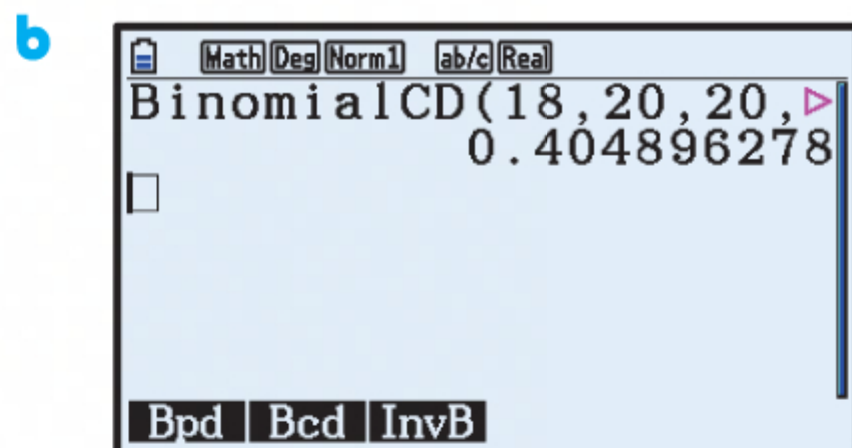
$$\therefore P(\text{test cancelled}) = P(X \geq 6) \\ \approx 0.807$$

- 6 Let X be the number of successful shots from the free throw line.

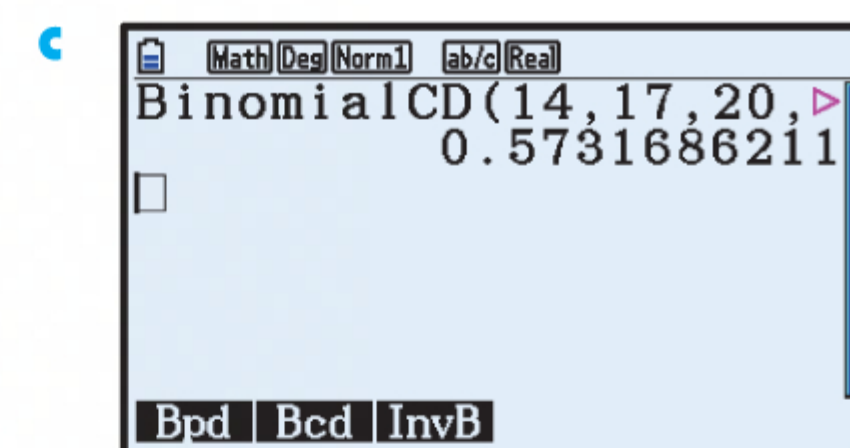
$n = 20$, so $X = 0, 1, 2, 3, \dots$, or 20 , and $p = 85\% = 0.85$

$$\therefore X \sim B(20, 0.85)$$

a
$$P(X = 20) = \binom{20}{20} (0.85)^{20} (0.15)^0 \\ \approx 0.0388$$



$$P(X \geq 18) \approx 0.405$$



$$P(14 \leq X \leq 17) \approx 0.573$$

- 7 For Jelena to win a set of 6 games to 4, she must win 5 of the first 9 games, and then win the 10th game.

Let X be the number of games Jelena wins in the first 9 games.

$n = 9$, so $X = 0, 1, 2, 3, \dots$, or 9

Now, Martina beats Jelena in 2 games out of 3, so the probability of Jelena winning a game is

$$p = 1 - \frac{2}{3} = \frac{1}{3}.$$

$$\therefore X \sim B(9, \frac{1}{3})$$

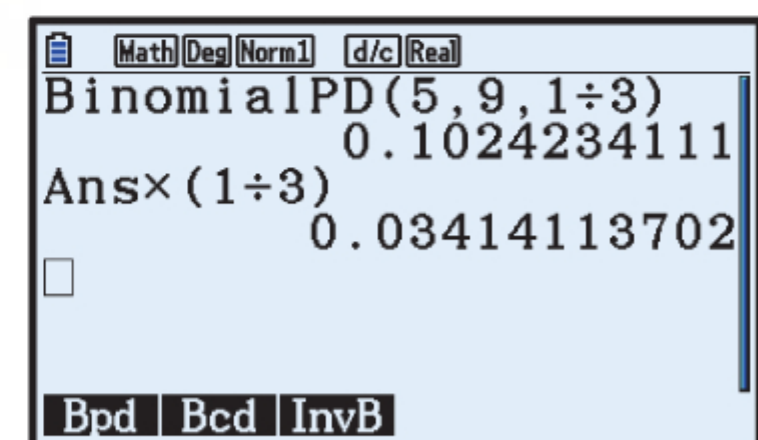
So, $P(\text{J wins 6 games to 4})$

$$= P(\text{J wins 5 of first 9 games}) \times P(\text{J wins 10th game})$$

$$= P(X = 5) \times \frac{1}{3}$$

$$\approx 0.1024 \times \frac{1}{3}$$

$$\approx 0.0341$$

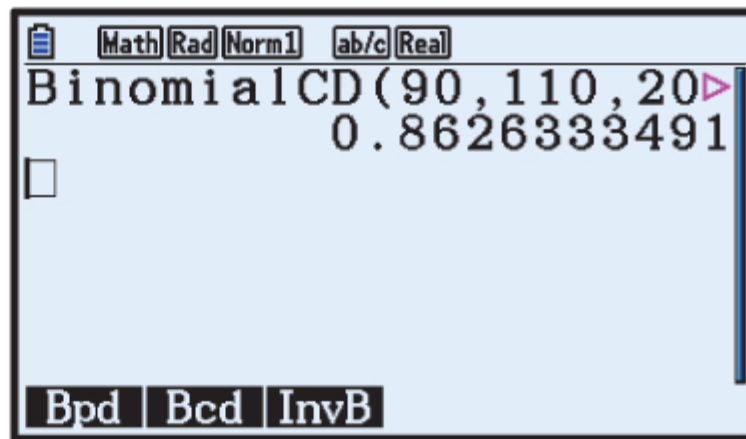


- 8 Let X be the number of heads.

$n = 200$, so $X = 0, 1, 2, 3, \dots$, or 200 , and $p = \frac{1}{2}$

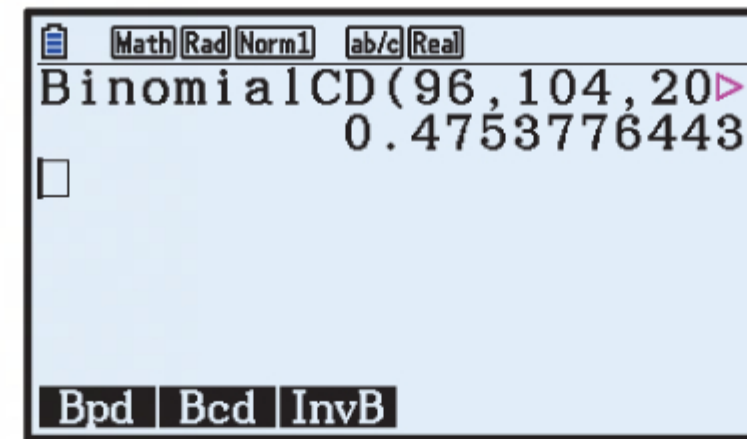
$$\therefore X \sim B(200, \frac{1}{2})$$

a



$$P(90 \leq X \leq 110) \approx 0.863$$

b



$$P(95 < X < 105) = P(96 \leq X \leq 104) \\ \approx 0.475$$

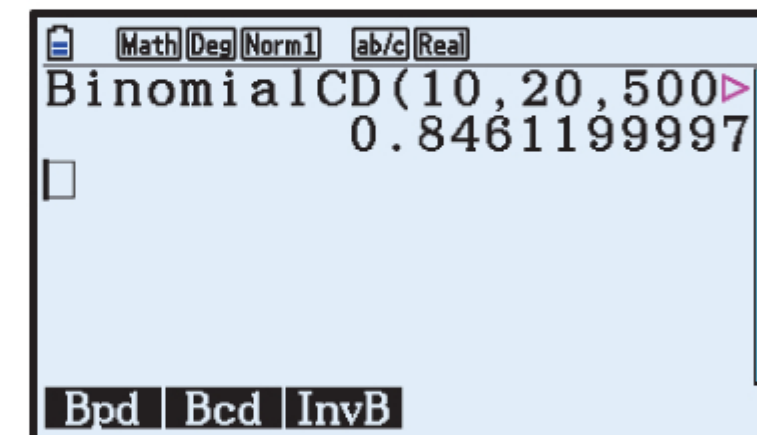
- 9 a $P(\text{rolling double sixes}) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

- b Let X be the number of double sixes rolled.

$n = 500$, so $X = 0, 1, 2, 3, \dots$, or 500 , and $p = \frac{1}{36}$

$$\therefore X \sim B(500, \frac{1}{36})$$

$$P(10 \leq X \leq 20) \approx 0.846$$

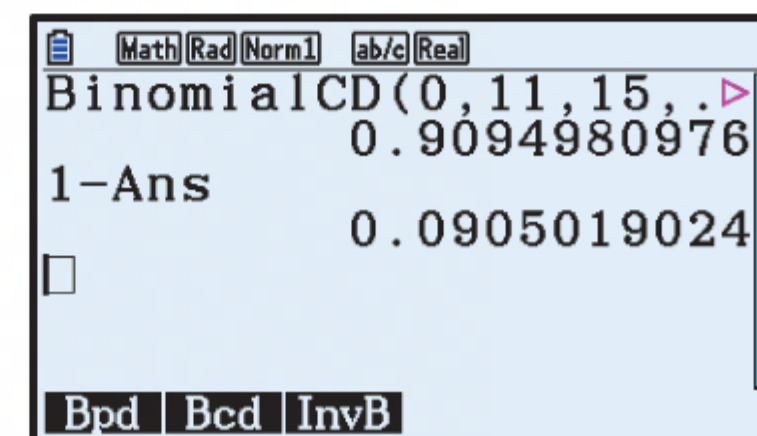


- 10 Let X be the number of traffic lights Shelley has stopped at.

$n = 15$, so $X = 0, 1, 2, 3, \dots$, or 15 , and $p = 0.6$

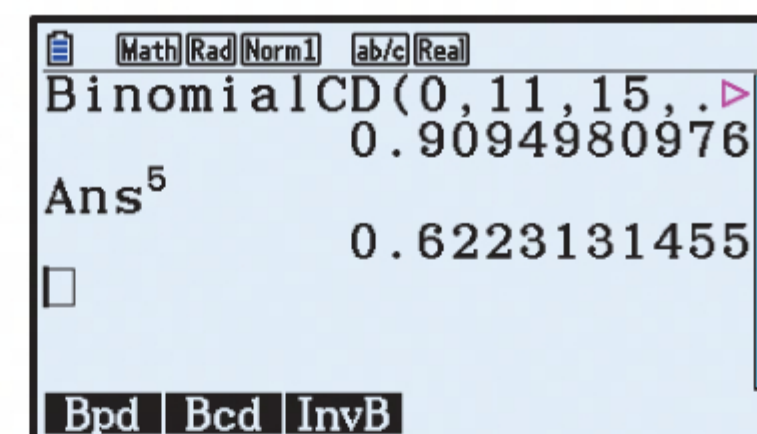
$$\therefore X \sim B(15, 0.6)$$

- a $P(\text{Shelley will be late}) = P(X > 11) \\ = 1 - P(X \leq 11) \\ \approx 0.0905$



- b $P(\text{Shelley will be on time}) = P(X \leq 11) \\ \approx 0.909$

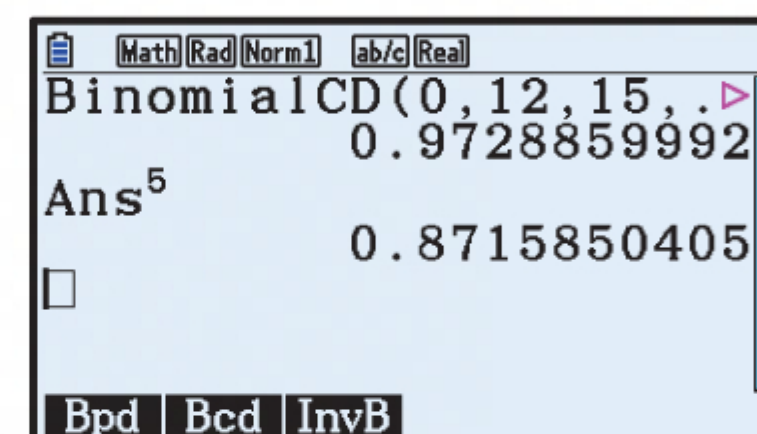
$$P(\text{Shelley will be on time all 5 days}) = [P(X \leq 11)]^5 \\ \approx 0.622$$



- c $P(\text{Shelley will be on time}) = P(X \leq 12) \\ \approx 0.973$

$$P(\text{Shelley will be on time all 5 days}) = [P(X \leq 12)]^5 \\ \approx 0.872$$

\therefore yes, the probability that Shelley is on time for work each day of a 5 day week is now about 87.2%.



- 11** Let X be the number of solar components which fail.
 $n = 20$, so $X = 0, 1, 2, 3, \dots$, or 20 , and $p = 0.85$
 $\therefore X \sim B(20, 0.85)$

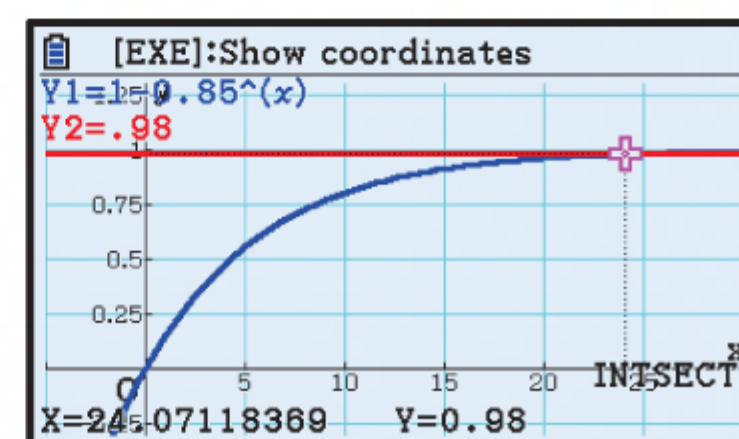
a $P(\text{hot water unit fails within one year}) = P(\text{all 20 components fail})$
 $= P(X = 20)$
 $= (0.85)^{20}$
 ≈ 0.0388

b $P(\text{hot water unit with } n \text{ components fails within one year}) = (0.85)^n$
 $\therefore P(\text{hot water unit with } n \text{ components is operating after one year}) = 1 - (0.85)^n$
 \therefore we need to find the smallest integer n such that $1 - (0.85)^n \geq 0.98$

Using technology, $1 - (0.85)^n = 0.98$ when
 $n \approx 24.1$ components.

\therefore at least 25 solar components are needed.

Check: When $n = 25$, the probability that at least one component will still work is
 $1 - (0.85)^{25} \approx 0.983 > 0.98$ ✓

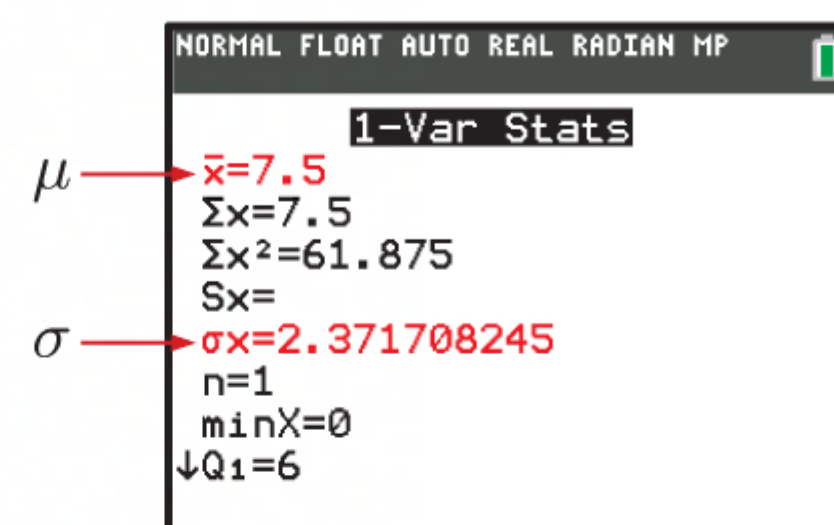


INVESTIGATION 2

THE MEAN AND STANDARD DEVIATION OF A BINOMIAL DISTRIBUTION

1 $X \sim B(30, 0.25)$

Consult the graphics calculator instructions by clicking on the icon in the Investigation box if you need help obtaining the result shown.



	$p = 0.1$	$p = 0.25$	$p = 0.5$	$p = 0.7$
$n = 10$	$\mu = 1$ $\sigma \approx 0.9487$	$\mu = 2.5$ $\sigma \approx 1.3693$	$\mu = 5$ $\sigma \approx 1.5811$	$\mu = 7$ $\sigma \approx 1.4491$
$n = 30$	$\mu = 3$ $\sigma \approx 1.6432$	$\mu = 7.5$ $\sigma \approx 2.3717$	$\mu = 15$ $\sigma \approx 2.7386$	$\mu = 21$ $\sigma \approx 2.5100$
$n = 50$	$\mu = 5$ $\sigma \approx 2.1213$	$\mu = 12.5$ $\sigma \approx 3.0619$	$\mu = 25$ $\sigma \approx 3.5355$	$\mu = 35$ $\sigma \approx 3.2404$

3	$p = 0.1$	$p = 0.25$	$p = 0.5$	$p = 0.7$
$n = 10$	$np = 1$ $\sqrt{np(1-p)}$ ≈ 0.9487	$np = 2.5$ $\sqrt{np(1-p)}$ ≈ 1.3693	$np = 5$ $\sqrt{np(1-p)}$ ≈ 1.5811	$np = 7$ $\sqrt{np(1-p)}$ ≈ 1.4491
$n = 30$	$np = 3$ $\sqrt{np(1-p)}$ ≈ 1.6432	$np = 7.5$ $\sqrt{np(1-p)}$ ≈ 2.3717	$np = 15$ $\sqrt{np(1-p)}$ ≈ 2.7386	$np = 21$ $\sqrt{np(1-p)}$ ≈ 2.5100
$n = 50$	$np = 5$ $\sqrt{np(1-p)}$ ≈ 2.1213	$np = 12.5$ $\sqrt{np(1-p)}$ ≈ 3.0619	$np = 25$ $\sqrt{np(1-p)}$ ≈ 3.5355	$np = 35$ $\sqrt{np(1-p)}$ ≈ 3.2404

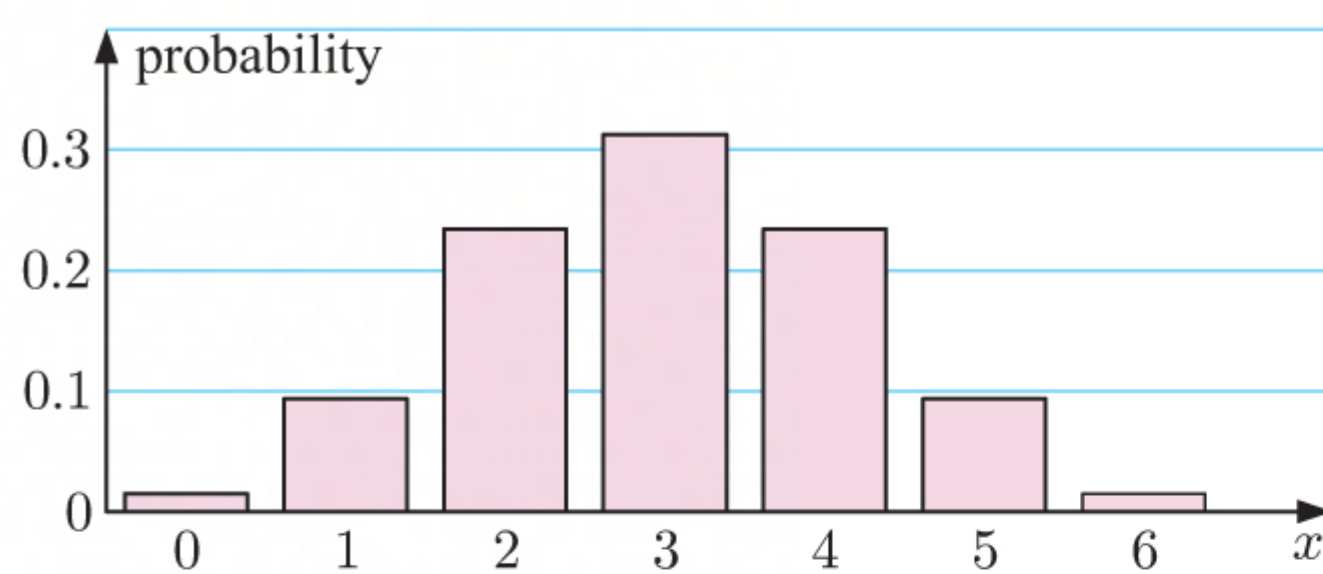
Our results in **2** and **3** agree with the formulae $\mu = np$ and $\sigma = \sqrt{np(1-p)}$.

EXERCISE 14F

1 a $X \sim B(6, 0.5)$

$$\begin{aligned}
 \text{i} \quad \mu &= np & \sigma &= \sqrt{np(1-p)} \\
 &= 6 \times 0.5 & &= \sqrt{6 \times 0.5 \times 0.5} \\
 &= 3 & &\approx 1.22
 \end{aligned}$$

ii	x_i	0	1	2	3	4	5	6
	$P(x_i)$	0.0156	0.0938	0.2344	0.3125	0.2344	0.0938	0.0156

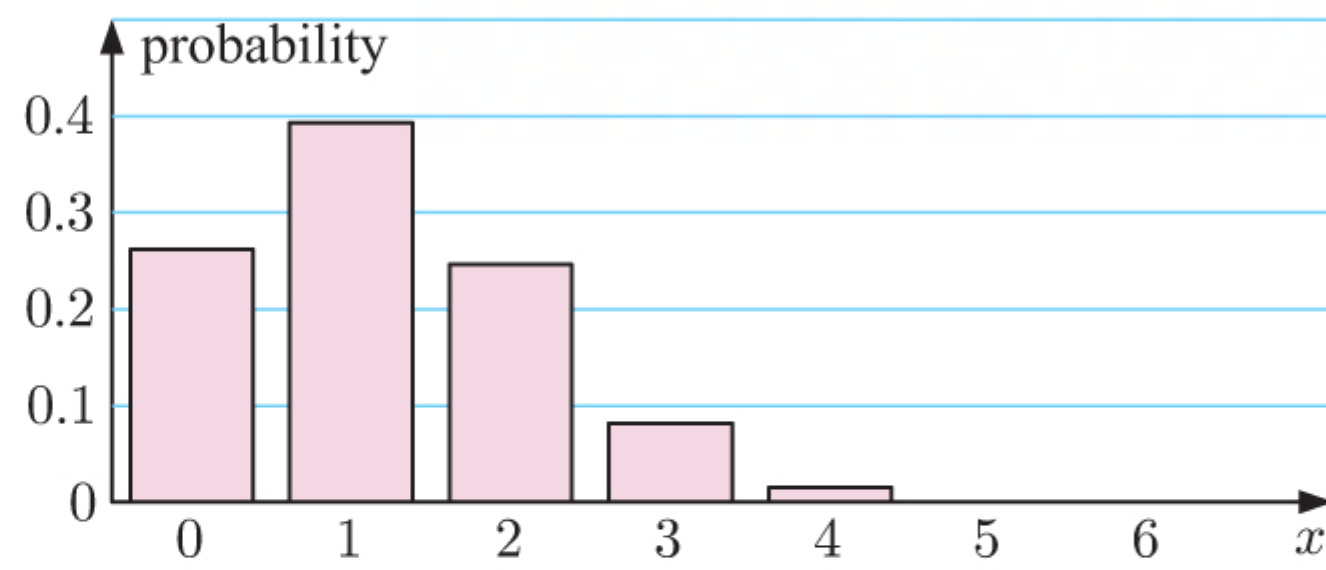


iii The distribution is symmetric.

b $X \sim B(6, 0.2)$

$$\begin{aligned}
 \text{i} \quad \mu &= np & \sigma &= \sqrt{np(1-p)} \\
 &= 6 \times 0.2 & &= \sqrt{6 \times 0.2 \times 0.8} \\
 &= 1.2 & &\approx 0.980
 \end{aligned}$$

x_i	0	1	2	3	4	5	6
$P(x_i)$	0.2621	0.3932	0.2458	0.0819	0.0154	0.0015	0.0001

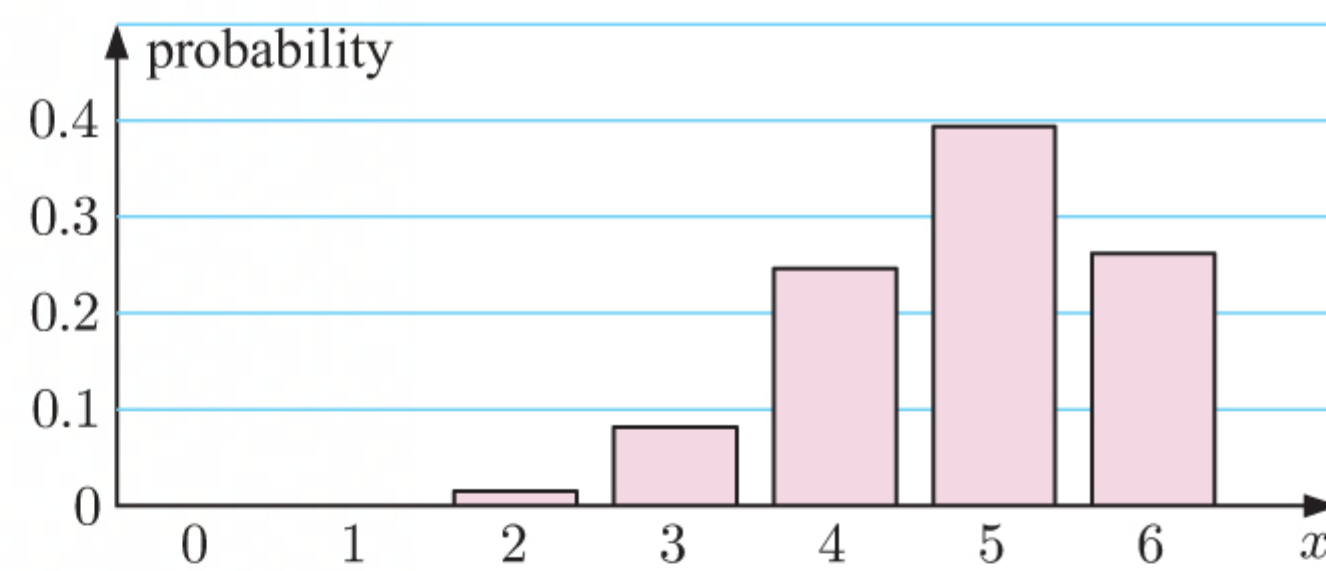


iii The distribution is positively skewed.

c $X \sim B(6, 0.8)$

$$\begin{aligned}
 \text{i } \mu &= np & \sigma &= \sqrt{np(1-p)} \\
 &= 6 \times 0.8 & &= \sqrt{6 \times 0.8 \times 0.2} \\
 &= 4.8 & &\approx 0.980
 \end{aligned}$$

x_i	0	1	2	3	4	5	6
$P(x_i)$	0.0001	0.0015	0.0154	0.0819	0.2458	0.3932	0.2621



iii The distribution is negatively skewed, and is the exact reflection of the distribution in b.

$$\begin{array}{lll}
 \text{2 } X \sim B(10, 0.5) & \text{mean } \mu = np & \text{and variance } \sigma^2 = np(1-p) \\
 & = 10 \times \frac{1}{2} & = 10 \times \frac{1}{2} \times \frac{1}{2} \\
 & = 5 & = 2.5
 \end{array}$$

3 a $X \sim B(30, 0.04)$

$$\begin{aligned}
 \mu_X &= np \\
 &= 30 \times 0.04 \\
 &= 1.2 \\
 \sigma_X &= \sqrt{np(1-p)} \\
 &= \sqrt{30 \times 0.04 \times 0.96} \\
 &\approx 1.07
 \end{aligned}$$

b $Y \sim B(30, 0.96)$

$$\begin{aligned}
 \mu_Y &= np \\
 &= 30 \times 0.96 \\
 &= 28.8 \\
 \sigma_Y &= \sqrt{np(1-p)} \\
 &= \sqrt{30 \times 0.96 \times 0.04} \\
 &\approx 1.07
 \end{aligned}$$

4 $X \sim B(30, 0.13)$

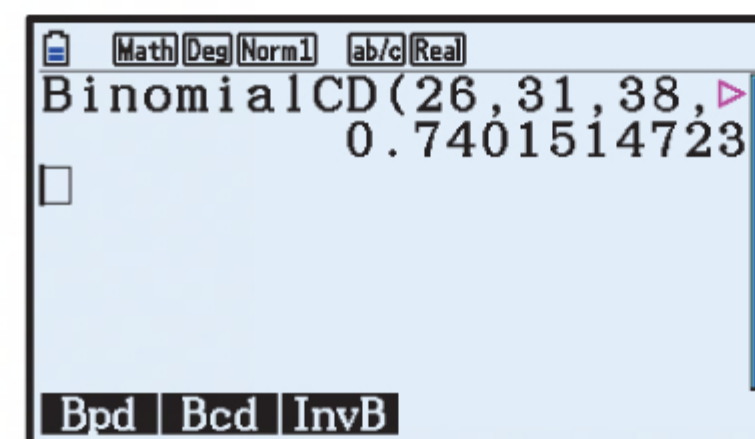
$$\begin{aligned}\mu &= np \\ &= 30 \times 0.13 \\ &= 3.9\end{aligned}\qquad \begin{aligned}\sigma &= \sqrt{np(1-p)} \\ &= \sqrt{30 \times 0.13 \times 0.87} \\ &\approx 1.84\end{aligned}$$

5 $X \sim B(38, 0.75)$

a
$$\begin{aligned}\mu &= np \\ &= 38 \times 0.75 \\ &= 28.5\end{aligned}\qquad \begin{aligned}\sigma &= \sqrt{np(1-p)} \\ &= \sqrt{38 \times 0.75 \times 0.25} \\ &\approx 2.67\end{aligned}$$

b
$$\begin{aligned}\mu - \sigma &\approx 28.5 - 2.67 \\ &\approx 25.8\end{aligned}\qquad \begin{aligned}\mu + \sigma &\approx 28.5 + 2.67 \\ &\approx 31.2\end{aligned}$$

$$\therefore P(\mu - \sigma < X < \mu + \sigma) = P(26 \leq X \leq 31) \approx 0.740$$

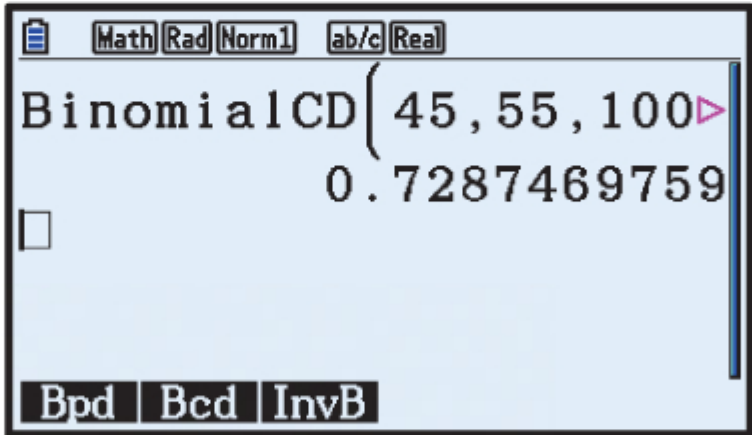


6 $X \sim B(100, \frac{1}{2}), Y \sim B(300, \frac{1}{6})$

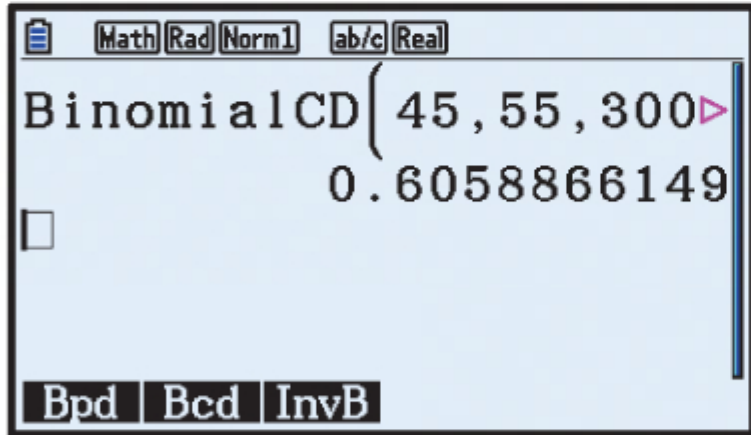
a
$$\begin{aligned}\mu_X &= np \\ &= 100 \times \frac{1}{2} \\ &= 50\end{aligned}\qquad \begin{aligned}\mu_Y &= np \\ &= 300 \times \frac{1}{6} \\ &= 50\end{aligned}$$

b
$$\begin{aligned}\sigma_X &= \sqrt{np(1-p)} \\ &= \sqrt{100 \times \frac{1}{2} \times \frac{1}{2}} \\ &= \sqrt{25} \\ &= 5\end{aligned}\qquad \begin{aligned}\sigma_Y &= \sqrt{np(1-p)} \\ &= \sqrt{300 \times \frac{1}{6} \times \frac{5}{6}} \\ &\approx 6.45\end{aligned}$$

- c** X is more likely to lie between 45 and 55 inclusive because the standard deviation of X is lower than that of Y , which means there are more values of X which lie close to the mean.

d i 

$$P(45 \leq X \leq 55) \approx 0.729$$

ii 

$$P(45 \leq Y \leq 55) \approx 0.606$$

REVIEW SET 14A

- 1**
- a** The number of attempts to pass a driving test is a discrete random variable.
 - b** The length of time before a phone loses its battery charge is a continuous random variable.
 - c** The number of phone calls made before a salesperson has sold 3 products is a discrete random variable.

2 a i

x	1	2	3
$P(X = x)$	0.6	0.25	0.15

$$\sum_{x=1}^3 P(X = x) = 0.6 + 0.25 + 0.15 = 1$$

Since $\sum_{x=1}^3 P(X = x) = 1$ and $0 \leq P(X = x) \leq 1$ for all x , it is a valid probability distribution.

ii

x	0	2	5	10
$P(X = x)$	0.3	0.5	0.1	0.2

$$\sum p_i = 0.3 + 0.5 + 0.1 + 0.2 = 1.1$$

Since $\sum p_i > 1$, it is not a valid probability distribution.

iii

x	0	1	2	3
$P(X = x)$	0.4	-0.2	0.35	0.45

Since $P(X = 1) = -0.2 < 0$, this is not a valid probability distribution.

iv

x	2	3	4	5
$P(X = x)$	0.25	0.25	0.25	0.25

$$\sum_{x=2}^5 P(X = x) = 0.25 + 0.25 + 0.25 + 0.25 = 1$$

Since $\sum_{x=2}^5 P(X = x) = 1$ and $0 \leq P(X = x) \leq 1$ for all x , it is a valid probability distribution.

v

x	2	3
$P(X = x)$	0.7	0.3

$$\sum_{x=2}^3 P(X = x) = 0.7 + 0.3 = 1$$

Since $\sum_{x=2}^3 P(X = x) = 1$ and $0 \leq P(X = x) \leq 1$ for all x , it is a valid probability distribution.

vi

x	0	1
$P(X = x)$	0.28	0.72

$$\sum_{x=0}^1 P(X = x) = 0.28 + 0.72 = 1$$

Since $\sum_{x=0}^1 P(X = x) = 1$ and $0 \leq P(X = x) \leq 1$ for all x , it is a valid probability distribution.

b The distribution in **a iv** is a uniform discrete random variable because $p_i = 0.25$ for each value of i .

3 a $P(X = x) = \frac{a}{x^2 + 1}$ for $x = 0, 1, 2, 3$

Since this is a probability mass function,

$$\begin{aligned}\sum_{x=0}^3 P(X = x) &= 1 \\ \therefore a + \frac{a}{2} + \frac{a}{5} + \frac{a}{10} &= 1 \\ \therefore 10a + 5a + 2a + a &= 10 \\ \therefore 18a &= 10 \\ \therefore a &= \frac{5}{9}\end{aligned}$$

x	0	1	2	3
$P(X = x)$	a	$\frac{a}{2}$	$\frac{a}{5}$	$\frac{a}{10}$

b $P(X \geq 1) = 1 - P(X = 0)$
 $= 1 - \frac{5}{9}$
 $= \frac{4}{9}$

4

x	0	1	2	3	4
$P(x)$	0.10	0.30	0.45	0.10	k

a Since this is a probability distribution, $\sum_{i=1}^n P(x_i) = 1$
 $\therefore 0.10 + 0.30 + 0.45 + 0.10 + k = 1$
 $\therefore 0.95 + k = 1$
 $\therefore k = 0.05$

b $P(X \geq 3) = P(X = 3) + P(X = 4)$
 $= 0.10 + 0.05$
 $= 0.15$

c Since $P(X = 2)$ is the greatest probability, 2 is the mode of this distribution.

d $E(X) = \sum_{i=1}^n x_i p_i$
 $= 0(0.10) + 1(0.30) + 2(0.45) + 3(0.10) + 4(0.05)$
 $= 0 + 0.3 + 0.9 + 0.3 + 0.2$
 $= 1.7$

5 a X is a discrete random variable because it has a set of distinct possible values.

b $X = 0, 1$, or 2

c

1st draw	2nd draw	Outcome	X	Probability
	G	GG	2	$\frac{3}{5} \times \frac{2}{4} = \frac{3}{10}$
	Y	GY	1	$\frac{3}{5} \times \frac{2}{4} = \frac{3}{10}$
	G	YG	1	$\frac{2}{5} \times \frac{3}{4} = \frac{3}{10}$
	Y	YY	0	$\frac{2}{5} \times \frac{1}{4} = \frac{1}{10}$

x	0	1	2
$P(x)$	$\frac{1}{10}$	$\frac{3}{5}$	$\frac{3}{10}$

$$\begin{aligned}
 \text{d } E(X) &= \sum_{i=1}^n x_i p_i \\
 &= \left(0 \times \frac{1}{10}\right) + \left(1 \times \frac{3}{5}\right) + \left(2 \times \frac{3}{10}\right) \\
 &= \frac{6}{5} \\
 &= 1.2 \text{ green balls}
 \end{aligned}$$

6 X has probability table:

x	1	3	4	6
$P(x)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{2}{6}$

$$\begin{aligned}
 E(X) &= \sum_{i=1}^n x_i p_i \\
 &= 1\left(\frac{1}{6}\right) + 3\left(\frac{2}{6}\right) + 4\left(\frac{1}{6}\right) + 6\left(\frac{2}{6}\right) \\
 &= \frac{23}{6} \\
 &\approx 3.83
 \end{aligned}$$

7 **a** Let X denote the amount of money Lakshmi wins from one roll.
 X has probability table:

x	2	4	6	8	10	12
$P(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$\begin{aligned}
 E(X) &= \sum_{i=1}^n x_i p_i \\
 &= \left(2 \times \frac{1}{6}\right) + \left(4 \times \frac{1}{6}\right) + \left(6 \times \frac{1}{6}\right) + \left(8 \times \frac{1}{6}\right) + \left(10 \times \frac{1}{6}\right) + \left(12 \times \frac{1}{6}\right) \\
 &= 7
 \end{aligned}$$

\therefore Lakshmi can expect to win \$7 from one roll of the die.

b Expected gain = \$7 - \$8 = -\$1.

So, Lakshmi should not play many games as she would lose \$1 per game in the long run.

8 The number of trials is $n = 5$.

The probability of success (kicking a goal) is $p = 80\% = 0.8$.

Let X be the number of goals scored, and G represent scoring a goal.

$\therefore X \sim B(5, 0.8)$

$$\begin{aligned}
 \text{a } &P(3 \text{ goals then misses twice}) \\
 &= P(GGGG'G') \\
 &= (0.8)^3(1 - 0.8)^2 \\
 &= 0.02048
 \end{aligned}$$

$$\begin{aligned}
 \text{b } &P(3 \text{ goals and misses twice}) \\
 &= P(3 \text{ goals}) \\
 &= P(X = 3) \\
 &= \binom{5}{3} (0.8)^3(0.2)^2 \\
 &= 10 \times 0.02048 \\
 &= 0.2048
 \end{aligned}$$

9 $P(x) = a(x^2 - 8x)$ where $x = 0, 1, 2, 3, \dots, 8$

a X has probability table:

x	0	1	2	3	4	5	6	7	8
$P(x)$	0	$-7a$	$-12a$	$-15a$	$-16a$	$-15a$	$-12a$	$-7a$	0

If this is a probability distribution then $\sum_{i=1}^n P(x_i) = 1$

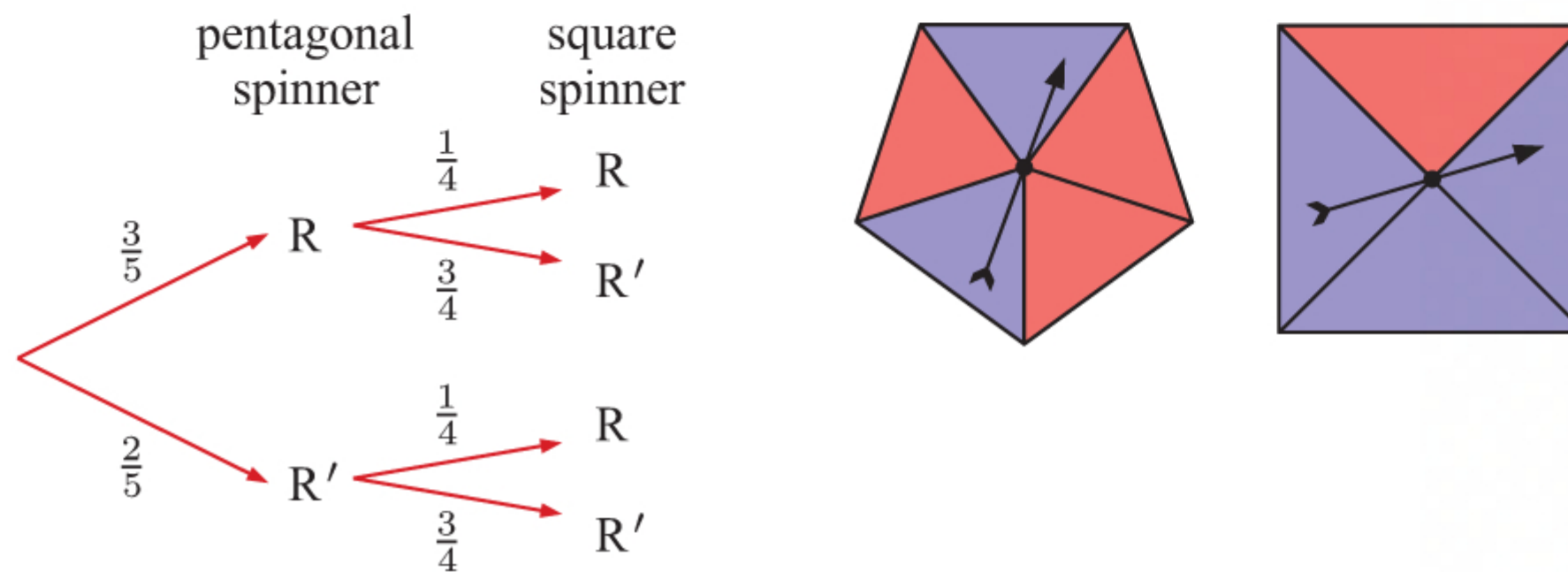
$$\therefore 0 + (-7a) + (-12a) + (-15a) + (-16a) + (-15a) + (-12a) + (-7a) + 0 = 1$$

$$\therefore -84a = 1$$

$$\therefore a = -\frac{1}{84}$$

b $E(X) = 0(0) + 1\left(\frac{7}{84}\right) + 2\left(\frac{12}{84}\right) + 3\left(\frac{15}{84}\right) + 4\left(\frac{16}{84}\right) + 5\left(\frac{15}{84}\right) + 6\left(\frac{12}{84}\right) + 7\left(\frac{7}{84}\right) + 8(0)$
 $= 4$ marsupials

10 a



b $P(\text{exactly one red}) = P(RR') + P(R'R)$
 $= \frac{3}{5} \times \frac{3}{4} + \frac{2}{5} \times \frac{1}{4}$
 $= \frac{9}{20} + \frac{1}{10}$
 $= \frac{11}{20}$

c i $X \sim B\left(10, \frac{11}{20}\right)$

ii $n = 10, p = \frac{11}{20}$

$$P(X = 1) = \binom{10}{1} \left(\frac{11}{20}\right)^1 \left(\frac{9}{20}\right)^9 \approx 0.00416$$

$$P(X = 9) = \binom{10}{9} \left(\frac{11}{20}\right)^9 \left(\frac{9}{20}\right)^1 \approx 0.0207$$

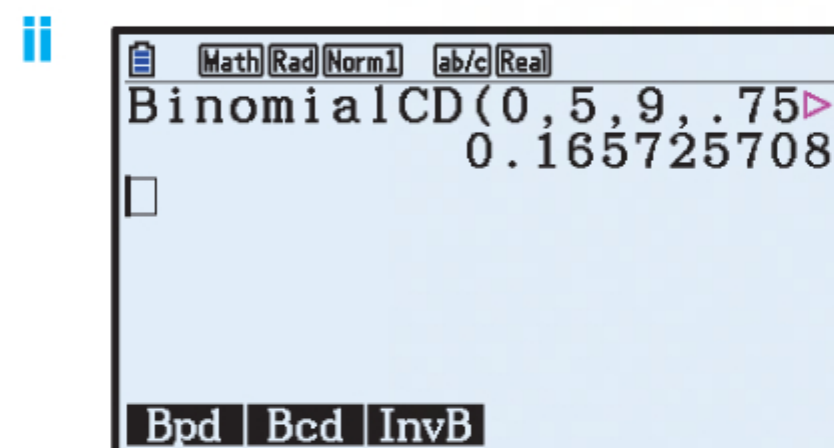
\therefore it is more likely that exactly one red will occur 9 times.

- 11** Let X denote the number of players who turn up to a game.

$n = 9$, so $X = 0, 1, 2, 3, \dots$, or 9 , and $p = 75\% = 0.75$

$\therefore X \sim B(9, 0.75)$

a i $P(X = 9) = \binom{9}{9}(0.75)^9(0.25)^0$
 ≈ 0.0751



$$\begin{aligned} P(\text{forfeit}) &= P(X < 6) \\ &= P(X \leq 5) \\ &\approx 0.166 \end{aligned}$$

- b** The team is expected to forfeit $30 \times 0.1657 \approx 4.97$ games throughout the season.

- 12** Let X denote the number of batteries that are defective.

$n = 20$, so $X = 0, 1, 2, 3, \dots$, or 20 , and $p = \frac{3}{100}$

$\therefore X \sim B(20, \frac{3}{100})$

a $P(X = 0) = \binom{20}{0} \left(\frac{3}{100}\right)^0 \left(\frac{97}{100}\right)^{20}$
 ≈ 0.544

b $P(X \geq 1) = 1 - P(X = 0)$
 $\approx 1 - 0.544$
 ≈ 0.456

REVIEW SET 14B

1

x	0	1	2	3	4	5
$P(X = x)$	0.07	0.14	k	0.46	0.08	0.02

- a** The random variable X represents the number of hits that Sally has in a softball match.
 $X = 0, 1, 2, 3, 4$, or 5

b i Since this is a probability distribution, $\sum_{x=0}^5 P(X = x) = 1$
 $\therefore 0.07 + 0.14 + k + 0.46 + 0.08 + 0.02 = 1$
 $\therefore k + 0.77 = 1$
 $\therefore k = 0.23$

ii $P(X \geq 2) = 1 - P(X \leq 1)$
 $= 1 - (0.07 + 0.14)$
 $= 0.79$

iii $P(1 \leq X \leq 3) = P(X = 1) + P(X = 2) + P(X = 3)$
 $= 0.14 + 0.23 + 0.46$
 $= 0.83$

- c It is most likely that Sally will have 3 hits in one softball match, so 3 hits is the mode.

$$p_1 = 0.07$$

$$p_1 + p_2 = 0.07 + 0.14 = 0.21$$

$$p_1 + p_2 + p_3 = 0.07 + 0.14 + 0.23 = 0.44$$

$$p_1 + p_2 + p_3 + p_4 = 0.07 + 0.14 + 0.23 + 0.46 = 0.9$$

Since $p_1 + p_2 + p_3 + p_4 \geq 0.5$, the median is 3 hits.

2 a $P(x) = \frac{e^x}{1+e}, \quad x = 0, 1$

$$P(0) = \frac{1}{1+e}, \quad P(1) = \frac{e}{1+e}$$

Both of these obey $0 \leq P(x_i) \leq 1$, and $\sum_{i=1}^n P(x_i) = \frac{1}{1+e} + \frac{e}{1+e} = 1$

$\therefore P(x)$ is a valid probability mass function.

b $P(x) = \frac{x^2 + x}{40}, \quad x = 1, 2, 3, 4$

$$P(1) = \frac{1+1}{40} = \frac{2}{40}, \quad P(2) = \frac{4+2}{40} = \frac{6}{40}, \quad P(3) = \frac{9+3}{40} = \frac{12}{40}, \quad P(4) = \frac{16+4}{40} = \frac{20}{40}$$

All of these obey $0 \leq P(x_i) \leq 1$, and $\sum_{i=1}^n P(x_i) = \frac{2}{40} + \frac{6}{40} + \frac{12}{40} + \frac{20}{40} = 1$

$\therefore P(x)$ is a valid probability mass function.

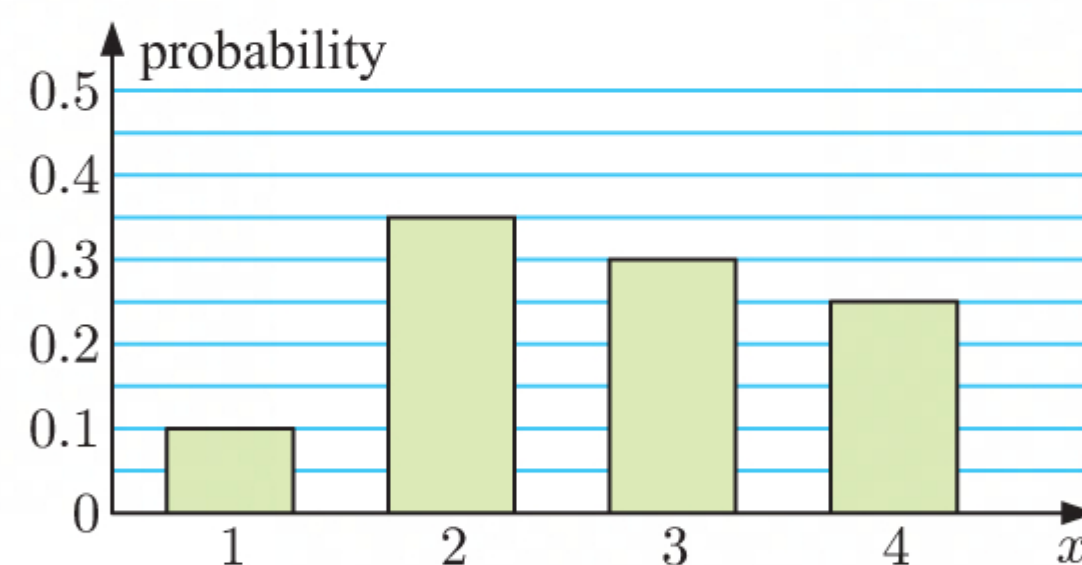
- 3 a 2 has the highest probability of occurring, so this is the mode of the distribution.

b $p_1 = 0.1$

$$p_1 + p_2 = 0.1 + 0.35 = 0.45$$

$$p_1 + p_2 + p_3 = 0.1 + 0.35 + 0.3 = 0.75$$

Since $p_1 + p_2 + p_3 \geq 0.5$, the median is 3.



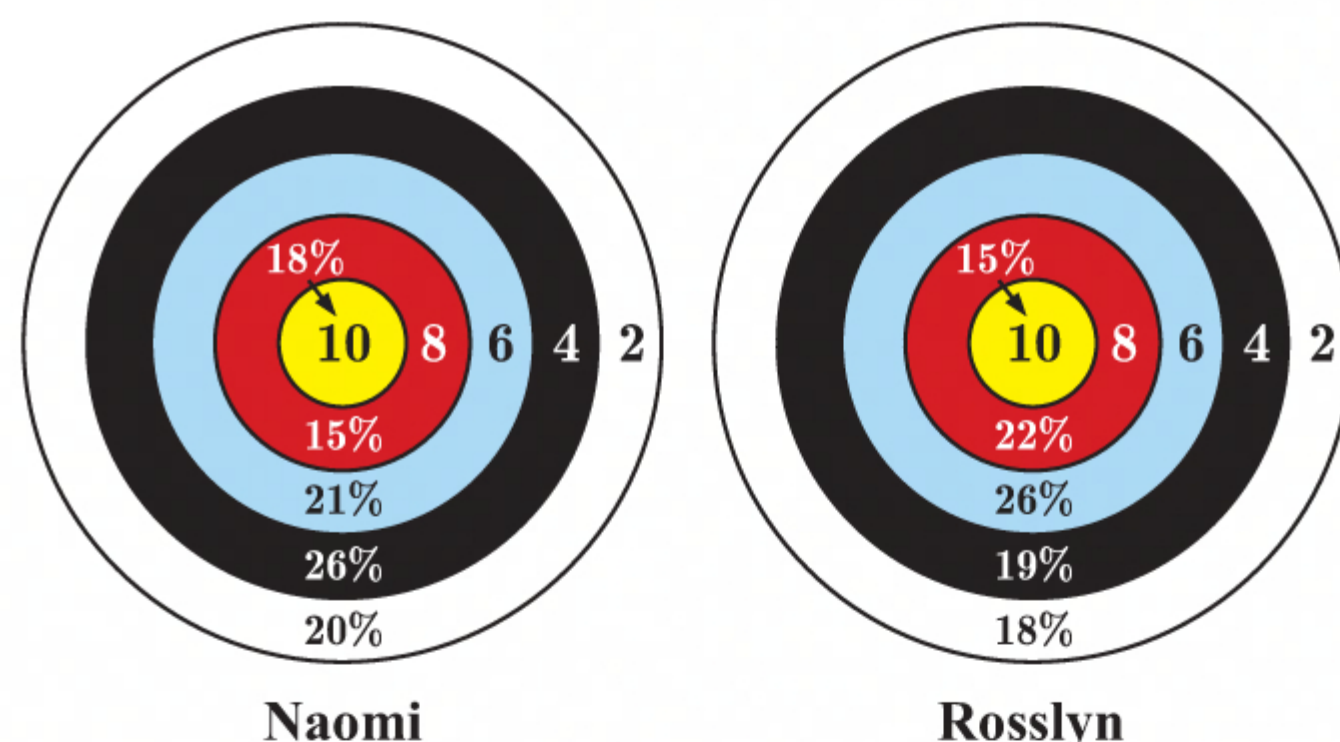
c $E(X) = (1 \times 0.1) + (2 \times 0.35) + (3 \times 0.3) + (4 \times 0.25)$
 $= 2.7$

- 4 Let N be the number of points Naomi scores per shot.
 Let R be the number of points Rosslyn scores per shot.

a i $P(N = 10) = 0.18$

$$P(R = 10) = 0.15$$

\therefore Naomi is more likely to score 10 points on a single shot.



$$\begin{aligned}
 \text{ii } P(N \geq 6) &= P(N = 6) + P(N = 8) + P(N = 10) \\
 &= 0.21 + 0.15 + 0.18 \\
 &= 0.54
 \end{aligned}$$

$$\begin{aligned}
 P(R \geq 6) &= P(R = 6) + P(R = 8) + P(R = 10) \\
 &= 0.26 + 0.22 + 0.15 \\
 &= 0.63
 \end{aligned}$$

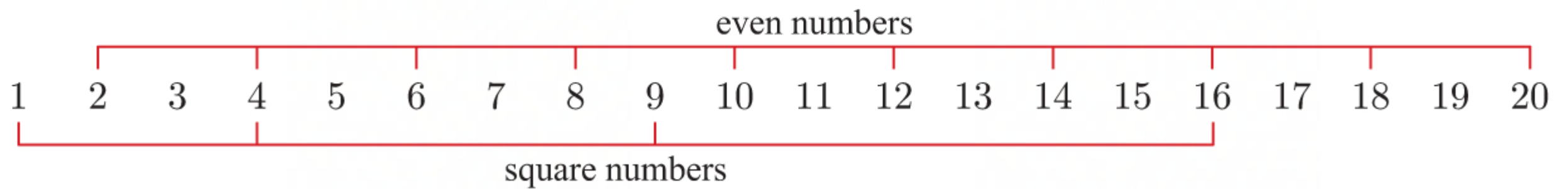
\therefore Rosslyn is more likely to score at least 6 points.

$$\begin{aligned}
 \text{b } E(N) &= (2 \times 0.2) + (4 \times 0.26) + (6 \times 0.21) + (8 \times 0.15) + (10 \times 0.18) \\
 &= 5.7 \text{ points}
 \end{aligned}$$

$$\begin{aligned}
 E(R) &= (2 \times 0.18) + (4 \times 0.19) + (6 \times 0.26) + (8 \times 0.22) + (10 \times 0.15) \\
 &= 5.94 \text{ points}
 \end{aligned}$$

\therefore in the long run, Rosslyn is expected to score more points per shot.

5



Let X denote the number written on the ticket drawn.

$$\begin{aligned}
 \text{a i } P(\text{player wins \$3}) &= P(X \text{ is even but not square}) \\
 &= \frac{8}{20} \\
 &= \frac{2}{5}
 \end{aligned}$$

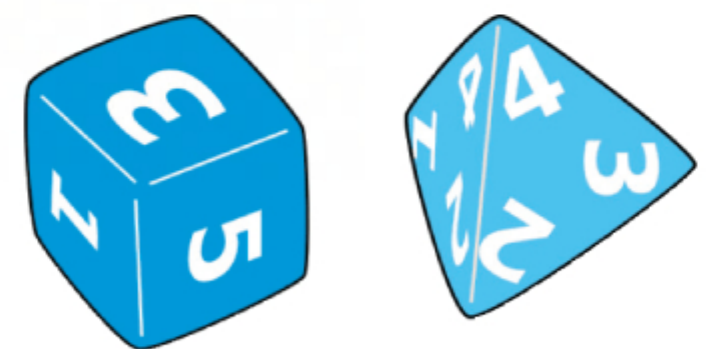
$$\begin{aligned}
 \text{ii } P(\text{player wins \$6}) &= P(X \text{ is square but not even}) \\
 &= \frac{2}{20} \\
 &= \frac{1}{10}
 \end{aligned}$$

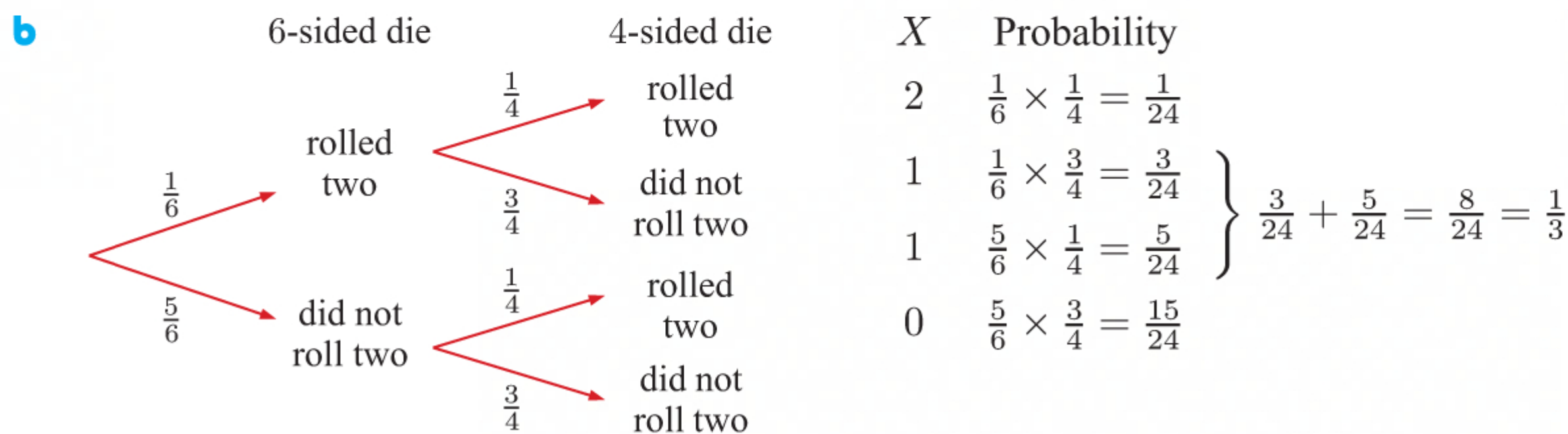
$$\begin{aligned}
 \text{iii } P(\text{player wins \$9}) &= P(X \text{ is even and square}) \\
 &= \frac{2}{20} \\
 &= \frac{1}{10}
 \end{aligned}$$

$$\begin{aligned}
 \text{b The expected gain of one game is } E(X) &= \left(0 \times \frac{8}{20}\right) + \left(3 \times \frac{2}{5}\right) + \left(6 \times \frac{1}{10}\right) + \left(9 \times \frac{1}{10}\right) \\
 &= \frac{27}{10} \\
 &= \$2.70 \text{ per game}
 \end{aligned}$$

To make the game fair, the game must cost the same as the expected gain, so \$2.70 should be charged each game.

- 6 a X is not a binomial random variable because the probability of rolling a two is not the same for each die.





x	0	1	2
$P(X = x)$	$\frac{15}{24}$	$\frac{1}{3}$	$\frac{1}{24}$

c $E(X) = 0\left(\frac{15}{24}\right) + 1\left(\frac{1}{3}\right) + 2\left(\frac{1}{24}\right) = \frac{5}{12}$

- 7** Since this is a probability distribution, $\sum p_i = 1$
 $\therefore 0.2 + a + 0.3 + b = 1$
 $\therefore b = 0.5 - a \quad \dots (*)$

x_i	1	2	3	4
p_i	0.2	a	0.3	b

Now, $E(X) = 2.8$
 $\therefore (1 \times 0.2) + (2 \times a) + (3 \times 0.3) + (4 \times b) = 2.8$
 $\therefore 0.2 + 2a + 0.9 + 4(0.5 - a) = 2.8 \quad \{\text{using } (*)\}$
 $\therefore 2a + 2 - 4a = 1.7$
 $\therefore -2a = -0.3$
 $\therefore a = 0.15 \text{ and } b = 0.35$

8

x	0	1	2	3	4
$P(X = x)$	0.1	0.2	0.3	0.3	0.1

a $E(X) = 0(0.1) + 1(0.2) + 2(0.3) + 3(0.3) + 4(0.1)$
 $= 2.1$

- b** If Caleb drops the ball y times, he catches the ball $4 - y$ times.

$\therefore P(Y = y) = P(X = 4 - y)$

So, the probability distribution of Y is:

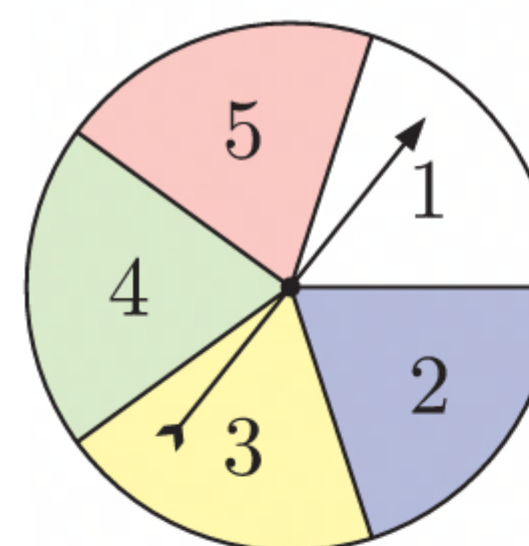
y	0	1	2	3	4
$P(Y = y)$	0.1	0.3	0.3	0.2	0.1

$\therefore E(Y) = 0(0.1) + 1(0.3) + 2(0.3) + 3(0.2) + 4(0.1)$
 $= 1.9$

- 9 a** The probability of success (spinning a 3) is the same for each spin, and the number of spins is fixed.

b $\mu = np$
 $= 20 \times \frac{1}{5}$
 $= 4$

$\sigma = \sqrt{np(1-p)}$
 $= \sqrt{20 \times \frac{1}{5} \times \frac{4}{5}}$
 $= \frac{4}{\sqrt{5}} \approx 1.79$



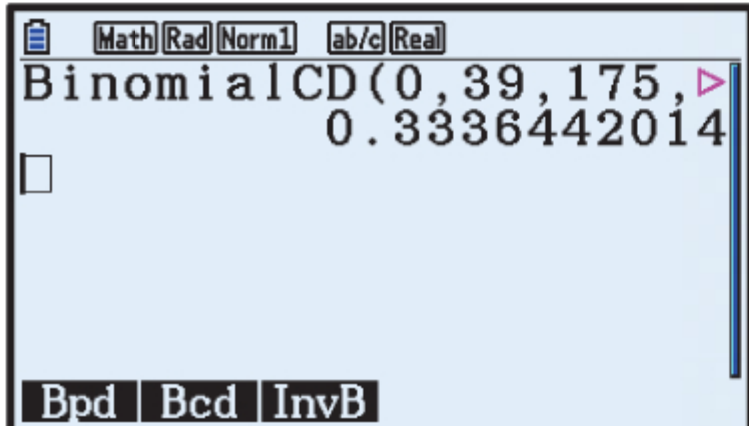
- 10** Let X be the number of visitors who make a voluntary donation upon entry.

$n = 175$, so $X = 0, 1, 2, 3, \dots$, or 175 , and $p = 24\% = 0.24$

$$\therefore X \sim B(175, 0.24)$$

a $E(X) = \mu = np$
 $= 175 \times 0.24$
 $= 42$ donations

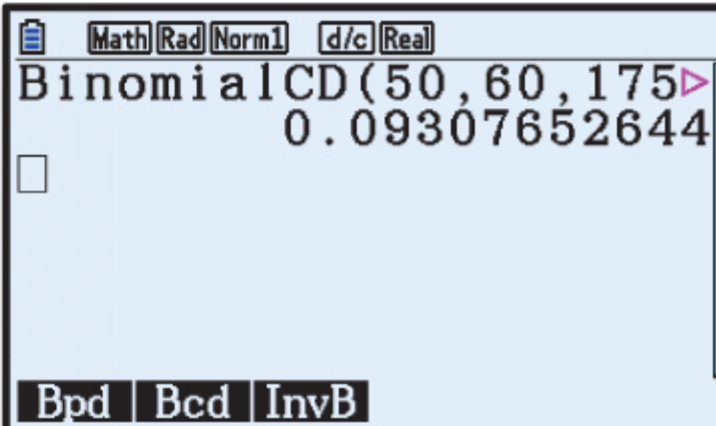
b i



BinomialCDF(0,39,175,0.24)
 0.3336442014

$$P(X < 40) = P(X \leq 39) \\ \approx 0.334$$

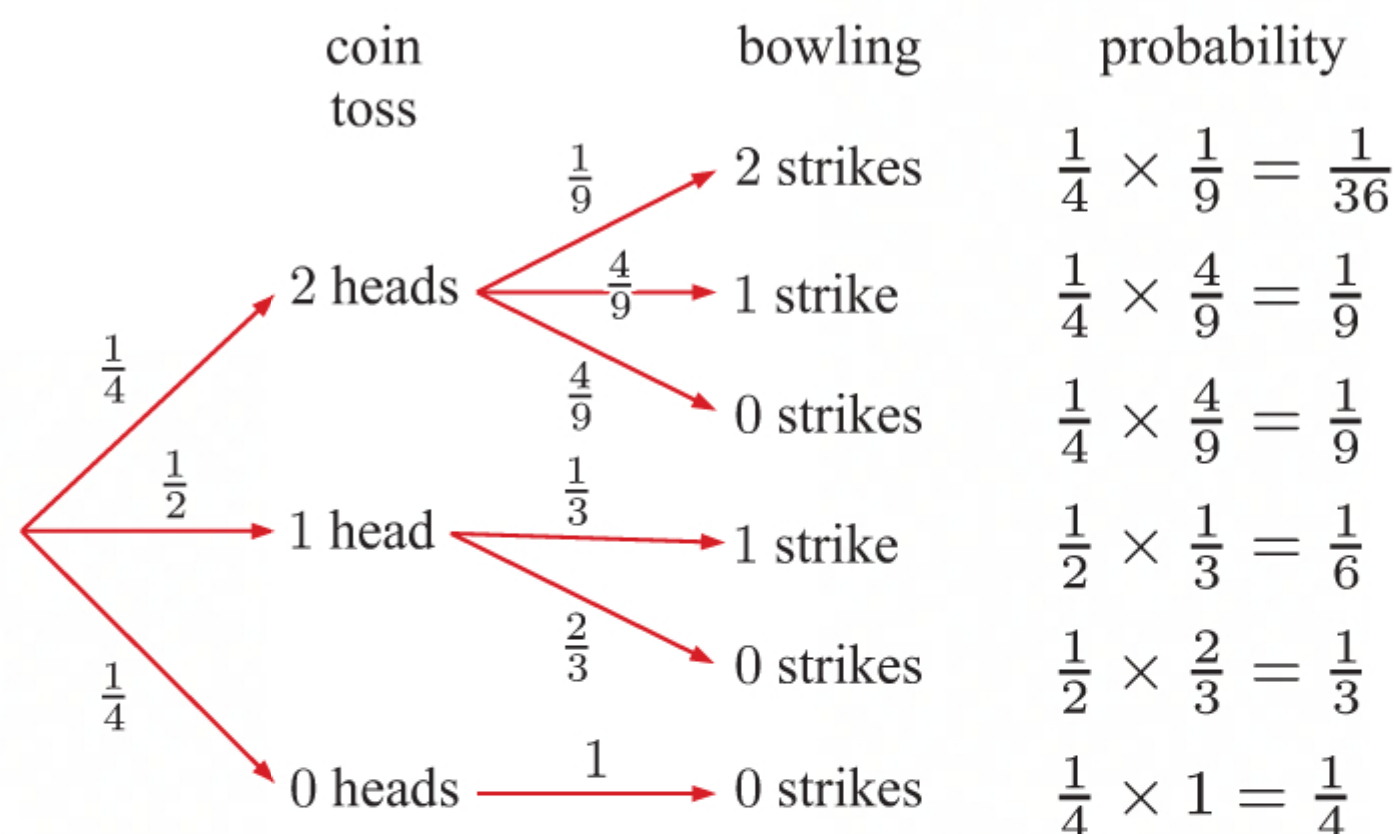
ii



BinomialCDF(50,60,175,0.24)
 0.09307652644

$$P(50 \leq X \leq 60) \approx 0.0931$$

- 11 a**



b $P(X = 0) = \frac{1}{9} + \frac{1}{3} + \frac{1}{4} = \frac{25}{36}$ $P(X = 1) = \frac{1}{9} + \frac{1}{6} = \frac{5}{18}$ $P(X = 2) = \frac{1}{36}$

x	0	1	2
$P(X = x)$	$\frac{25}{36}$	$\frac{5}{18}$	$\frac{1}{36}$

c The expected return per game is $E(X) = (0 \times \frac{25}{36}) + (10 \times \frac{5}{18}) + (20 \times \frac{1}{36})$
 $= \frac{10}{3}$ dollars
 $\approx \$3.33$

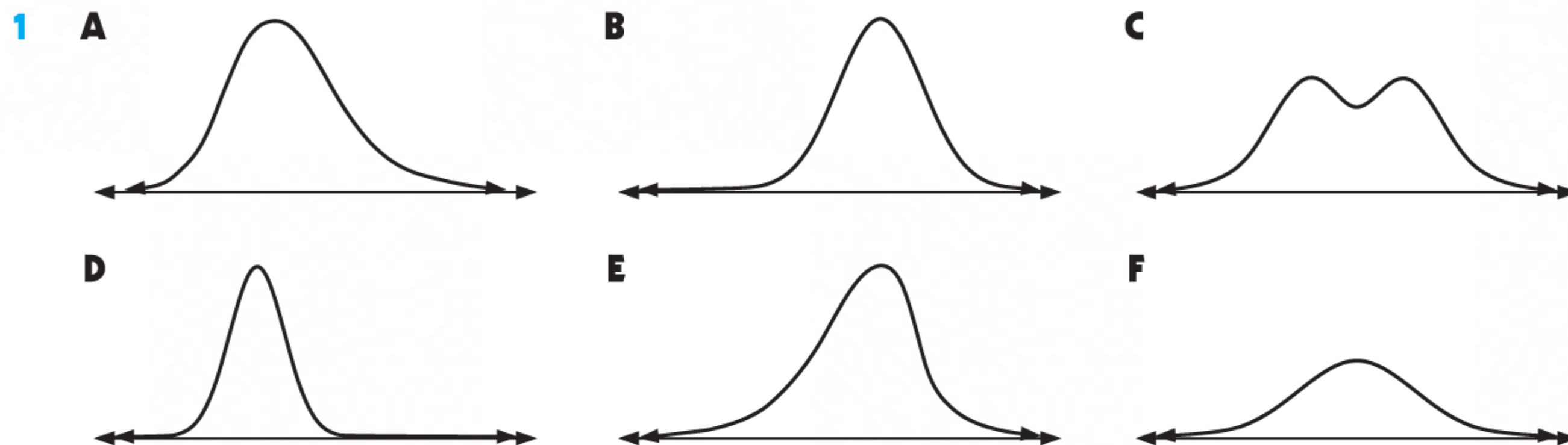
d Suvi's expected gain per game $\approx \$3.33 - \5
 $\approx -\$1.67$

\therefore Suvi should not play the game many times as she is expected to lose \$1.67 per game on average.

Chapter 15

THE NORMAL DISTRIBUTION

EXERCISE 15A.1



Distributions **B**, **D**, and **F** are symmetrical and bell-shaped.

∴ **B**, **D**, and **F** appear to be normally distributed.

- 2 Most measurements in each situation will be centred about the mean, with random variation about the mean explained by some of the factors listed below.

a The diameters may be affected by:

- the type of lathe used
- the steadiness of the woodworker's hand
- the operating speed of the lathe.

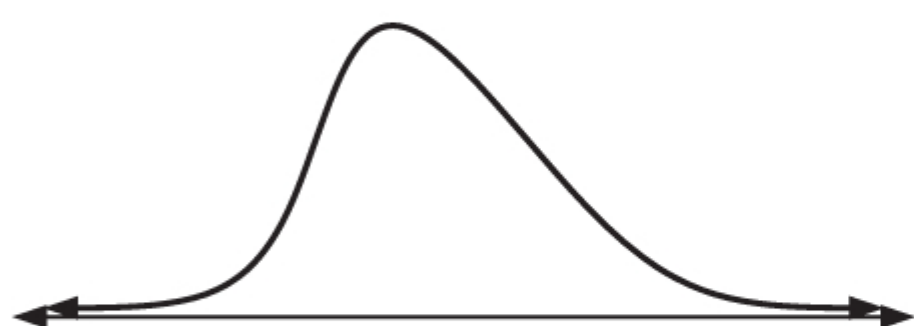
b The scores may be affected by:

- the time spent studying
- natural ability (for example, memory, learning ability)
- general knowledge.

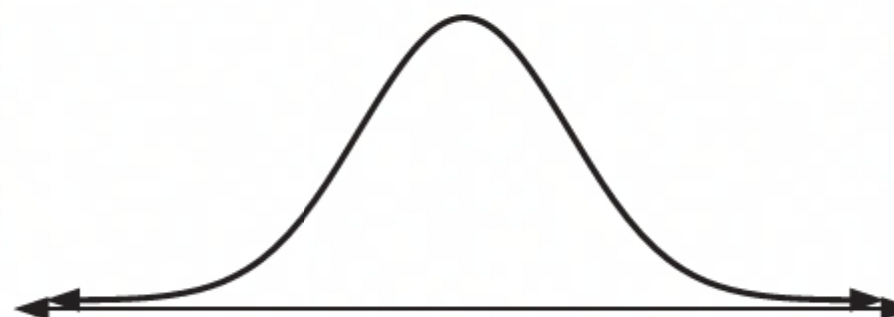
c The times may be affected by:

- weather conditions
- walking speed
- physical fitness
- traffic.

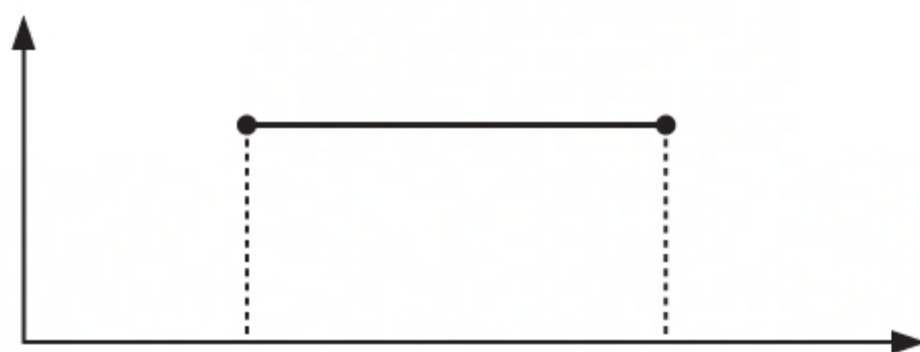
- 3 **a** The variable is not likely to be normally distributed as it is more likely that there would be more people younger than the mean age than there are older. The distribution may be positively skewed.



- b** The variable is likely to be normally distributed as the long jumper is likely to jump the same distance consistently, but it will vary due to factors such as the speed at which the long jumper runs before the jump, and the positioning of their body before hitting the sand.



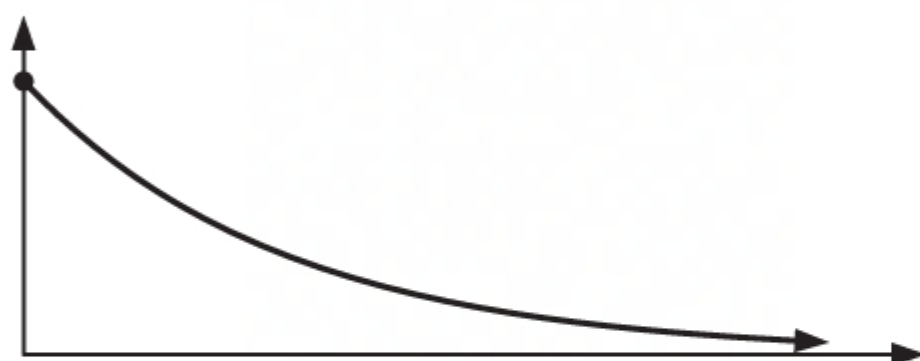
- c** The variable is not likely to be normally distributed as each number has the same chance of being drawn. The distribution should be uniform.



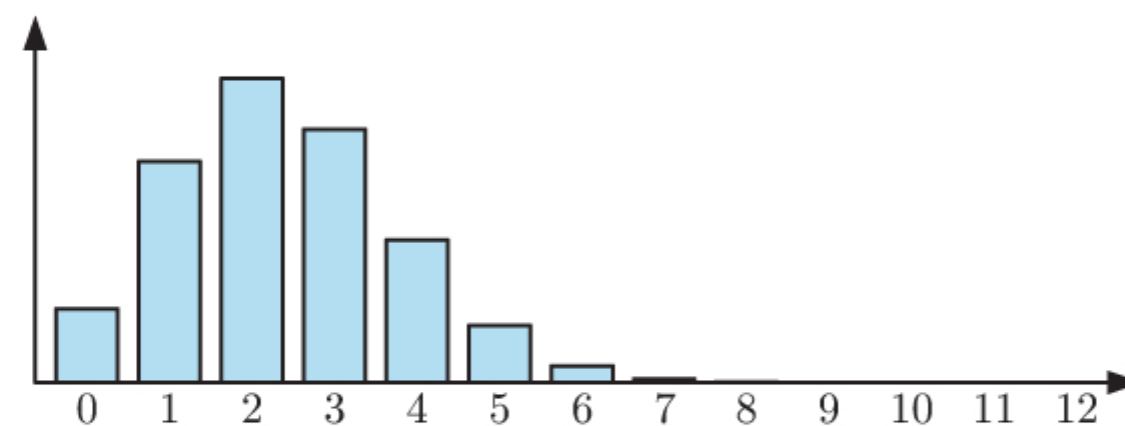
- d** The variable is likely to be normally distributed as the lengths of the carrots will be generally centred around the mean, but will vary due to factors such as soil quality, different weather conditions, harvest times, and so on.



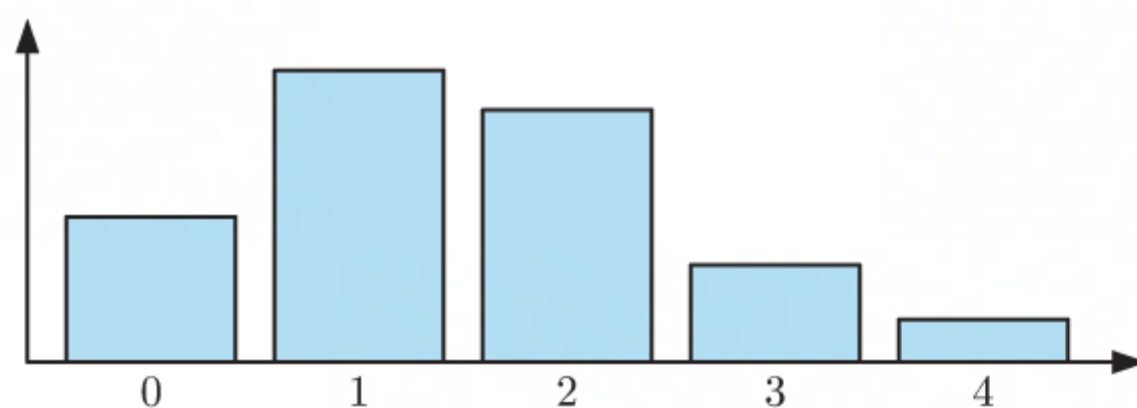
- e** The variable is not likely to be normally distributed. People are most likely to be served quite quickly. The distribution is likely to be negatively skewed.



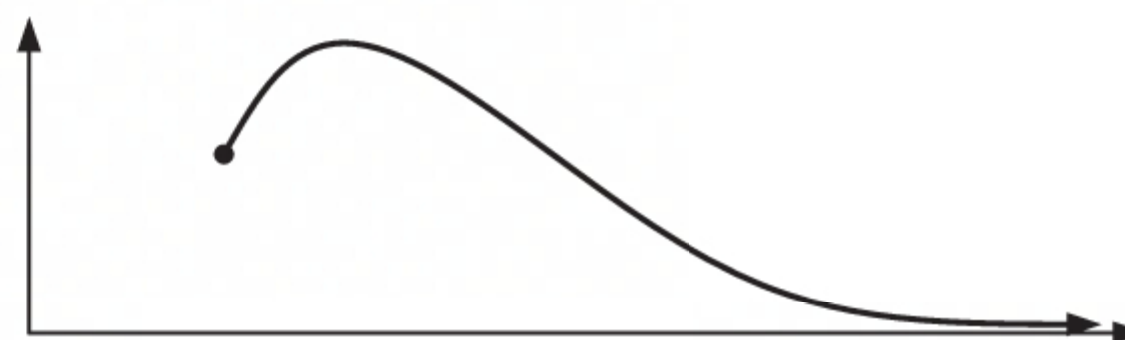
- f** The variable is not likely to be normal as it is a discrete variable. Each egg has the same probability of being brown, so the distribution is binomial.



- g** The variable is not likely to be normally distributed as it is a discrete variable. Most families will have 0 - 2 children, and there will be much fewer families with more than 2 children. The distribution will be positively skewed.



- h** The variable is not likely to be normally distributed as there will tend to be many more shorter buildings than tall buildings in a city. The distribution will be positively skewed.



INVESTIGATION 1

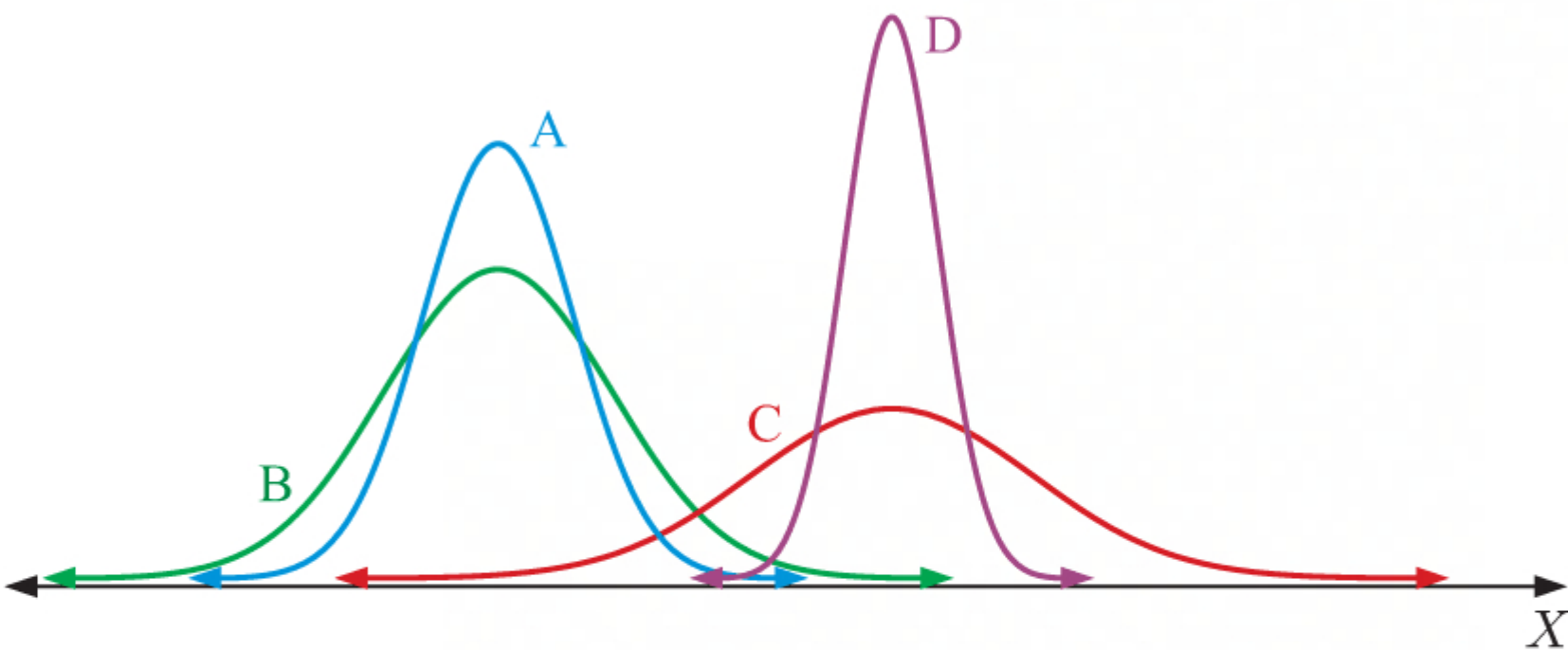
PROPERTIES OF THE NORMAL CURVE

- μ controls where the centre of the distribution is. As μ changes, the curve is translated horizontally, which is reasonable since μ is the mean and a measure of centre.
 σ controls the shape of the curve. As σ increases, the curve becomes flatter and more spread out, which is reasonable since σ is the standard deviation and a measure of spread.
- The curve has a vertical line of symmetry $x = \mu$.
- The function is never negative. This is important because a probability density function can never be negative.

- 4 As $x \rightarrow \pm\infty$, the curve approaches zero from above. The x -axis is a horizontal asymptote.
- 5 The area under the curve should remain constant as we change μ and σ , as the area under a probability density function must be 1.

EXERCISE 15A.2

1

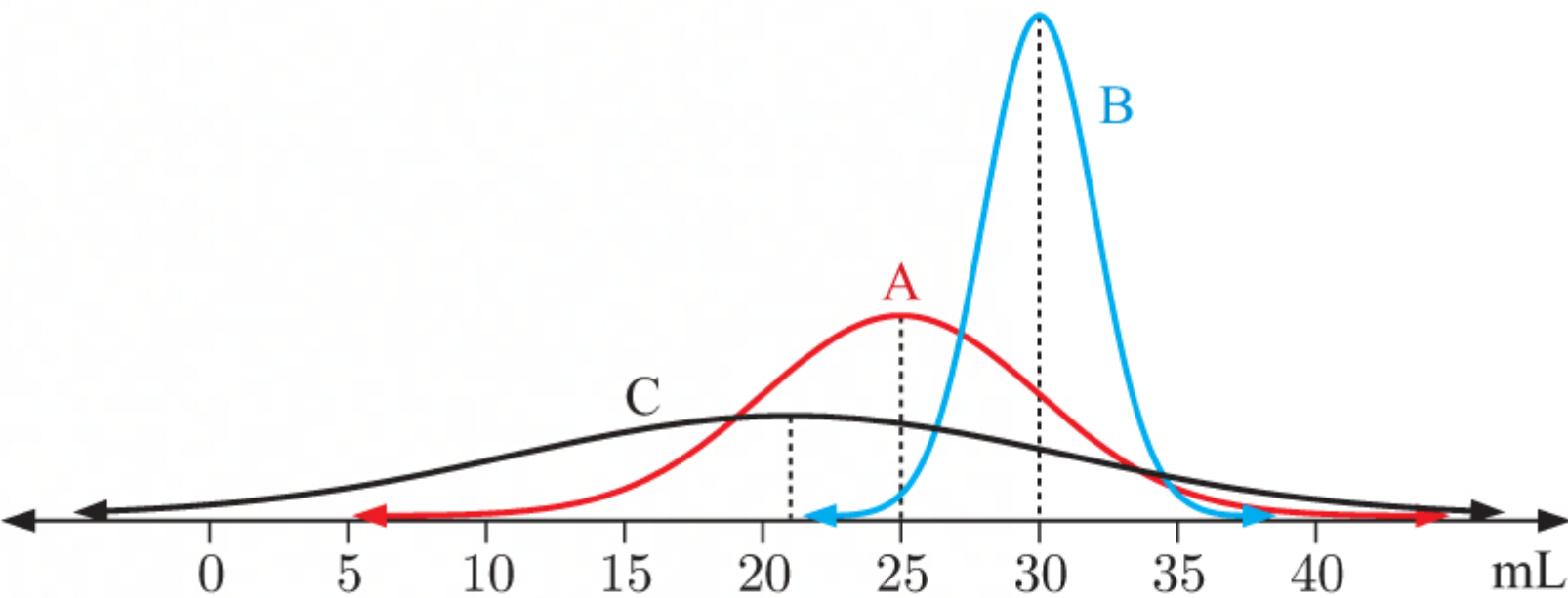


A and B have the same mean, and C and D have the same mean. The mean of A and B is lower than the mean of C and D. B has a greater spread, and hence a larger standard deviation than A. Similarly, C has a larger standard deviation than D.

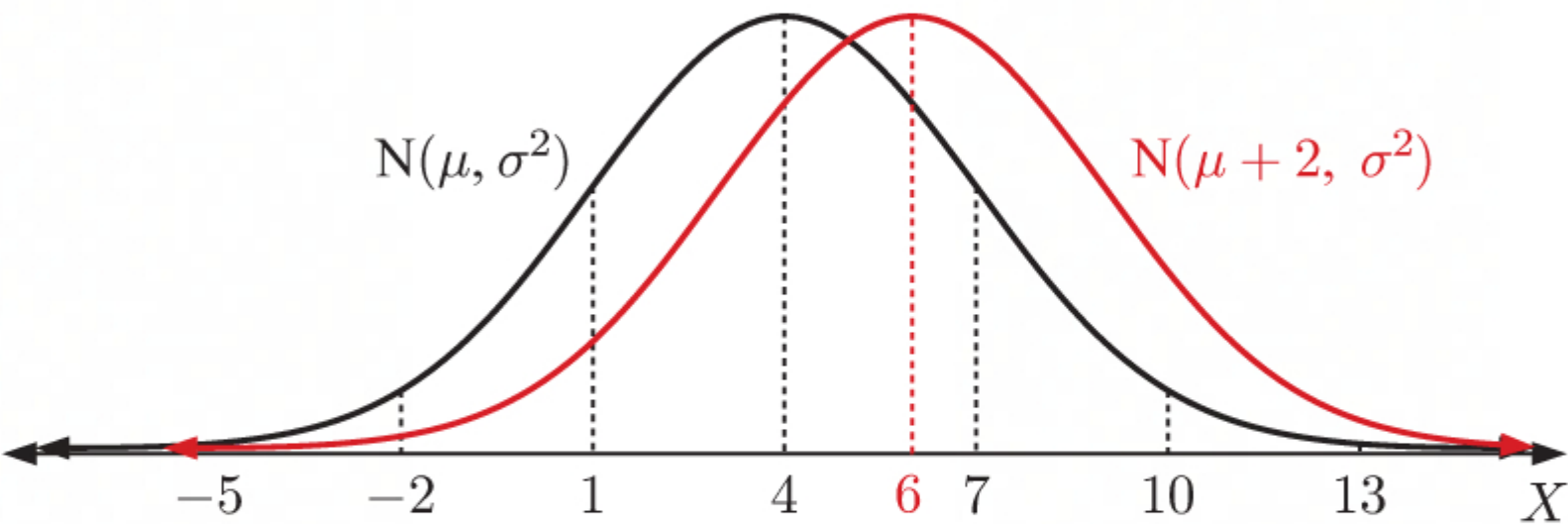
- a $\mu = 5, \sigma = 2$ corresponds to B
- b $\mu = 15, \sigma = 0.5$ corresponds to D
- c $\mu = 5, \sigma = 1$ corresponds to A
- d $\mu = 15, \sigma = 3$ corresponds to C

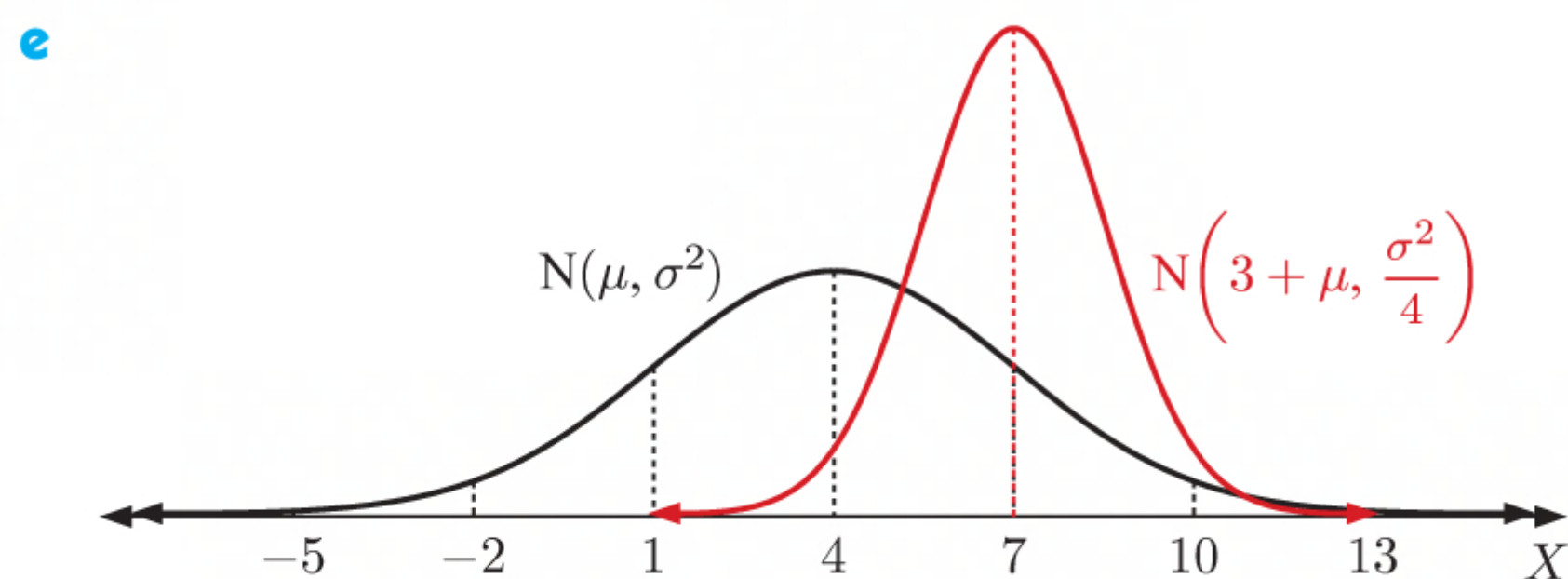
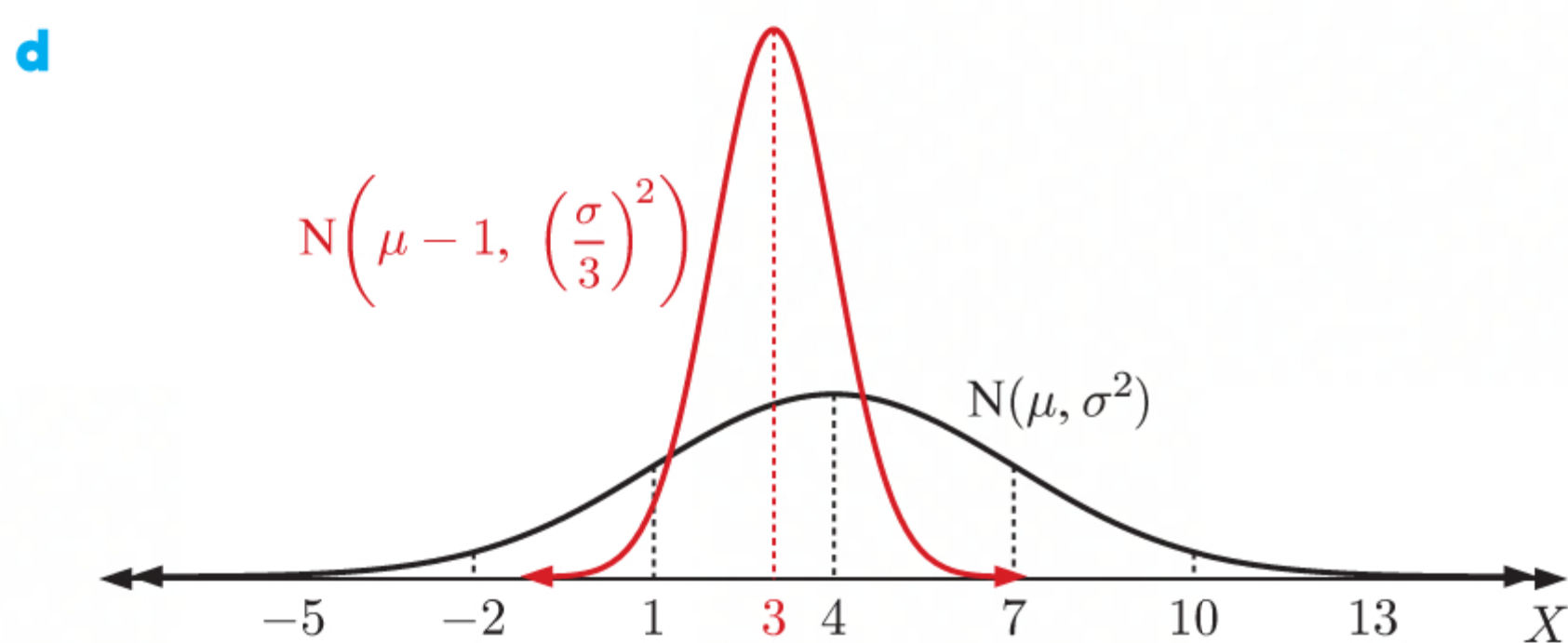
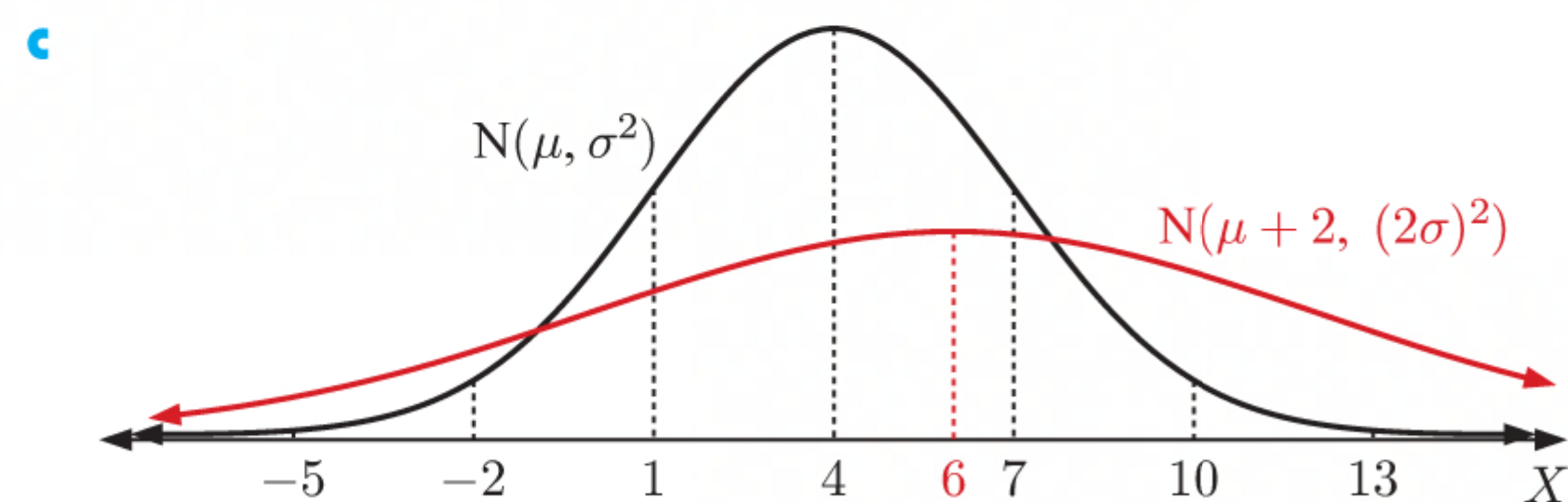
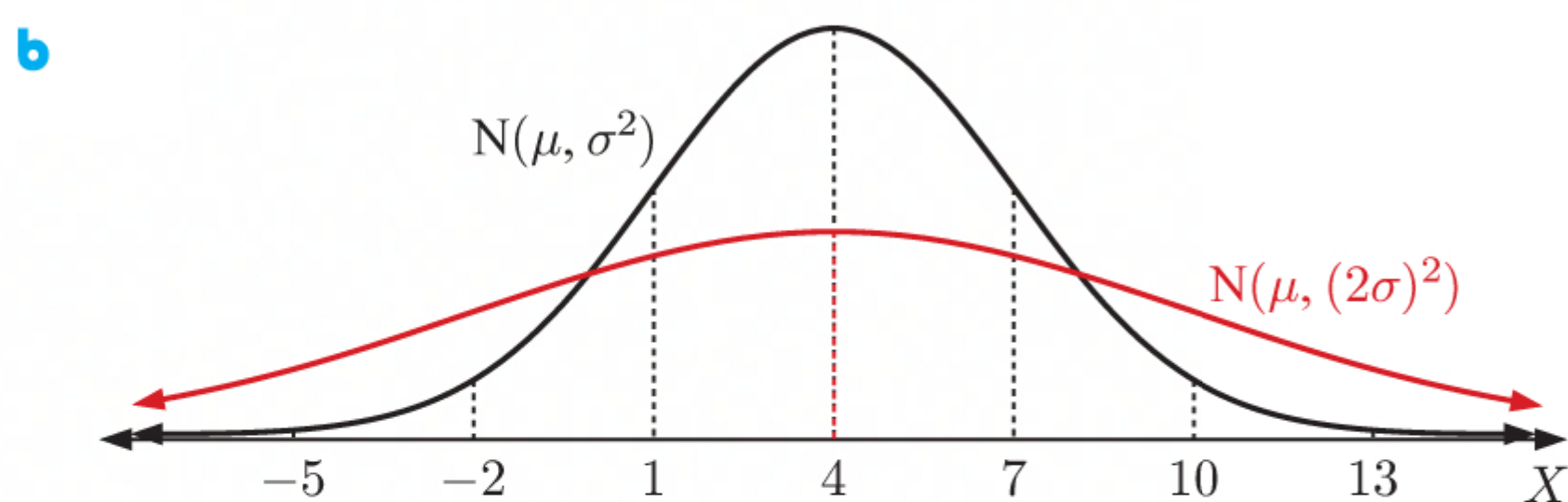
2

Distribution	Mean (mL)	Standard deviation (mL)
A	25	5
B	30	2
C	21	10



3 a





EXERCISE 15B.1

1 $X \sim N(30, 5^2)$

a **i** The value which is 2 standard deviations above the mean $= 30 + 2 \times 5$
 $= 40$

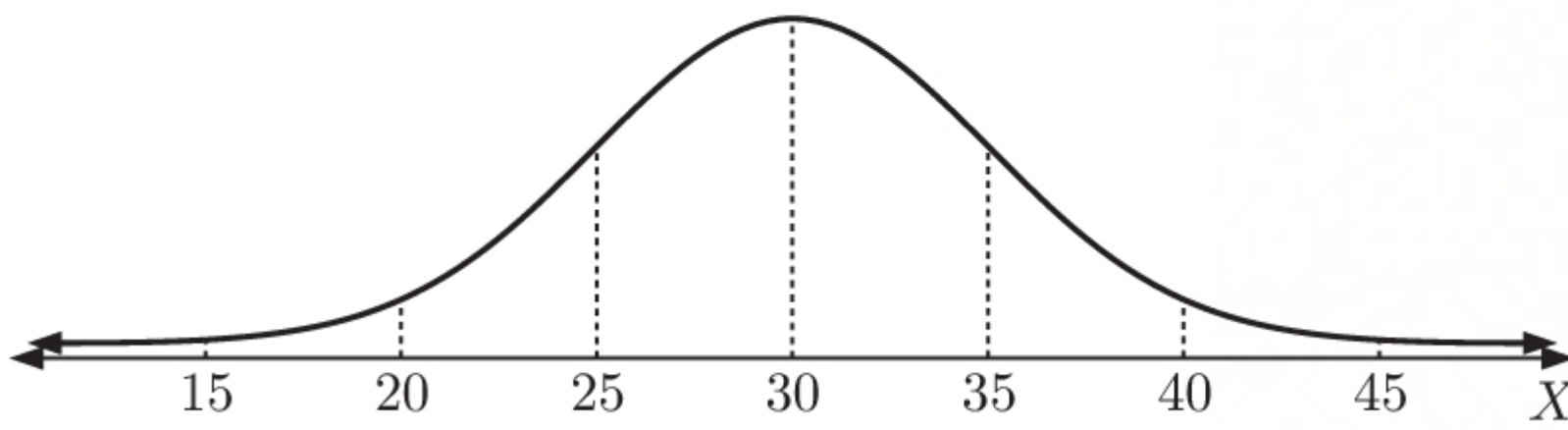
ii The value which is 1 standard deviation below the mean $= 30 - 5$
 $= 25$

b **i** $35 = 30 + 5$
 $\therefore 35$ is 1 standard deviation above the mean.

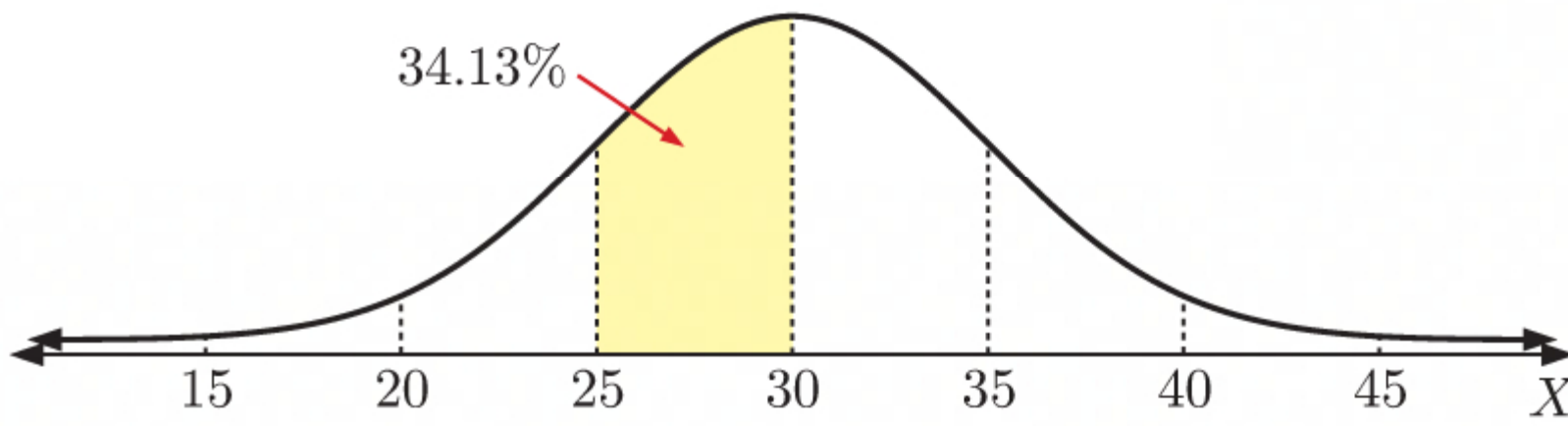
ii $20 = 30 - 2 \times 5$
 $\therefore 20$ is 2 standard deviations below the mean.

iii $45 = 30 + 3 \times 5$
 $\therefore 45$ is 3 standard deviations above the mean.

c

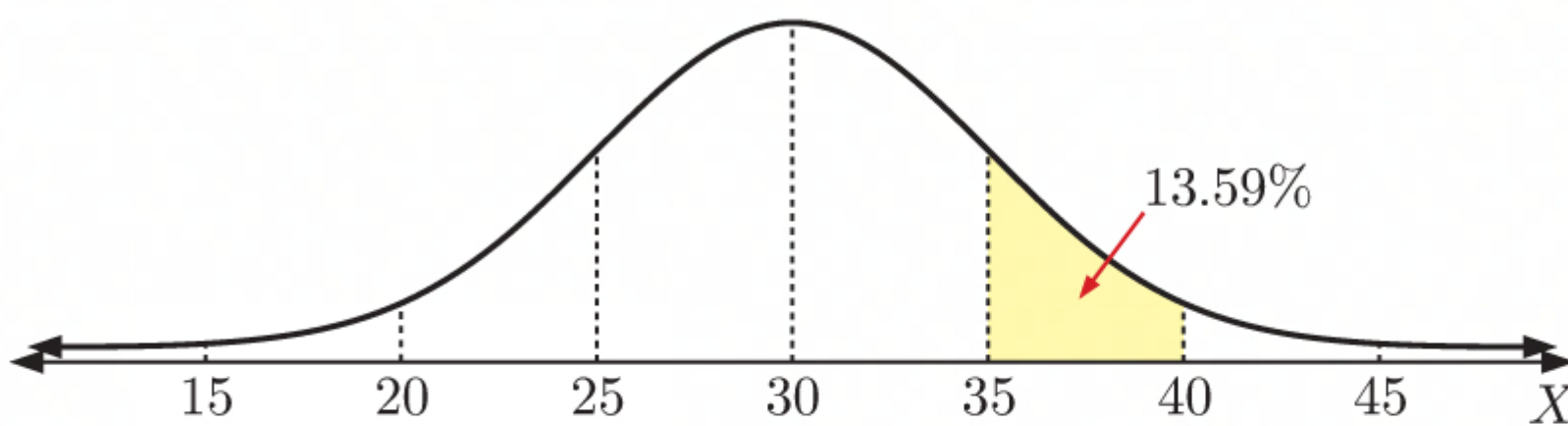


d



About 34.13% of the values of X are between 25 and 30.

e

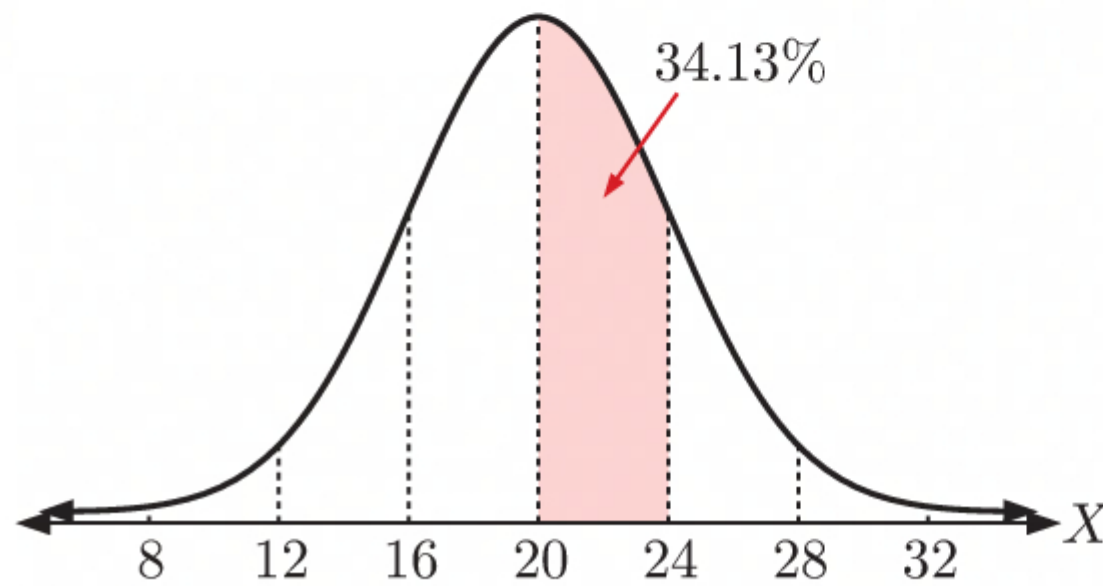


About 13.59% of the values of X are between 35 and 40.

\therefore the probability that a randomly selected member of the population will measure between 35 and 40 is approximately 0.1359.

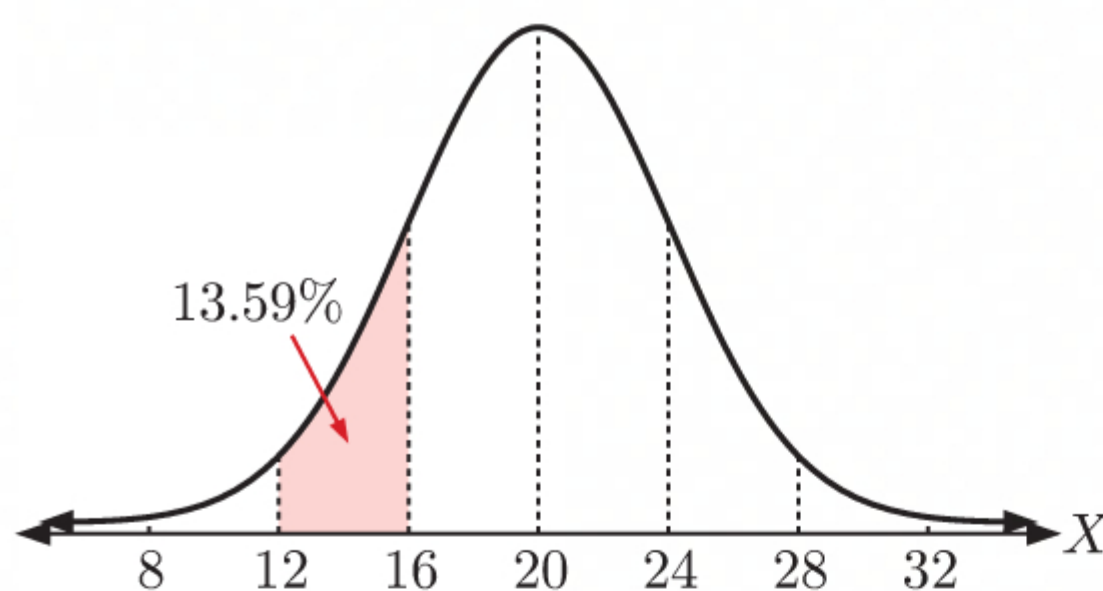
2 a $\mu = 20, \sigma = 4$

b i



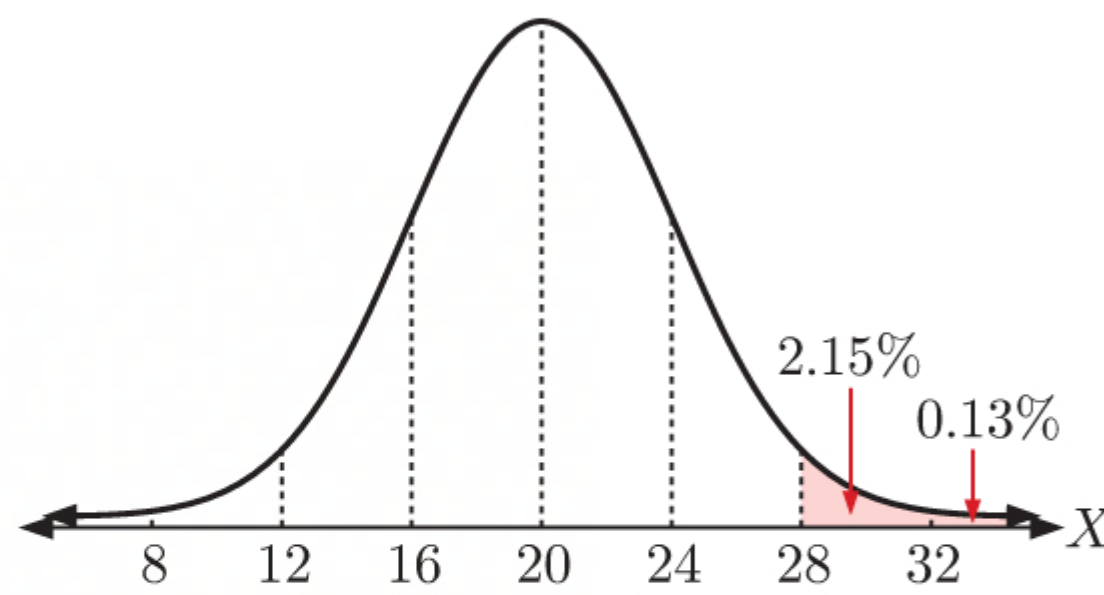
About 34.13% of X values are between 20 and 24.

ii



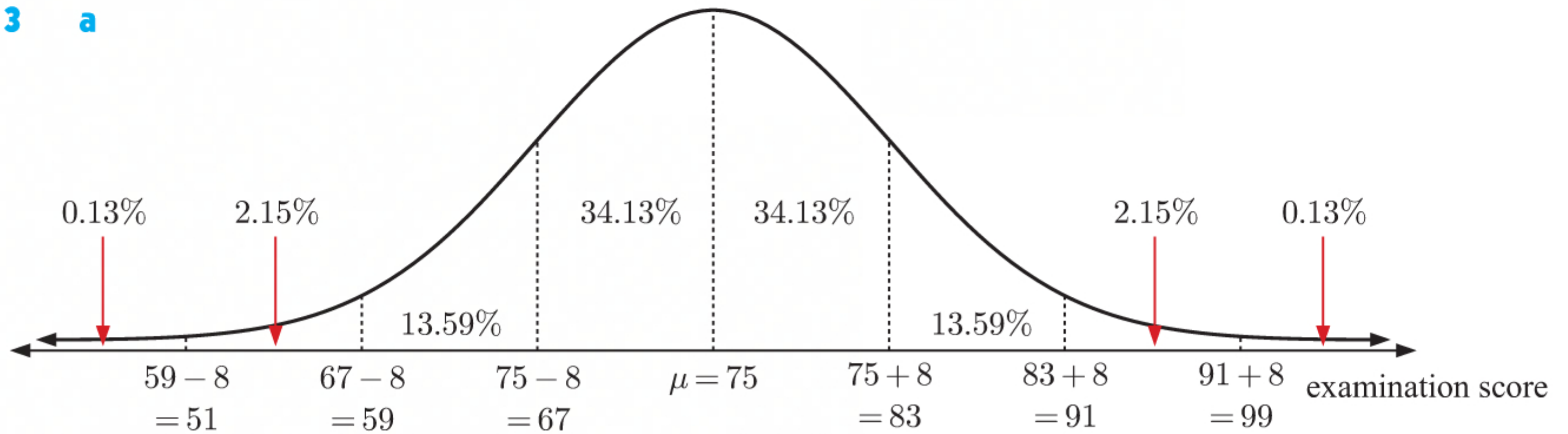
About 13.59% of X values are between 12 and 16.

iii

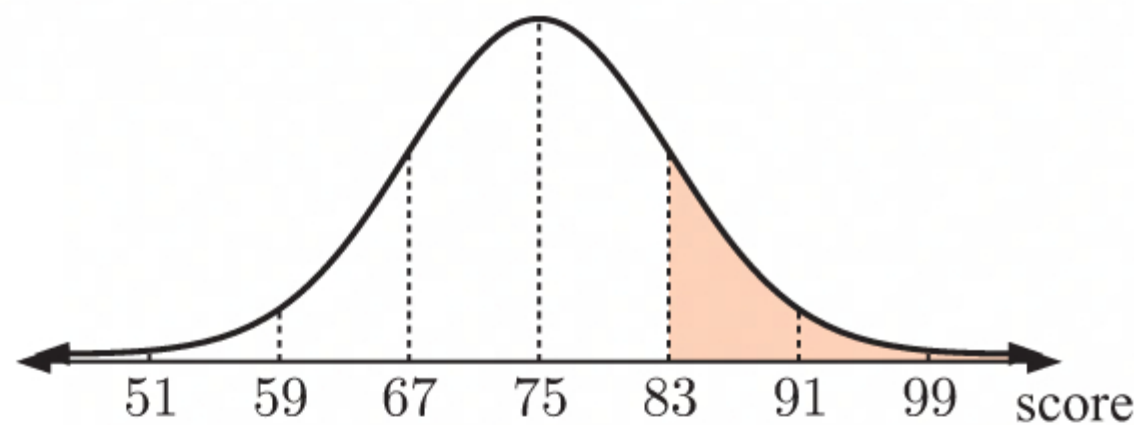


About $2.15\% + 0.13\% = 2.28\%$ of X values are greater than 28.

3 a



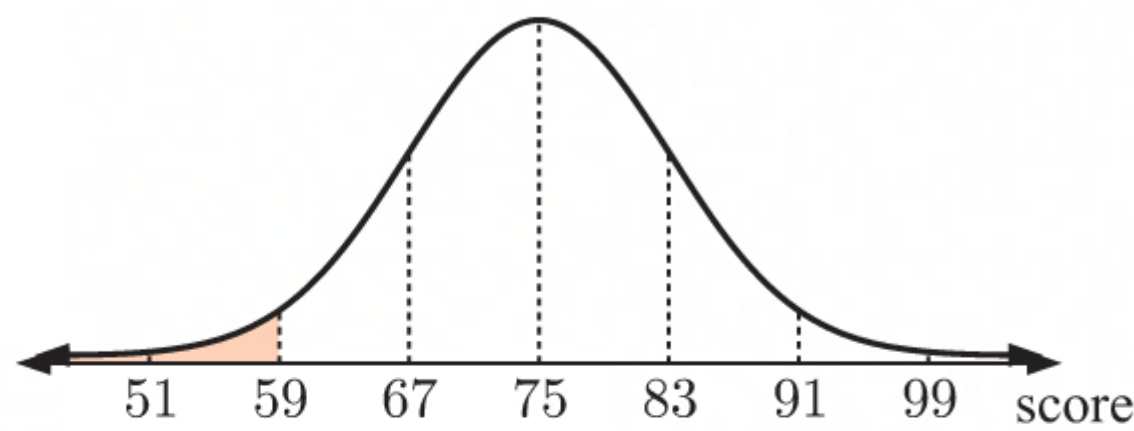
b i



About $13.59\% + 2.15\% + 0.13\%$
 $= 15.87\%$

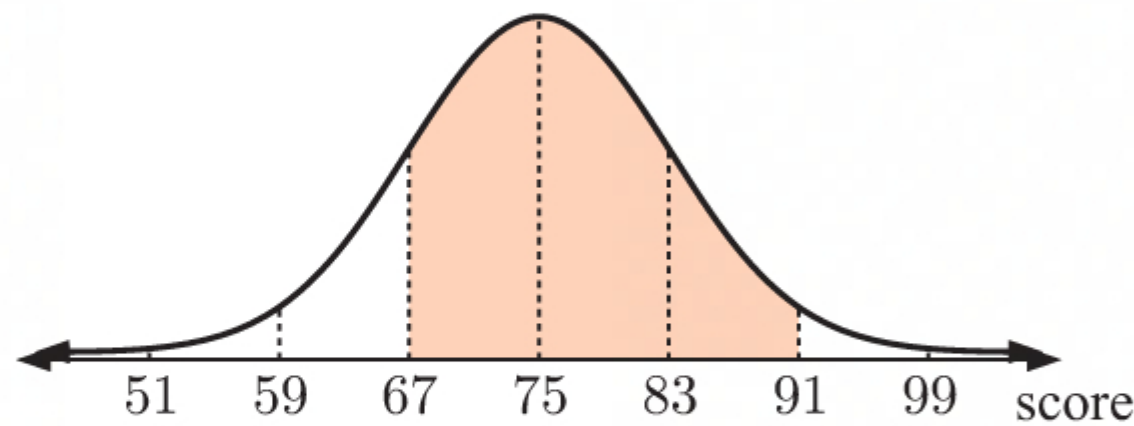
of students would be expected to have scored more than 83.

ii



About $2.15\% + 0.13\% = 2.28\%$ of students would be expected to have scored less than 59.

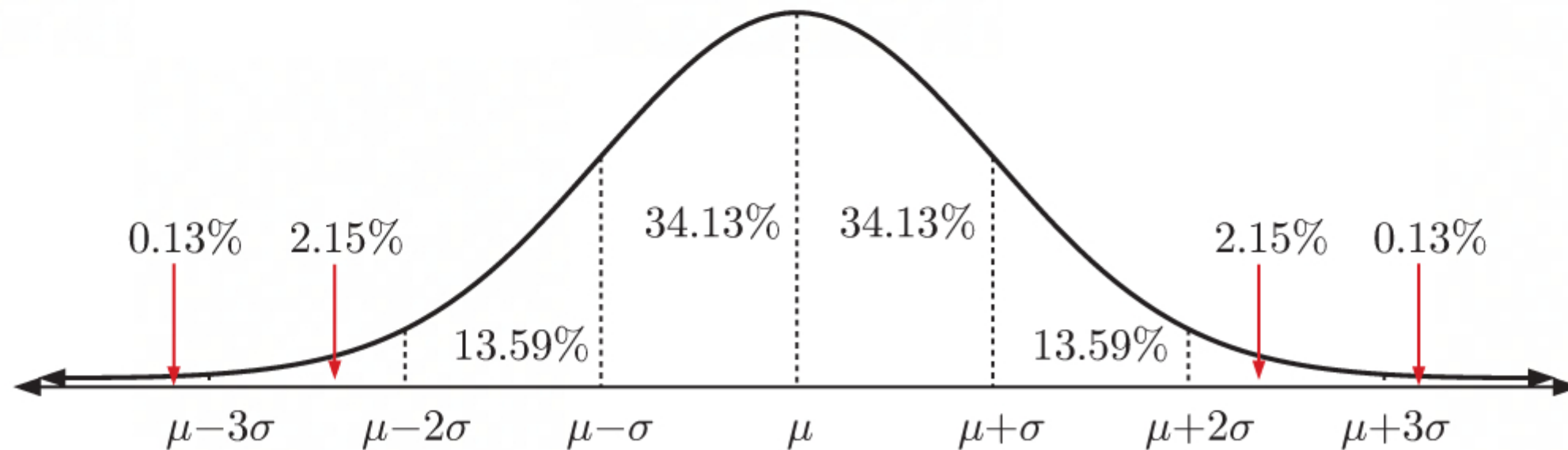
iii



About $34.13\% + 34.13\% + 13.59\%$
 $= 81.85\%$

of students would be expected to have scored between 67 and 91.

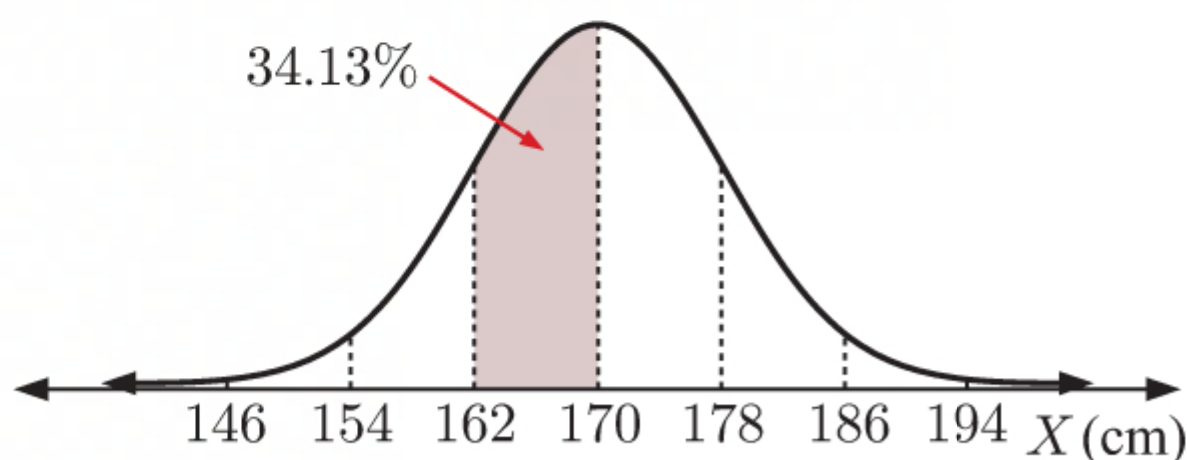
4



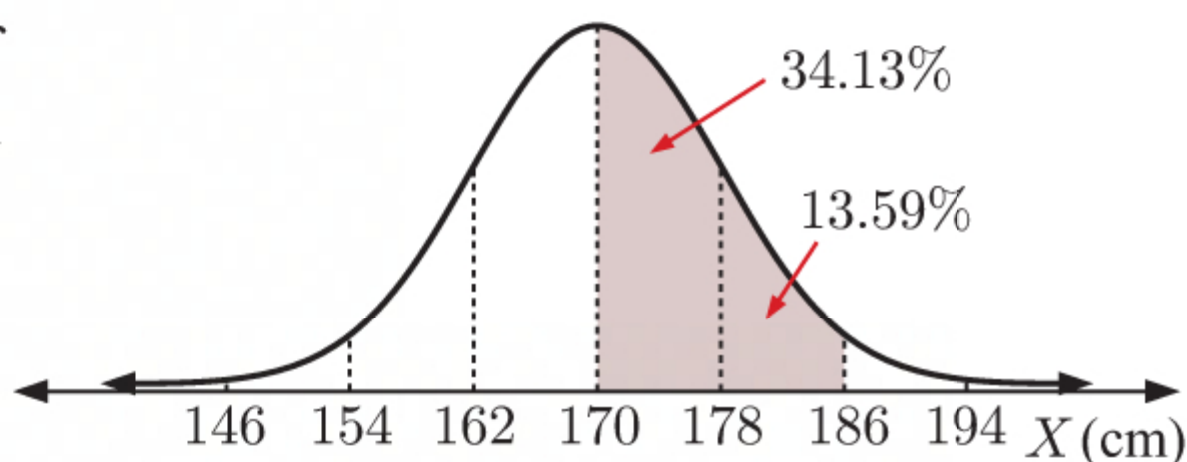
a $P(\text{value between } \mu - \sigma \text{ and } \mu + \sigma)$
 $\approx 0.3413 + 0.3413$
 ≈ 0.6826

b $P(\text{value} > \mu + 2\sigma)$
 $\approx 0.0215 + 0.0013$
 ≈ 0.0228

- 5 a i** About 34.13% of female students have a height between 162 cm and 170 cm.

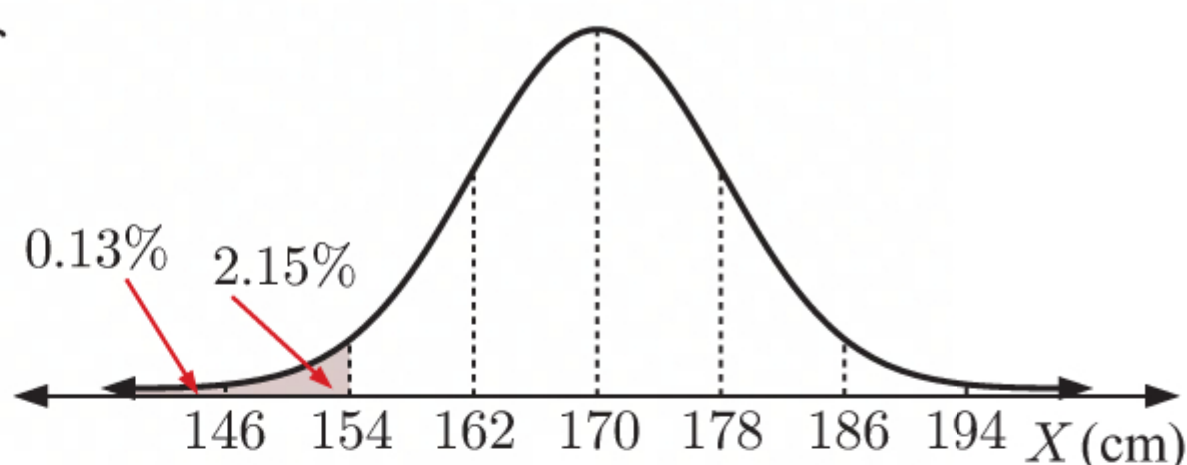


- ii** About $34.13\% + 13.59\% = 47.72\%$ of female students have a height between 170 cm and 186 cm.



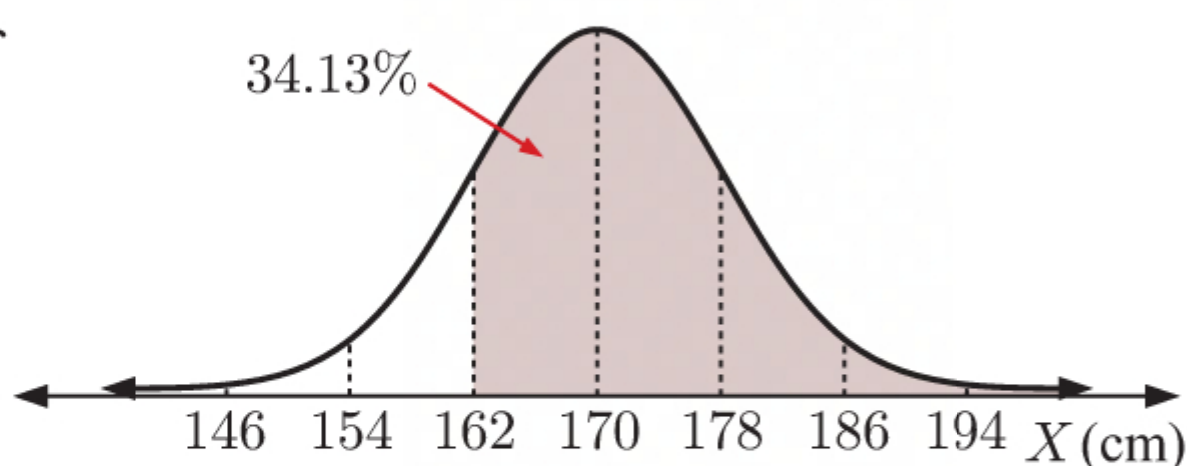
- b i** About $2.15\% + 0.13\% = 2.28\%$ of female students have a height less than 154 cm.

$$\therefore P(\text{height is less than 154 cm}) \approx 0.0228$$

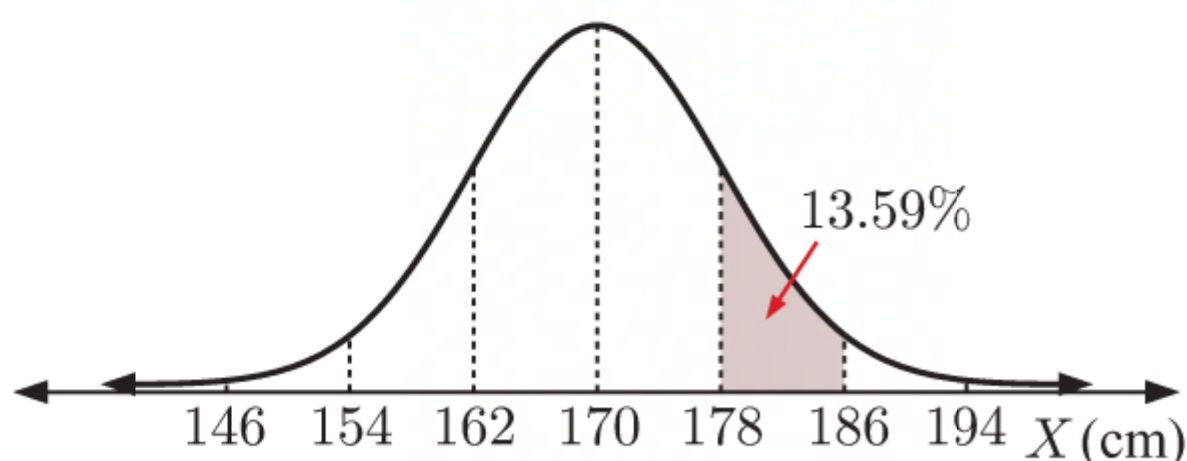


- ii** About $34.13\% + 50\% = 84.13\%$ of female students have a height greater than 162 cm.

$$\therefore P(\text{height is greater than 162 cm}) \approx 0.8413$$



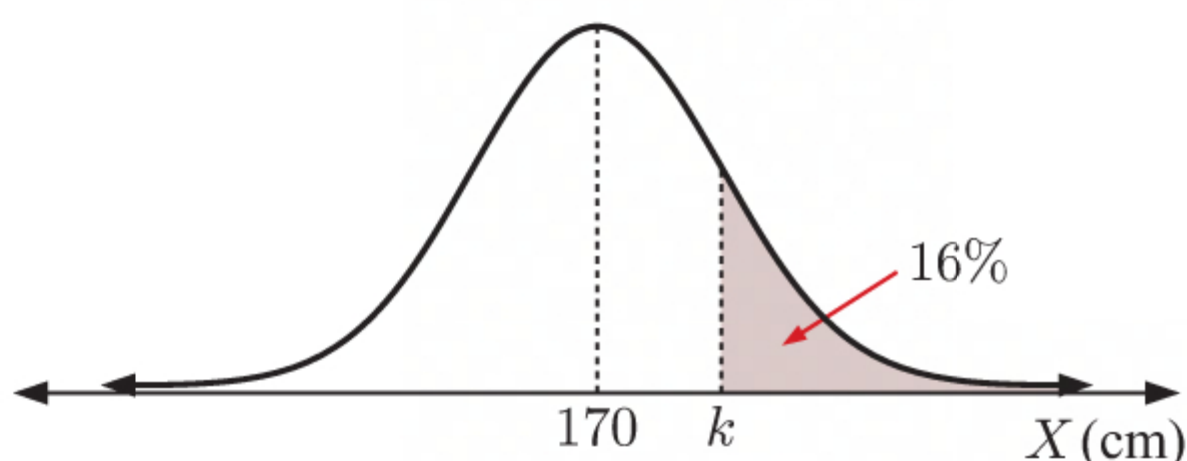
- c** About 13.59% of the female students have a height between 178 cm and 186 cm.
So, we would expect about 13.59% of $500 \approx 68$ students to have a height between 178 cm and 186 cm.



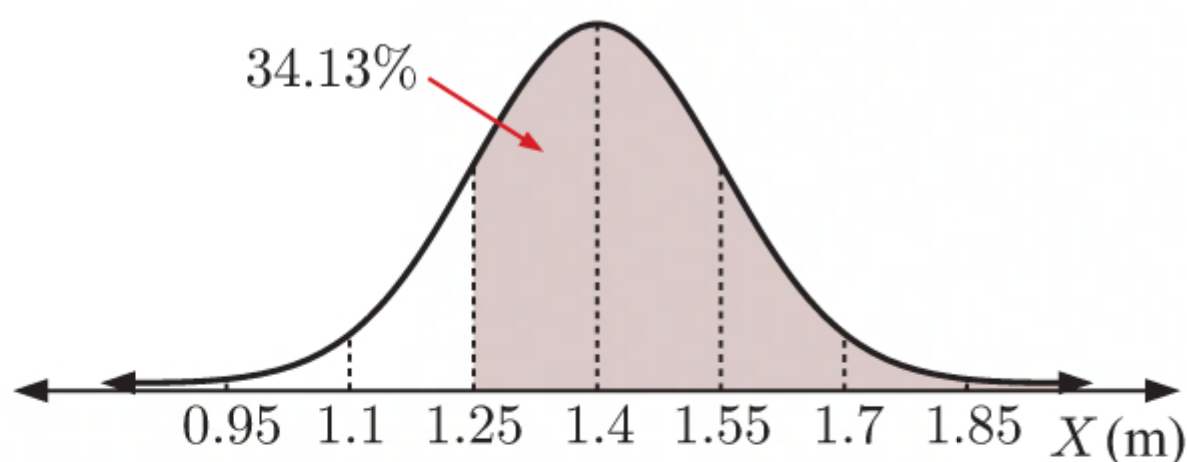
- d** Approximately 16% of data lies more than one standard deviation above the mean.

$\therefore k$ is about σ above the mean μ

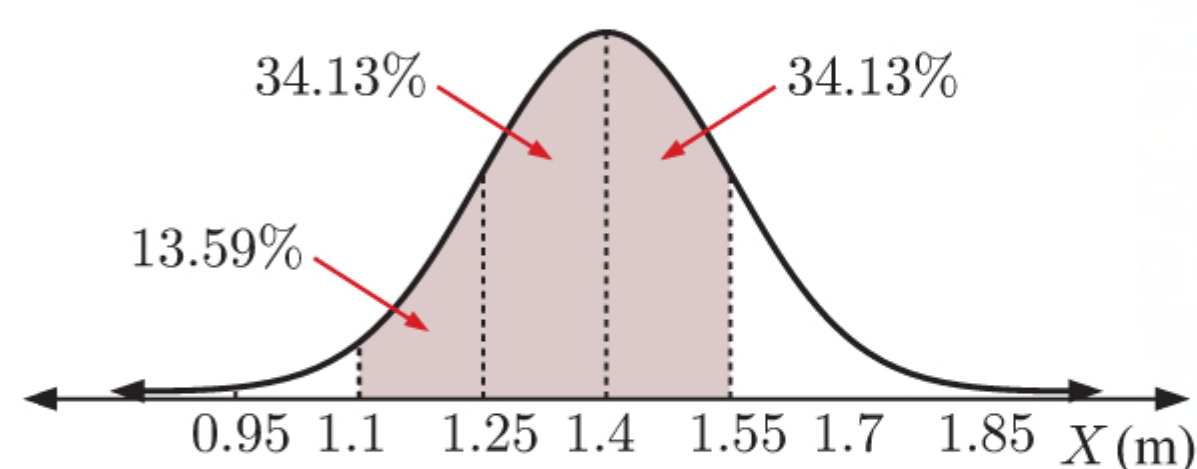
$$\therefore k \approx 170 + 8 \approx 178$$



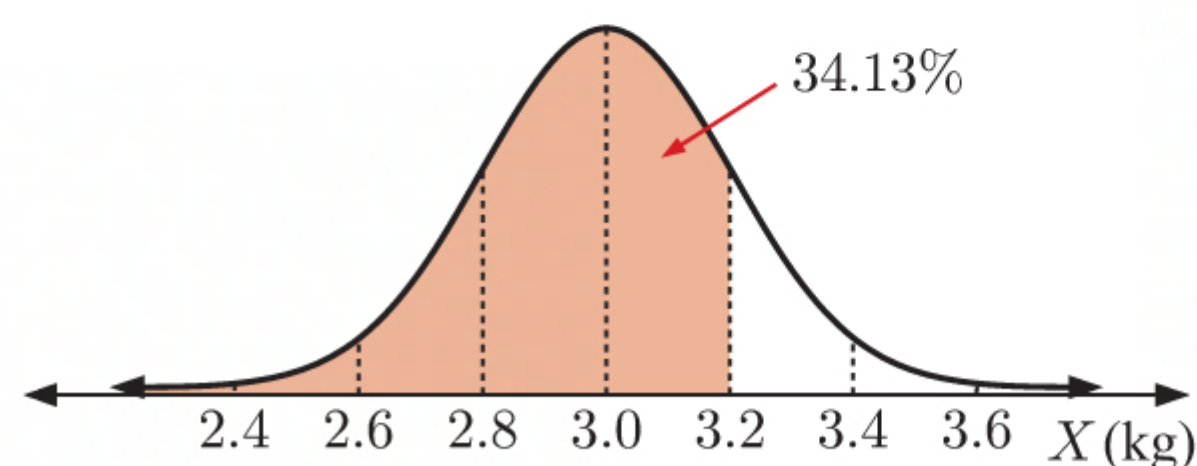
- 6 a** About $34.13\% + 50\% = 84.13\%$ of adult female frilled sharks measure more than 1.25 m long.



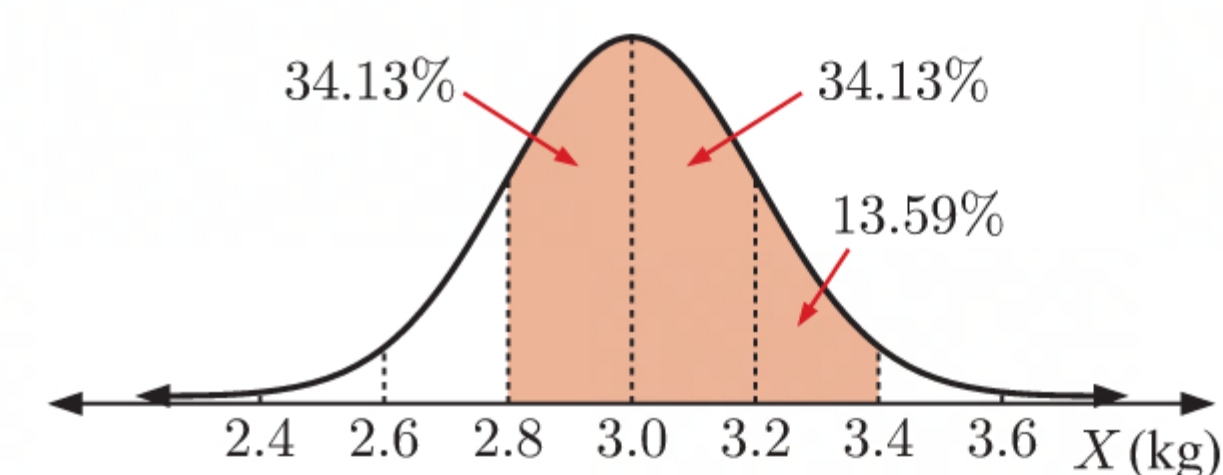
- b** About $13.59\% + 34.13\% + 34.13\%$
 $= 81.85\%$ of adult female frilled sharks
 measure between 1.1 m and 1.55 m long.



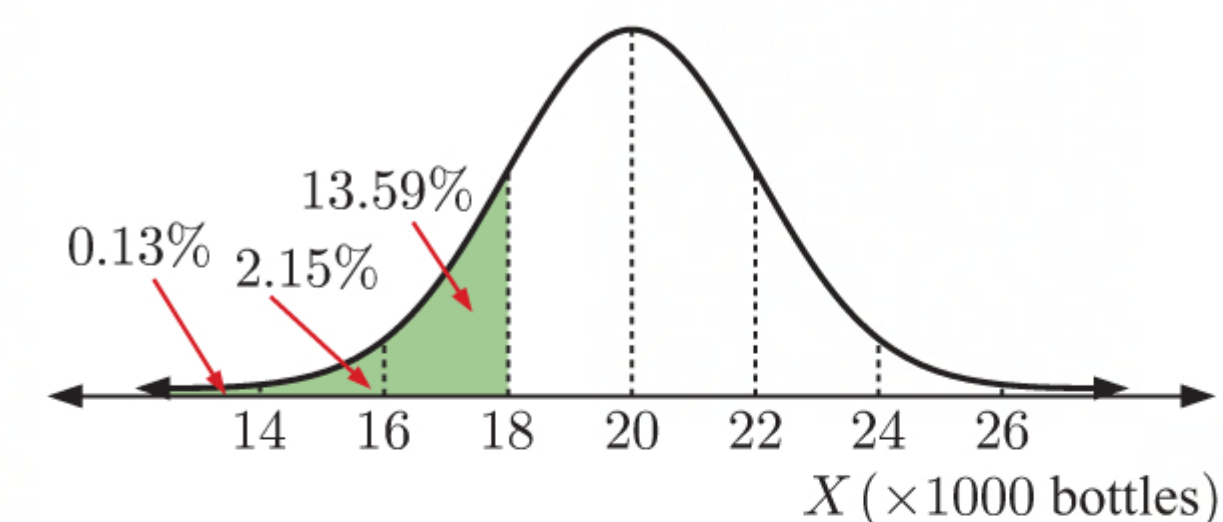
- 7 a** About $50\% + 34.13\% = 84.13\%$ of babies
 born weighed less than 3.2 kg.
 So, about $84.13\% \times 545 \approx 459$ babies born
 weighed less than 3.2 kg.



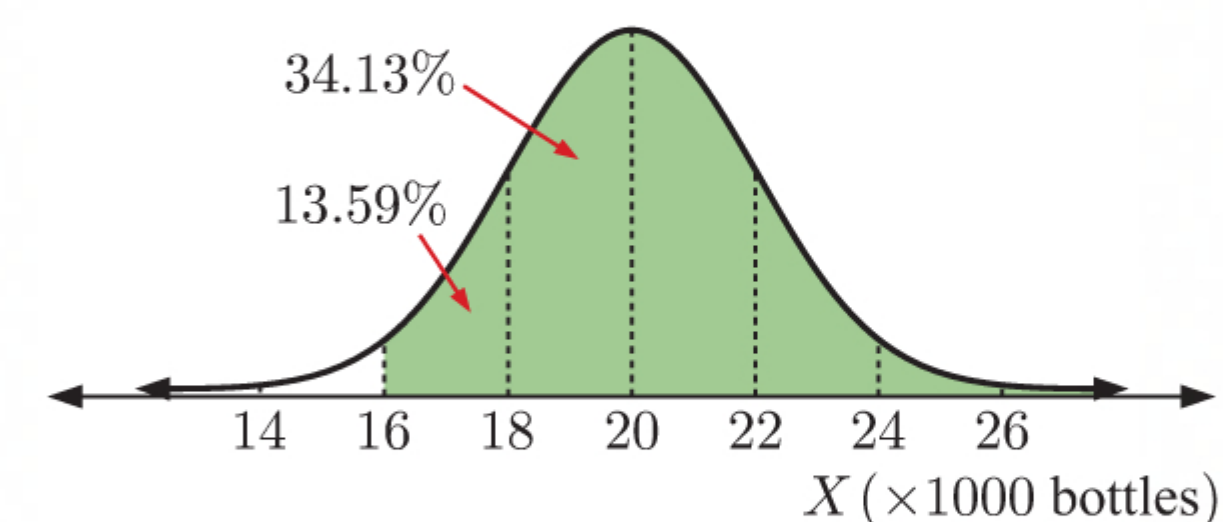
- b** About $34.13\% + 34.13\% + 13.59\%$
 $= 81.85\%$ of babies born weighed
 between 2.8 kg and 3.4 kg.
 So, about $81.85\% \times 545 \approx 446$ babies born
 weighed between 2.8 kg and 3.4 kg.



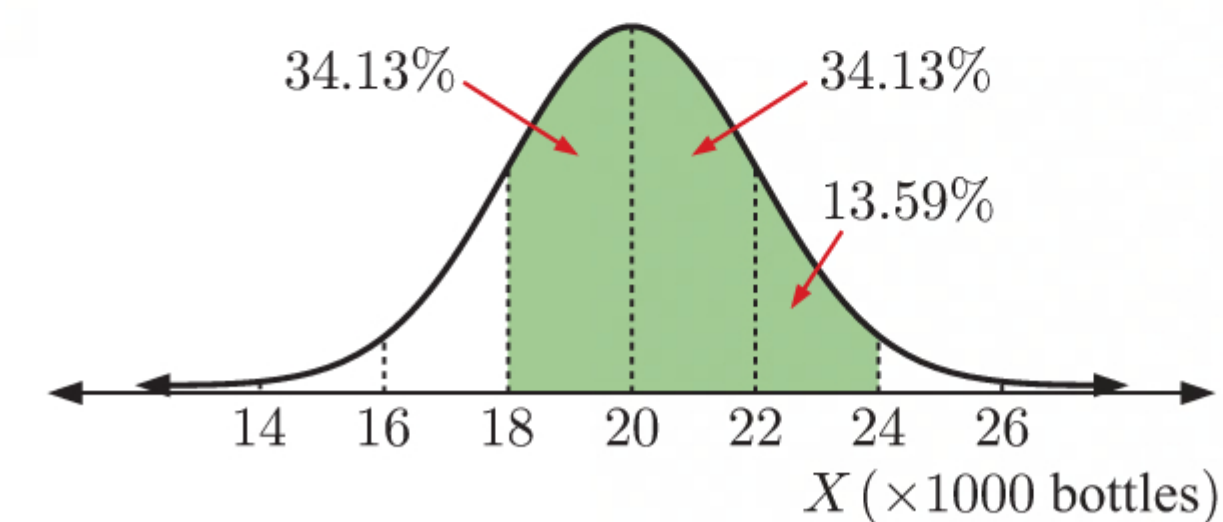
- 8 a** Under 18 000 bottles are filled on about
 $0.13\% + 2.15\% + 13.59\% = 15.87\%$ of days.
 So, under 18 000 bottles are filled on about
 $15.87\% \times 260 \approx 41$ days.



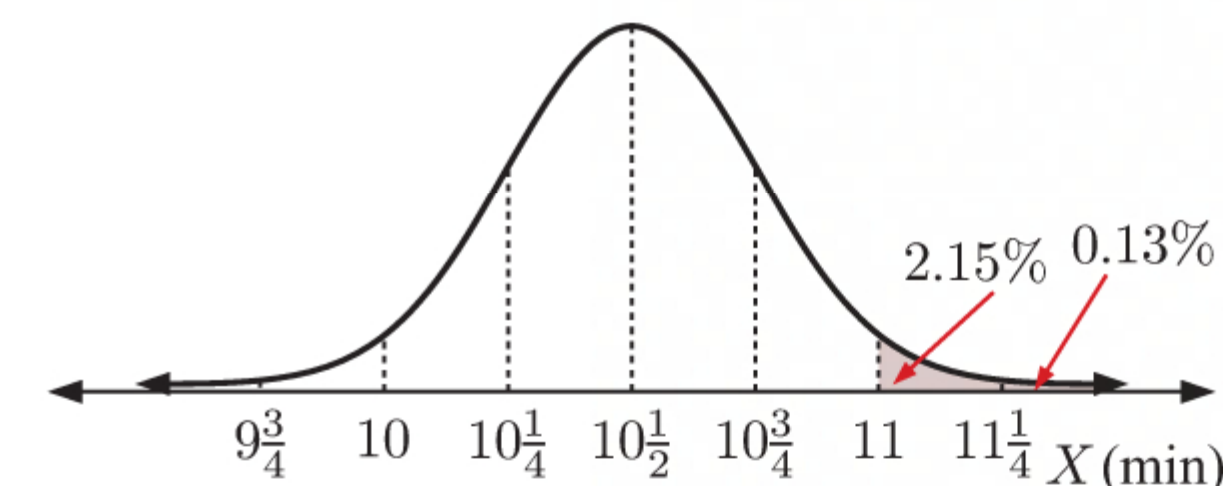
- b** Over 16 000 bottles are filled on about
 $13.59\% + 34.13\% + 50\% = 97.72\%$ of days.
 So, over 16 000 bottles are filled on about
 $97.72\% \times 260 \approx 254$ days.



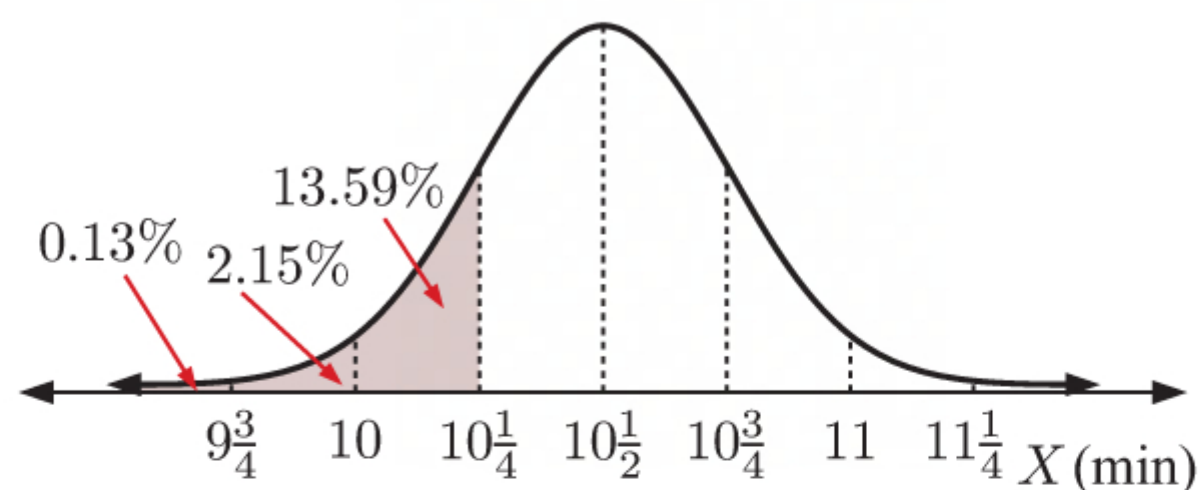
- c** Between 18 000 and 24 000 bottles are filled
 on about $34.13\% + 34.13\% + 13.59\%$
 $= 81.85\%$ of days.
 So, between 18 000 and 24 000 bottles are
 filled on about $81.85\% \times 260 \approx 213$ days.



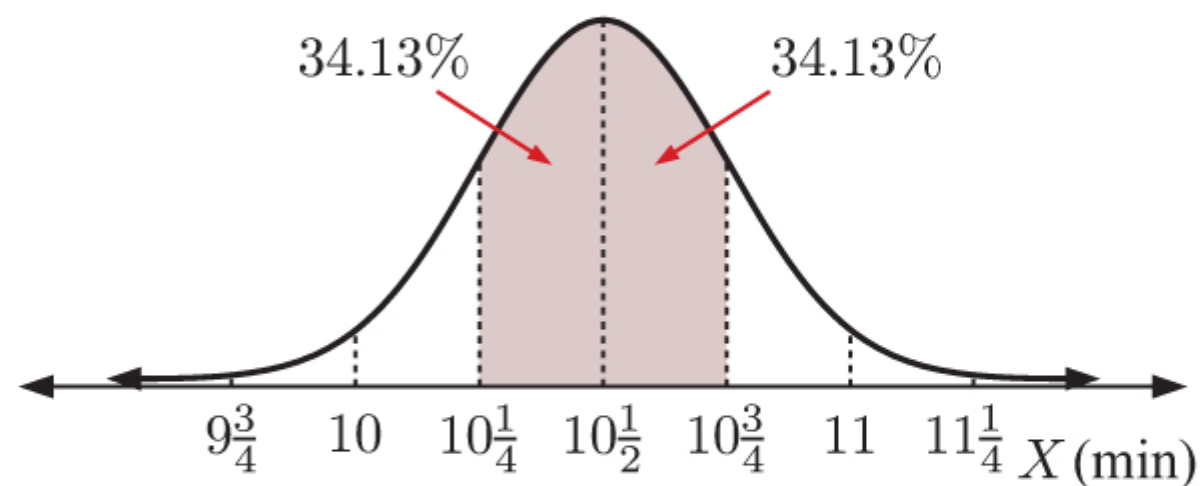
- 9 a** About $2.15\% + 0.13\% = 2.28\%$ of
 competitors completed the race in a time
 longer than 11 minutes.
 So, about $2.28\% \times 200 \approx 5$ competitors
 completed the race in a time longer than
 11 minutes.



- b** About $0.13\% + 2.15\% + 13.59\% = 15.87\%$ of competitors completed the race in a time less than 10 minutes 15 seconds.
So, about $15.87\% \times 200 \approx 32$ competitors completed the race in a time less than 10 minutes 15 seconds.



- c** About $34.13\% + 34.13\% = 68.26\%$ of competitors completed the race in a time between 10 minutes 15 seconds and 10 minutes 45 seconds.
So, about $68.26\% \times 200 \approx 137$ competitors completed the race in a time between 10 minutes 15 seconds and 10 minutes 45 seconds.



- 10 a** Approximately 84% of data is more than one standard deviation below the mean, and $34.13\% + 50\% \approx 84\%$.

\therefore 152 grams is about σ below the mean μ

$$\therefore \mu \approx 152 + \sigma \quad \dots (1)$$

Approximately 16% of data is more than one standard deviation above the mean, and $13.59\% + 2.15\% + 0.13\% \approx 16\%$.

\therefore 200 grams is σ above the mean μ

$$\therefore \mu \approx 200 - \sigma \quad \dots (2)$$

Equating (1) and (2) gives: $152 + \sigma \approx 200 - \sigma$

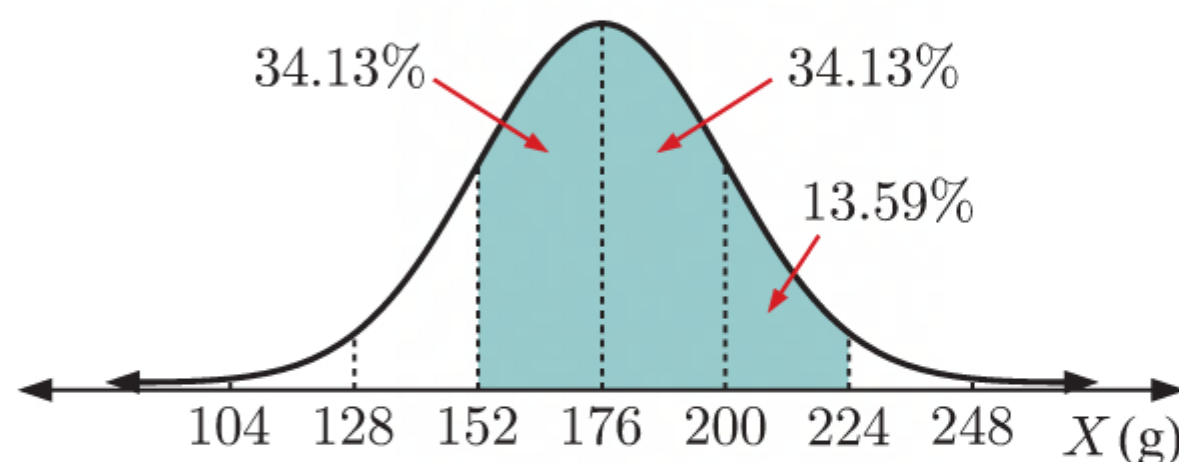
$$\therefore 2\sigma \approx 48$$

$$\therefore \sigma \approx 24$$

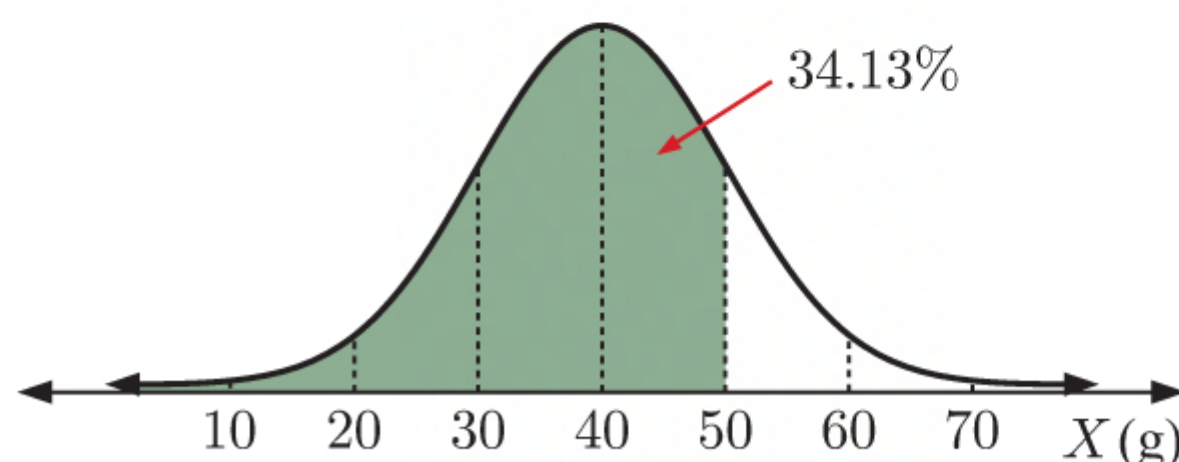
$$\text{and } \mu \approx 200 - 24 \quad \{\text{using (2)}\} \\ \approx 176$$

So, $\mu \approx 176$ grams and $\sigma \approx 24$ grams.

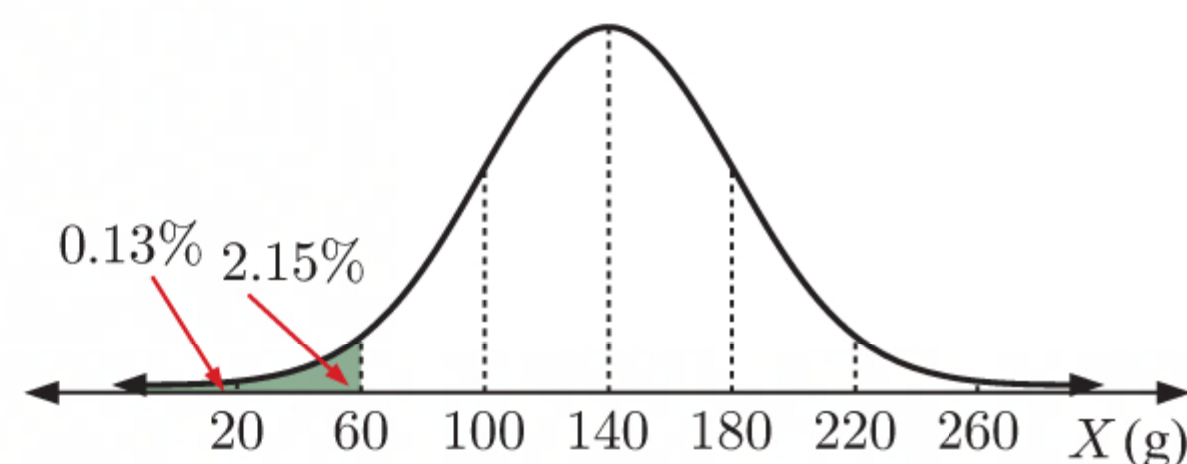
- b** About $34.13\% + 34.13\% + 13.59\% = 81.85\%$ of the oranges weigh between 152 grams and 224 grams.



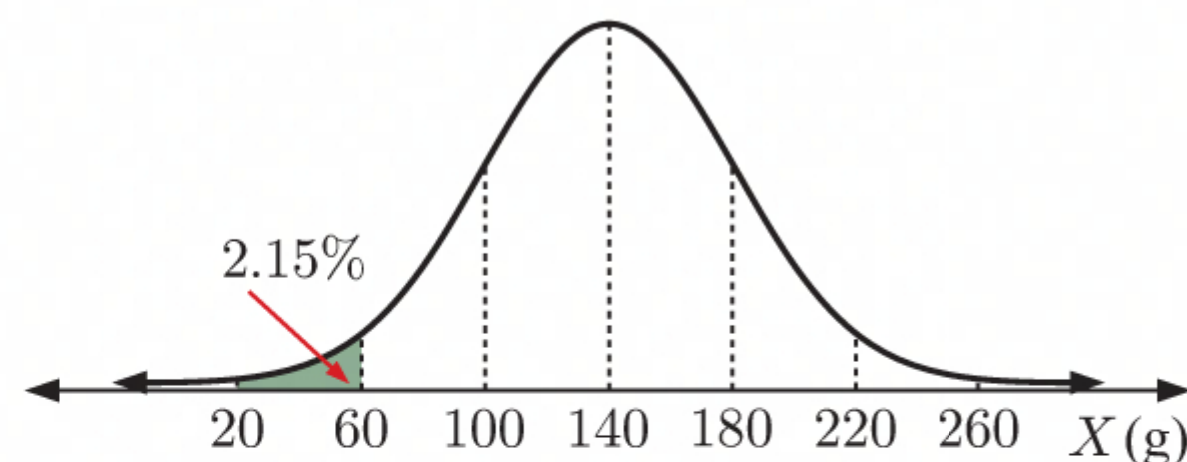
- 11 a i** About $50\% + 34.13\% = 84.13\%$ of radishes grown without fertiliser will have weights less than 50 grams.



- ii About $0.13\% + 2.15\% = 2.28\%$ of radishes grown with fertiliser will have weights less than 60 grams.

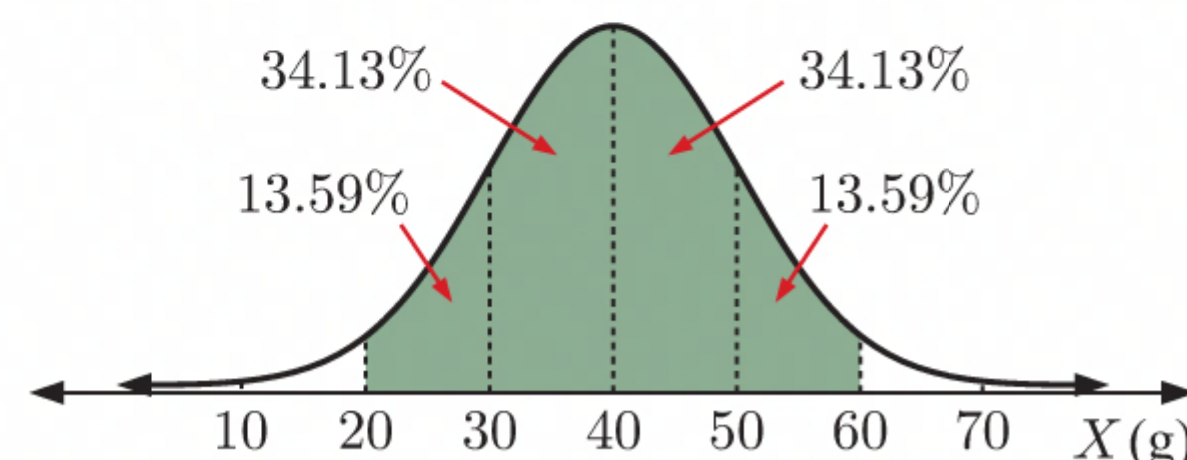


- b i About 2.15% of radishes grown with fertiliser will have weights between 20 g and 60 g.



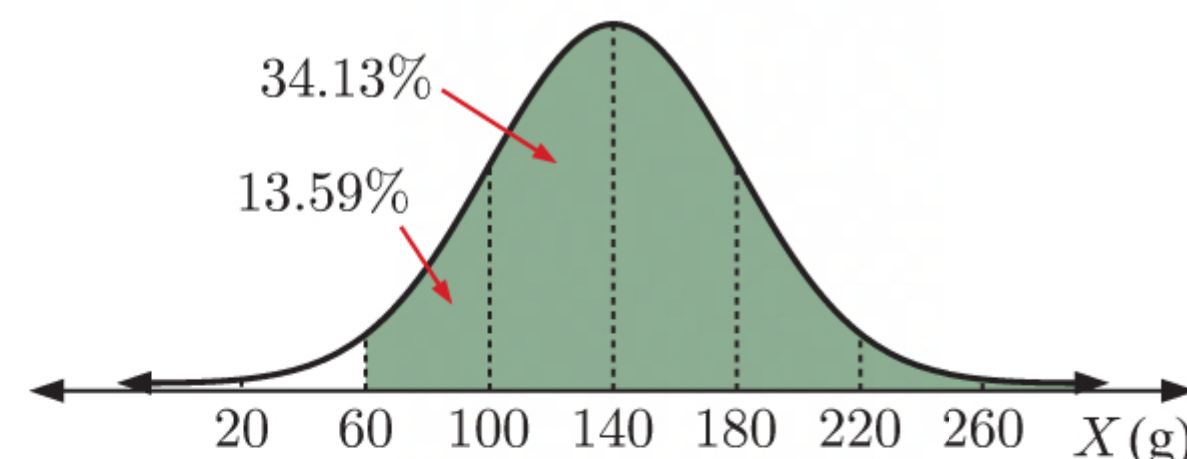
$\therefore P(\text{radish grown with fertiliser weighs between 20 g and 60 g}) \approx 0.0215$

- ii About $13.59\% + 34.13\% + 34.13\% + 13.59\% = 95.44\%$ of radishes grown without fertiliser will have weights between 20 g and 60 g.



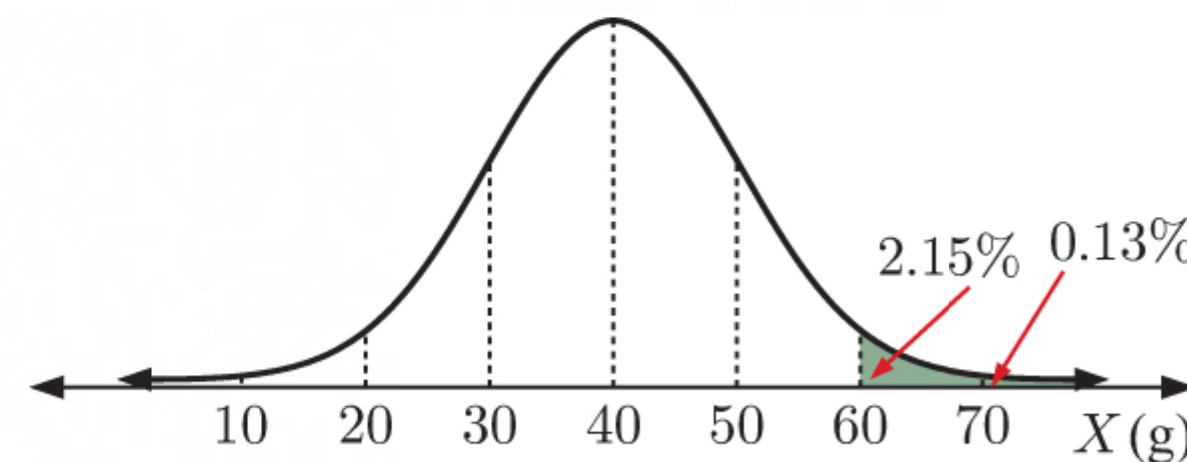
$\therefore P(\text{radish grown without fertiliser weighs between 20 g and 60 g}) \approx 0.9544$

- c About $13.59\% + 34.13\% + 50\% = 97.72\%$ of radishes grown with fertiliser will have weights more than 60 g.



$\therefore P(\text{radish grown with fertiliser weighs more than 60 g}) \approx 0.9772$

About $2.15\% + 0.13\% = 2.28\%$ of radishes grown without fertiliser will have weights more than 60 g.



$\therefore P(\text{radish grown without fertiliser weighs more than 60 g}) \approx 0.0228$

$$\begin{aligned}
 & P(\text{both radishes weigh more than 60 g}) \\
 &= P(\text{radish grown with fertiliser weighs more than 60 g}) \\
 &\quad \times P(\text{radish grown without fertiliser weighs more than 60 g}) \\
 &\approx 0.9772 \times 0.0228 \\
 &\approx 0.0223
 \end{aligned}$$

EXERCISE 15B.2

1 $X \sim N(60, 5^2)$

- a To find $P(60 \leq X \leq 65)$, we set the lower bound to 60 and the upper bound to 65.

```

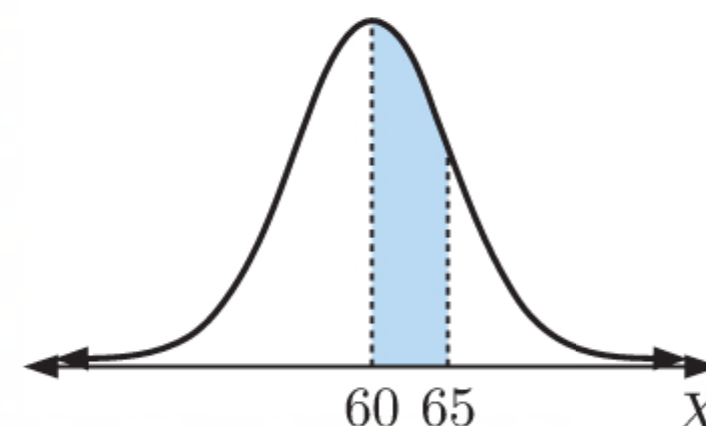
Normal C.D
Data :Variable
Lower :60
Upper :65
σ :5
μ :60
Save Res:None
[None] LIST

```

```

Normal C.D
p =0.34134474
z:Low=0
z:Up =1

```



$$P(60 \leq X \leq 65) \approx 0.341$$

- b To find $P(62 \leq X \leq 67)$, we set the lower bound to 62 and the upper bound to 67.

```

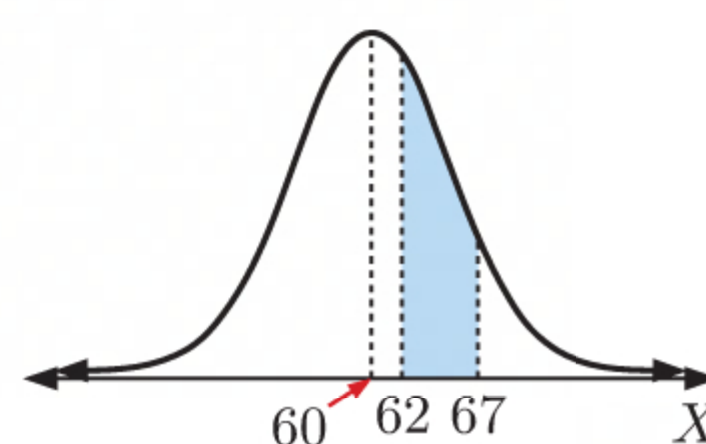
Normal C.D
Data :Variable
Lower :62
Upper :67
σ :5
μ :60
Save Res:None
[None] LIST

```

```

Normal C.D
p =0.26382159
z:Low=0.4
z:Up =1.4

```



$$P(62 \leq X \leq 67) \approx 0.264$$

- c To find $P(X \geq 64)$, we use a very high value such as 10^{99} to represent the upper bound.

```

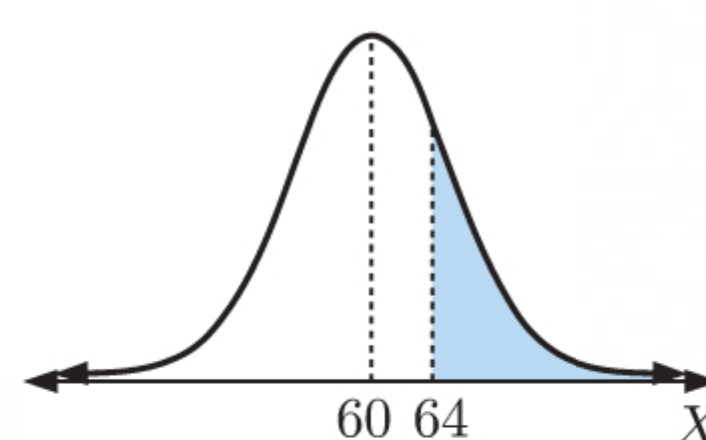
Normal C.D
Data :Variable
Lower :64
Upper :1E+99
σ :5
μ :60
Save Res:None
[None] LIST

```

```

Normal C.D
p =0.21185539
z:Low=0.8
z:Up =2E+98

```



$$P(X \geq 64) \approx 0.212$$

- d To find $P(X \leq 68)$, we use a very low value such as -10^{99} to represent the lower bound.

```

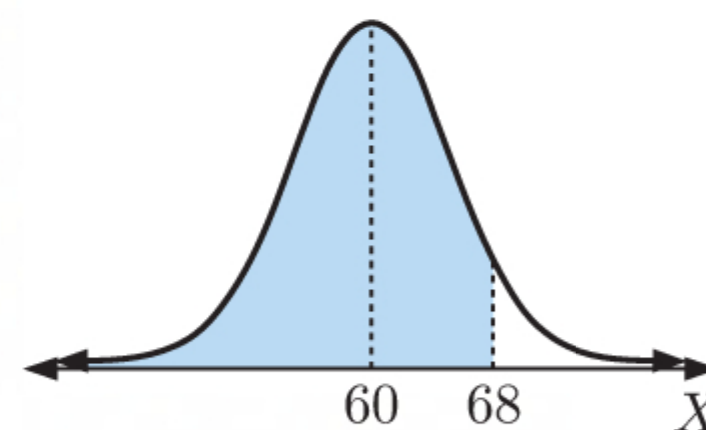
Normal C.D
Data :Variable
Lower :-1E+99
Upper :68
σ :5
μ :60
Save Res:None
[None] LIST

```

```

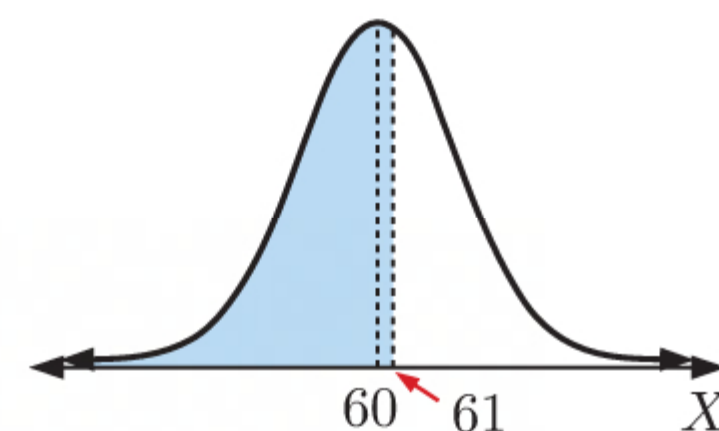
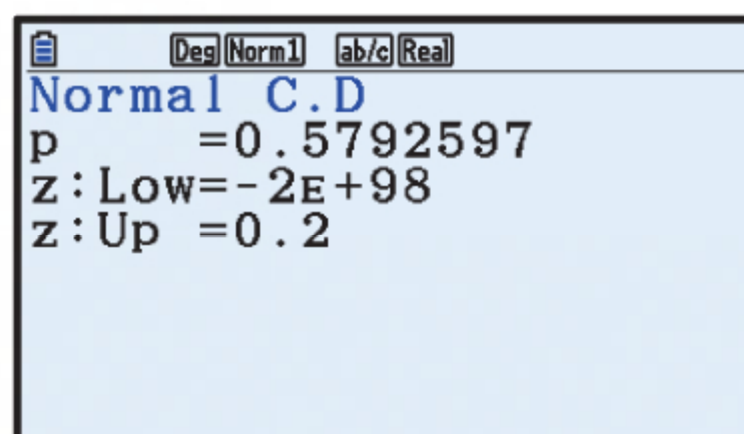
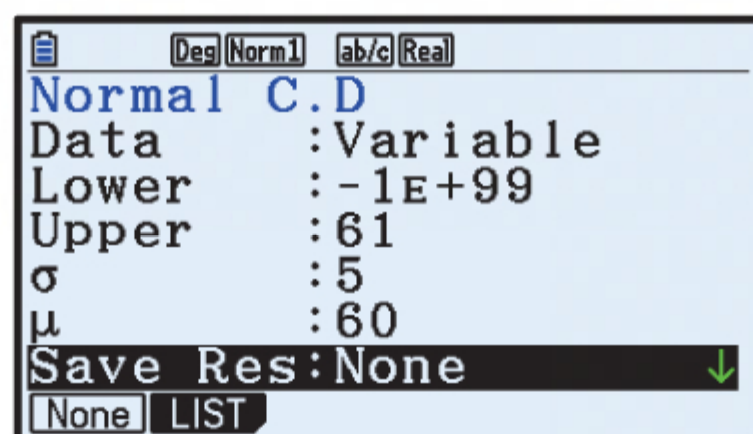
Normal C.D
p =0.9452007
z:Low=-2E+98
z:Up =1.6

```



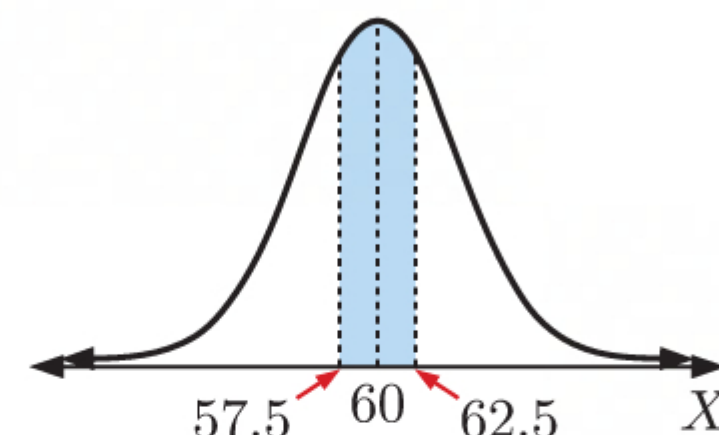
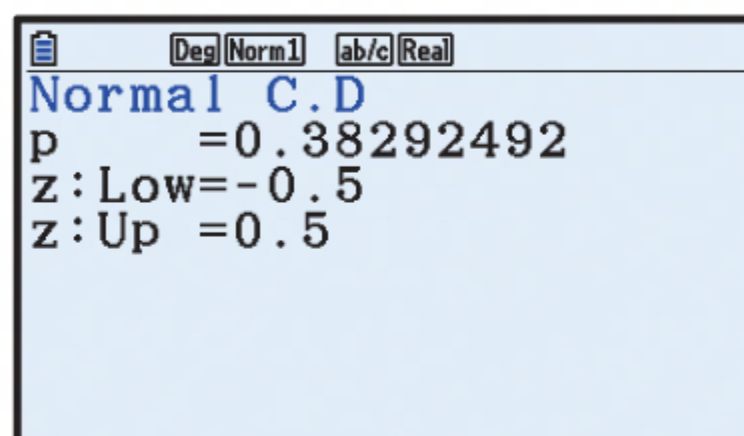
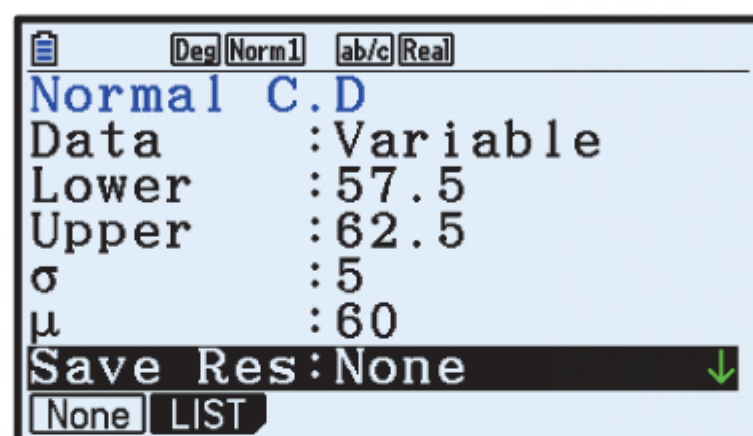
$$P(X \leq 68) \approx 0.945$$

- e To find $P(X \leq 61)$, we use a very low value such as -10^{99} to represent the lower bound.



$$P(X \leq 61) \approx 0.579$$

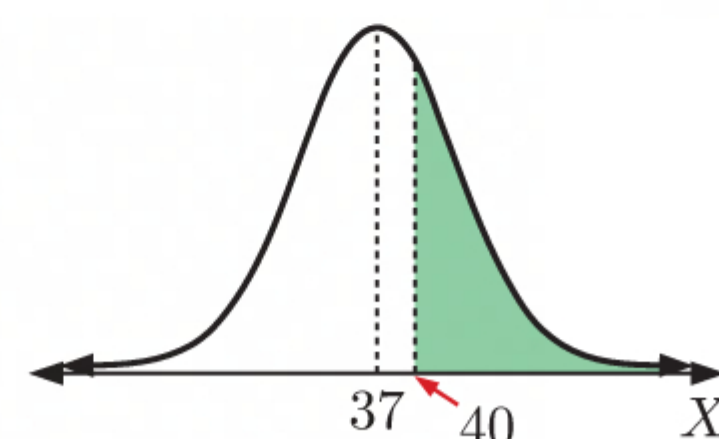
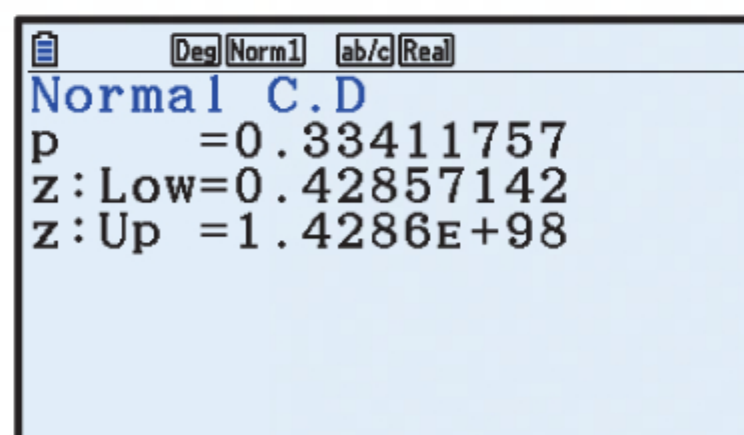
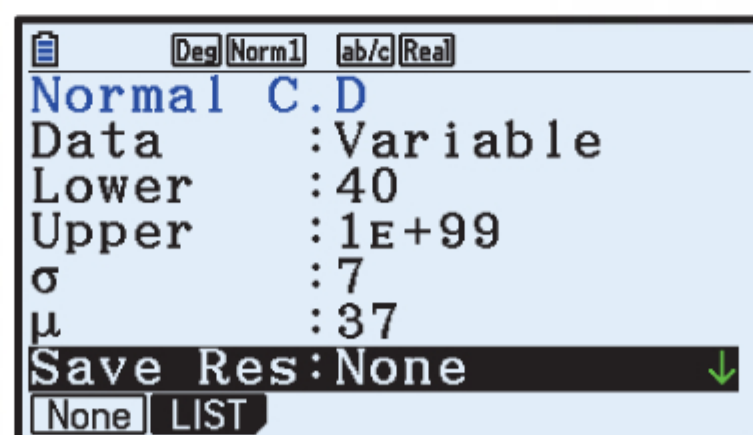
- f To find $P(57.5 \leq X \leq 62.5)$, we set the lower bound to 57.5 and the upper bound to 62.5.



$$P(57.5 \leq X \leq 62.5) \approx 0.383$$

2 $X \sim N(37, 7^2)$

- a To find $P(X > 40)$, we use a very high value such as 10^{99} to represent the upper bound.



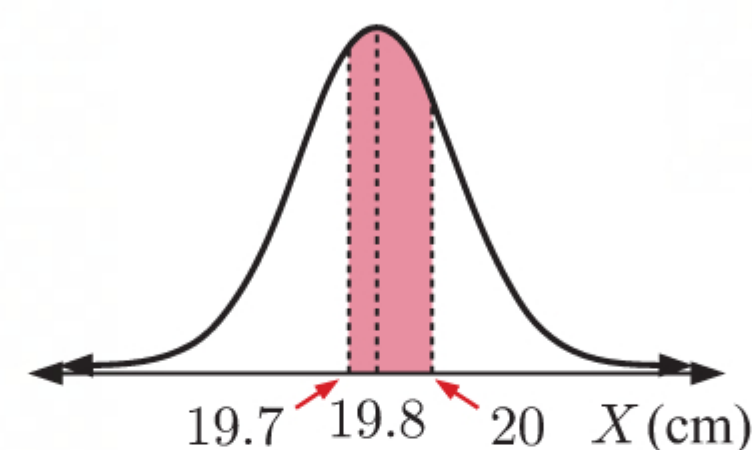
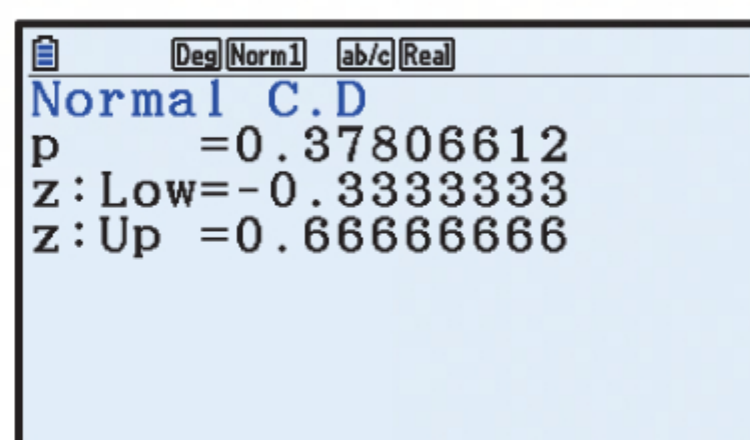
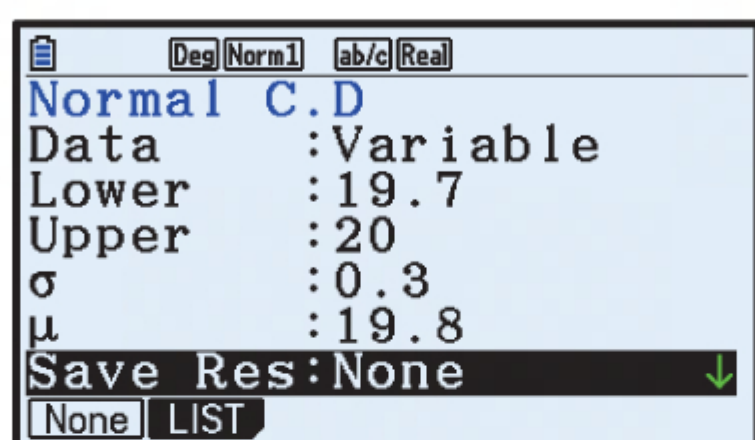
$$P(X > 40) \approx 0.334$$

- b Since the mean of the distribution is 37, then $P(X > 37) = 0.5$

$$\begin{aligned} \therefore P(37 \leq X \leq 40) &= P(X > 37) - P(X > 40) \\ &\approx 0.5 - 0.334 \\ &\approx 0.166 \end{aligned}$$

3 Let X cm be the length of a randomly selected bolt.

$$\therefore X \sim N(19.8, 0.3^2)$$



$$P(19.7 < X < 20) \approx 0.378$$

- 4 Let $X \text{ km h}^{-1}$ be the speed of a randomly selected car.

$$\therefore X \sim N(46.3, 7.4^2)$$

a

```

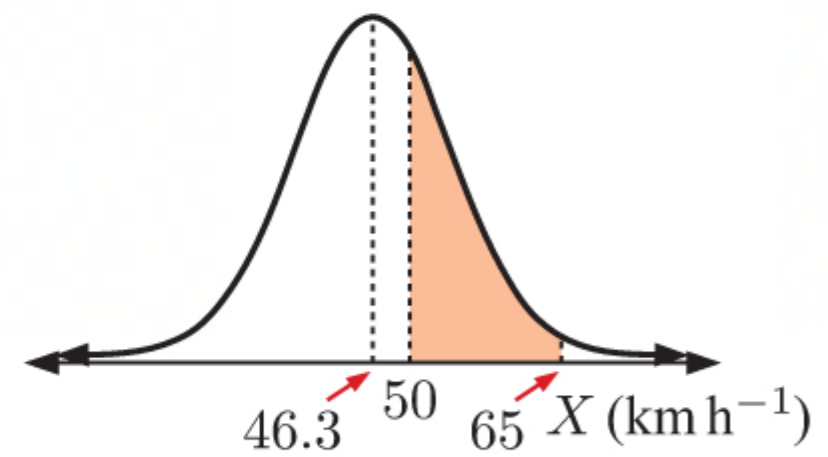
Normal C.D
Data      :Variable
Lower     :50
Upper     :65
σ         :7.4
μ         :46.3
Save Res:None
[None] LIST

```

```

Normal C.D
p         =0.3027859
z:Low=0.5
z:Up      =2.52702703

```



$$P(50 < X < 65) \approx 0.303$$

b

```

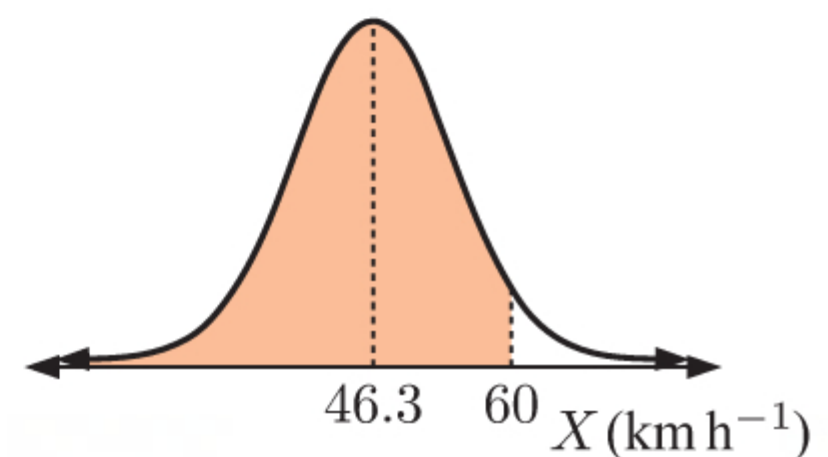
Normal C.D
Data      :Variable
Lower     :-1E+99
Upper     :60
σ         :7.4
μ         :46.3
Save Res:None
[None] LIST

```

```

Normal C.D
p         =0.96794048
z:Low=-1.351E+98
z:Up      =1.85135135

```



$$P(X < 60) \approx 0.968$$

c

```

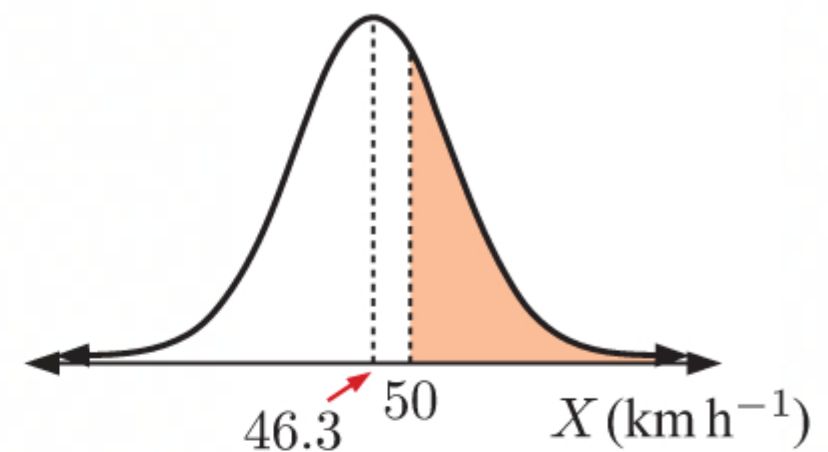
Normal C.D
Data      :Variable
Lower     :50
Upper     :1E+99
σ         :7.4
μ         :46.3
Save Res:None
[None] LIST

```

```

Normal C.D
p         =0.30853753
z:Low=0.5
z:Up      =1.3514E+98

```



$$P(X > 50) \approx 0.309$$

- 5 Let $X \text{ cm}$ be the length of a randomly selected eel.

$$\therefore X \sim N(41, 5.5^2)$$

a

```

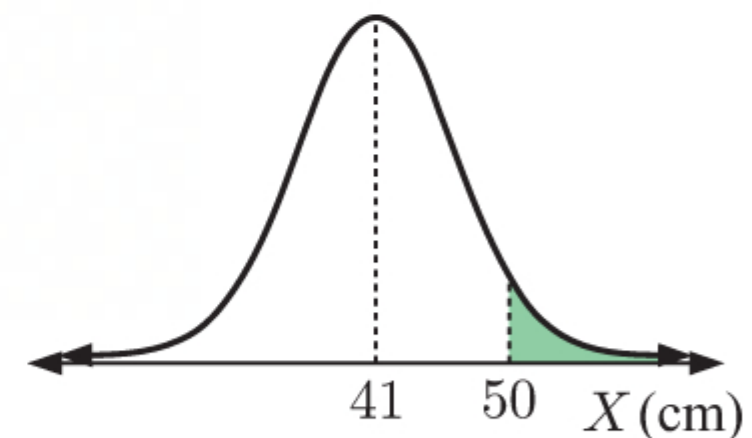
Normal C.D
Data      :Variable
Lower     :50
Upper     :1E+99
σ         :5.5
μ         :41
Save Res:None
[None] LIST

```

```

Normal C.D
p         =0.05088175
z:Low=1.63636364
z:Up      =1.8182E+98

```



$$P(X \geq 50) \approx 0.0509$$

b

```

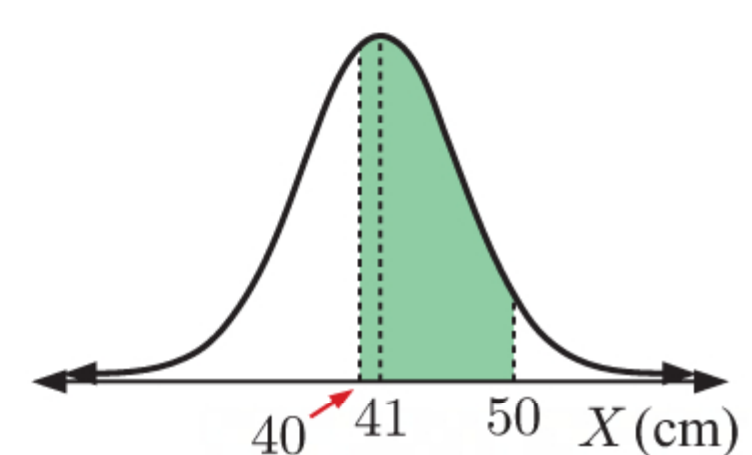
Normal C.D
Data      :Variable
Lower     :40
Upper     :50
σ         :5.5
μ         :41
Save Res:None
[None] LIST

```

```

Normal C.D
p         =0.52125554
z:Low=-0.1818181
z:Up      =1.63636364

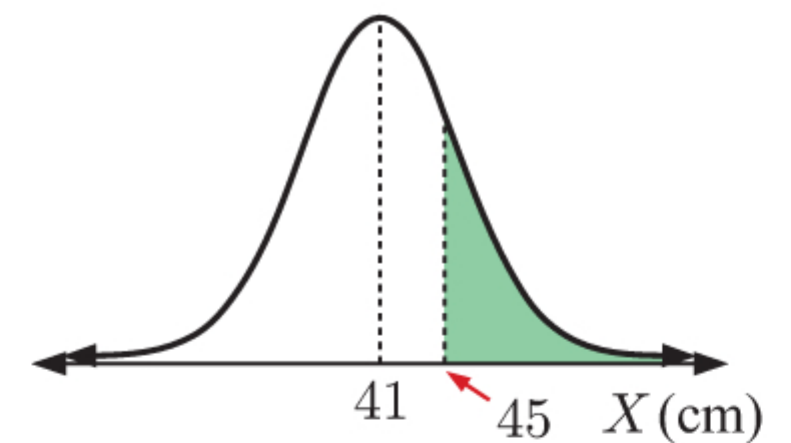
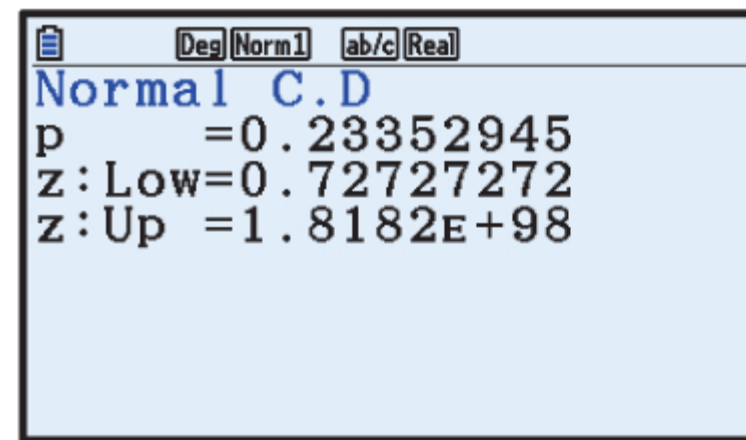
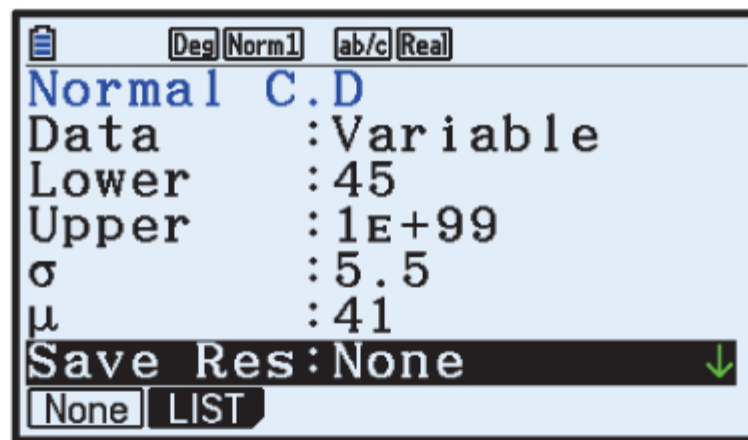
```



$$P(40 < X < 50) \approx 0.521$$

\therefore about 52.1% of eels measure between 40 cm and 50 cm long.

c



$$P(X \geq 45) \approx 0.234$$

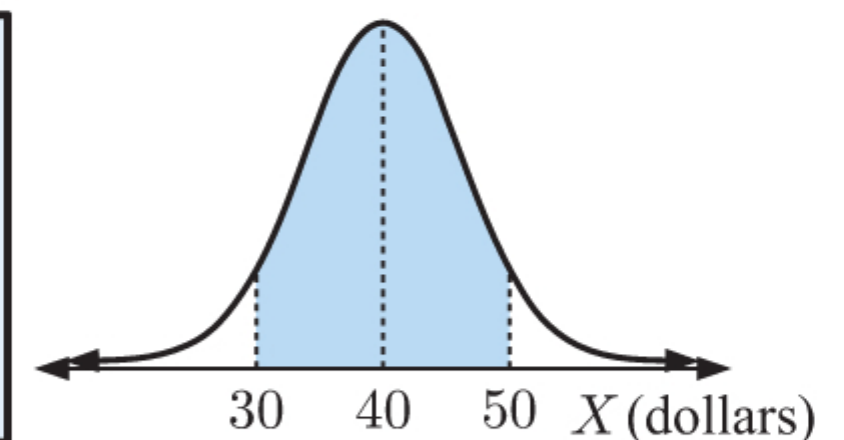
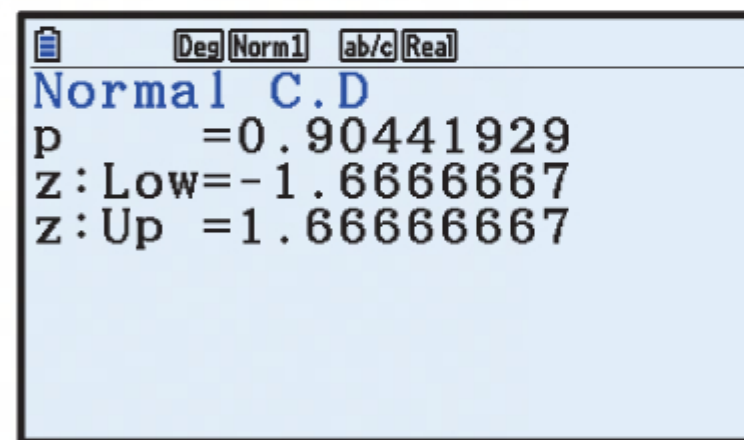
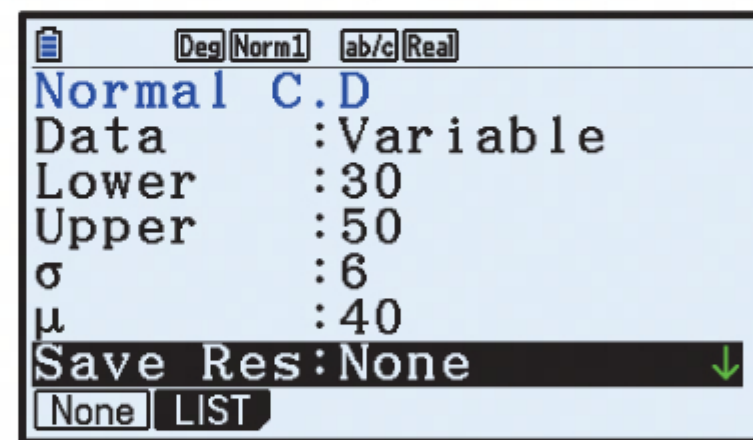
\therefore we would expect about $0.234 \times 200 \approx 47$ eels to measure at least 45 cm in length.

- 6 Let X be the amount collected by Max in a randomly selected week.

$$\therefore X \sim N(40, 6^2)$$

a

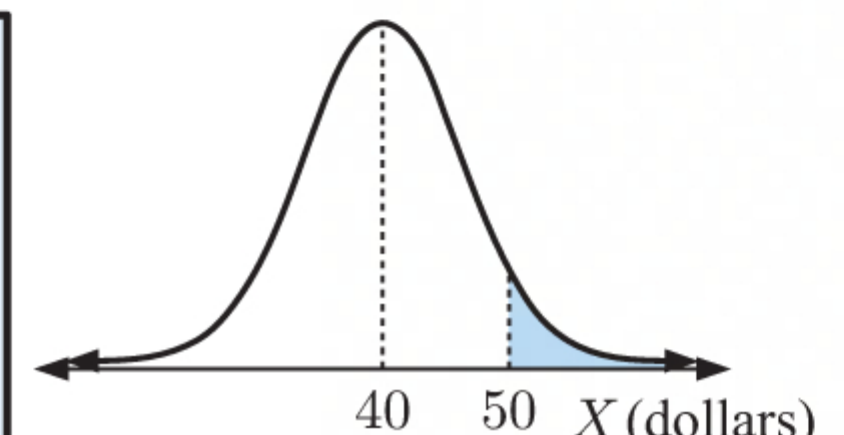
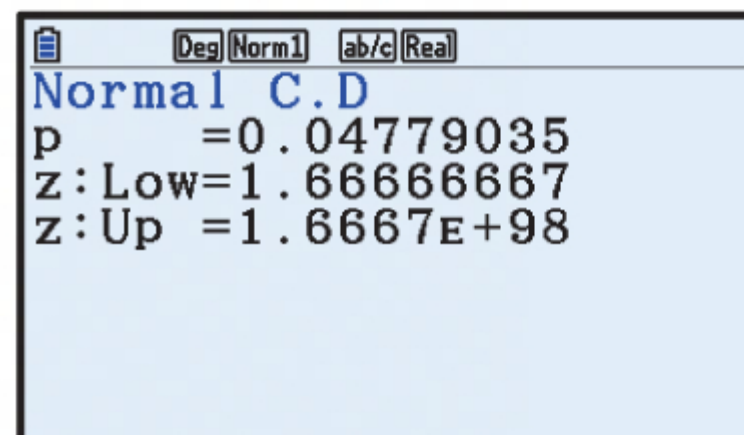
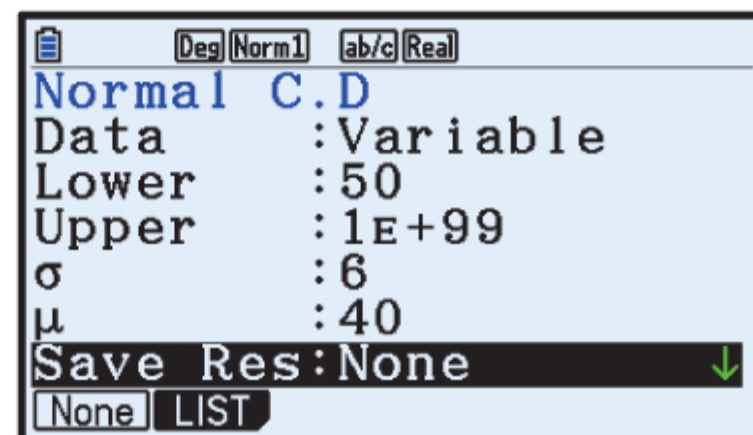
i



$$P(30 < X < 50) \approx 0.904$$

\therefore Max would expect to collect between \$30 and \$50 in about 90.4% of weeks.

ii



$$P(X \geq 50) \approx 0.0478$$

\therefore Max would expect to collect at least \$50 in about 4.78% of weeks.

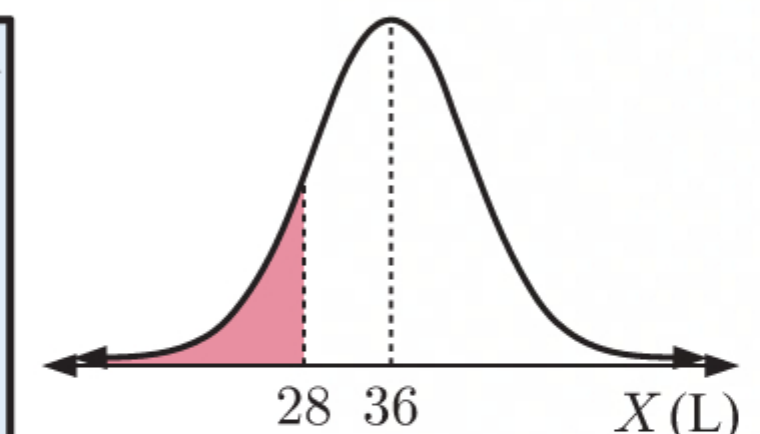
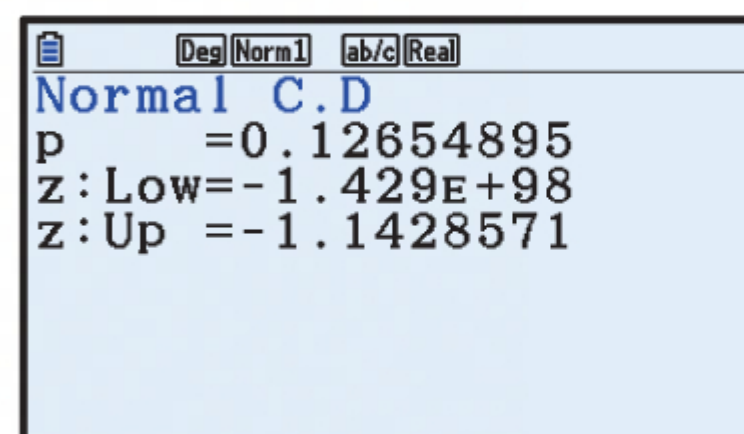
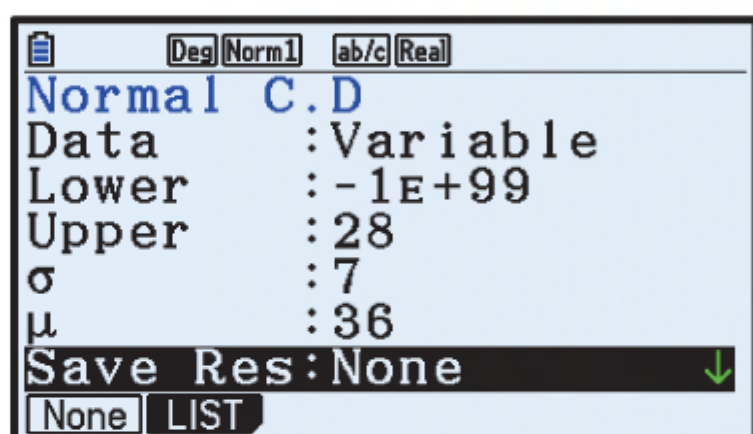
- b There are about 52 weeks in a year, and the average weekly collection is \$40, so in 2 years we would expect Max to collect about $2 \times 52 \times \$40 = \4160 .

- 7 Let X L be the amount of petrol bought by a randomly selected customer.

$$\therefore X \sim N(36, 7^2)$$

a

i



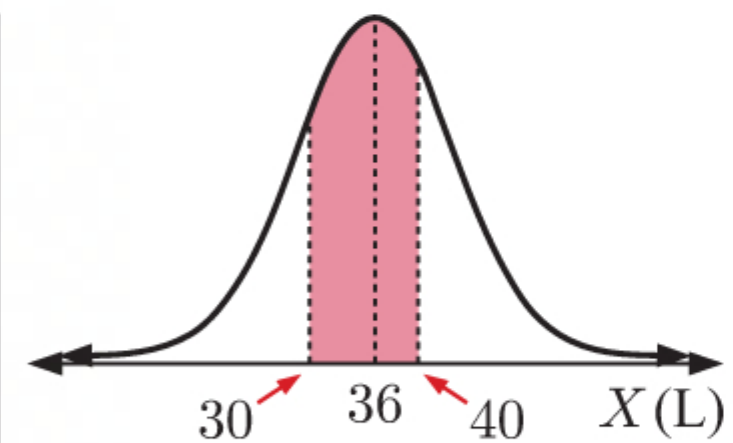
$$P(X < 28) \approx 0.127$$

\therefore about 12.7% of customers buy less than 28 L of petrol.

ii

Normal C.D
Data : Variable
Lower : 30
Upper : 40
σ : 7
μ : 36
Save Res: None

Normal C.D
p = 0.52046244
z: Low = -0.8571428
z: Up = 0.57142857



$$P(30 < X < 40) \approx 0.520$$

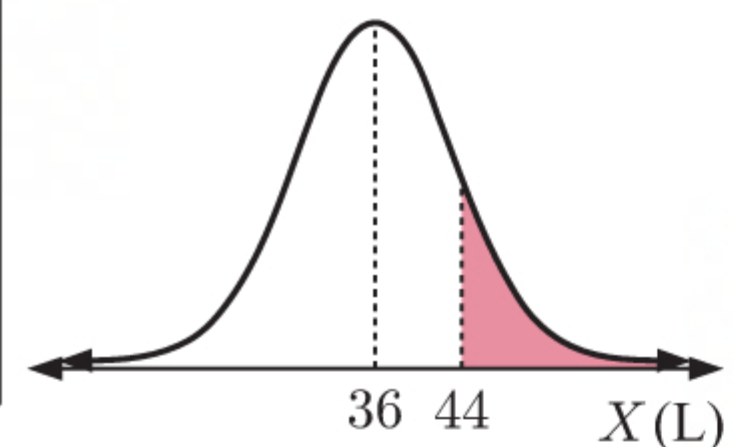
\therefore about 52.0% of customers buy between 30 L and 40 L of petrol.

- b** i We would expect the petrol station to sell about $36 \text{ L} \times 600 = 21.6 \text{ kL}$ of petrol.

ii

Normal C.D
Data : Variable
Lower : 44
Upper : $1\text{E}+99$
σ : 7
μ : 36
Save Res: None
None LIST

Normal C.D
p = 0.12654895
z: Low = 1.14285714
z: Up = $1.4286\text{E}+98$



$$P(X \geq 44) \approx 0.127$$

\therefore we would expect about $0.127 \times 600 \approx 76$ customers to buy at least 44 L of petrol.

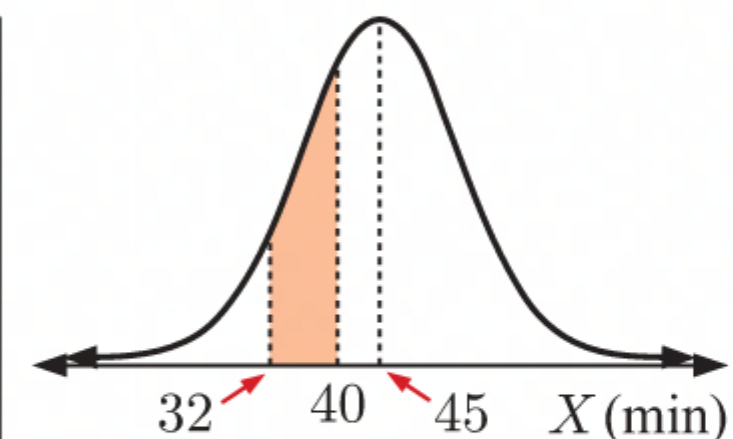
- 8** Let X minutes be the amount of time Enrique spends at the gym in a day, and Y minutes be the amount of time Damien spends at the gym in a day.

$$\therefore X \sim N(45, 9^2), \quad Y \sim N(45, 6^2)$$

a i

Normal C.D
Data : Variable
Lower : 32
Upper : 40
σ : 9
μ : 45
Save Res: None
None LIST

Normal C.D
p = 0.21495036
z: Low = -1.4444444
z: Up = -0.5555555



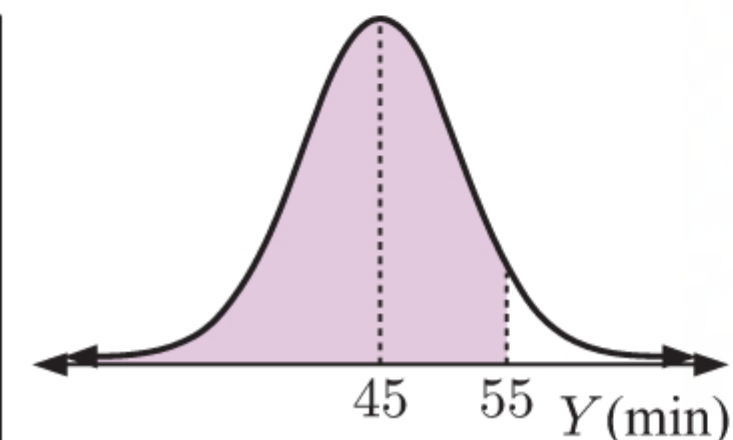
$$P(32 < X < 40) \approx 0.215$$

\therefore Enrique will spend between 32 and 40 minutes at the gym on about 21.5% of days.

ii

Normal C.D
Data : Variable
Lower : $-1\text{E}+99$
Upper : 55
σ : 6
μ : 45
Save Res: None
None LIST

Normal C.D
p = 0.95220964
z: Low = $-1.667\text{E}+98$
z: Up = 1.66666667

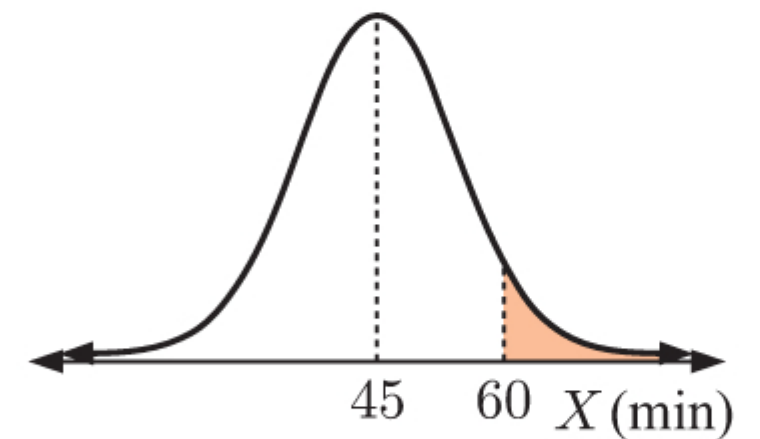
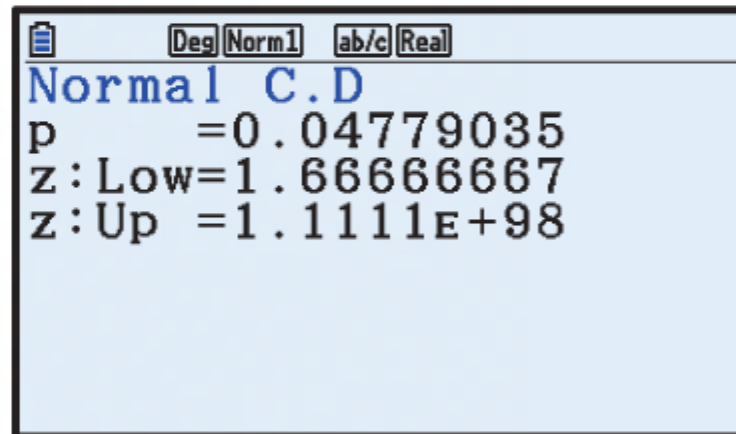
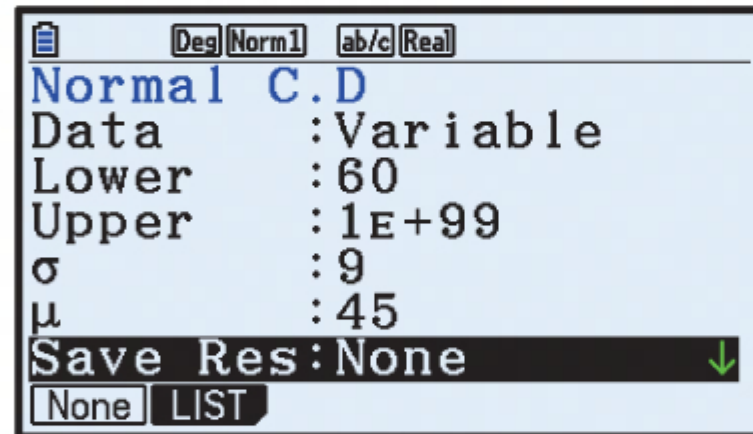


$$P(Y < 55) \approx 0.952$$

\therefore Damien will spend less than 55 minutes at the gym on about 95.2% of days.

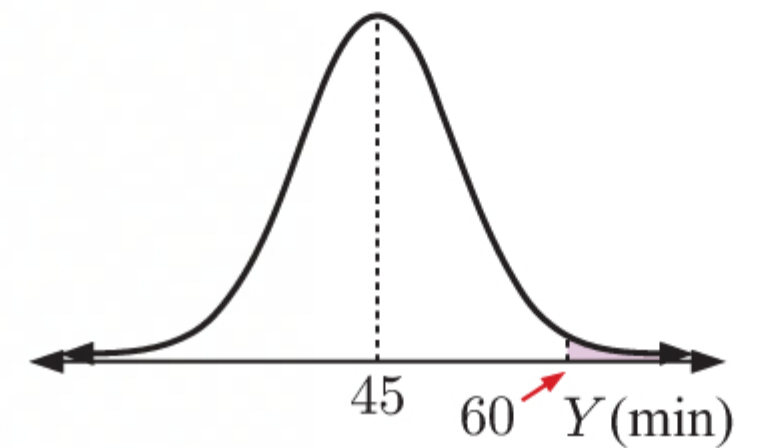
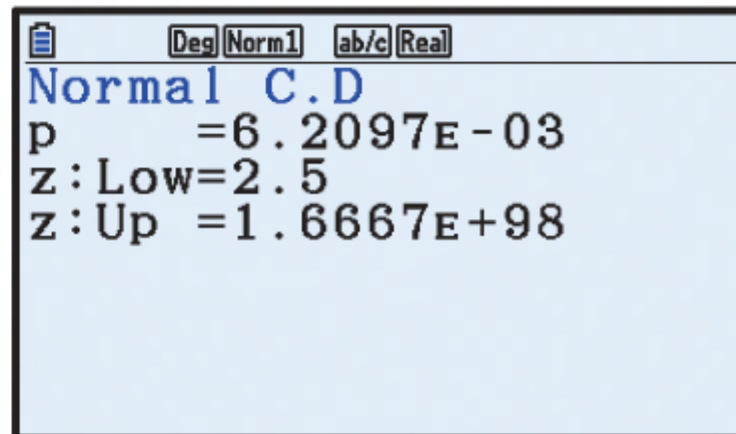
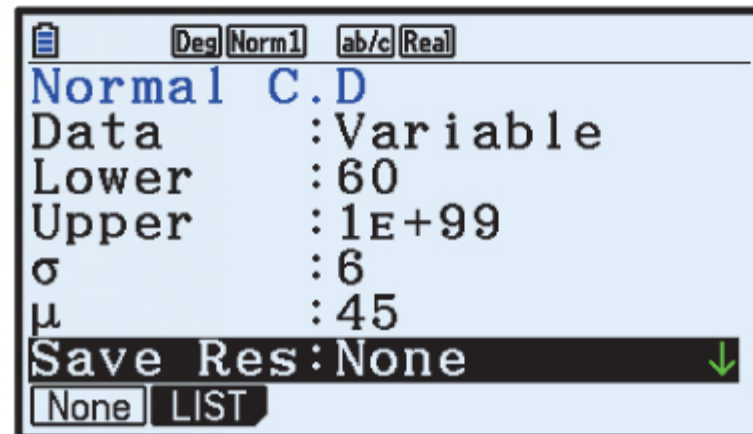
- b** i Enrique is more likely to spend at least 1 hour at the gym. The mean of both of their times is 45 minutes, but Enrique has a greater standard deviation, and so is more likely to exceed 1 hour.
- ii Damien is more likely to spend between 40 and 50 minutes at the gym. Damien has the smaller standard deviation and is more likely to stay between 40 and 50 minutes, which is close to the mean of 45 minutes.

c i Enrique:



$$P(X > 60) \approx 0.0478$$

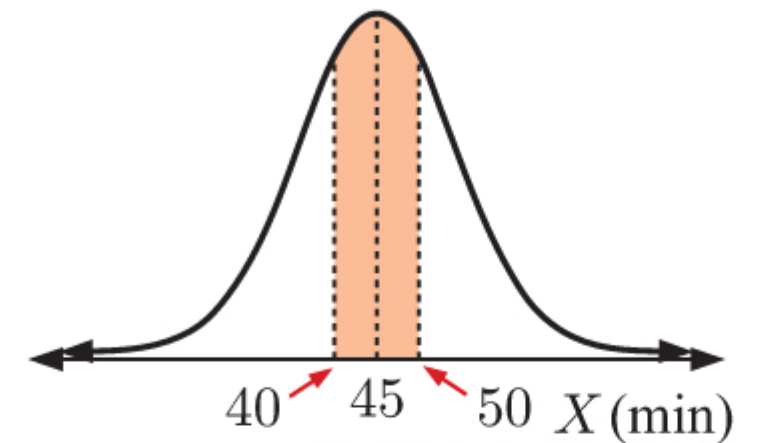
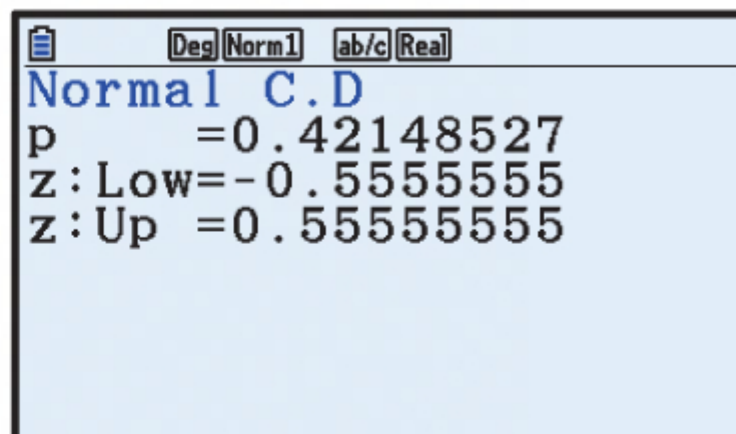
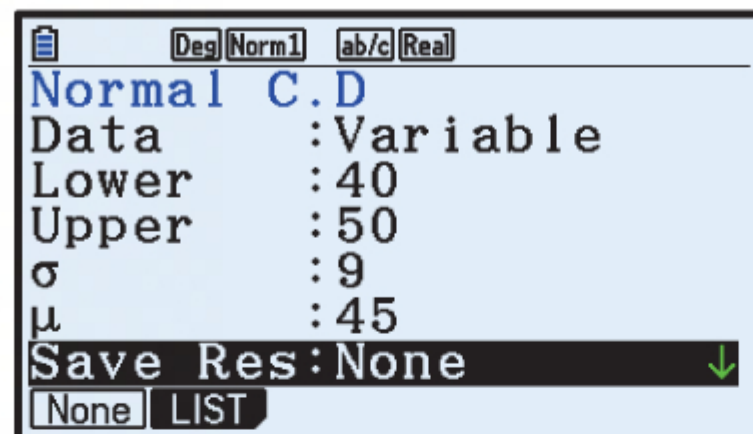
Damien:



$$P(Y > 60) \approx 0.00621$$

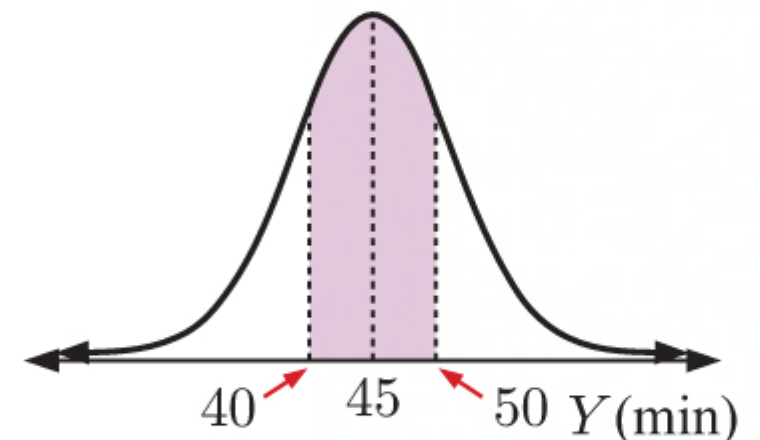
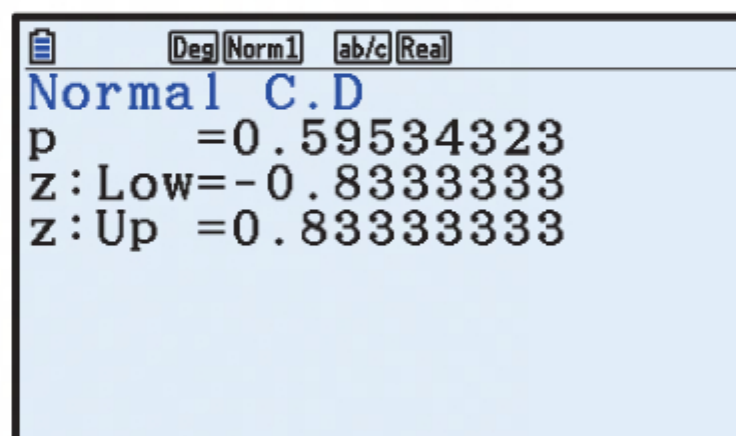
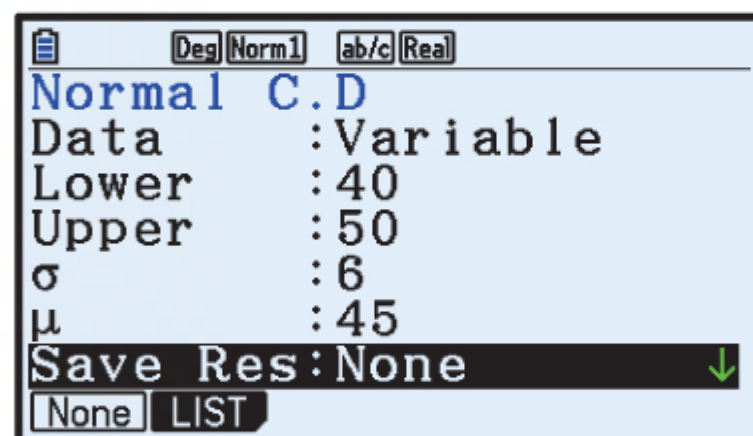
\therefore Enrique is more likely to spend at least 1 hour at the gym.

ii Enrique:



$$P(40 < X < 50) \approx 0.421$$

Damien:



$$P(40 < Y < 50) \approx 0.595$$

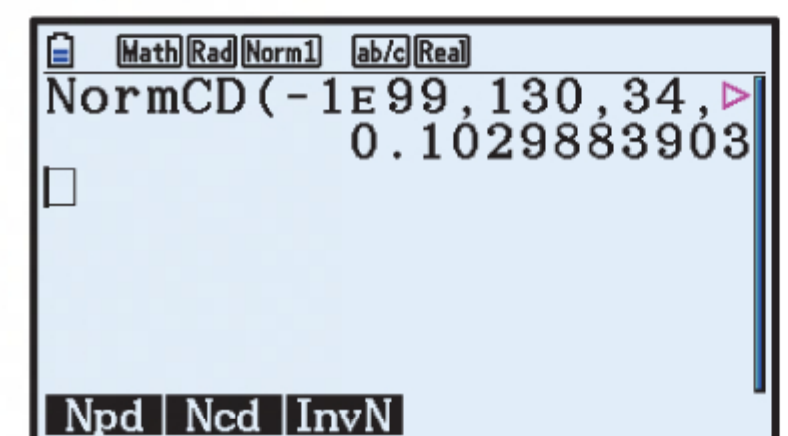
\therefore Damien is more likely to spend between 40 and 50 minutes at the gym.

9 a Let X grams be the weight of a randomly selected apple.

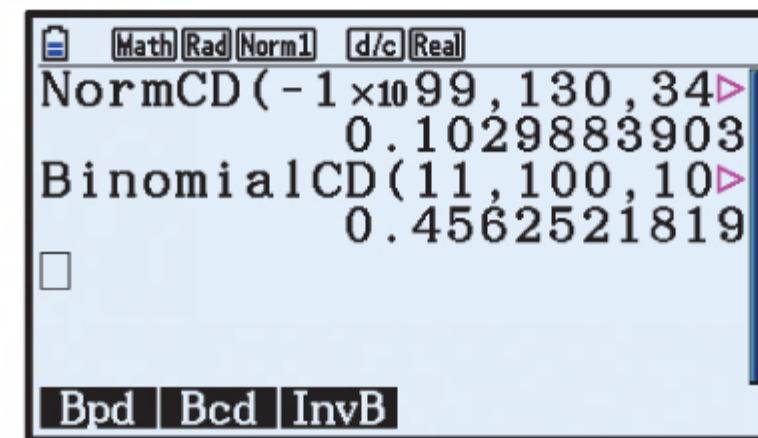
$$X \sim N(173, 34^2)$$

$$\therefore P(X < 130) \approx 0.10299 \approx 0.103$$

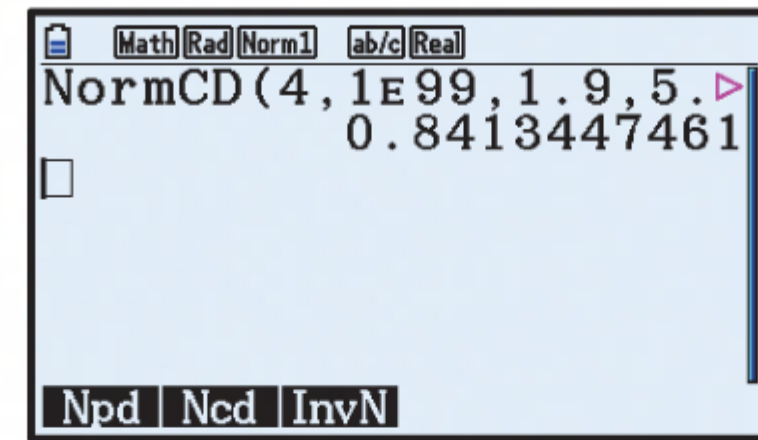
\therefore about 10.3% of apples from the crop were too small to sell.



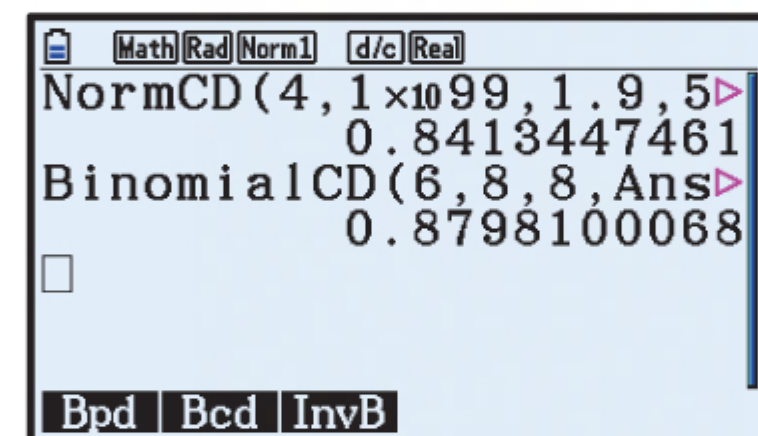
- b** Let Y be the number of apples which were too small to sell.
 $Y \sim B(100, 0.10299)$
 $\therefore P(Y > 10) = P(Y \geq 11)$
 ≈ 0.456



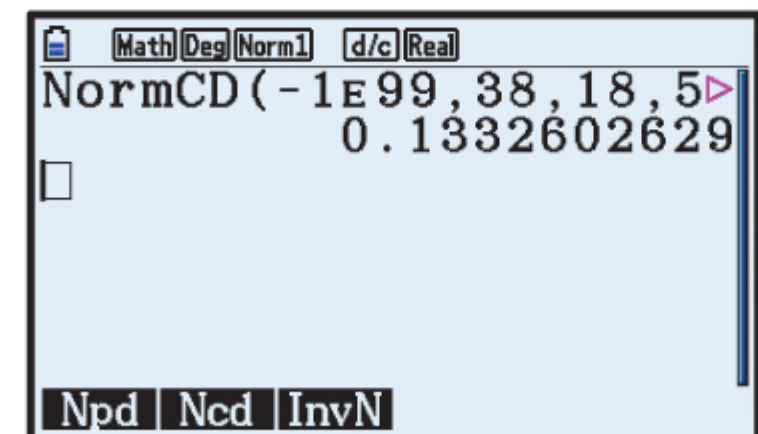
- 10 a** Let X units be the drop in blood pressure of a randomly selected patient.
 $X \sim N(5.9, 1.9^2)$
 $\therefore P(X > 4) \approx 0.84134$
 ≈ 0.841
 \therefore about 84.1% of patients show a drop of more than 4 units.



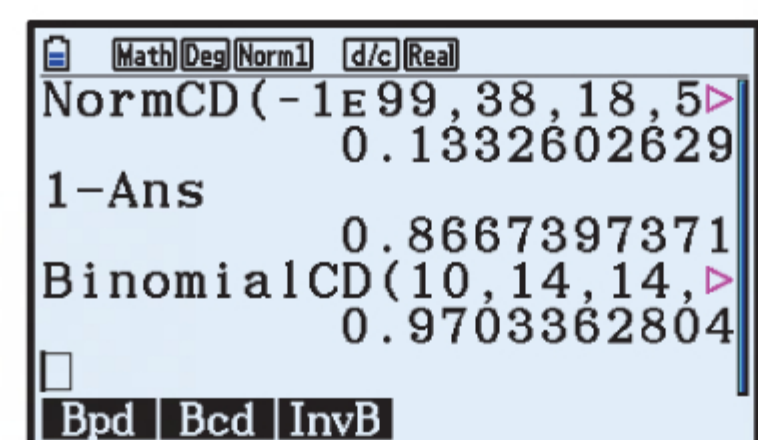
- b** Let Y be the number of patients with a drop of more than 4 units.
 $Y \sim B(8, 0.84134)$
 $\therefore P(Y \geq 6) \approx 0.880$



- 11 a** Let X cm be the length of a randomly selected red snapper.
 $X \sim N(58, 18^2)$
 $\therefore P(X < 38) \approx 0.13326$
 ≈ 0.133



- b** Let Y be the number of red snapper that are long enough to keep.
 From **a**, the probability that a single snapper can be *kept* is about $1 - 0.13326 \approx 0.86674$.
 $Y \sim B(14, 0.86674)$
 $\therefore P(Y \geq 10) \approx 0.970$

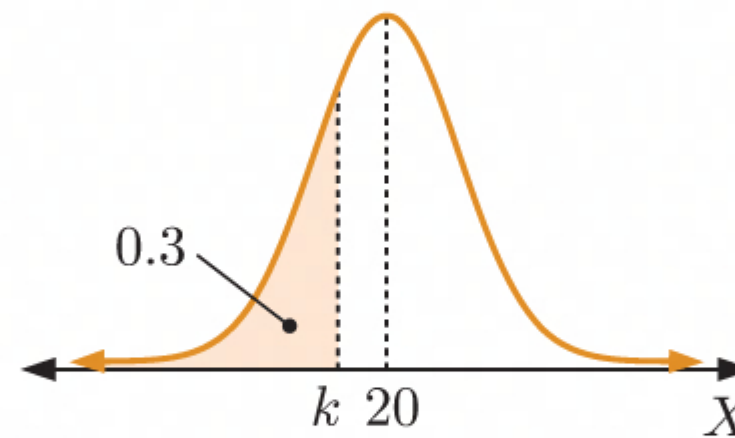


EXERCISE 15C**1 a**

Rad Norm1 ab/c Real	
Inverse Normal	
Data	:Variable
Tail	:Left
Area	:0.3
σ	:3
μ	:20
Save Res:None	
None	LIST

Rad Norm1 ab/c Real	
Inverse Normal	
xInv=18.4267985	

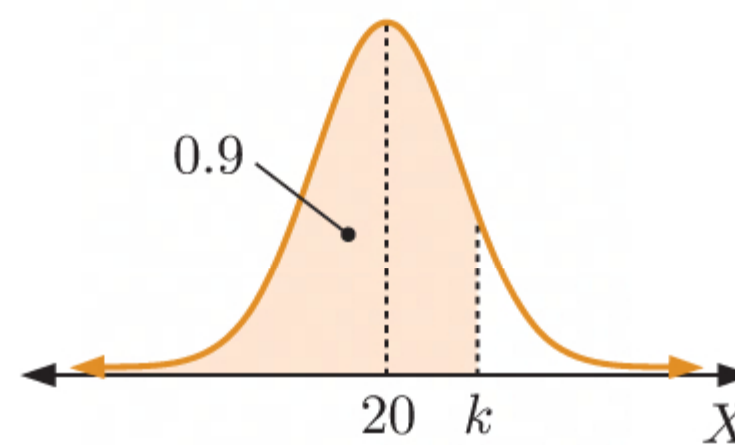
If $P(X \leq k) = 0.3$
 then $k \approx 18.4$

**b**

Rad Norm1 ab/c Real	
Inverse Normal	
Data	:Variable
Tail	:Left
Area	:0.9
σ	:3
μ	:20
Save Res:None	
None	LIST

Rad Norm1 ab/c Real	
Inverse Normal	
xInv=23.8446547	

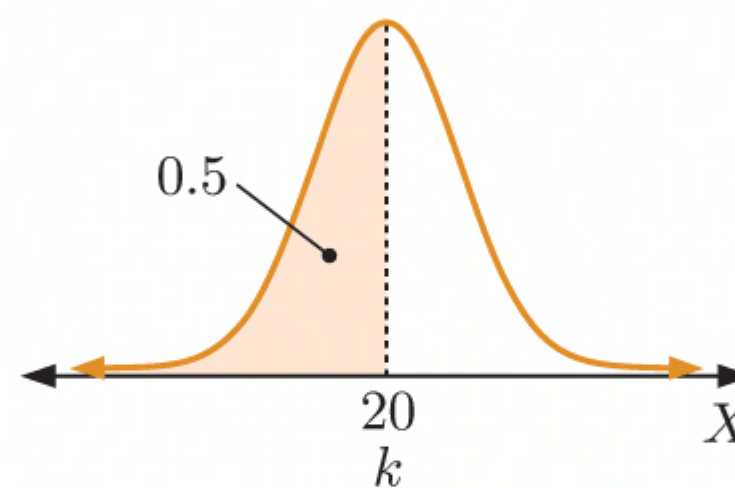
If $P(X \leq k) = 0.9$
 then $k \approx 23.8$

**c**

Rad Norm1 ab/c Real	
Inverse Normal	
Data	:Variable
Tail	:Left
Area	:0.5
σ	:3
μ	:20
Save Res:None	
None	LIST

Rad Norm1 ab/c Real	
Inverse Normal	
xInv=20	

If $P(X \leq k) = 0.5$
 then $k = 20$

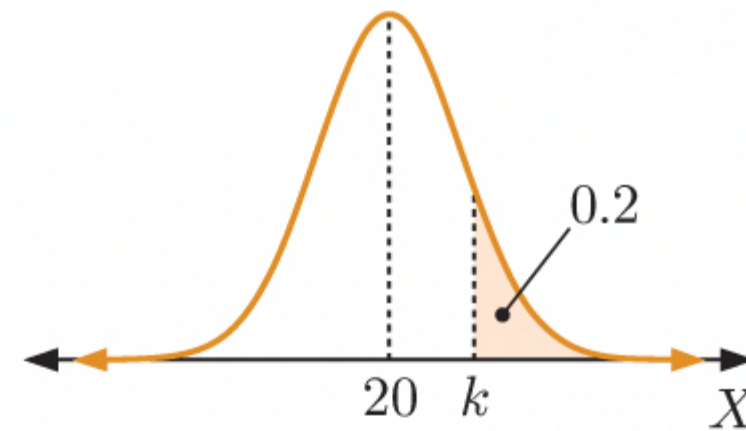


d

Rad Norm1 ab/c Real	
Inverse Normal	
Data	:Variable
Tail	:Right
Area	:0.2
σ	:3
μ	:20
Save Res:None	
None	LIST

Rad Norm1 ab/c Real	
Inverse Normal	
xInv=22.5248637	

If $P(X > k) = 0.2$
then $k \approx 22.5$

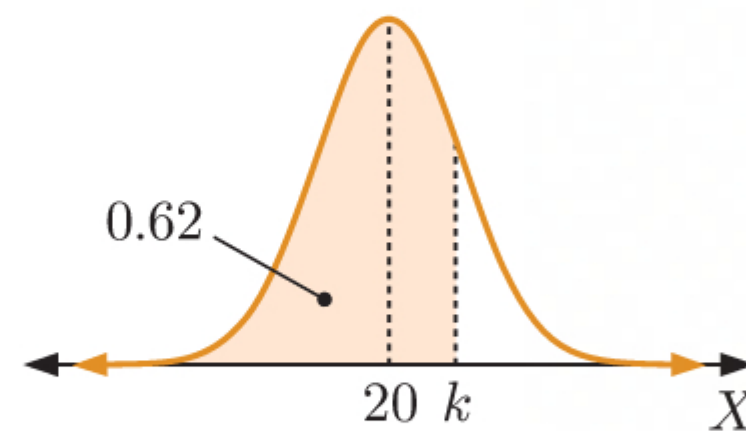


e

Rad Norm1 ab/c Real	
Inverse Normal	
Data	:Variable
Tail	:Left
Area	:0.62
σ	:3
μ	:20
Save Res:None	
None	LIST

Rad Norm1 ab/c Real	
Inverse Normal	
xInv=20.9164424	

If $P(X < k) = 0.62$
then $k \approx 20.9$

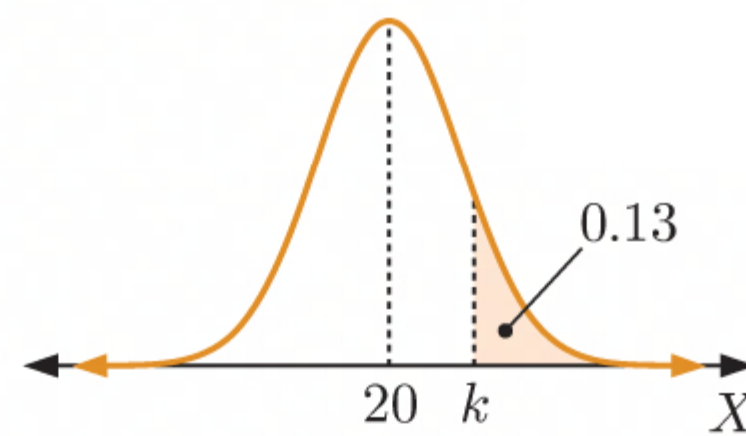


f

Rad Norm1 ab/c Real	
Inverse Normal	
Data	:Variable
Tail	:Right
Area	:0.13
σ	:3
μ	:20
Save Res:None	
None	LIST

Rad Norm1 ab/c Real	
Inverse Normal	
xInv=23.3791734	

If $P(X \geq k) = 0.13$
then $k \approx 23.4$

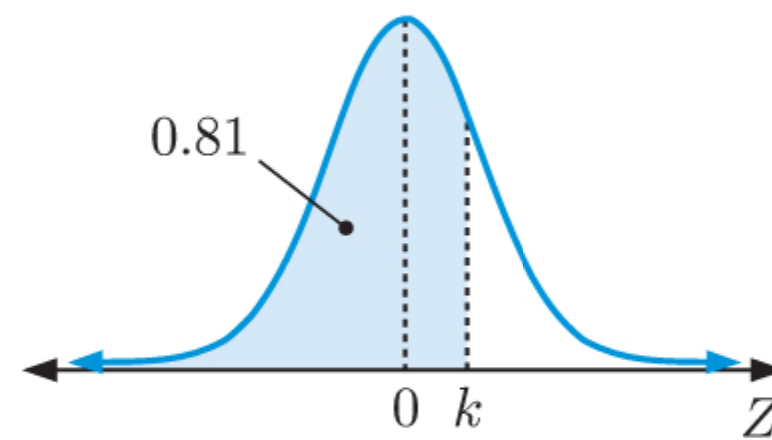


2 a

Rad Norm1 ab/c Real	
Inverse Normal	
Data	:Variable
Tail	:Left
Area	:0.81
σ	:1
μ	:0
Save Res:None	
None	LIST

Rad Norm1 ab/c Real	
Inverse Normal	
xInv=0.87789629	

If $P(Z \leq k) = 0.81$
 then $k \approx 0.878$

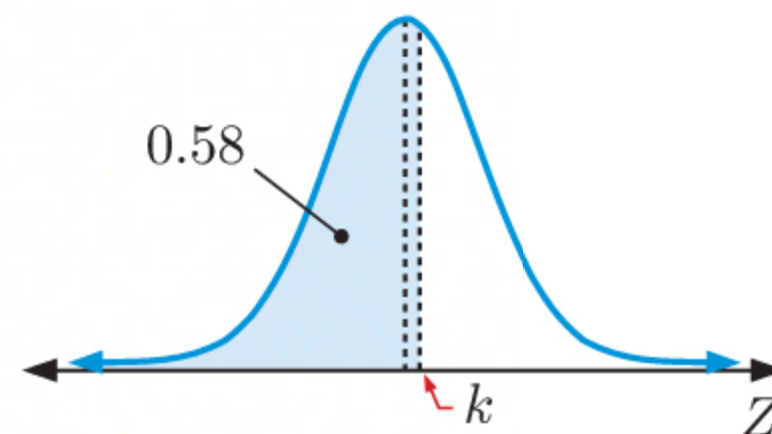


b

Rad Norm1 ab/c Real	
Inverse Normal	
Data	:Variable
Tail	:Left
Area	:0.58
σ	:1
μ	:0
Save Res:None	
None	LIST

Rad Norm1 ab/c Real	
Inverse Normal	
xInv=0.20189347	

If $P(Z \leq k) = 0.58$
 then $k \approx 0.202$

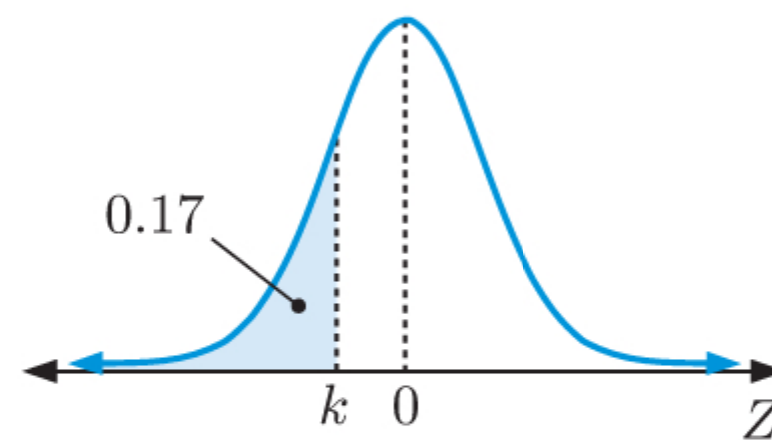


c

Rad Norm1 ab/c Real	
Inverse Normal	
Data	:Variable
Tail	:Left
Area	:0.17
σ	:1
μ	:0
Save Res:None	
None	LIST

Rad Norm1 ab/c Real	
Inverse Normal	
xInv=-0.9541652	

If $P(Z \leq k) = 0.17$
 then $k \approx -0.954$



d

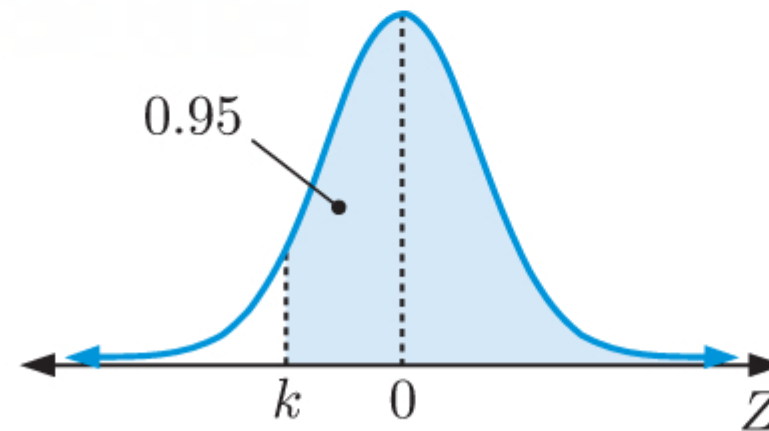
```

Inverse Normal
Data      :Variable
Tail      :Right
Area      :0.95
σ         :1
μ         :0
Save Res:None
None LIST
    
```

```

Inverse Normal
xInv=-1.6448536
    
```

If $P(Z \geq k) = 0.95$
 $\therefore k \approx -1.64$



e

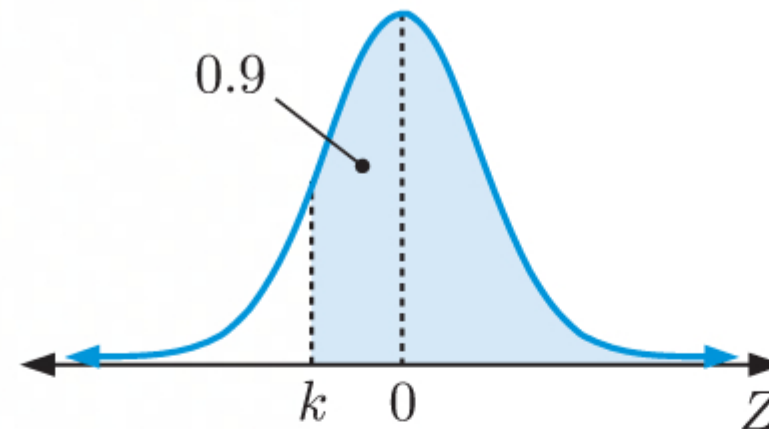
```

Inverse Normal
Data      :Variable
Tail      :Right
Area      :0.9
σ         :1
μ         :0
Save Res:None
None LIST
    
```

```

Inverse Normal
xInv=-1.2815516
    
```

If $P(Z \geq k) = 0.9$
 $\therefore k \approx -1.28$



f

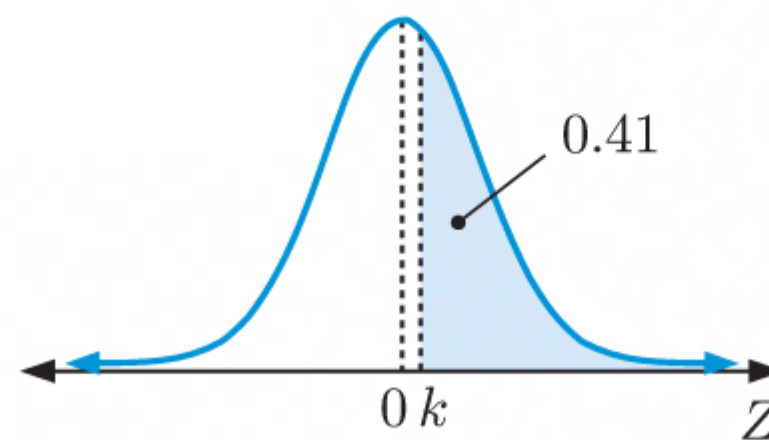
```

Inverse Normal
Data      :Variable
Tail      :Right
Area      :0.41
σ         :1
μ         :0
Save Res:None
None LIST
    
```

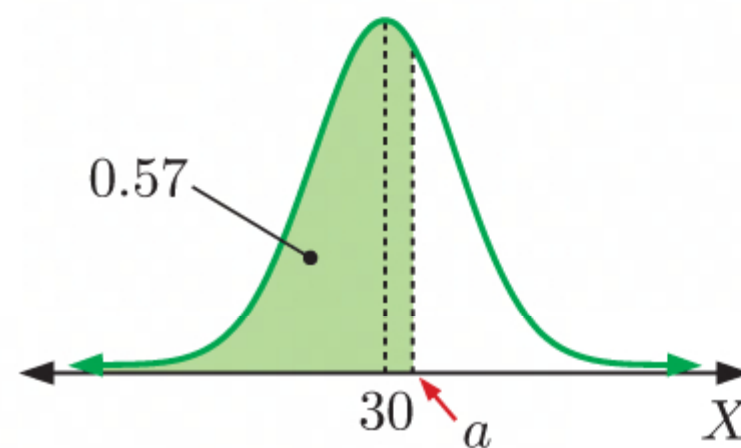
```

Inverse Normal
xInv=0.22754497
    
```

If $P(Z \geq k) = 0.41$
 $\therefore k \approx 0.228$



3 a $X \sim N(30, 5^2)$



$\therefore a > 30$

b $P(X \leq a) = 0.57$
 $\therefore a \approx 30.9$

Rad Norm1 ab/c Real	
Inverse Normal	
Data	:Variable
Tail	:Left
Area	:0.57
σ	:5
μ	:30
Save Res:None	
None	LIST

Rad Norm1 ab/c Real	
Inverse Normal	
xInv=30.8818708	

c **i** $P(X \geq a) = 1 - P(X \leq a)$
 $= 1 - 0.57$
 $= 0.43$

ii $P(30 \leq X \leq a) = P(X \leq a) - P(X \leq 30)$
 $= 0.57 - 0.5$
 $= 0.07$

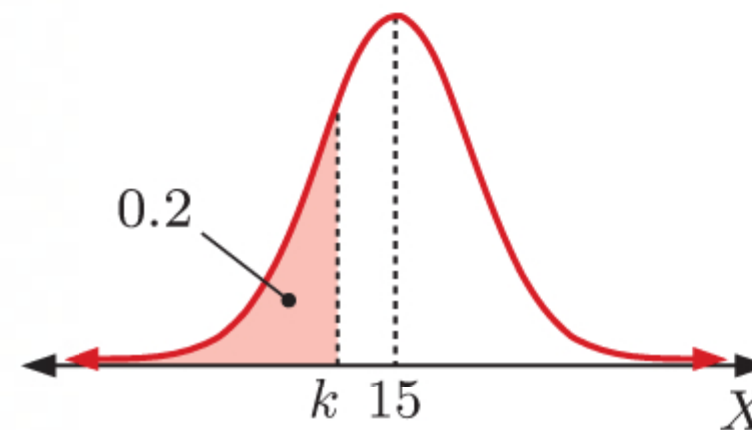
4 $X \sim N(15, 3^2)$

a

Rad Norm1 ab/c Real	
Inverse Normal	
Data	:Variable
Tail	:Left
Area	:0.2
σ	:3
μ	:15
Save Res:None	
None	LIST

Rad Norm1 ab/c Real	
Inverse Normal	
xInv=12.4751363	

If $P(X < k) = 0.2$
 then $k \approx 12.5$

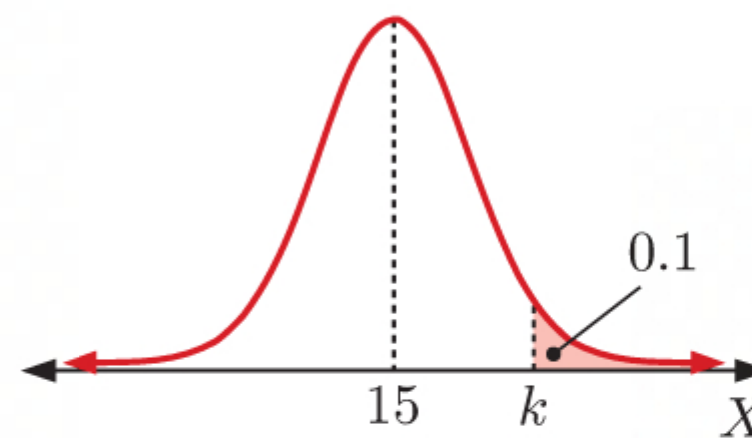


b

Rad Norm1 ab/c Real	
Inverse Normal	
Data	:Variable
Tail	:Right
Area	:0.1
σ	:3
μ	:15
Save Res:None	
None	LIST

Rad Norm1 ab/c Real	
Inverse Normal	
xInv=18.8446547	

If $P(X > k) = 0.1$
 then $k \approx 18.8$

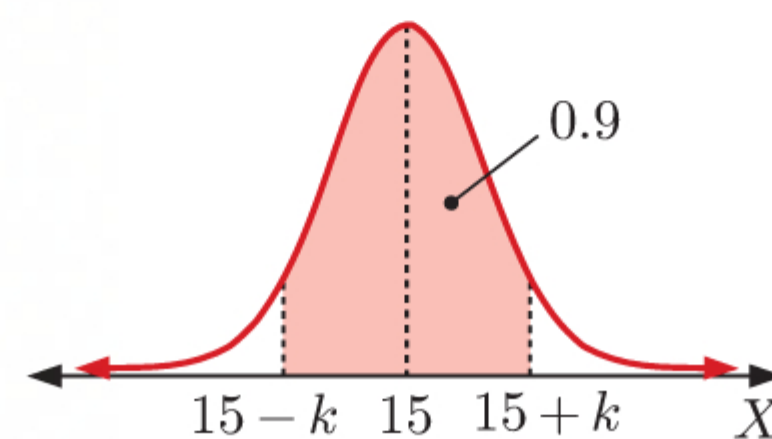


c

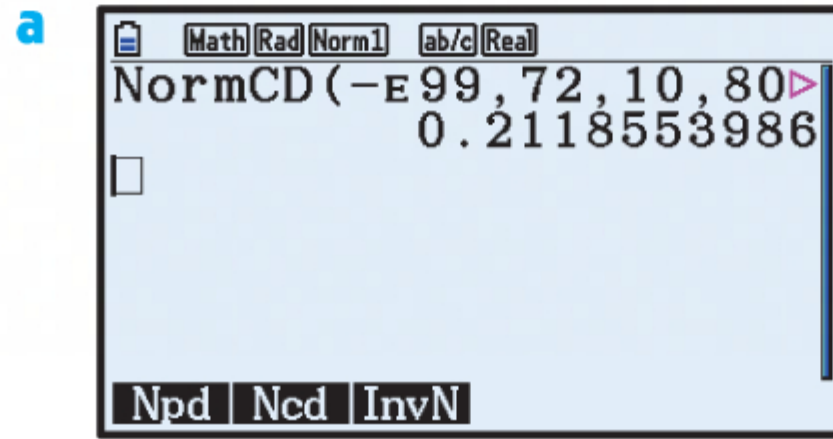
Rad Norm1 ab/c Real	
Inverse Normal	
Data	:Variable
Tail	:Central
Area	:0.9
σ	:3
μ	:15
Save Res:None	
None	LIST

Rad Norm1 ab/c Real	
Inverse Normal	
x1 Inv	=10.0654391
x2 Inv	=19.9345609

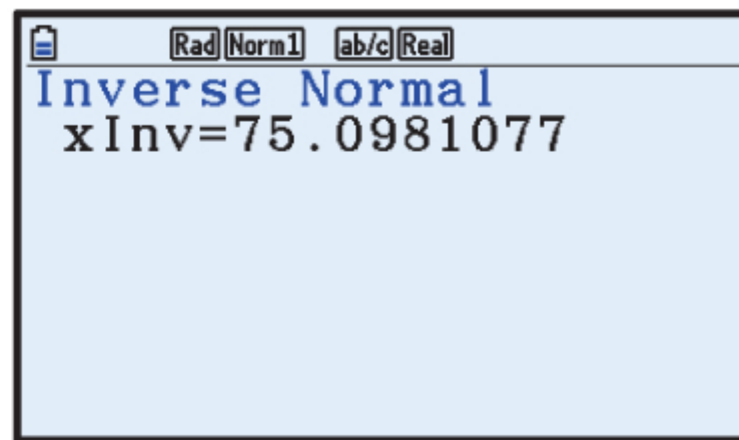
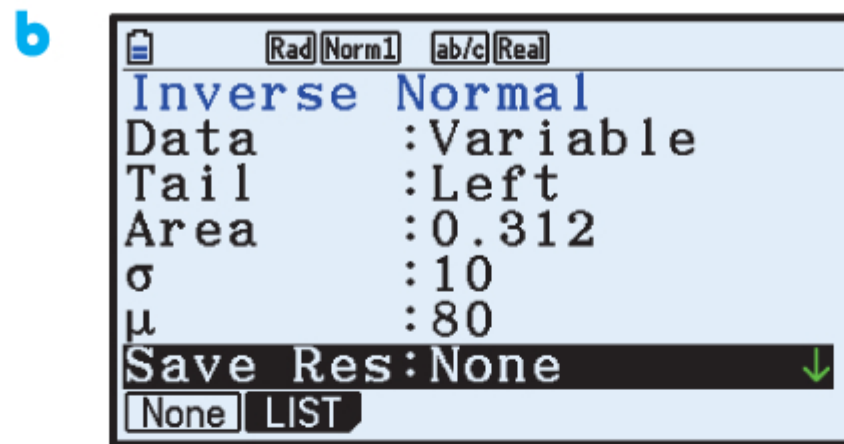
If $P(15 - k < X < 15 + k) = 0.9$
 then $15 - k \approx 10.07$
 or $15 + k \approx 19.93$
 $\therefore k \approx 4.93$



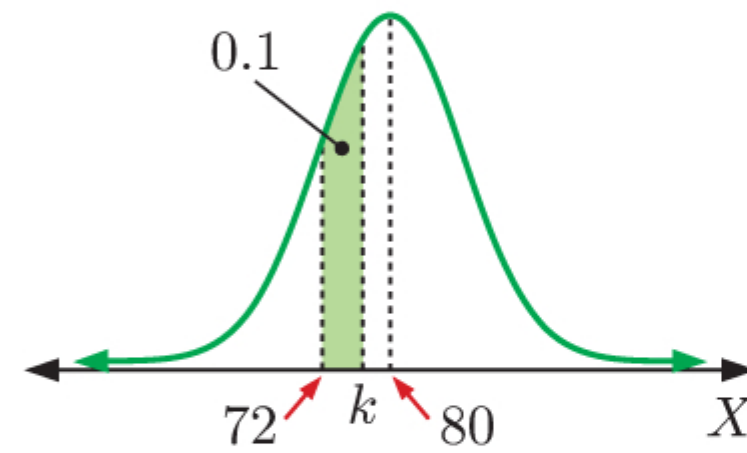
5 $X \sim N(80, 10^2)$



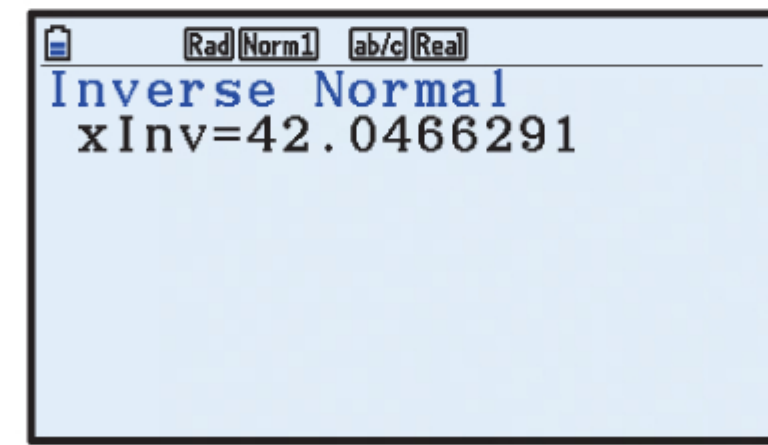
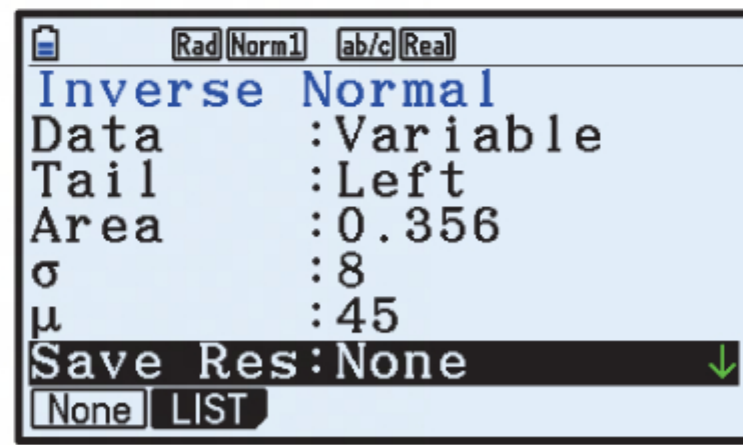
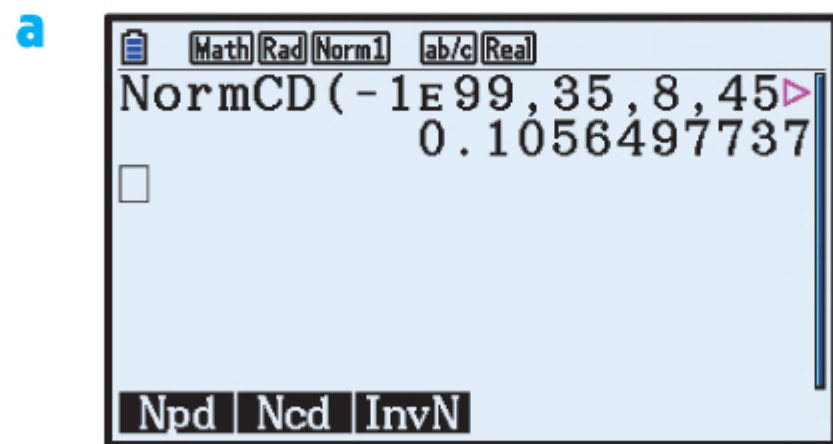
$$P(X \leq 72) \approx 0.212$$



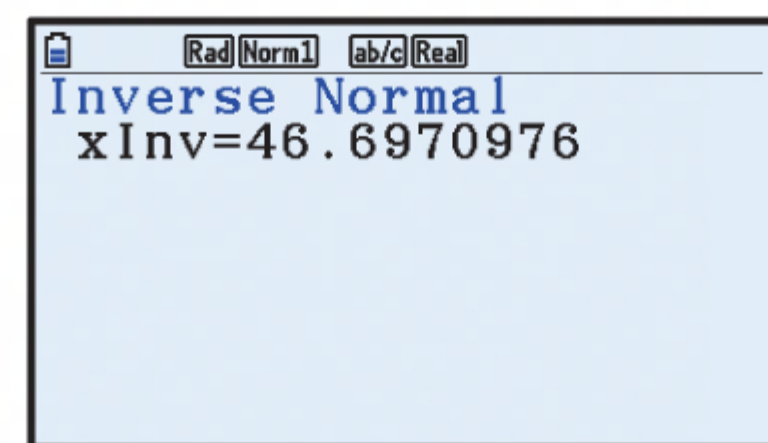
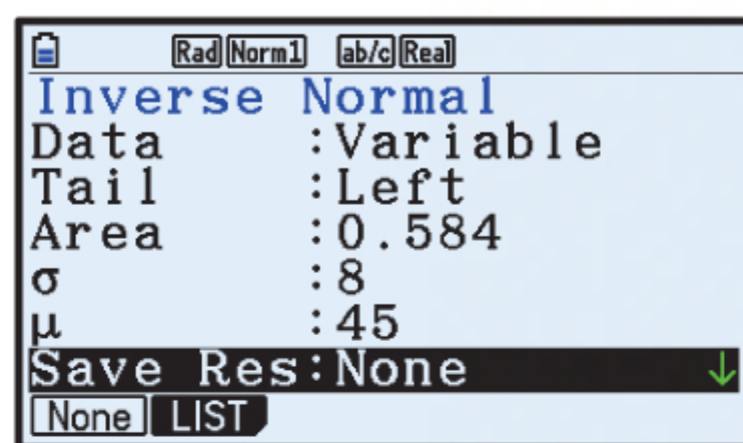
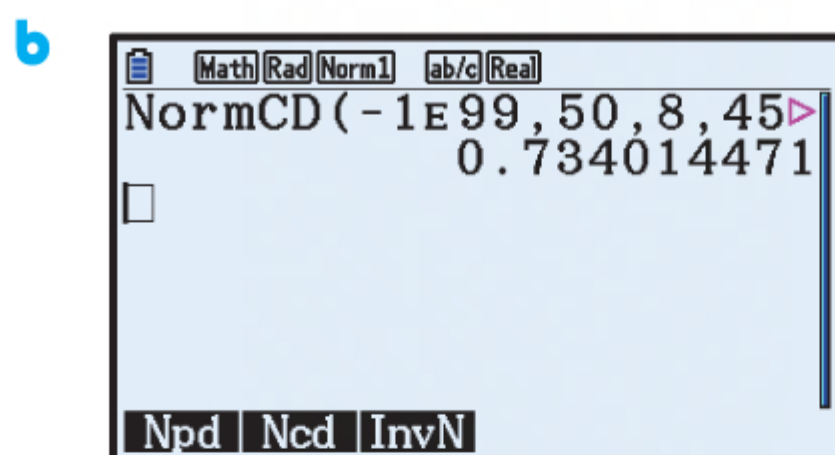
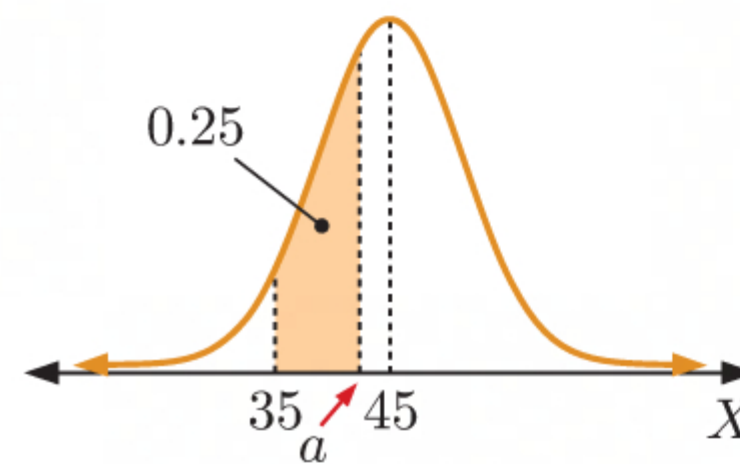
$$\begin{aligned} P(72 \leq X \leq k) &= 0.1 \\ \therefore P(X \leq k) - P(X \leq 72) &= 0.1 \\ \therefore P(X \leq k) - 0.212 &\approx 0.1 \quad \{\text{using a}\} \\ \therefore P(X \leq k) &\approx 0.312 \\ \therefore k &\approx 75.1 \end{aligned}$$



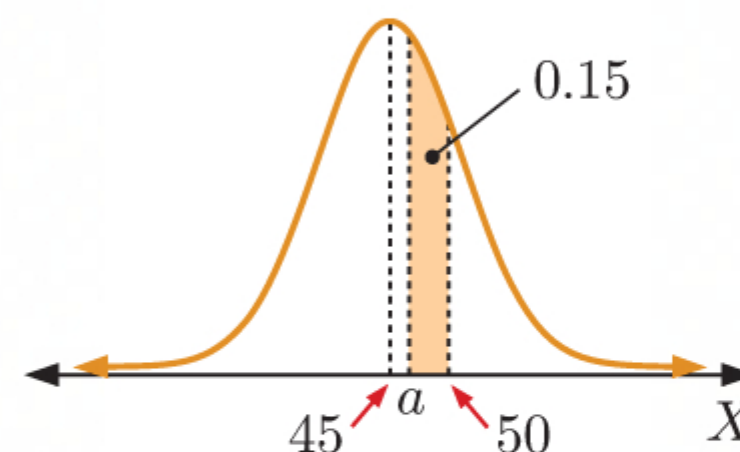
6 $X \sim N(45, 8^2)$

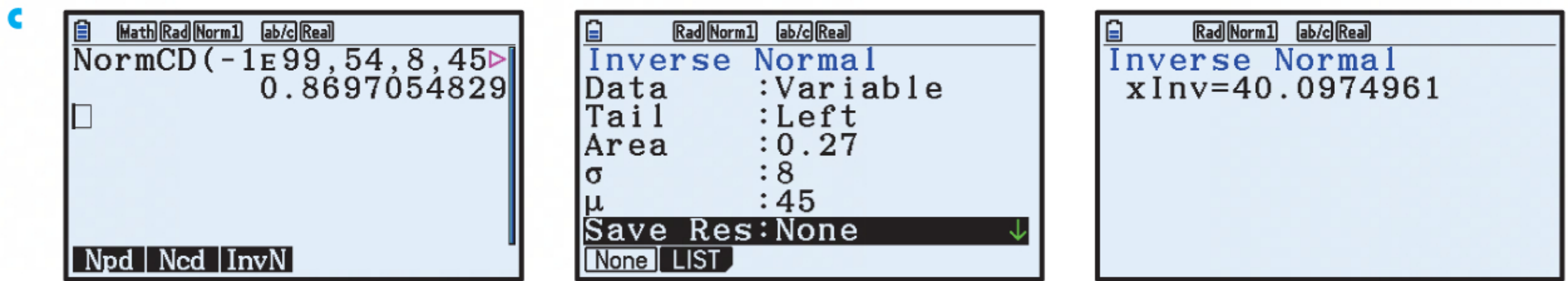


$$\begin{aligned} P(35 \leq X \leq a) &= 0.25 \\ \therefore P(X \leq a) - P(X \leq 35) &= 0.25 \\ \therefore P(X \leq a) - 0.106 &\approx 0.25 \\ \therefore P(X \leq a) &\approx 0.356 \\ \therefore a &\approx 42.0 \end{aligned}$$

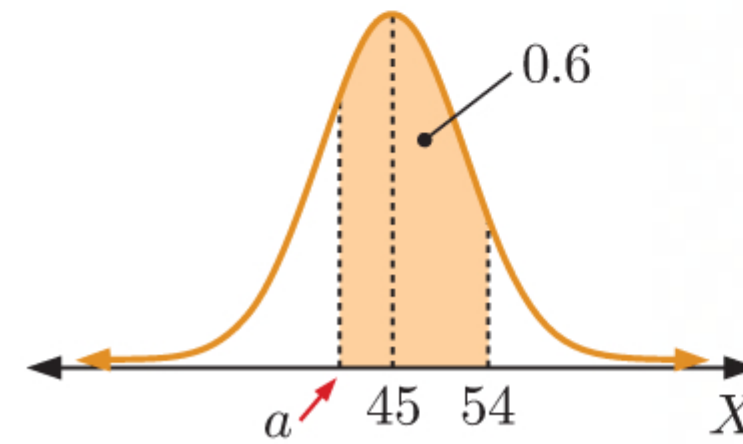
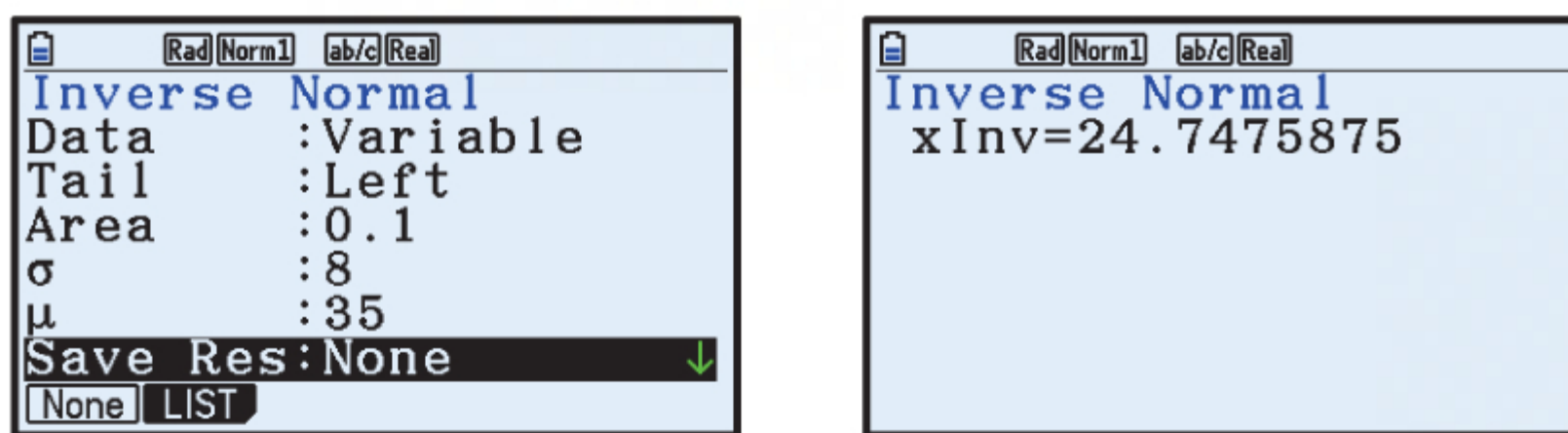


$$\begin{aligned} P(a \leq X \leq 50) &= 0.15 \\ \therefore P(X \leq 50) - P(X \leq a) &= 0.15 \\ \therefore 0.734 - P(X \leq a) &\approx 0.15 \\ \therefore P(X \leq a) &\approx 0.584 \\ \therefore a &\approx 46.7 \end{aligned}$$





$$\begin{aligned}
 P(a \leq X \leq 54) &= 0.6 \\
 \therefore P(X \leq 54) - P(X \leq a) &= 0.6 \\
 \therefore 0.870 - P(X \leq a) &\approx 0.6 \\
 \therefore P(X \leq a) &\approx 0.27 \\
 \therefore a &\approx 40.1
 \end{aligned}$$

**7**

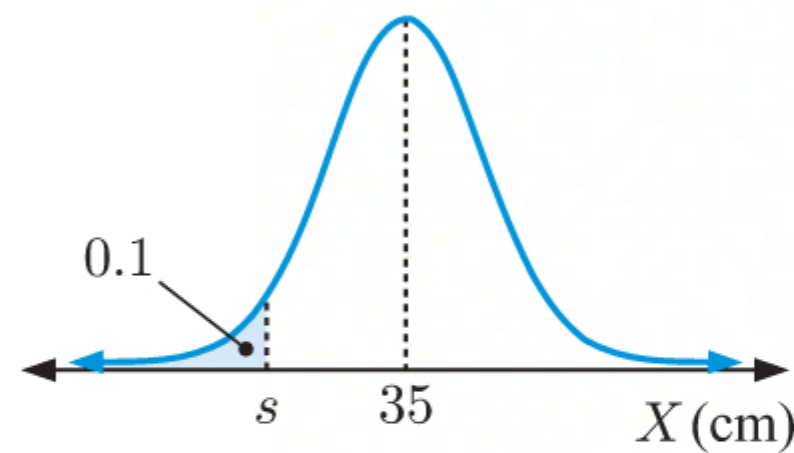
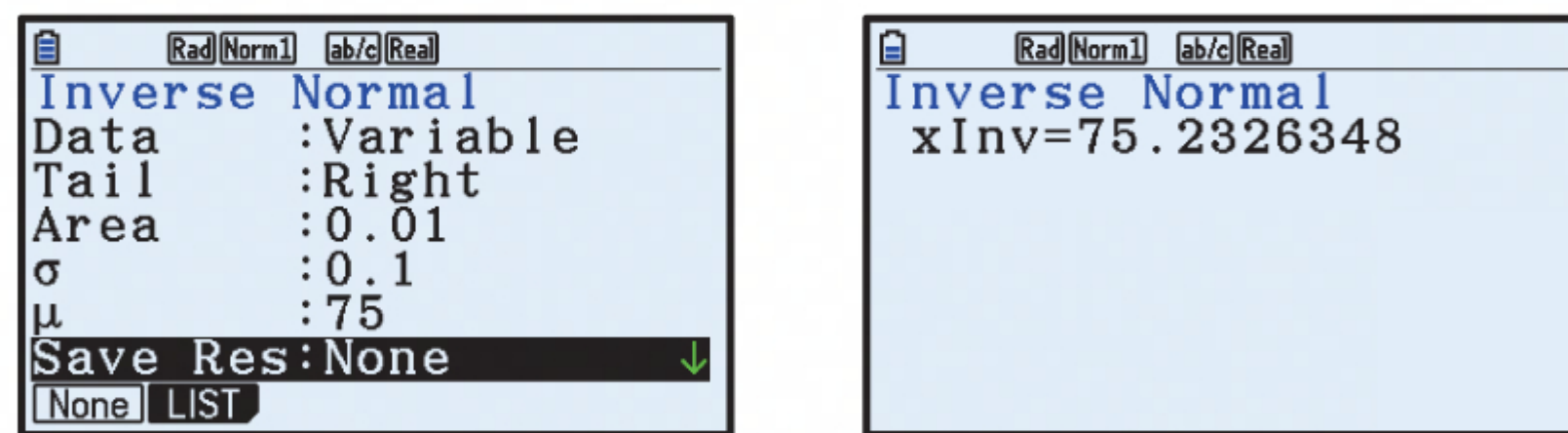
Let X cm be the length of a randomly selected fish, and s cm be the size of the smallest fish that can be harvested.

$$X \sim N(35, 8^2)$$

$$P(X < s) = 0.1$$

$$\therefore s \approx 24.7$$

\therefore the size of the smallest fish that can be harvested is about 24.7 cm.

**8**

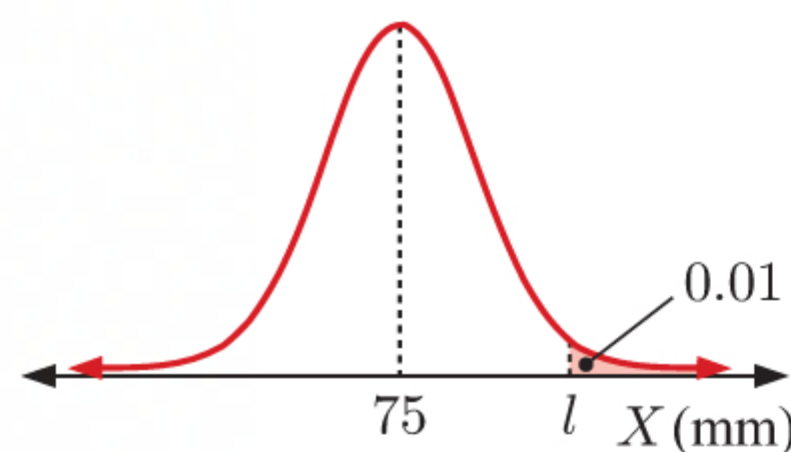
Let X mm be the length of a randomly selected screw, and l mm be the length of the smallest screw to be rejected.

$$X \sim N(75, 0.1^2)$$

$$P(X \geq l) = 0.01$$

$$\therefore l \approx 75.2$$

\therefore the length of the smallest screw to be rejected is about 75.2 mm.



9 $X \sim N(57, 10^2)$

a i

```

Deg Norm1 d/c Real
Inverse Normal
Tail :Right
Area :0.6
σ :10
μ :57
Save Res:None
Execute
None LIST

```

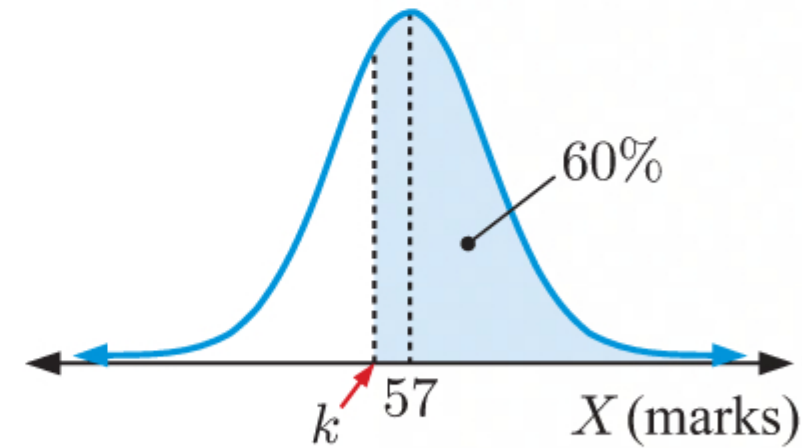
```

Deg Norm1 d/c Real
Inverse Normal
xInv=54.466529

```

$$P(X \geq k) = 0.6$$

$$\therefore k \approx 54.5$$



ii

```

Deg Norm1 d/c Real
Inverse Normal
Data :Variable
Tail :Right
Area :0.1
σ :10
μ :57
Save Res:None
None LIST

```

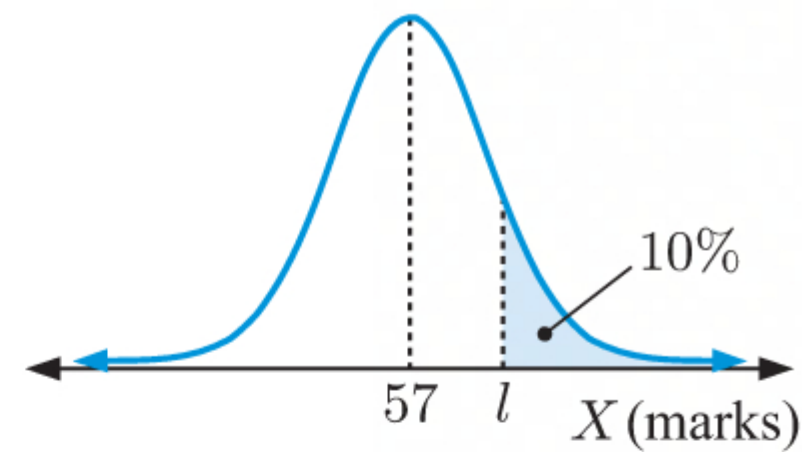
```

Deg Norm1 d/c Real
Inverse Normal
xInv=69.8155157

```

$$P(X \geq l) = 0.1$$

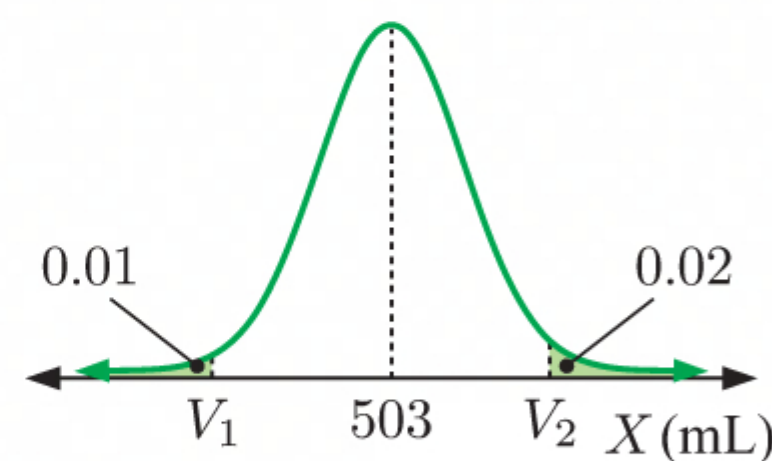
$$\therefore l \approx 69.8$$



b

$$\begin{aligned}
 P(k \leq X \leq l) &= P(X \leq l) - P(X < k) \\
 &= (1 - P(X > l)) - (1 - P(X \geq k)) \\
 &= (1 - 0.1) - (1 - 0.6) \\
 &= 0.9 - 0.4 \\
 &= 0.5 \quad \checkmark
 \end{aligned}$$

- 10 Let X mL be the volume in a randomly selected bottle.
 Let V_1 mL be the volume of the smallest bottle kept,
 and V_2 mL be the volume of the largest bottle kept.
 $X \sim N(503, 0.5^2)$



$$P(X \leq V_1) = 0.01$$

$$\therefore V_1 \approx 501.8$$

```

Rad Norm1 ab/c Real
Inverse Normal
Data :Variable
Tail :Left
Area :0.01
σ :0.5
μ :503
Save Res:None
None LIST

```

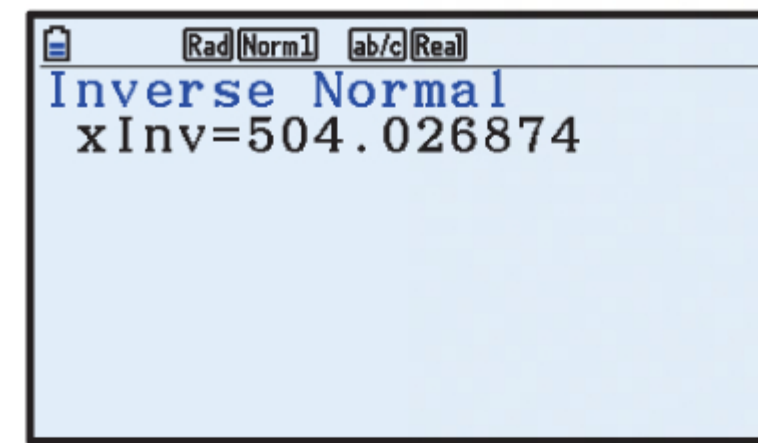
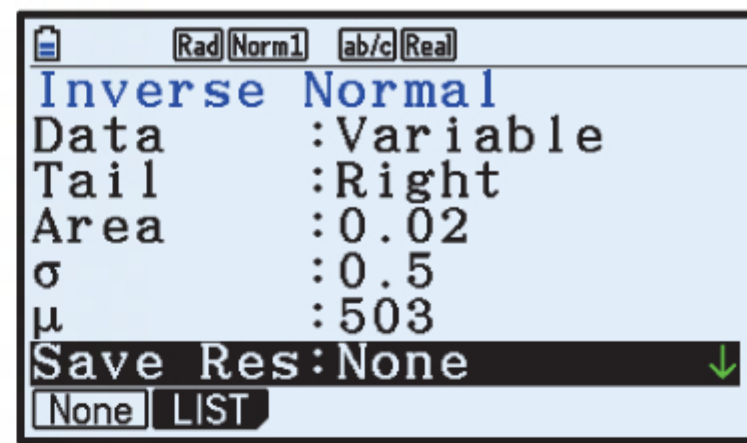
```

Rad Norm1 ab/c Real
Inverse Normal
xInv=501.836826

```


$$P(X \geq V_2) = 0.02$$

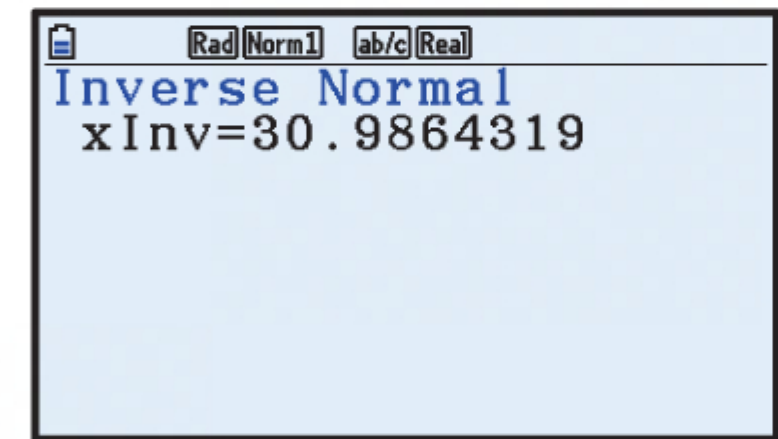
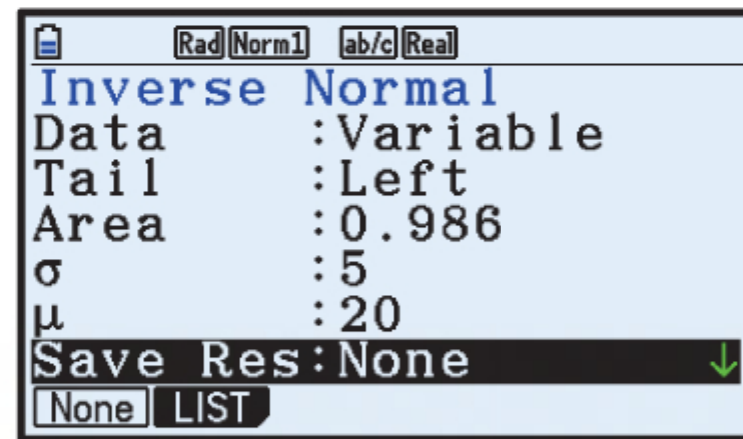
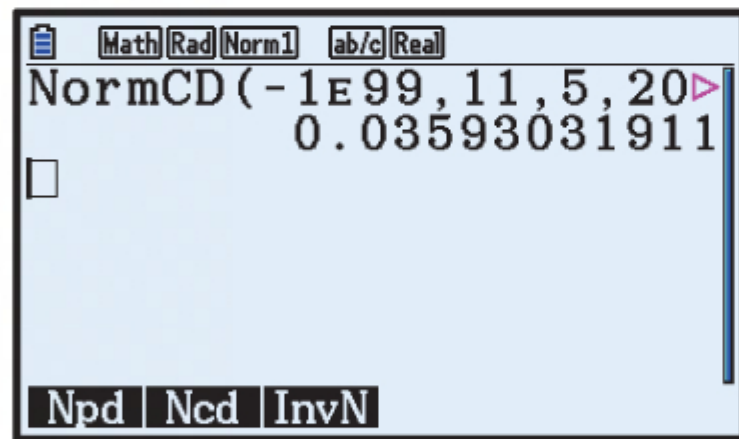
$$\therefore V_2 \approx 504.0$$



\therefore the bottles which are kept have volumes ranging from about 501.8 mL to 504.0 mL.

- 11** Let $X^\circ\text{C}$ be the temperature on a randomly selected morning, and $t^\circ\text{C}$ be the hottest temperature at which Abbey would still walk.

$$X \sim N(20, 5^2)$$



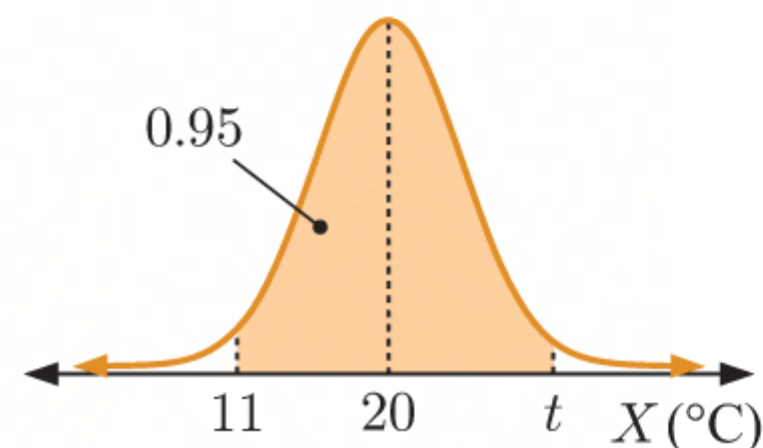
$$P(11 \leq X \leq t) = 0.95$$

$$\therefore P(X \leq t) - P(X \leq 11) = 0.95$$

$$\therefore P(X \leq t) - 0.0359 \approx 0.95$$

$$\therefore P(X \leq t) \approx 0.986$$

$$\therefore t \approx 31.0$$



\therefore the upper limit of Abbey's walking temperatures is about 31.0°C .

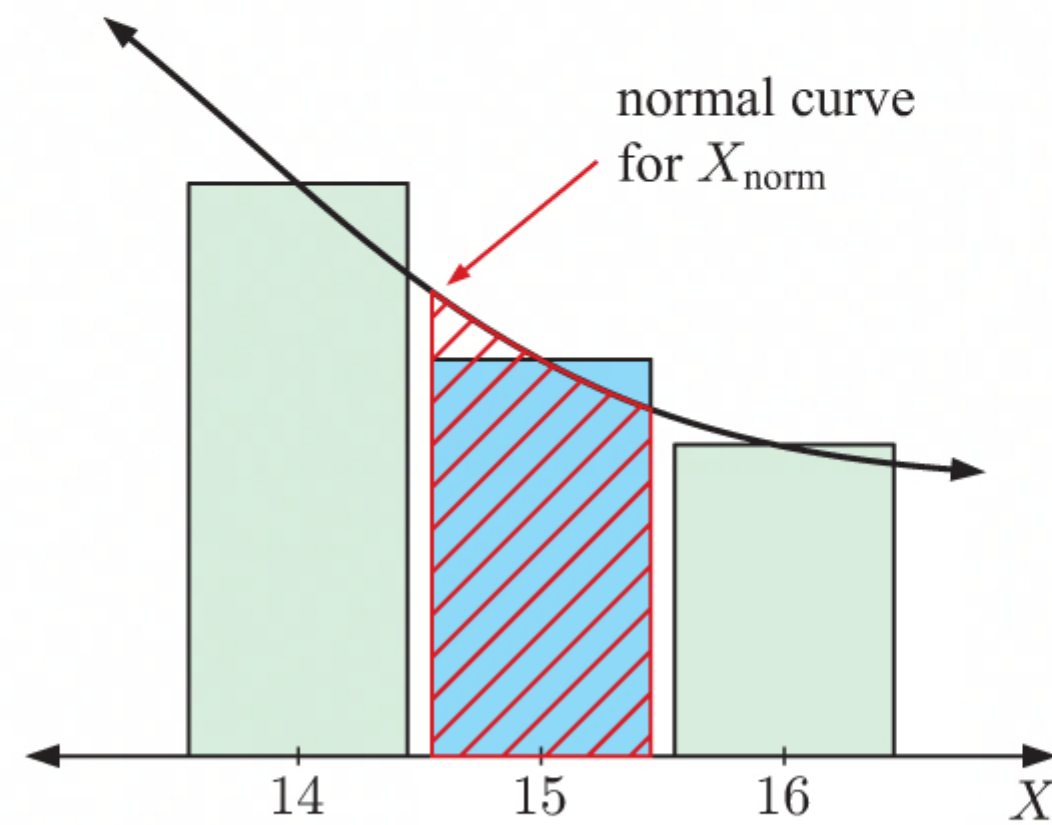
INVESTIGATION 3

THE NORMAL APPROXIMATION TO THE BINOMIAL DISTRIBUTION

- 1**
 - a** As n increases, the distribution of X approaches that of the normal distribution.
 - b** As n increases, the distribution of X approaches that of the normal distribution for all values of p used.
 - c** It is reasonable to approximate the binomial distribution with a normal distribution as long as n is sufficiently large. The distribution should be symmetrical about the most commonly occurring value.
 - d** We expect that $\mu = np$ and $\sigma = \sqrt{np(1-p)}$, like that of the binomial distribution.

$$\begin{array}{ll}
 \mathbf{2} \quad \mathbf{a} \quad \mu = np & \sigma = \sqrt{np(1-p)} \\
 & = \sqrt{50 \times 0.2 \times 0.8} \\
 & = 10 & = \sqrt{8} \\
 & & \approx 2.83
 \end{array}$$

- b** The blue shaded area represents $P(X = 15)$.
 The red shaded area represents $P(14.5 \leq X_{\text{norm}} \leq 15.5)$.
 The two areas are approximately equal.
 $\therefore P(X = 15) \approx P(14.5 \leq X_{\text{norm}} \leq 15.5)$.

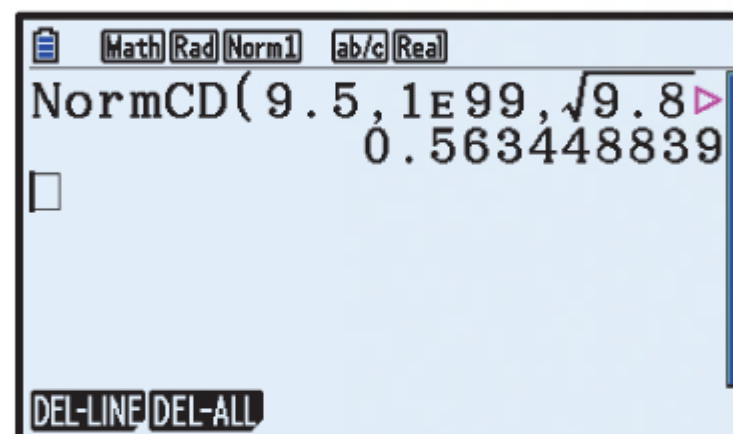


- c** $X_{\text{norm}} \sim N(10, (\sqrt{8})^2)$
- i** $P(X \leq 10) \approx P(X_{\text{norm}} \leq 10.5)$
 - ii** $P(X < 25) = P(X \leq 24) \approx P(X_{\text{norm}} \leq 24.5)$
 - iii** $P(10 \leq X < 25) = P(10 \leq X \leq 24) \approx P(9.5 \leq X_{\text{norm}} \leq 24.5)$

$$\begin{array}{lll}
 \mathbf{3} \quad X \sim B(500, 0.02) & \mu = np & \sigma = \sqrt{np(1-p)} \\
 & = 500 \times 0.02 & = \sqrt{500 \times 0.02 \times 0.98} \\
 & = 10 & = \sqrt{9.8} \\
 & & \approx 3.13
 \end{array}$$

$$X_{\text{norm}} \sim N(10, (\sqrt{9.8})^2)$$

$$\begin{aligned}
 P(X \geq 10) &\approx P(X_{\text{norm}} \geq 9.5) \\
 &\approx 0.563
 \end{aligned}$$

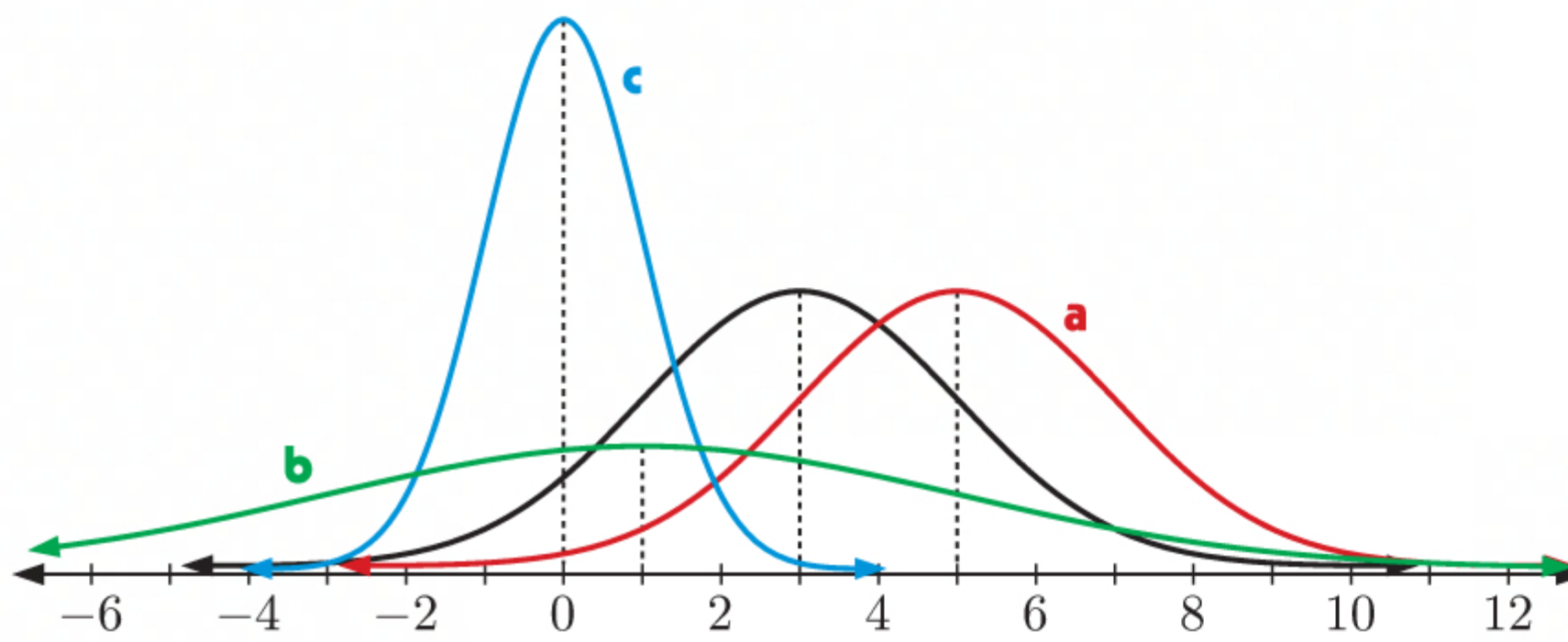


\therefore we estimate that the probability that at least 10 tyres in the sample will be unfit for sale is approximately 0.563.

REVIEW SET 15A

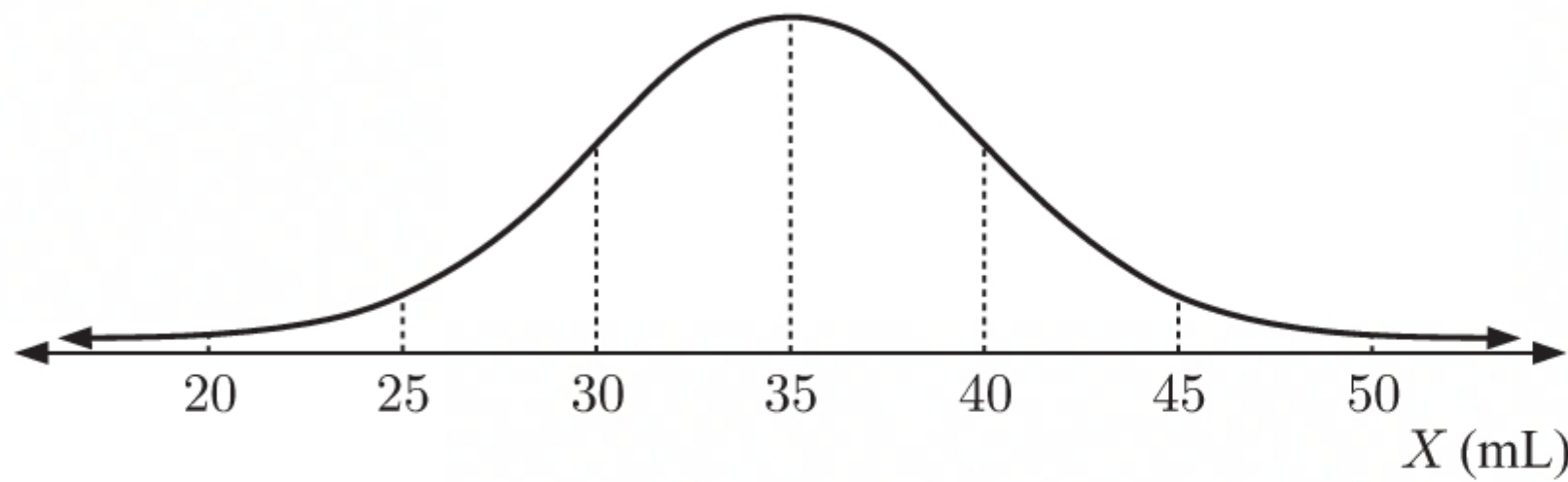
- 1 a** The distribution of times taken for students to read a novel is likely to be positively skewed, and hence not normal.
- b** The mean amount spent on groceries at a supermarket is likely to occur most often with variations around the mean occurring symmetrically as a result of random variation in the prices of items bought and/or the quantities of items bought (for example weights of fruits and vegetables). So, the distribution is likely to be normal.

2



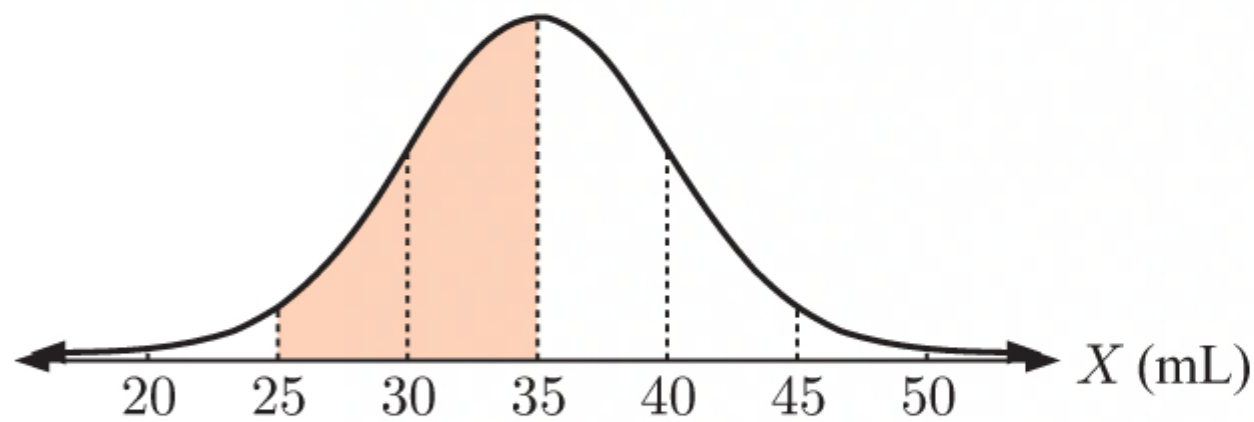
3

a



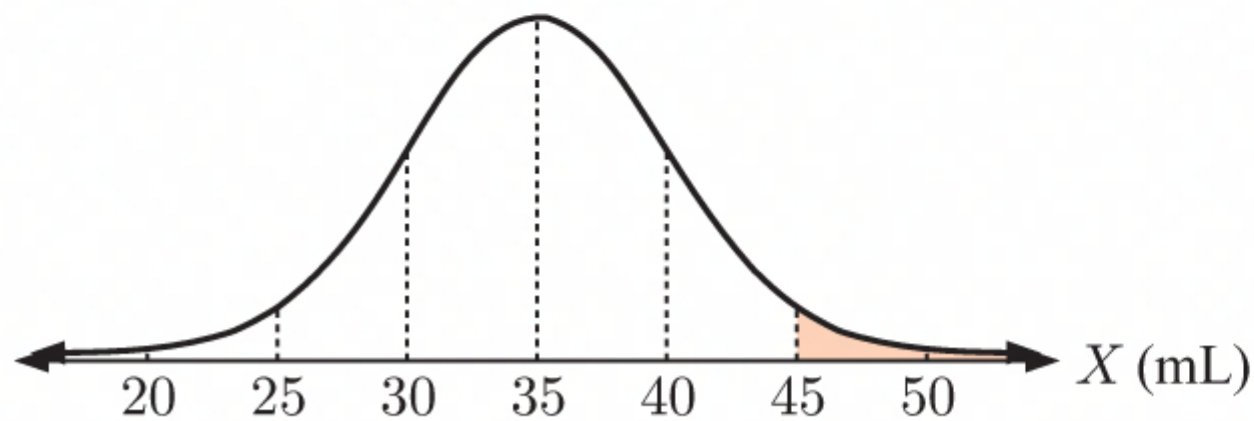
b

i



About $13.59\% + 34.13\% = 47.72\%$ of Simon's lemons will produce between 25 mL and 35 mL of juice.

ii

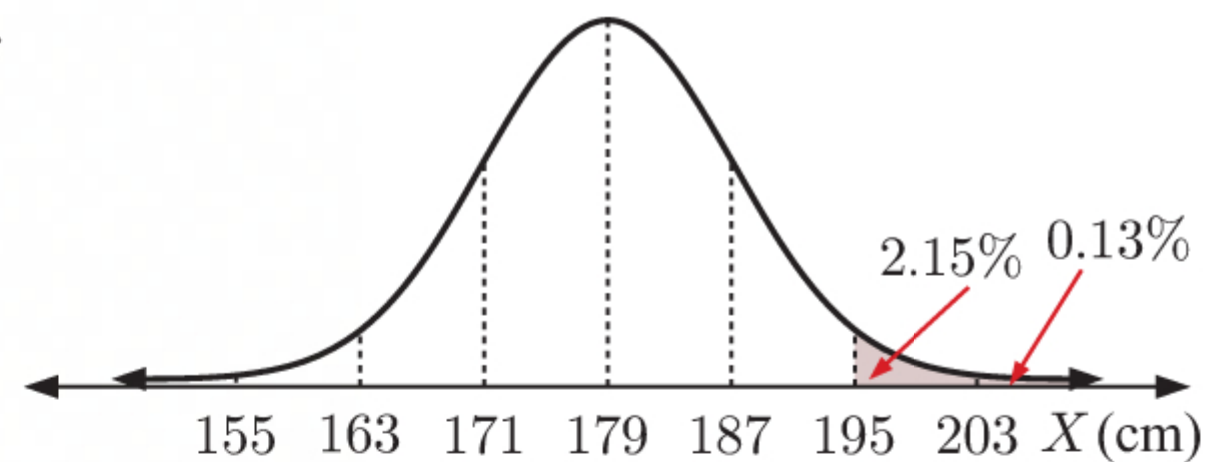


About $2.15\% + 0.13\% = 2.28\%$ of Simon's lemons will produce at least 45 mL of juice.

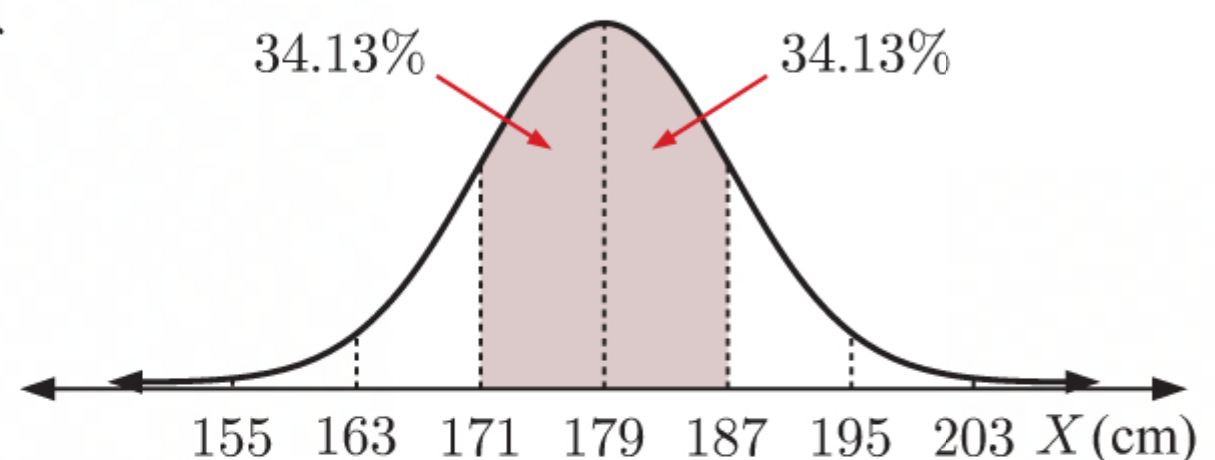
4 Let X cm be the height of a 17 year old boy.

$$X \sim N(179, 8^2)$$

a About $2.15\% + 0.13\% = 2.28\%$ of 17 year old boys have a height more than 195 cm.

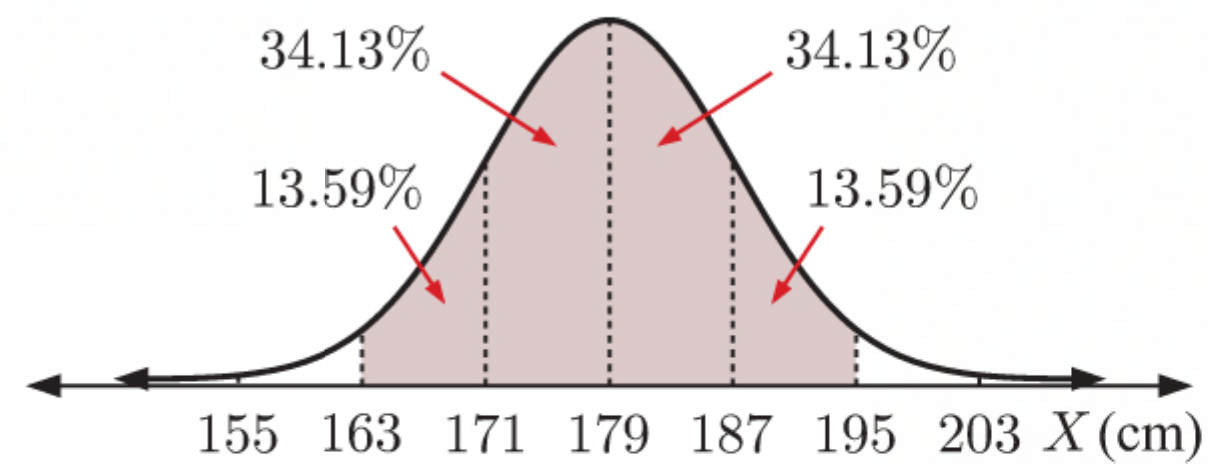


b About $34.13\% + 34.13\% = 68.26\%$ of 17 year old boys have a height between 171 cm and 187 cm.



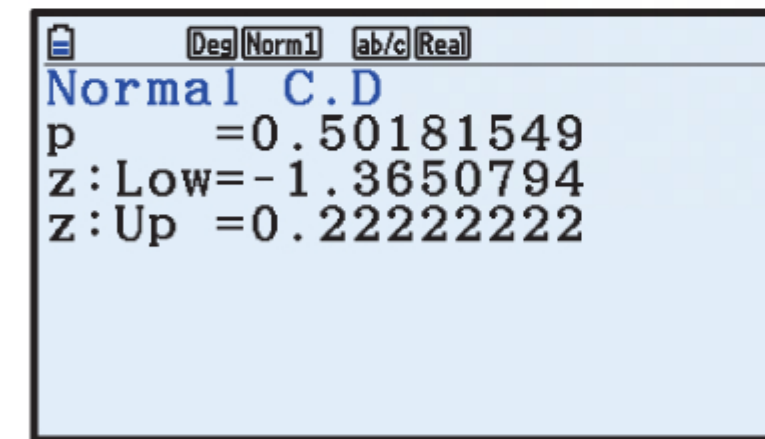
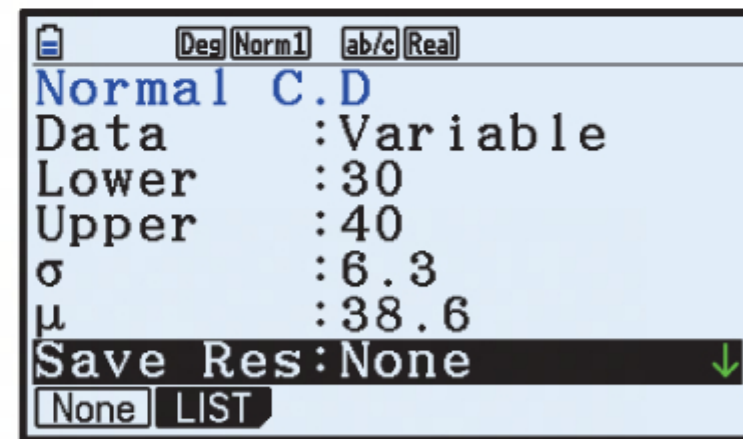
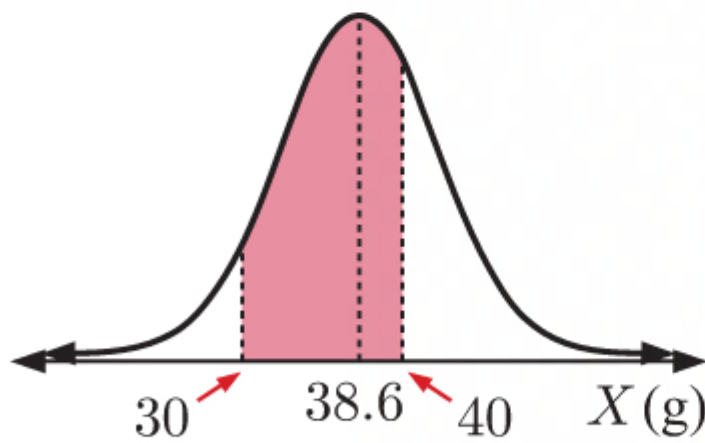
c About

$13.59\% + 34.13\% + 34.13\% + 13.59\%$
 $= 95.44\%$ of 17 year old boys have a
 height between 163 cm and 195 cm.



5 Let X grams be the weight of the edible part of a randomly selected Coffin Bay oyster.

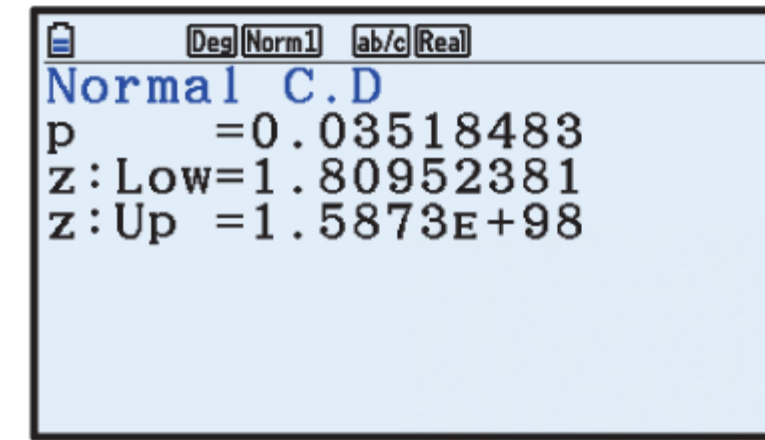
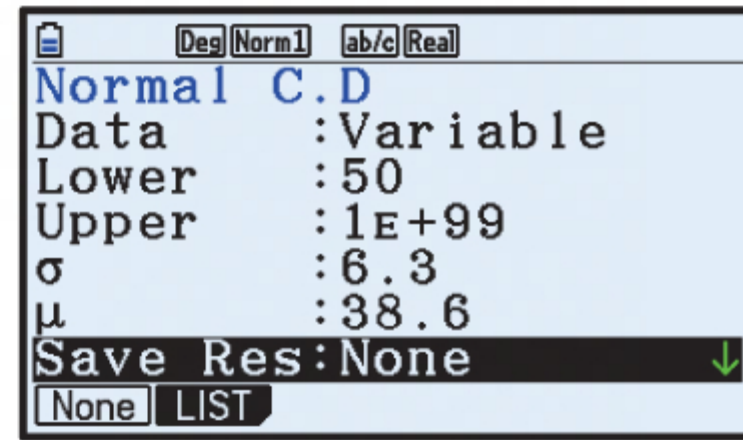
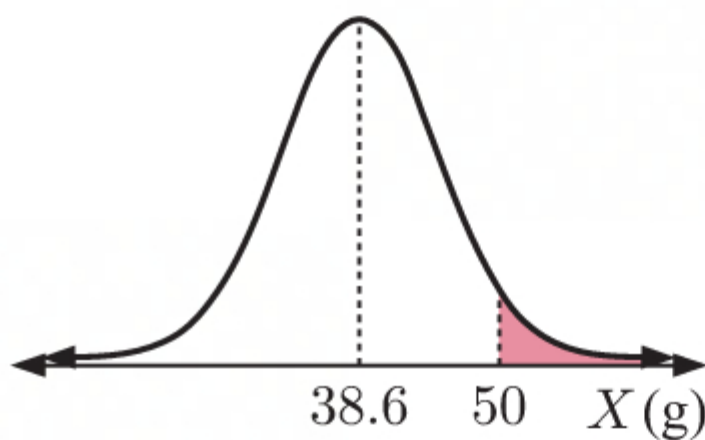
a



$$P(30 < X < 40) \approx 0.502 \approx 50.2\%$$

About 50.2% of oysters have an edible part that weighs between 30 g and 40 g.

b

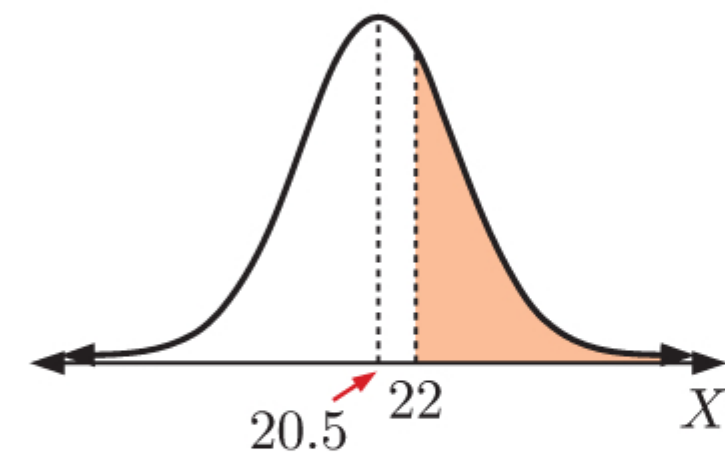
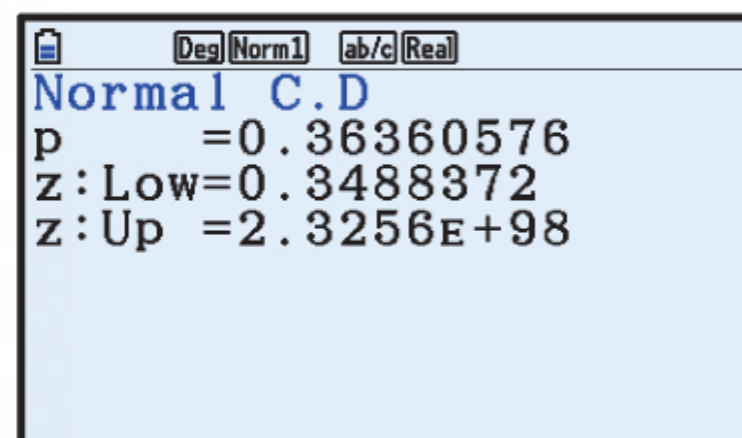
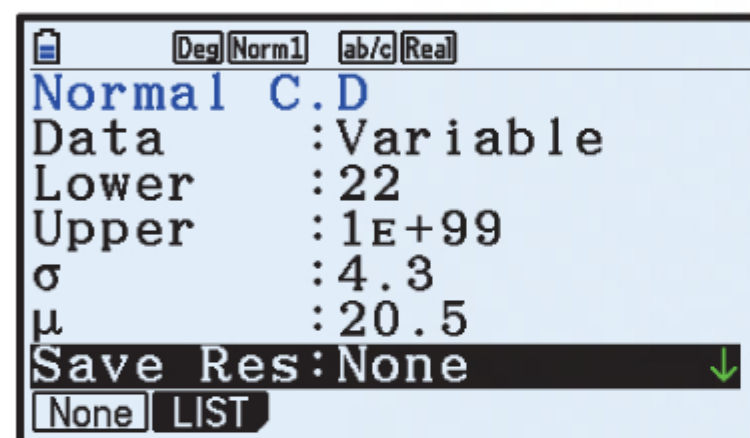


$$P(X > 50) \approx 0.0352$$

\therefore we would expect about $0.0352 \times 200 \approx 7$ oysters to have an edible part that weighs more than 50 g.

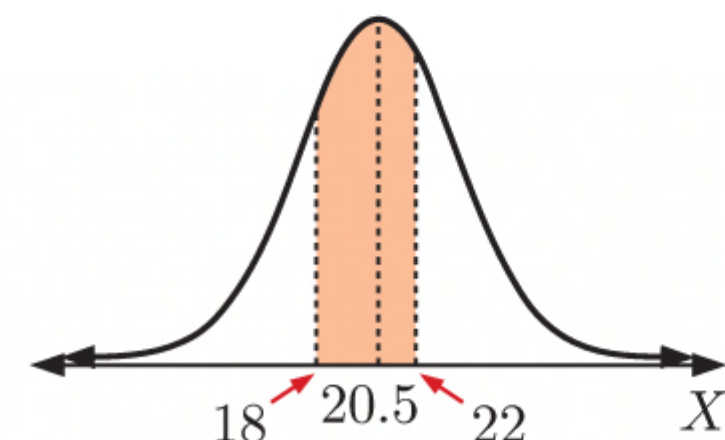
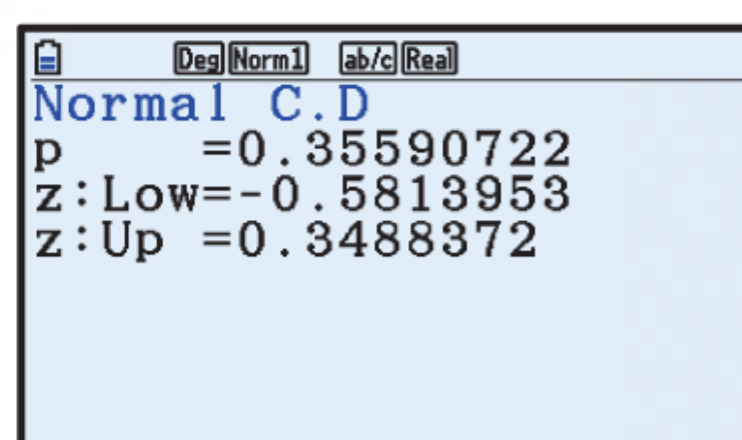
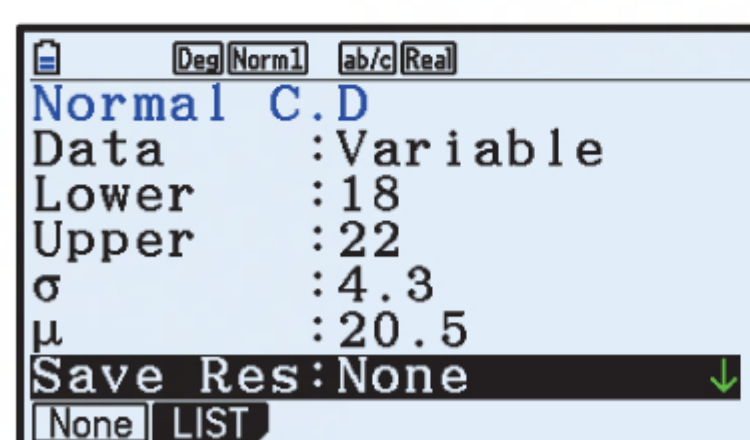
6 $X \sim N(20.5, 4.3^2)$

a

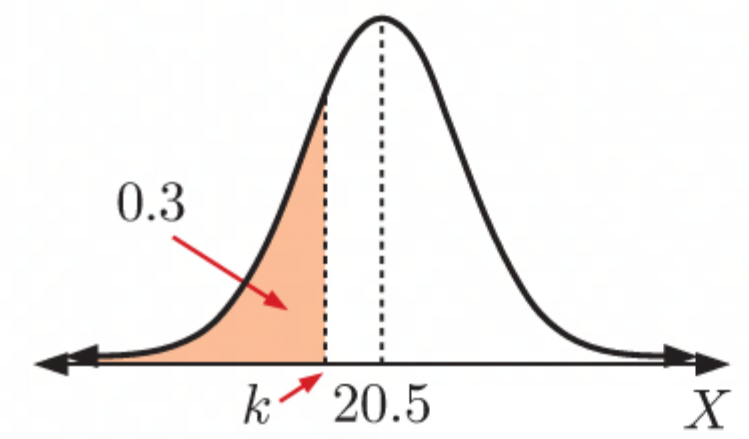
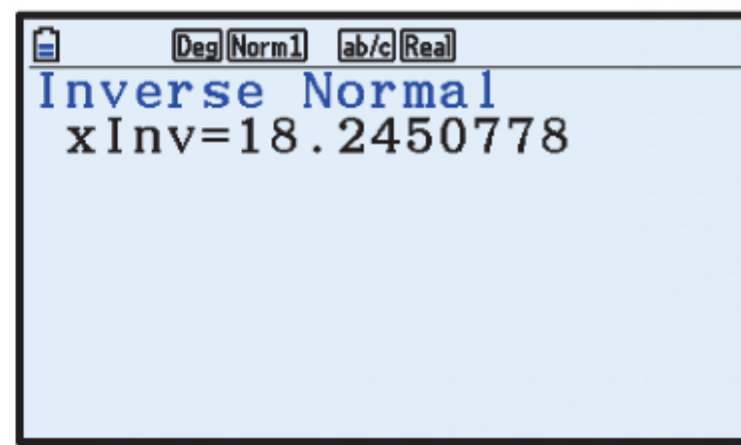
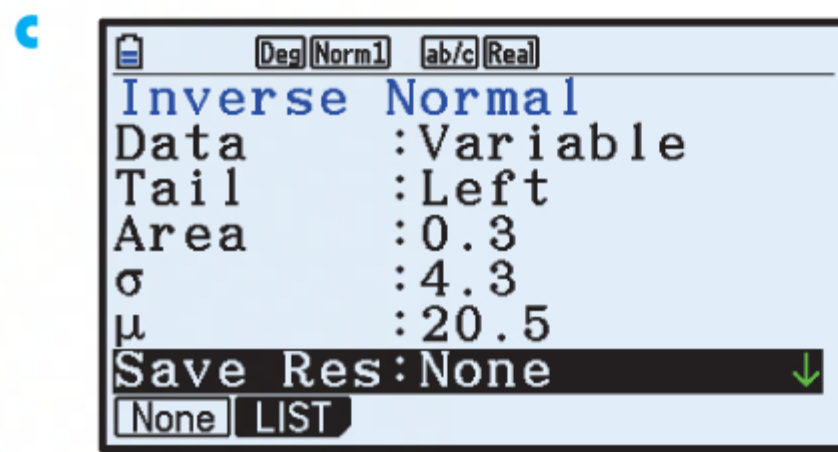


$$P(X \geq 22) \approx 0.364$$

b



$$P(18 \leq X \leq 22) \approx 0.356$$



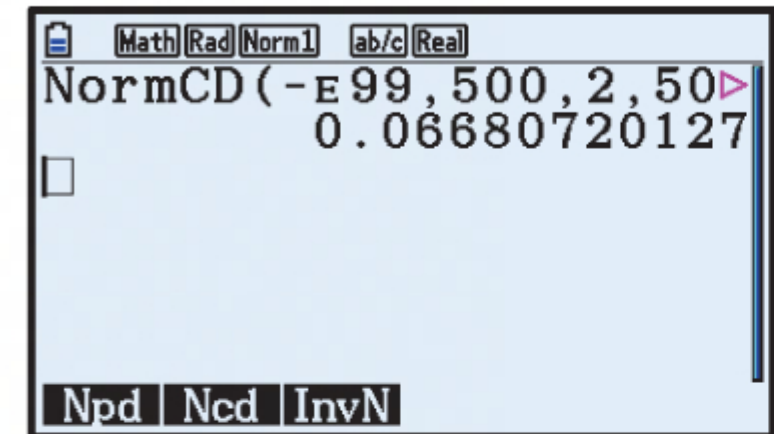
If $P(X \leq k) = 0.3$
then $k \approx 18.2$

7 a $X \sim N(503, 2^2)$

$$P(X < 500) \approx 0.066807$$

$$\approx 0.0668$$

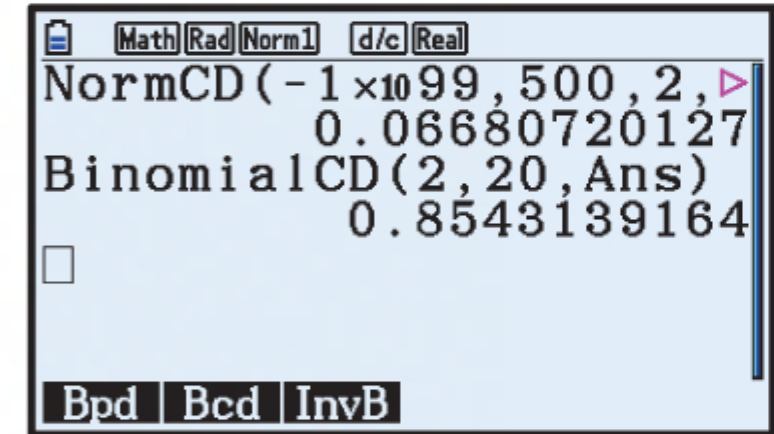
\therefore approximately 6.68% of the bags are underweight.



b Let Y be the number of bags which are underweight.

$$Y \sim B(20, 0.066807)$$

$$\therefore P(Y \leq 2) \approx 0.854$$

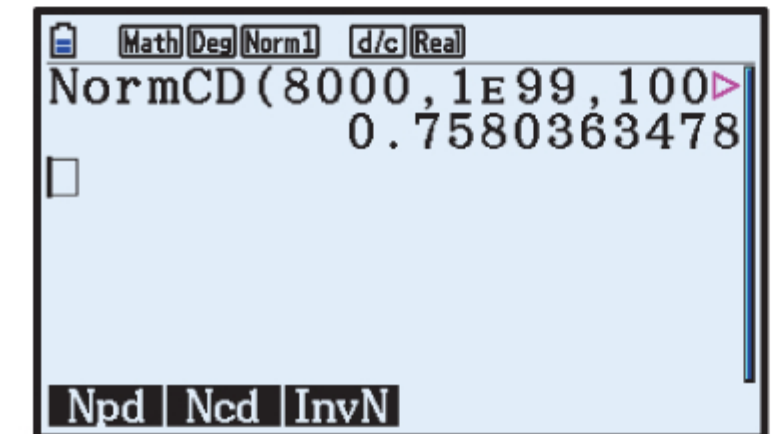


8 Let X kJ be the daily energy intake of a randomly selected Canadian adult.

$$X \sim N(8700, 1000^2)$$

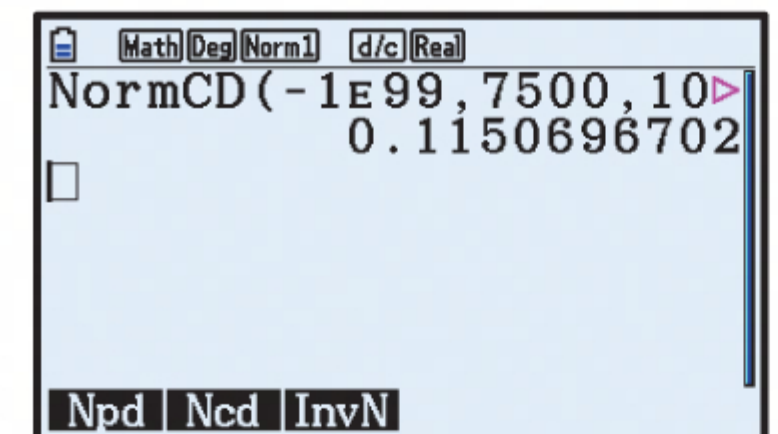
a $P(X > 8000) \approx 0.758$

\therefore about 75.8% of Canadian adults have a daily energy intake of more than 8000 kJ.



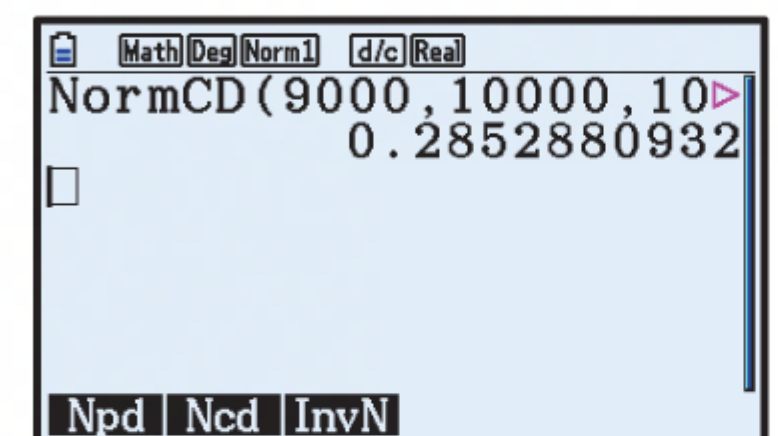
b $P(X < 7500) \approx 0.115$

\therefore about 11.5% of Canadian adults have a daily energy intake of less than 7500 kJ.



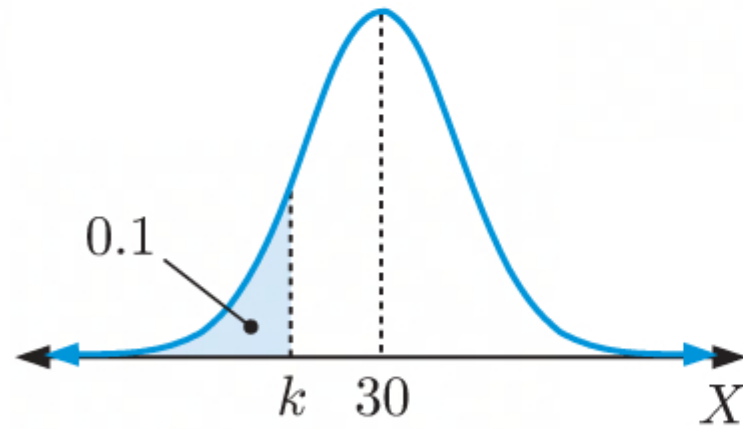
c $P(9000 < X < 10000) \approx 0.285$

\therefore about 28.5% of Canadian adults have a daily energy intake between 9000 kJ and 10000 kJ.

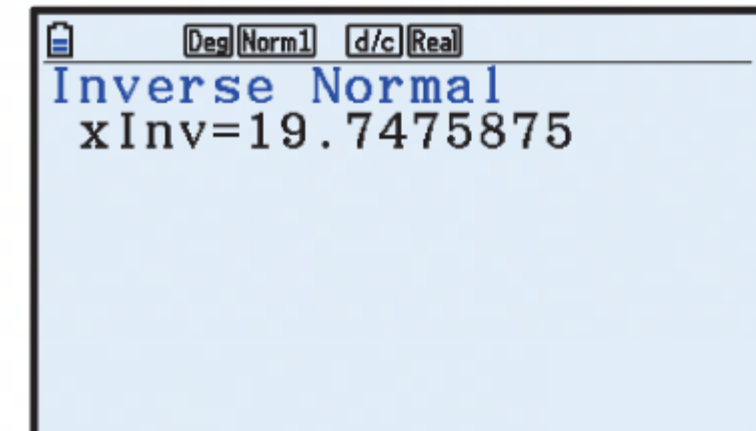
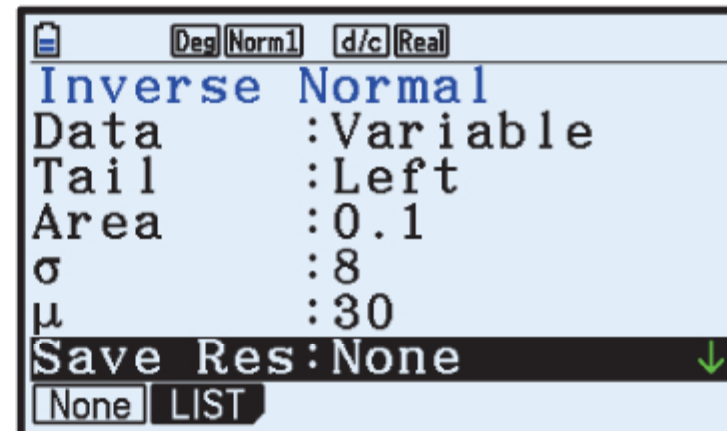


9 $X \sim N(30, 8^2)$

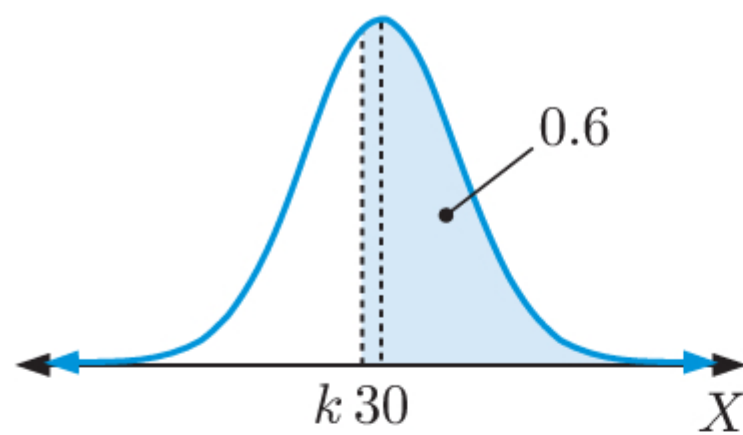
a



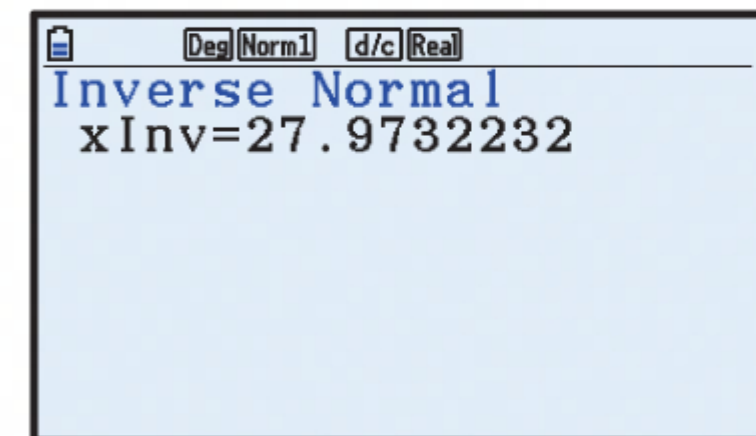
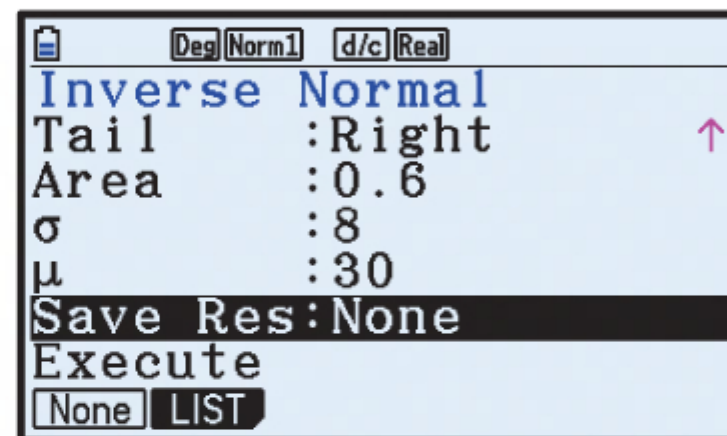
If $P(X \leq k) = 0.1$
then $k \approx 19.7$



b

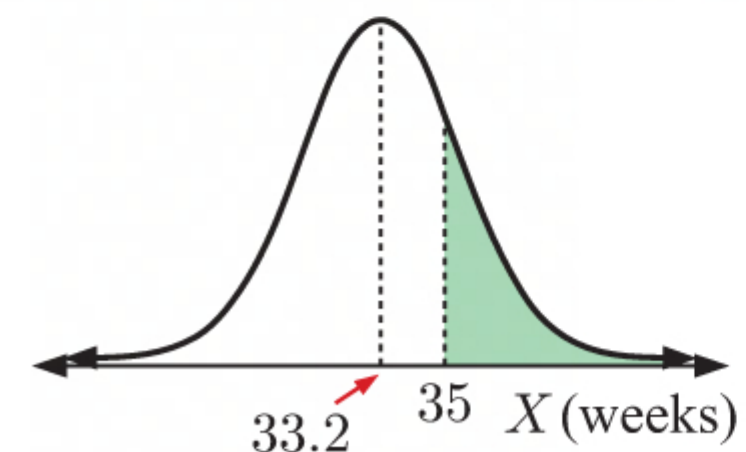
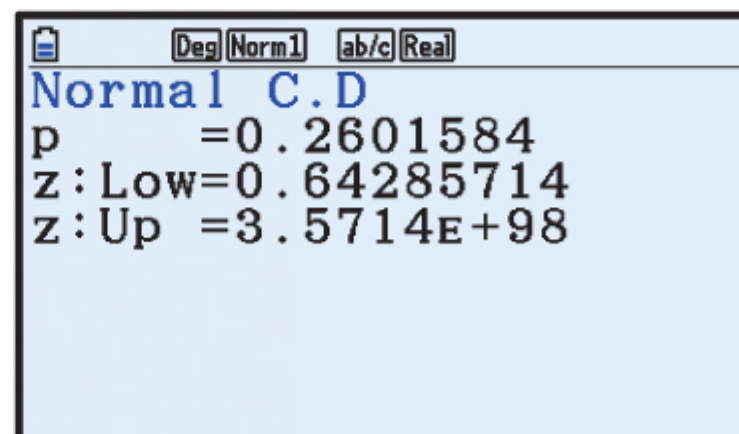
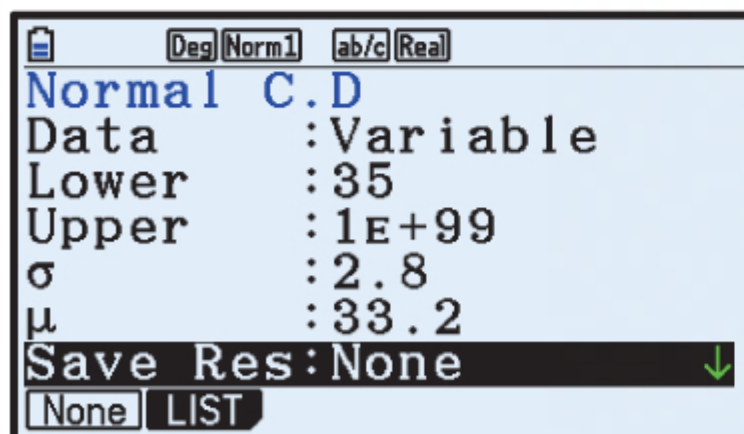


If $P(X \geq k) = 0.6$
then $k \approx 28.0$



- 10 Let X be the life of a randomly selected battery in weeks.
 $X \sim N(33.2, 2.8^2)$

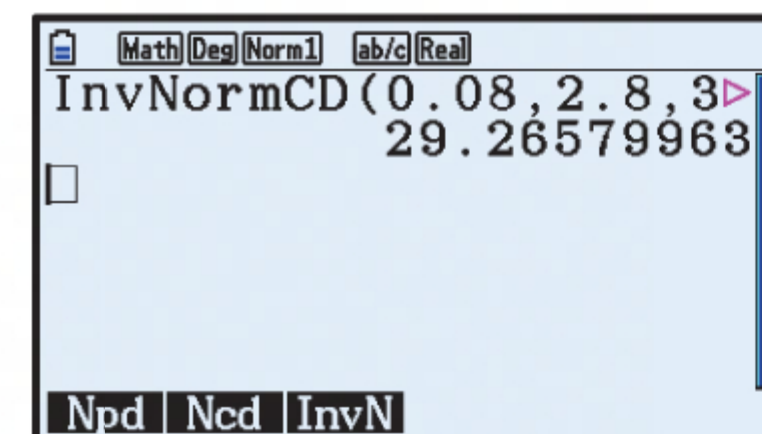
a



$$P(X \geq 35) \approx 0.260$$

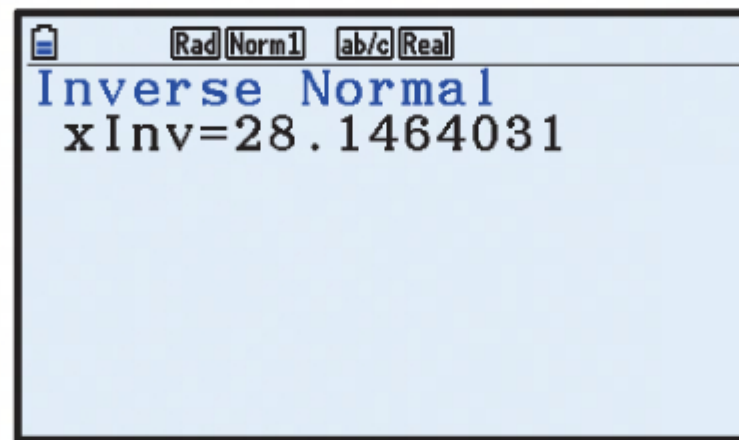
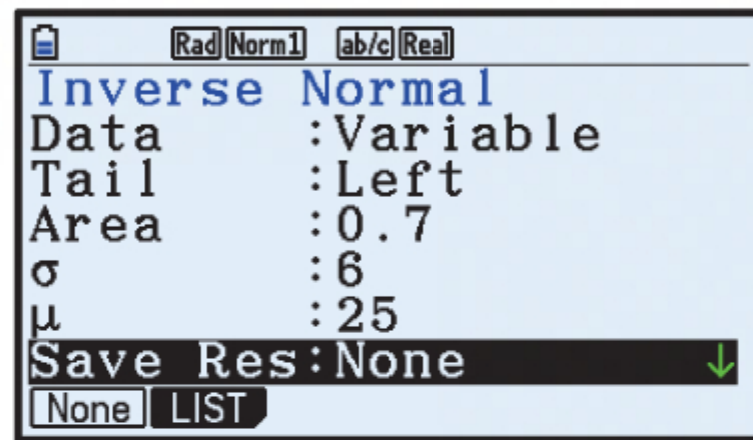
- b We need to find k such that
 $P(X \leq k) = 0.08$
 $\therefore k \approx 29.3$

So, the manufacturer can expect the batteries to last about 29.3 weeks before 8% of them fail.

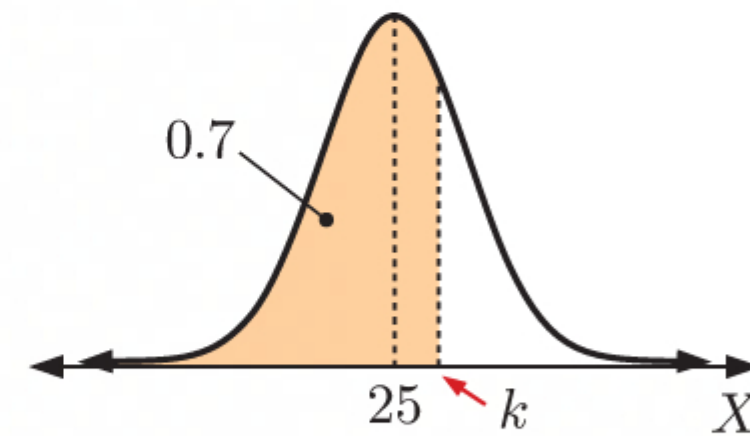


11 $X \sim N(25, 6^2)$

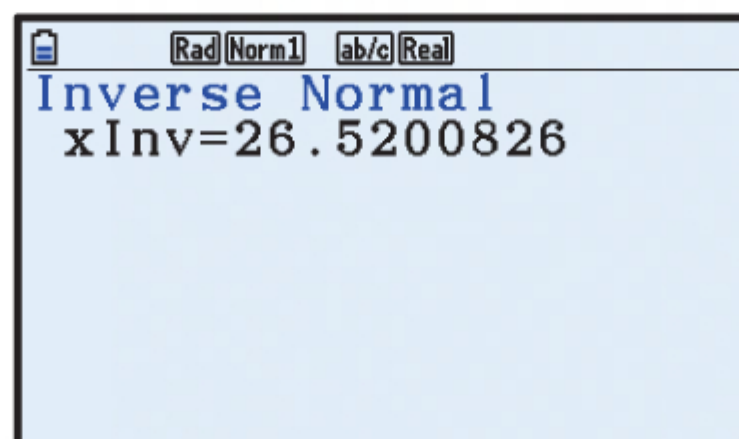
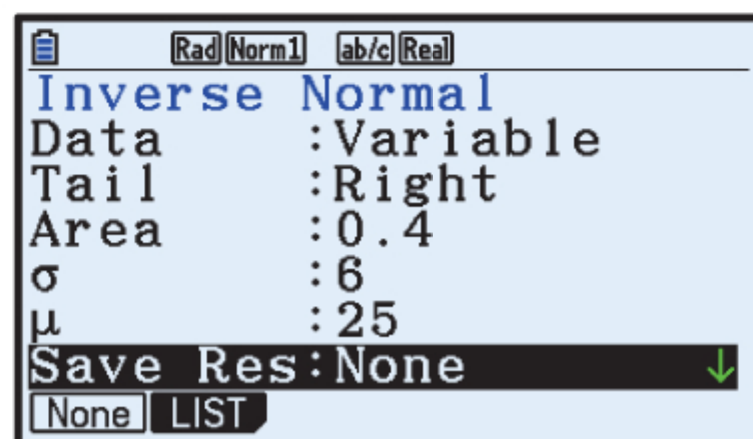
a



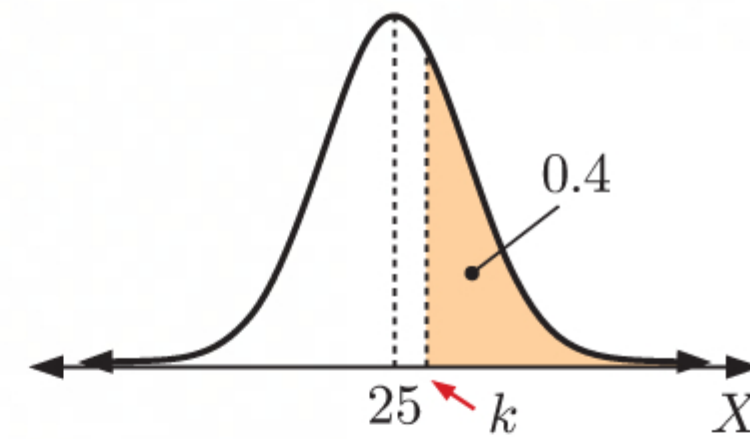
If $P(X \leq k) = 0.7$
then $k \approx 28.1$



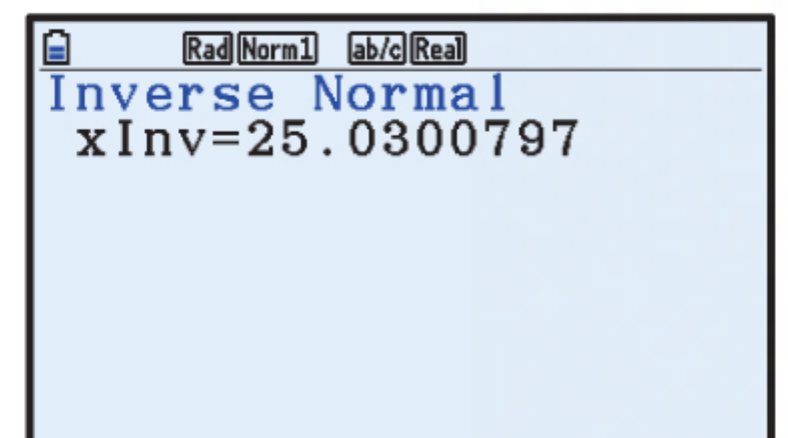
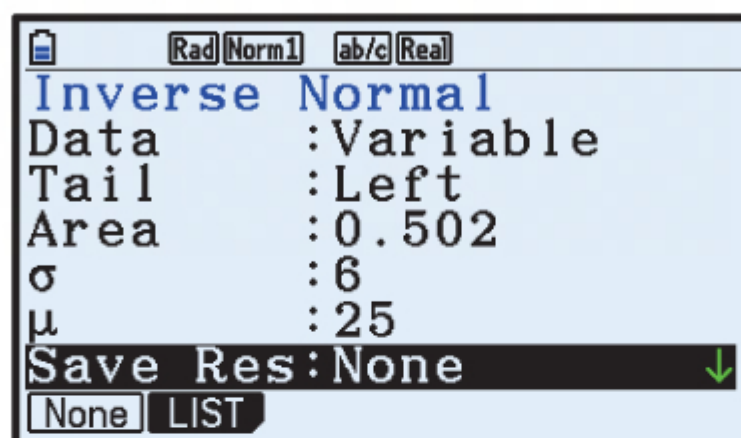
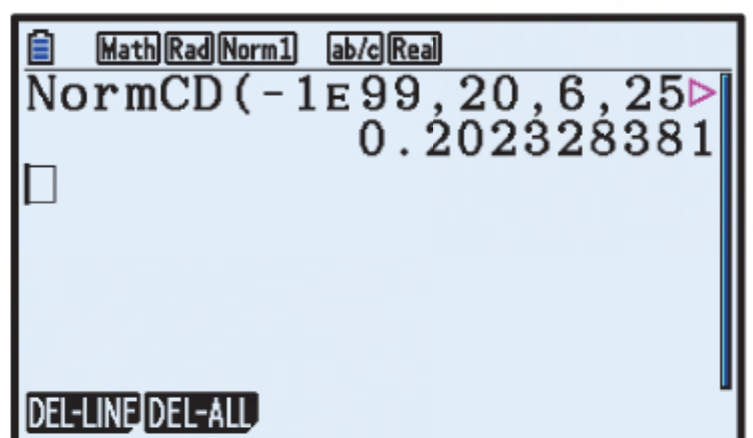
b

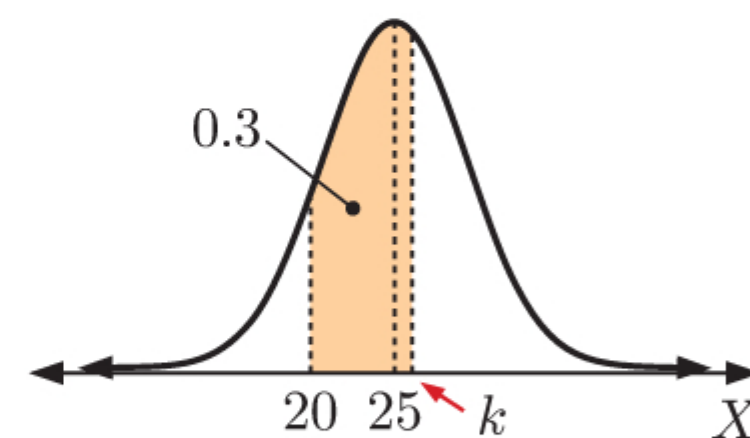


If $P(X \geq k) = 0.4$
then $k \approx 26.5$



c



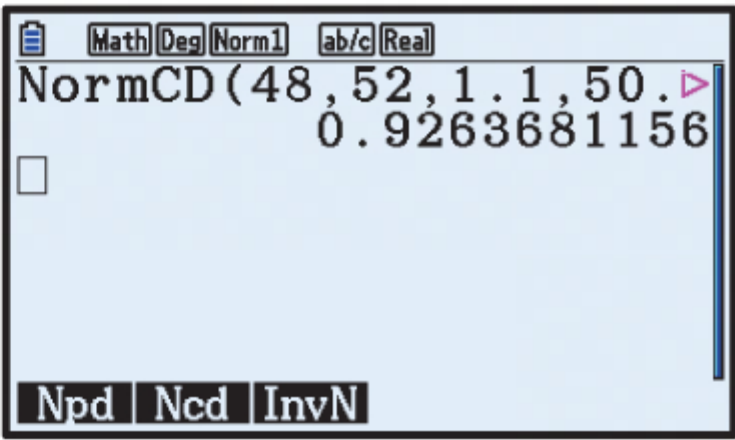
$$\begin{aligned} P(20 \leq X \leq k) &= 0.3 \\ \therefore P(X \leq k) - P(X \leq 20) &= 0.3 \\ \therefore P(X \leq k) - 0.202 &\approx 0.3 \\ \therefore P(X \leq k) &\approx 0.502 \\ \therefore k &\approx 25.0 \end{aligned}$$


12 a Let X_A cm be the length of a randomly selected nail produced by machine A, and X_B cm be the length of a randomly selected nail produced by machine B.

i $X_A \sim N(50.2, 1.1^2)$

$P(48 \leq X_A \leq 52) \approx 0.9264$

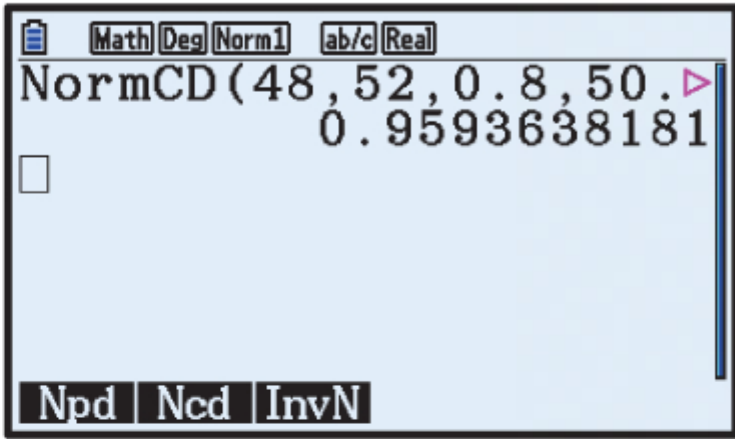
\therefore the probability that a nail from machine A needs to be rejected is about $1 - 0.9264 \approx 0.0736$.



ii $X_B \sim N(50.6, 0.8^2)$

$P(48 \leq X_B \leq 52) \approx 0.9594$

\therefore the probability that a nail from machine B needs to be rejected is about $1 - 0.9594 \approx 0.0406$.



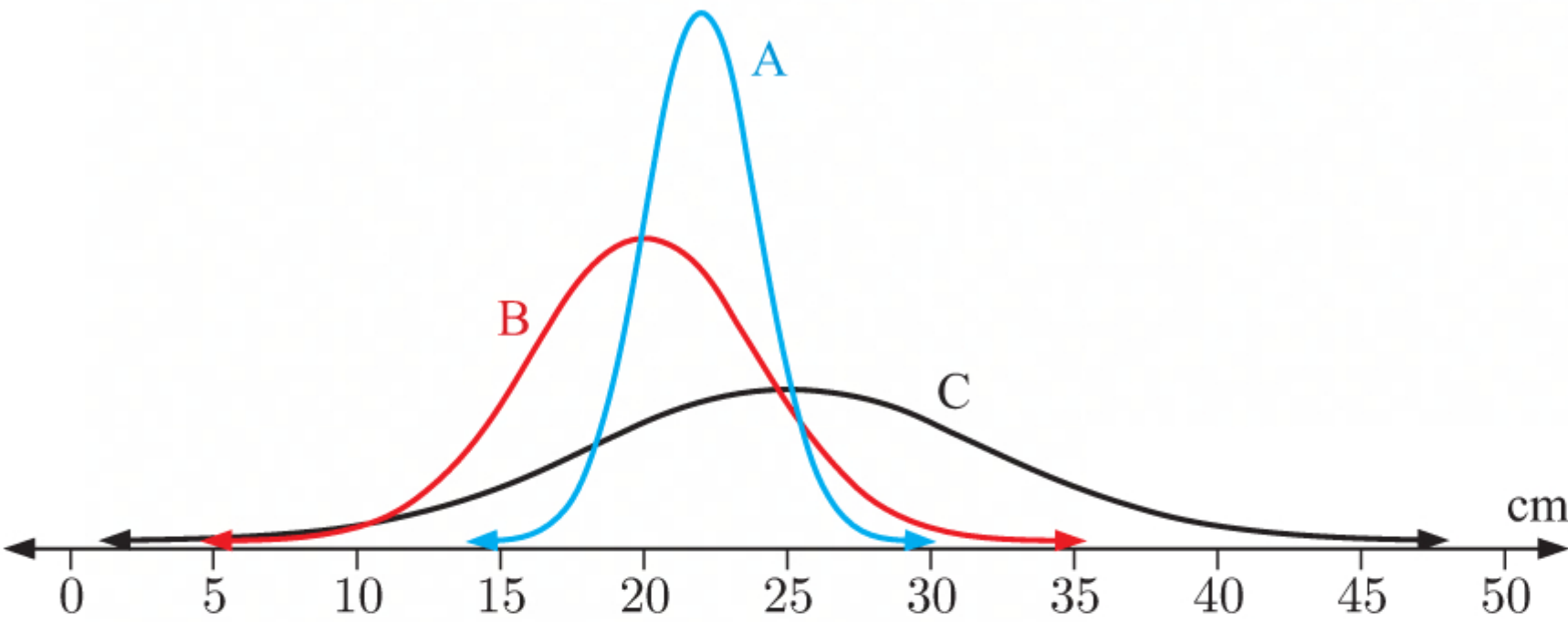
b
$$P(\text{made by machine A} \mid \text{rejected}) = \frac{P(\text{made by machine A} \cap \text{rejected})}{P(\text{rejected})}$$
$$\approx \frac{0.5 \times 0.0736}{0.5 \times 0.0736 + 0.5 \times 0.0406}$$
$$\approx 0.644$$

The probability that the nail was made by machine A *given* that it should be rejected is approximately 0.644.

REVIEW SET 15B

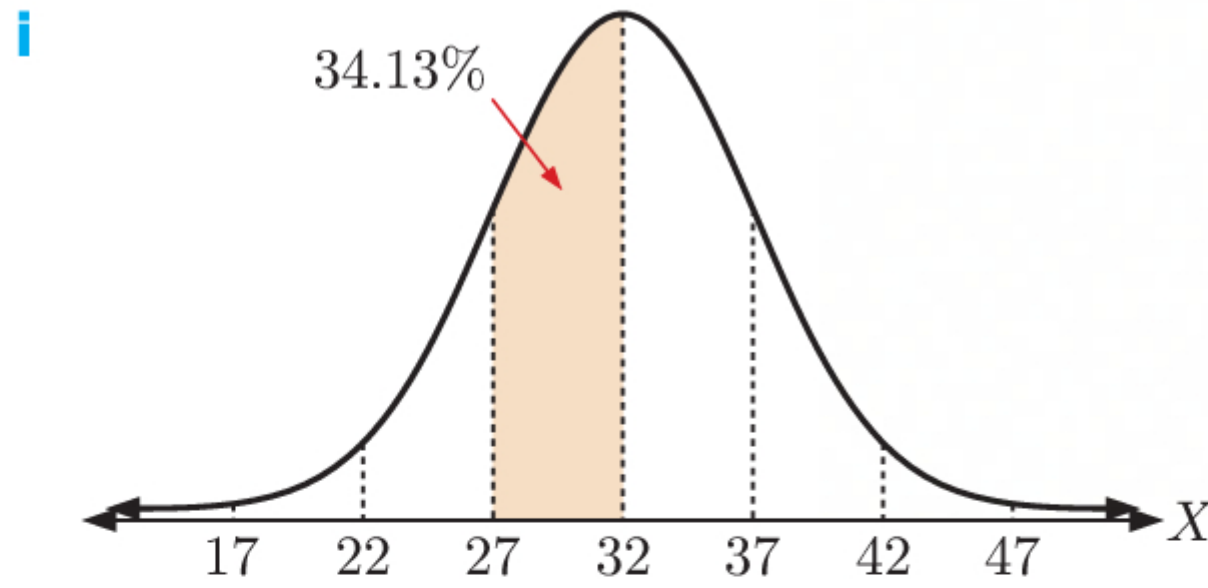
1

Distribution	Mean (cm)	Standard deviation (cm)
A	22	2
B	20	4
C	25	7

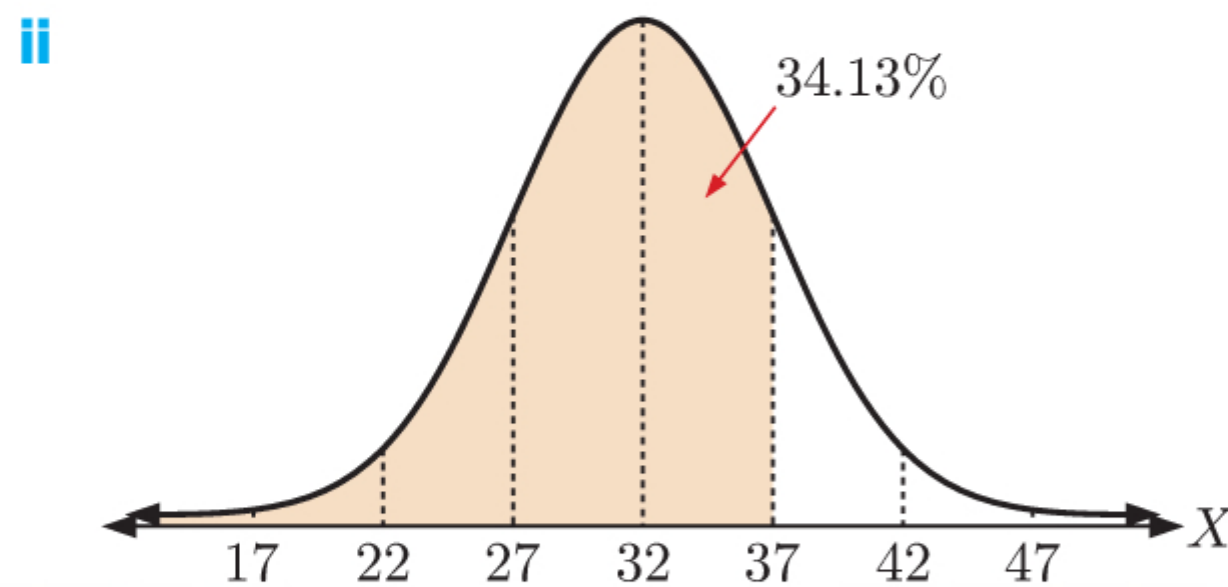


2 a The mean is $\mu = 32$, and the standard deviation is $\sigma = 5$.

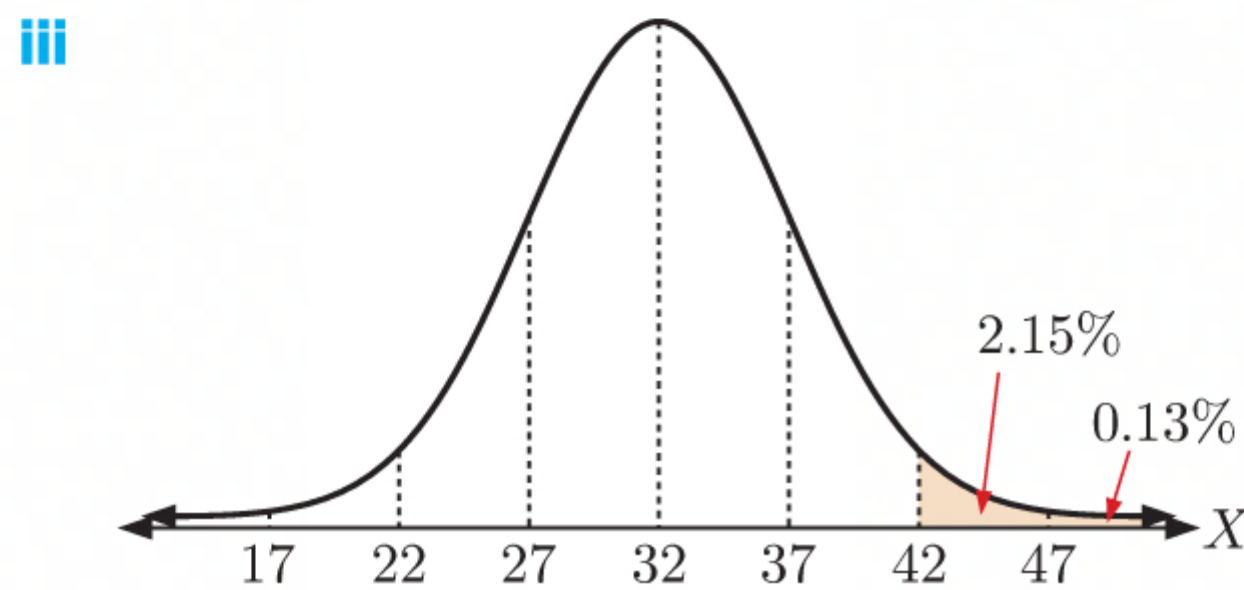
b $X \sim N(32, 5^2)$



About 34.13% of values of X are between 27 and 32.



About $50\% + 34.13\% = 84.13\%$ of values of X are less than 37.



About $2.15\% + 0.13\% = 2.28\%$ of values of X are greater than 42.

3 If $P(-k \leq Z \leq k) = 0.95$

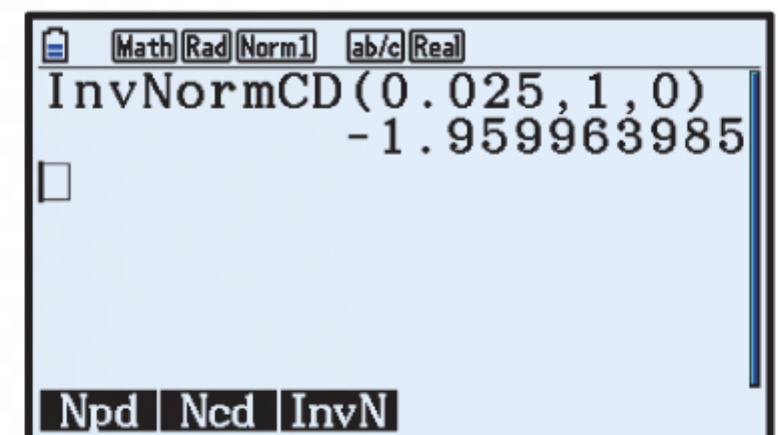
$$\therefore 1 - P(Z \leq -k) - P(Z \geq k) = 0.95$$

$$\therefore 1 - 2P(Z \leq -k) = 0.95 \quad \{\text{symmetry of the normal distribution}\}$$

$$\therefore 2P(Z \leq -k) = 0.05$$

$$\therefore P(Z \leq -k) = 0.025$$

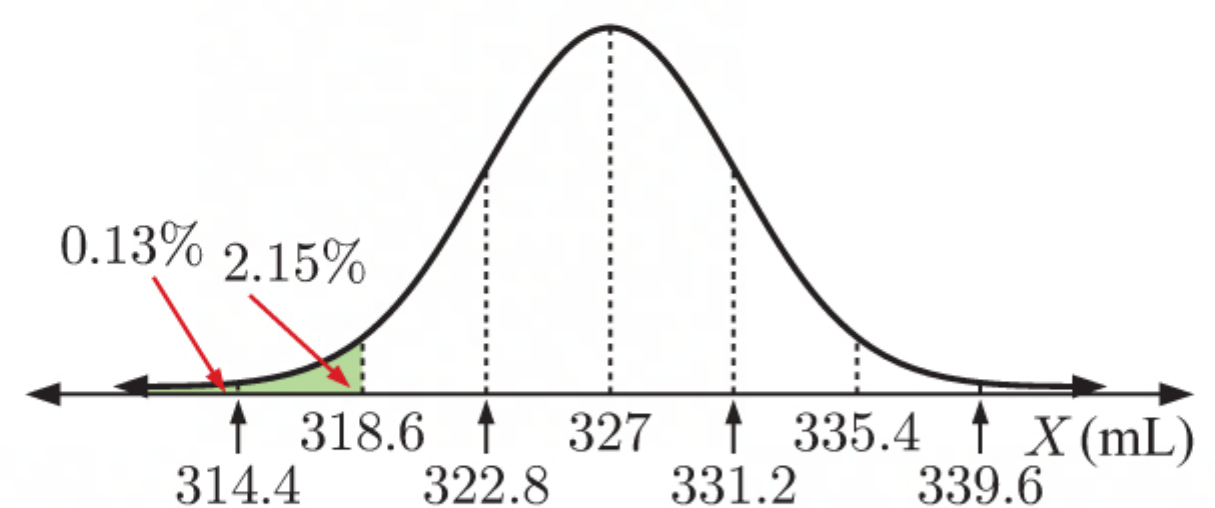
$$\therefore k \approx 1.96$$



4 Let X mL be the contents of a can.

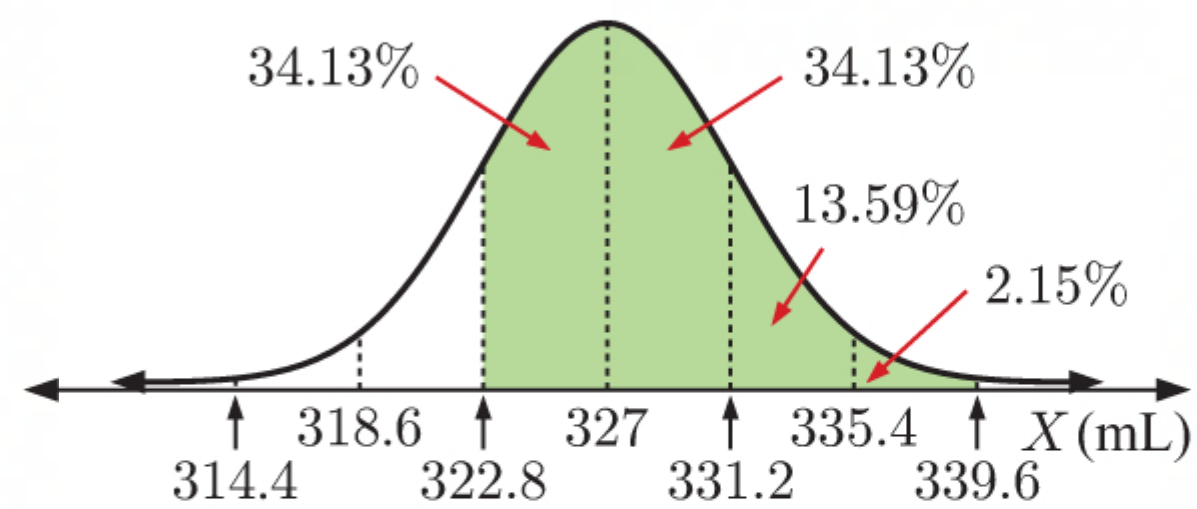
$$X \sim N(327, 4.2^2)$$

a i About $0.13\% + 2.15\% = 2.28\%$ of cans have contents less than 318.6 mL.



ii About

$$\begin{aligned}
 &34.13\% + 34.13\% + 13.59\% + 2.15\% \\
 &= 84.0\% \text{ of cans have contents} \\
 &\text{between } 322.8 \text{ mL and } 339.6 \text{ mL.}
 \end{aligned}$$

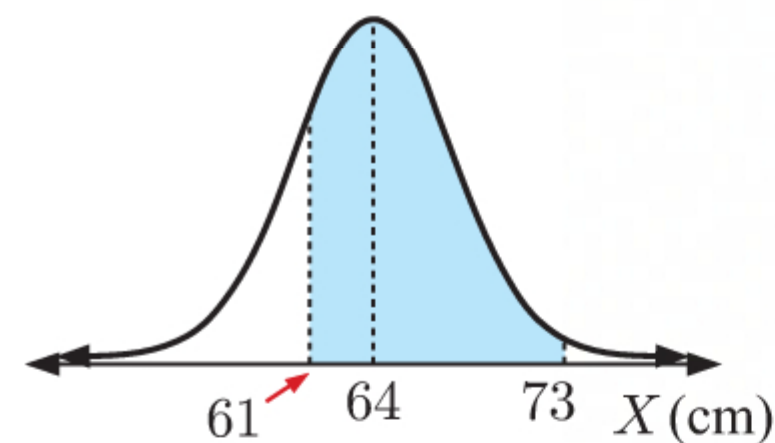
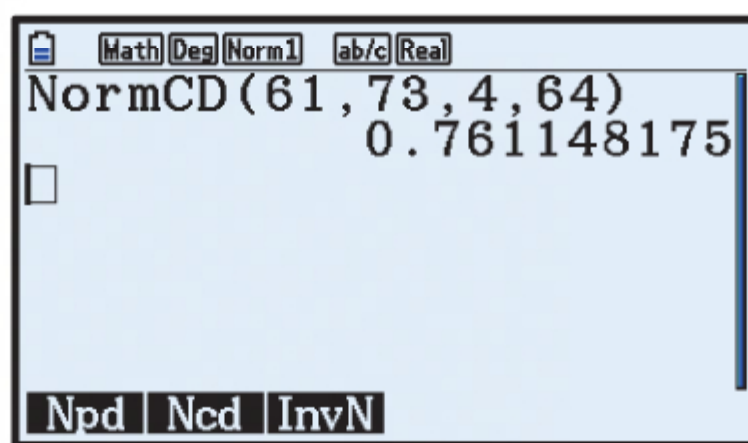


b About 34.13% of cans have contents between 327 mL and 331.2 mL.

$$\begin{aligned}
 &P(\text{contents between } 327 \text{ mL and } 331.2 \text{ mL}) \\
 &\approx 0.3413
 \end{aligned}$$

5 Let X cm be the arm length of a randomly selected 18 year old female.
 $X \sim N(64, 4^2)$

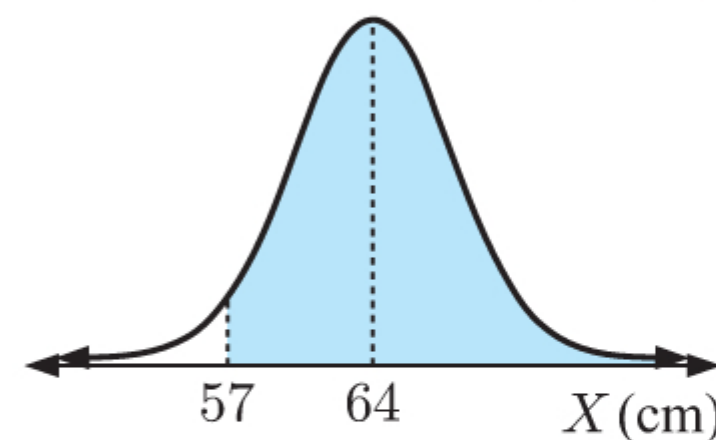
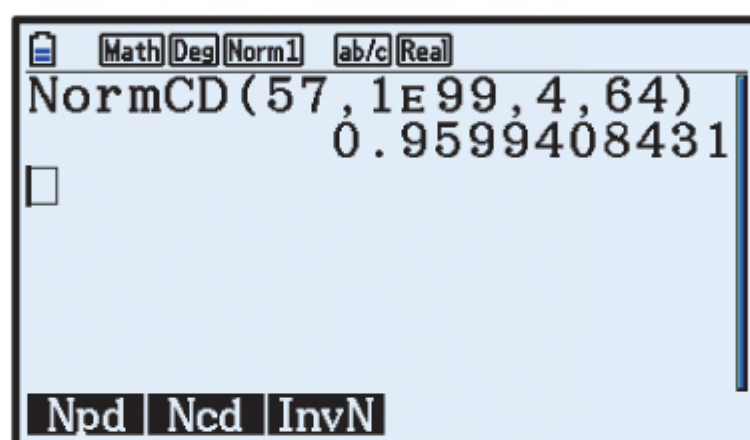
a i



$$P(61 < X < 73) \approx 0.761$$

\therefore approximately 76.1% of 18 year old females have an arm length between 61 cm and 73 cm.

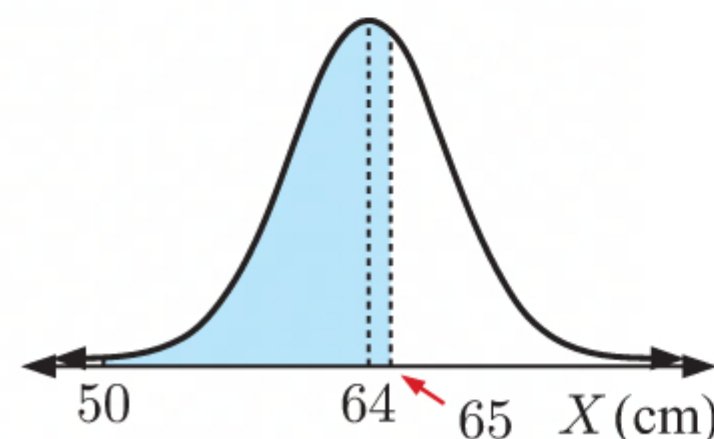
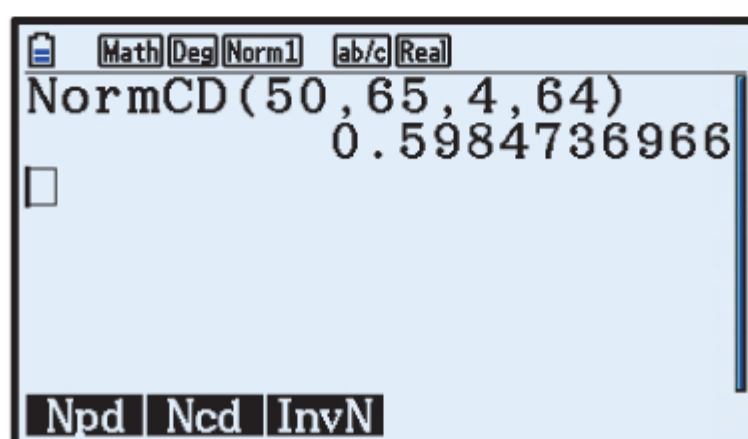
ii



$$P(X > 57) \approx 0.960$$

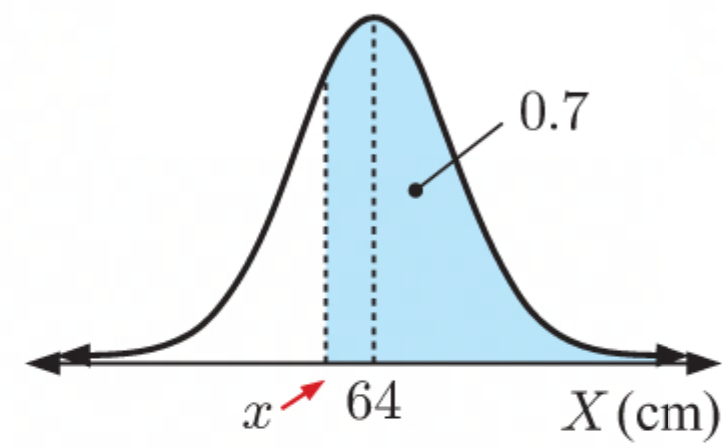
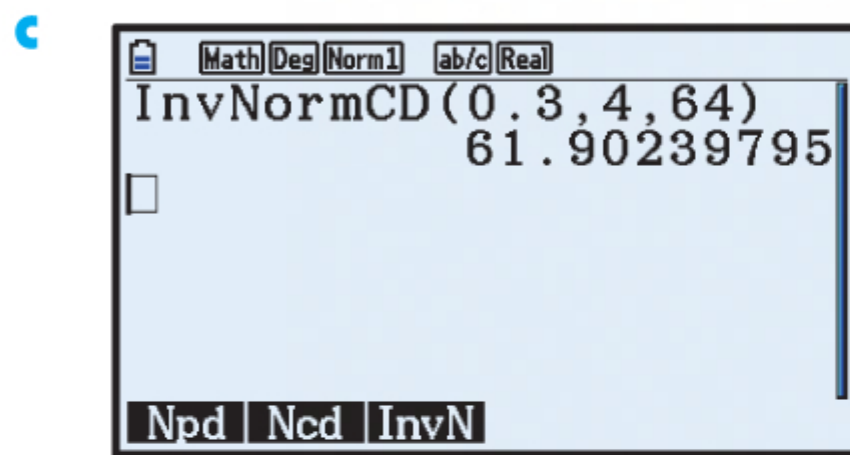
\therefore approximately 96.0% of 18 year old females have an arm length greater than 57 cm.

b



$$P(50 < X < 65) \approx 0.598$$

\therefore the probability that an 18 year old female has an arm length in the range 50 cm to 65 cm is approximately 0.598.

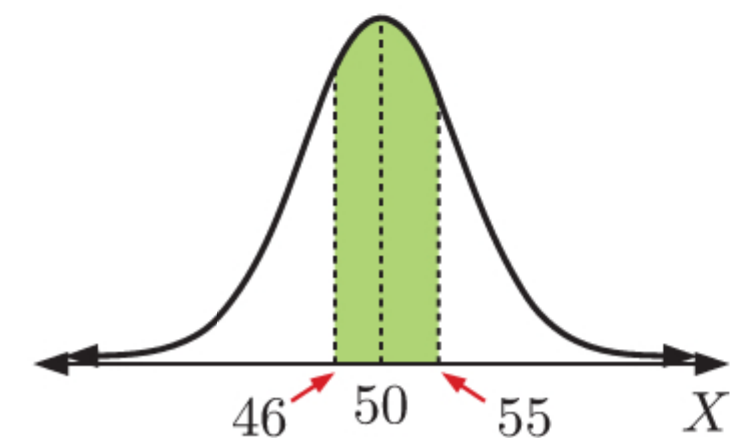
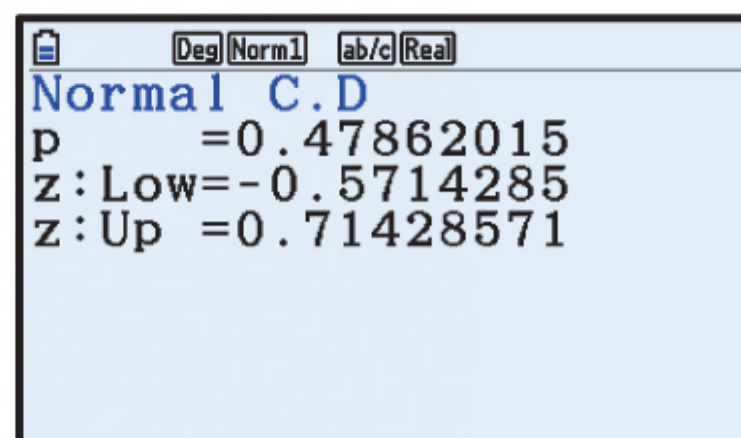
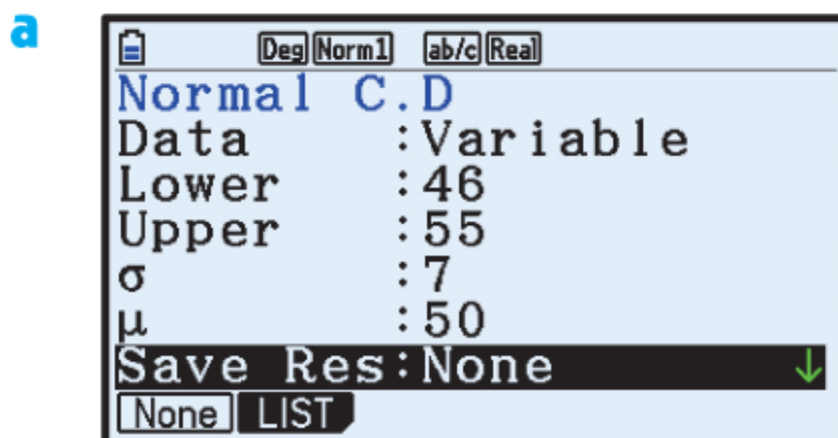


$$P(X > x) = 0.7$$

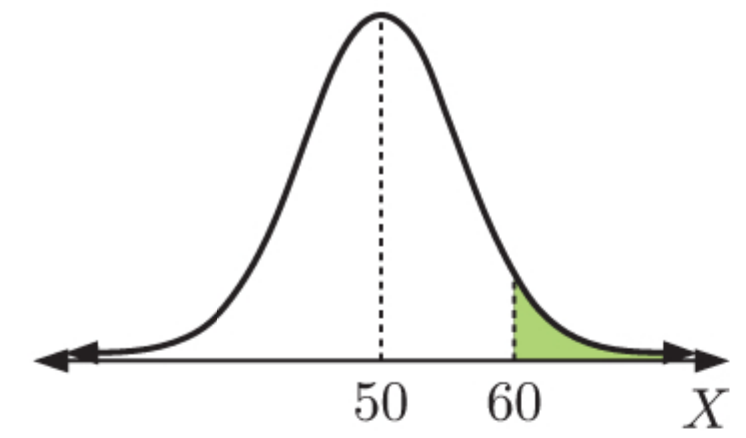
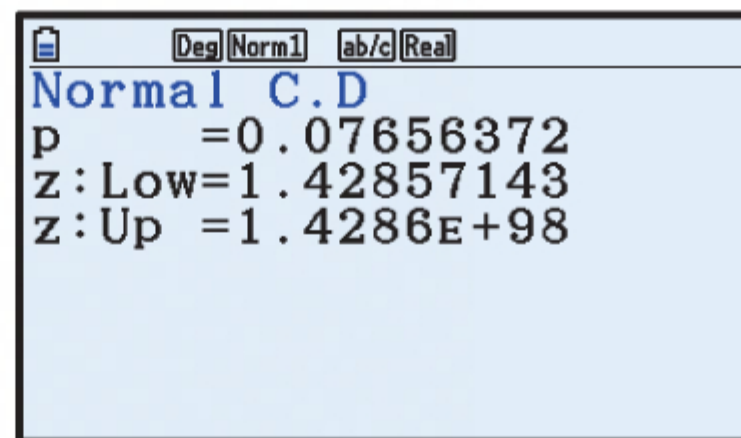
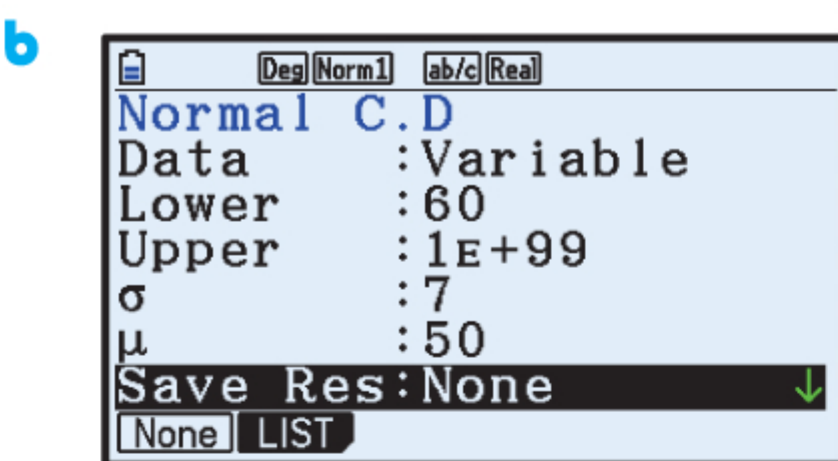
$$\therefore P(X < x) = 0.3$$

$$\therefore x \approx 61.9$$

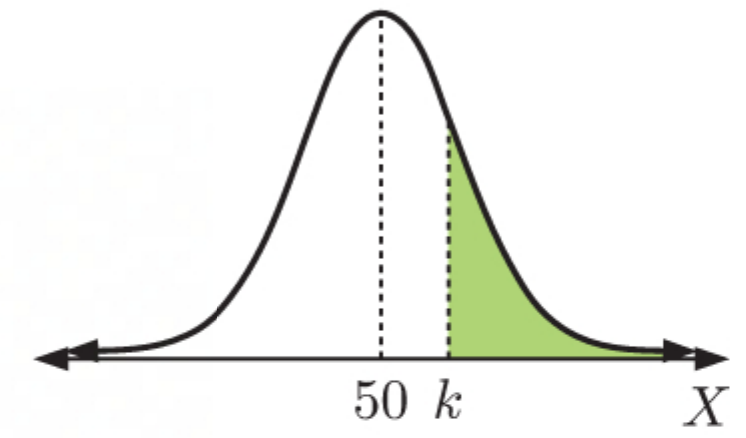
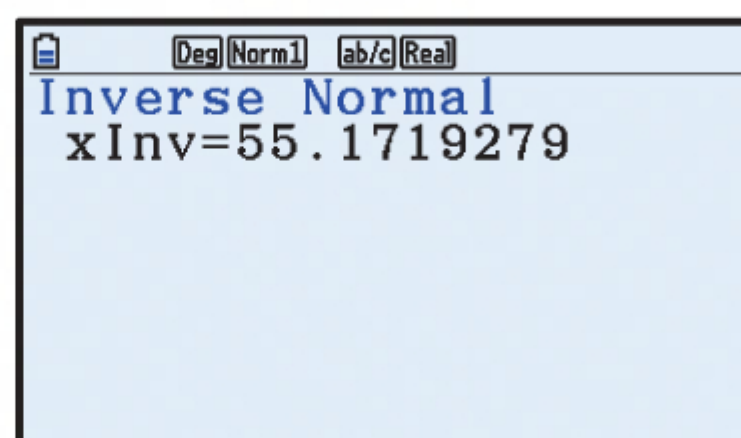
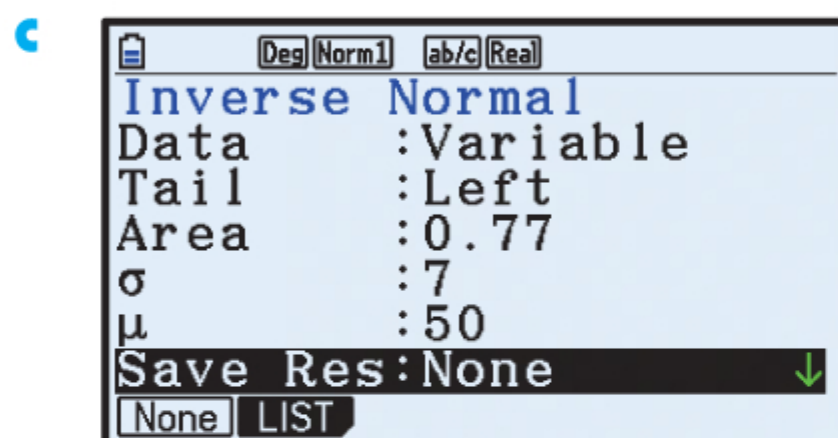
6 $X \sim N(50, 7^2)$



$$P(46 \leq X \leq 55) \approx 0.479$$



$$P(X \geq 60) \approx 0.0766$$



$$\text{If } P(X > k) = 0.23$$

$$\therefore P(X < k) = 0.77$$

$$\therefore k \approx 55.2$$

- 7 Let X seconds be the time a contestant holds their breath.

$$X \sim N(150, 12^2)$$

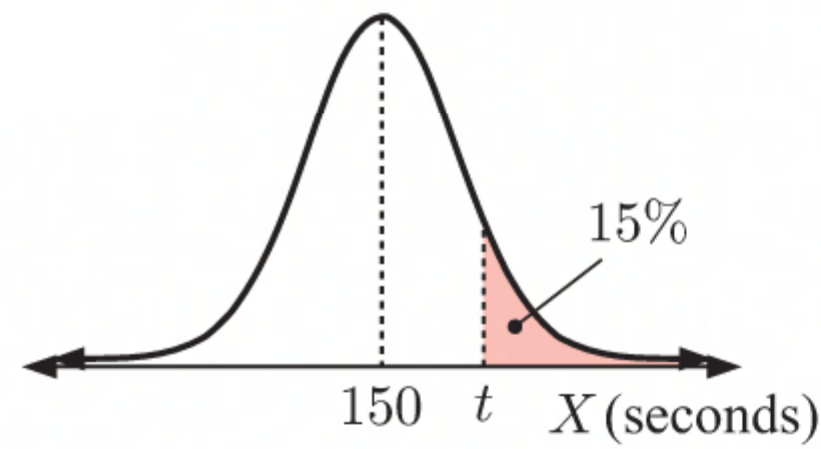
Deg Norm1 ab/c Real	
Inverse Normal	
Data	:Variable
Tail	:Left
Area	:0.85
σ	:12
μ	:150
Save Res:None	
None	LIST

Deg Norm1 ab/c Real	
Inverse Normal	
xInv=162.437201	

$$P(X > t) = 0.15$$

$$\therefore P(X < t) = 0.85$$

$$\therefore t \approx 162.4$$



To advance to the final round, a contestant would need to hold their breath for about 162 seconds.

- 8 $X \sim N(12, 2^2)$

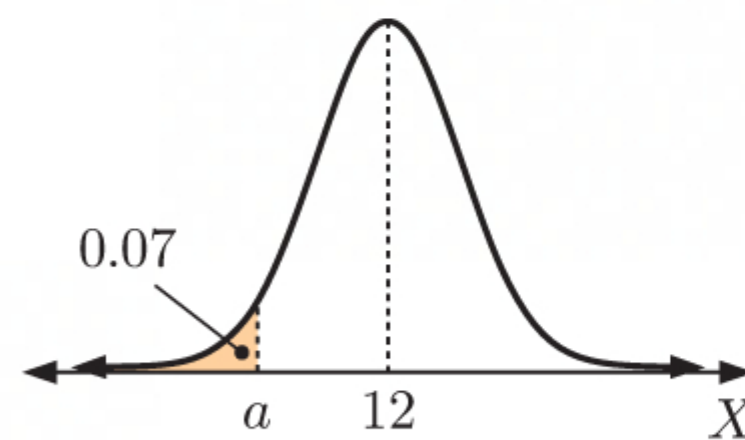
a

Rad Norm1 ab/c Real	
Inverse Normal	
Data	:Variable
Tail	:Left
Area	:0.07
σ	:2
μ	:12
Save Res:None	
None	LIST

Rad Norm1 ab/c Real	
Inverse Normal	
xInv=9.04841794	

$$\text{If } P(X < a) = 0.07$$

$$\text{then } a \approx 9.05$$



b

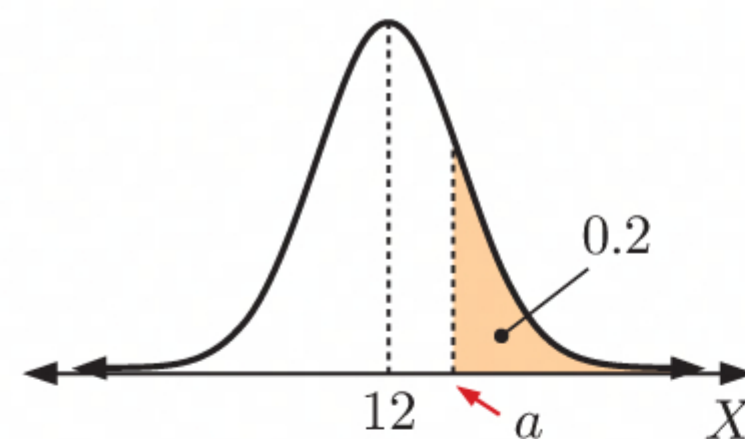
Rad Norm1 ab/c Real	
Inverse Normal	
Data	:Variable
Tail	:Left
Area	:0.8
σ	:2
μ	:12
Save Res:None	
None	LIST

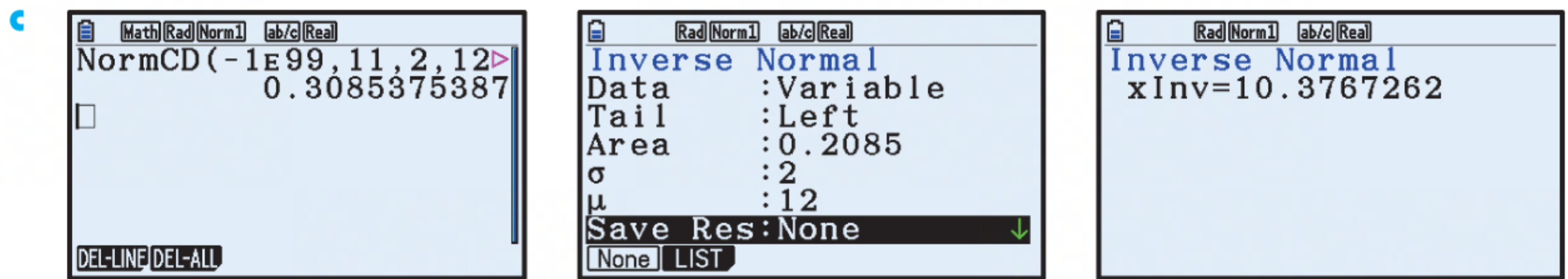
Rad Norm1 ab/c Real	
Inverse Normal	
xInv=13.6832425	

$$\text{If } P(X > a) = 0.2$$

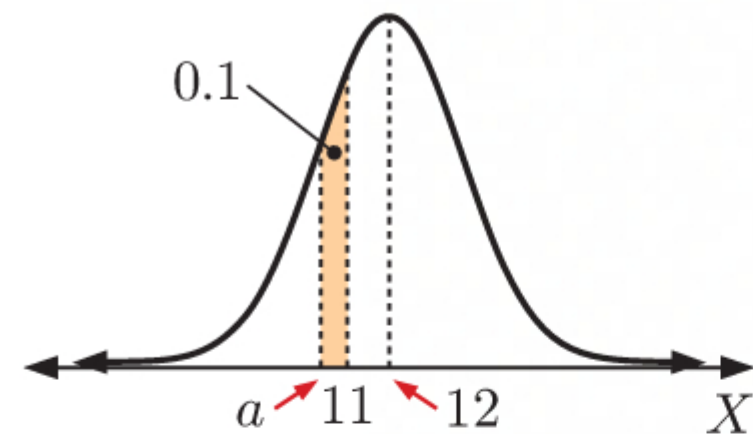
$$\therefore P(X < a) = 0.8$$

$$\therefore a \approx 13.7$$

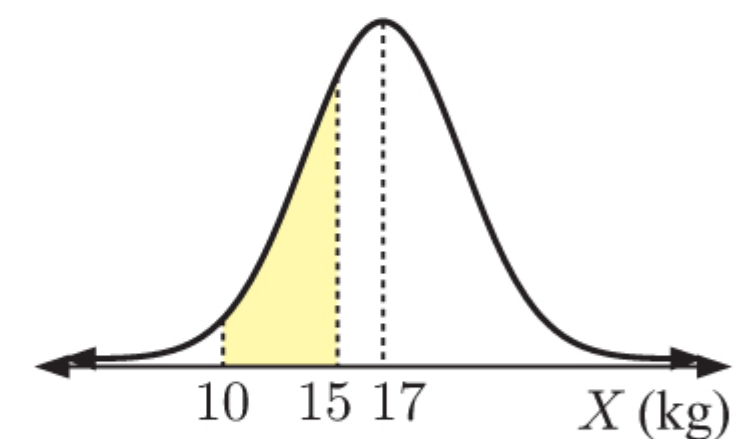
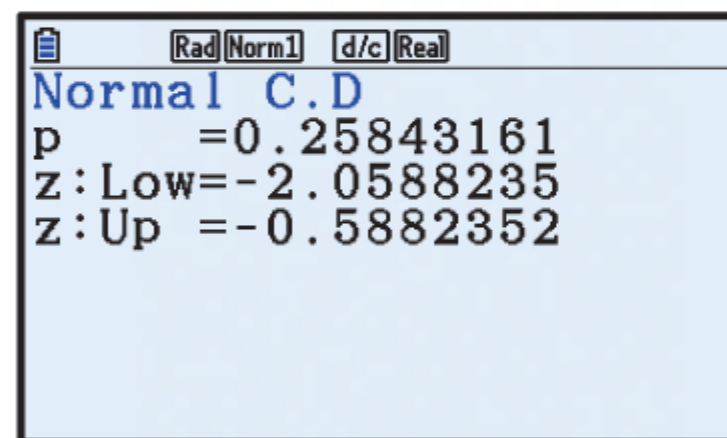
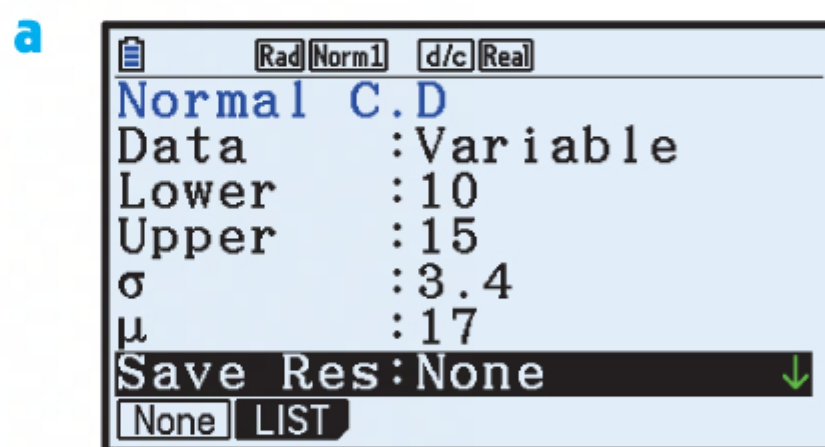




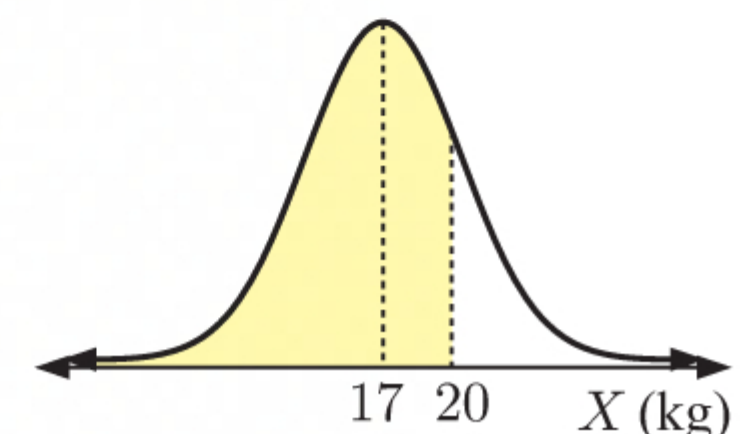
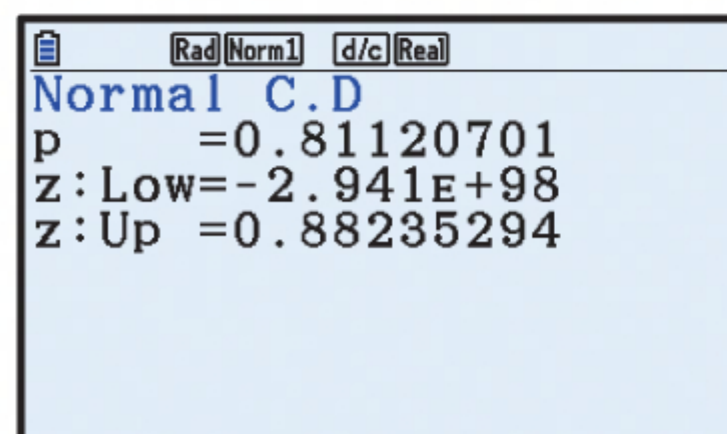
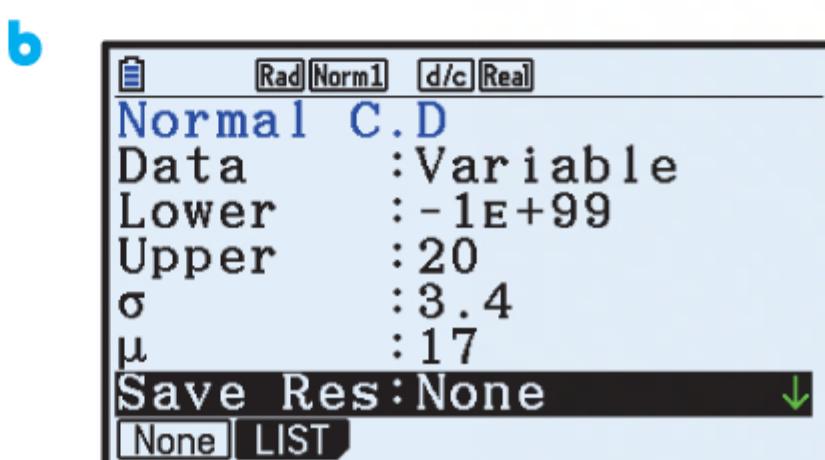
$$\begin{aligned}
 &\text{If } P(a \leq X \leq 11) = 0.1 \\
 \therefore P(X \leq 11) - P(X \leq a) &= 0.1 \\
 \therefore 0.3085 - P(X \leq a) &\approx 0.1 \\
 \therefore P(X \leq a) &\approx 0.2085 \\
 \therefore a &\approx 10.4
 \end{aligned}$$



- 9 Let X kg be the weight of a suitcase at the airport. $X \sim N(17, 3.4^2)$

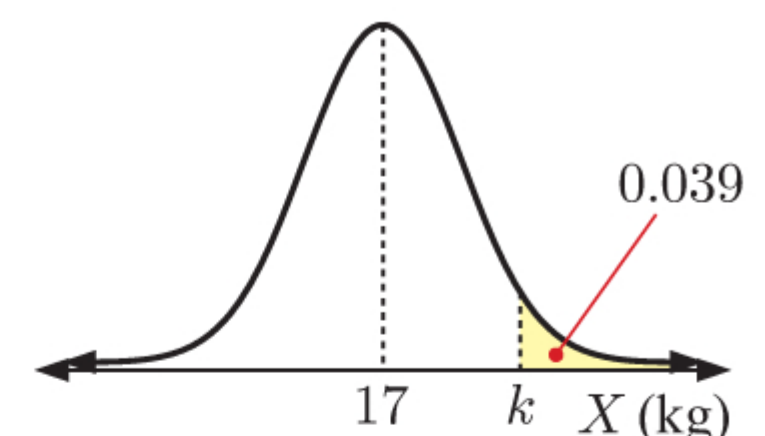
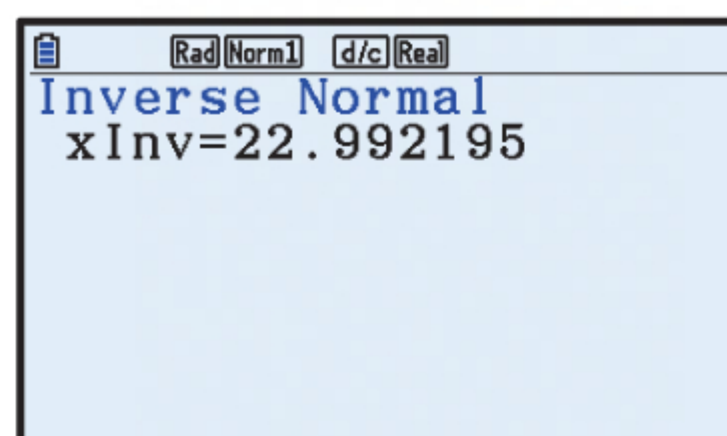
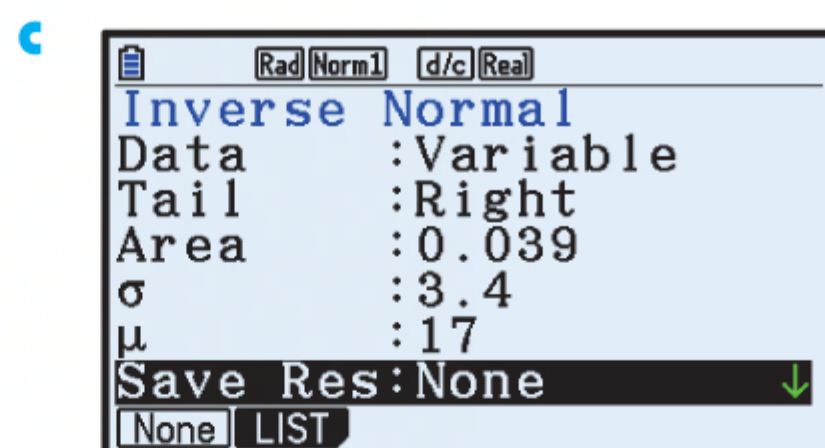


$$P(10 < X < 15) \approx 0.258$$



$$P(X < 20) \approx 0.811$$

\therefore we would expect about $0.811 \times 300 \approx 243$ suitcases to be lighter than 20 kg.



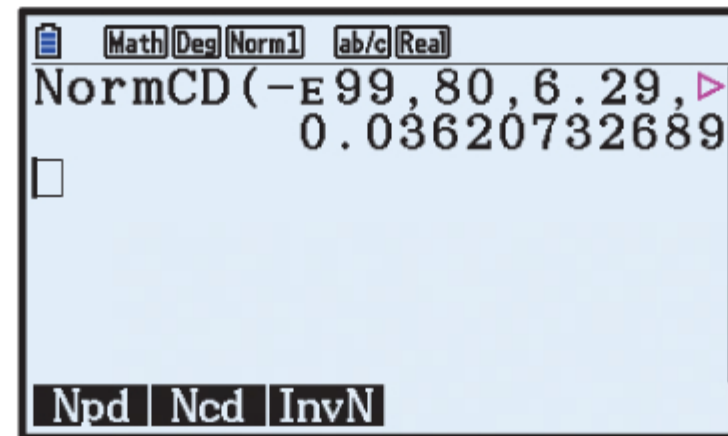
$$\begin{aligned}
 &\text{If } P(X > k) = 0.039 \\
 &\text{then } k \approx 23.0
 \end{aligned}$$

\therefore the maximum weight limit is ≈ 23.0 kg.

- 10 a** Let X_F be the weight in kilograms of a female ostrich and X_M be the weight in kilograms of a male ostrich.

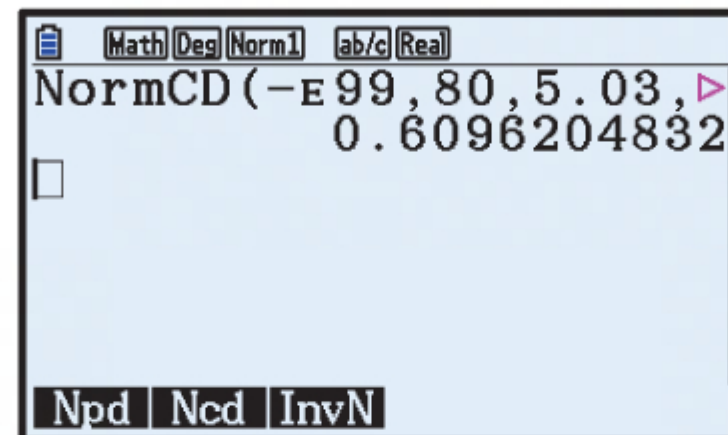
$$X_F \sim N(78.6, 5.03^2) \quad \text{and} \quad X_M \sim N(91.3, 6.29^2)$$

i $P(X_M < 80) \approx 0.0362$



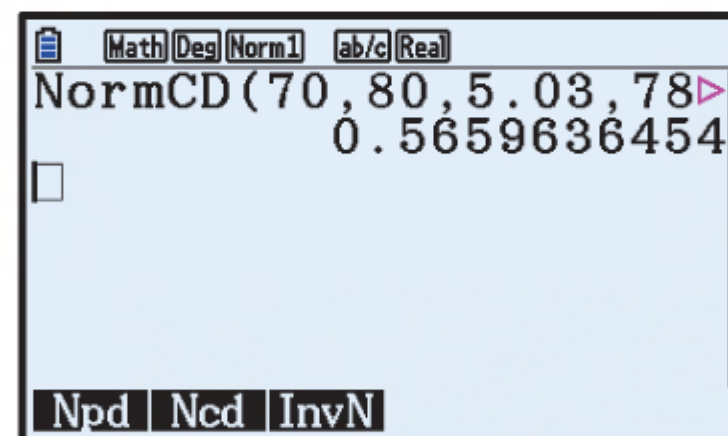
The probability that a randomly selected male ostrich will weigh less than 80 kg is about 0.0362.

ii $P(X_F < 80) \approx 0.610$



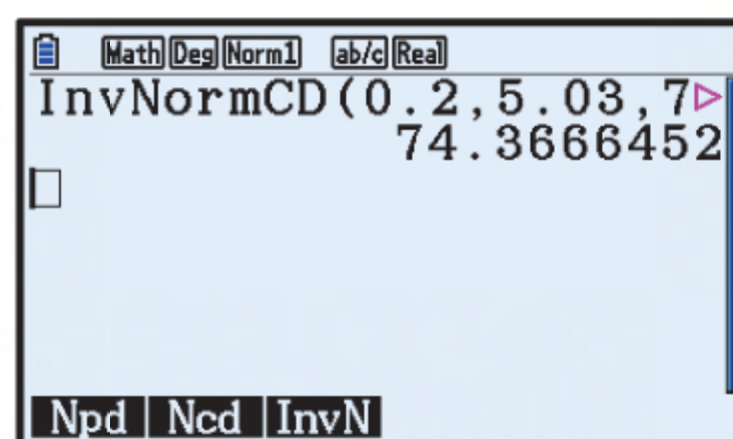
The probability that a randomly selected female ostrich will weigh less than 80 kg is about 0.610.

iii $P(70 < X_F < 80) \approx 0.566$



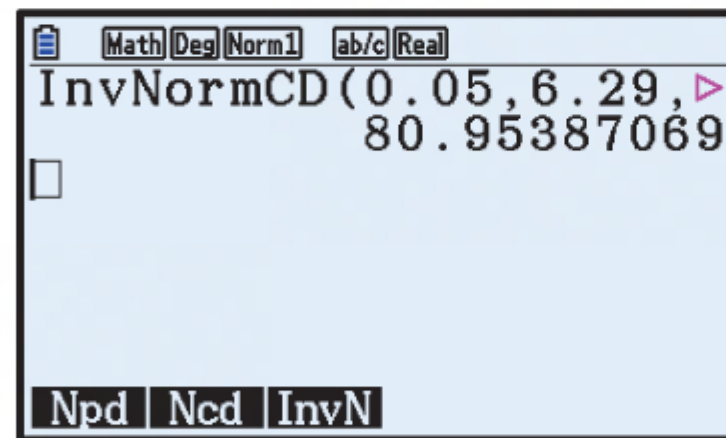
The probability that a randomly selected female ostrich will weigh between 70 and 80 kg is about 0.566.

b $P(X_F < k) = 0.2$
then $k \approx 74.4$

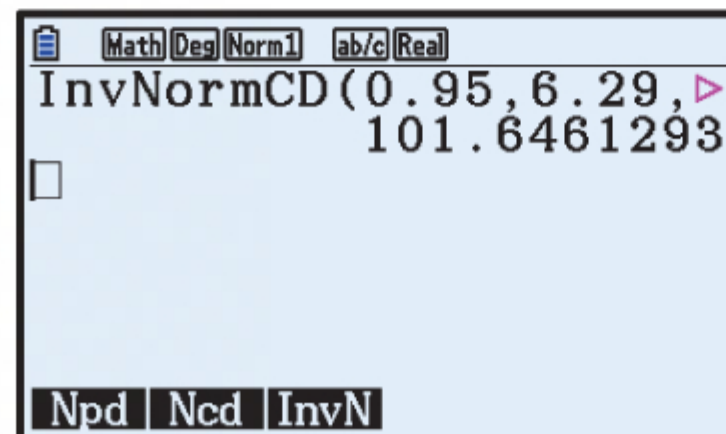


- c We need to find a and b such that $P(X_M < a) = 0.05$ and $P(X_M > b) = 0.05$.

If $P(X_M < a) = 0.05$
then $a \approx 81.0$

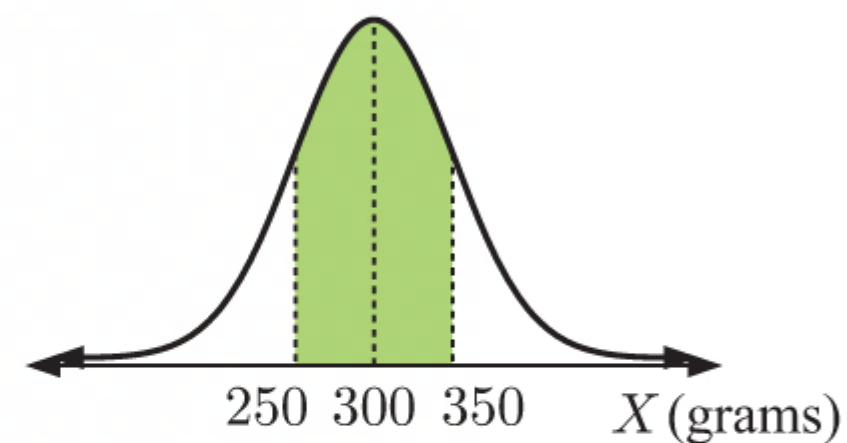
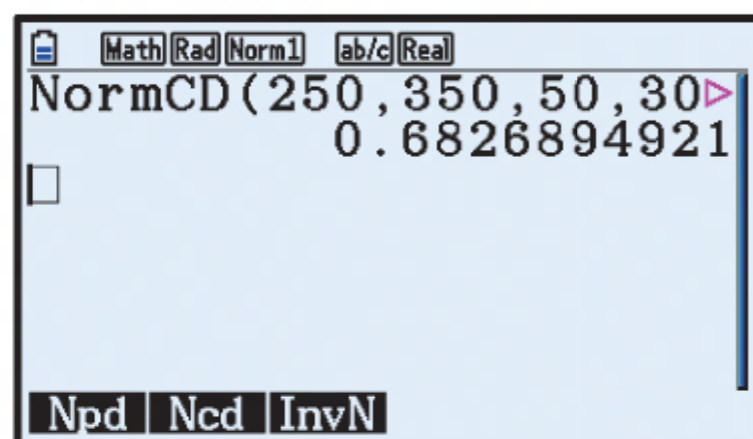


If $P(X_M > b) = 0.05$
then $P(X_M < b) = 0.95$
 $\therefore b \approx 102$



- d Probability that the ostrich weighs less than 80 kg
 $= P(\text{ostrich is female} \cap \text{less than 80 kg}) + P(\text{ostrich is male} \cap \text{less than 80 kg})$
 $\approx 0.82 \times 0.610 + 0.18 \times 0.0362$
 ≈ 0.506

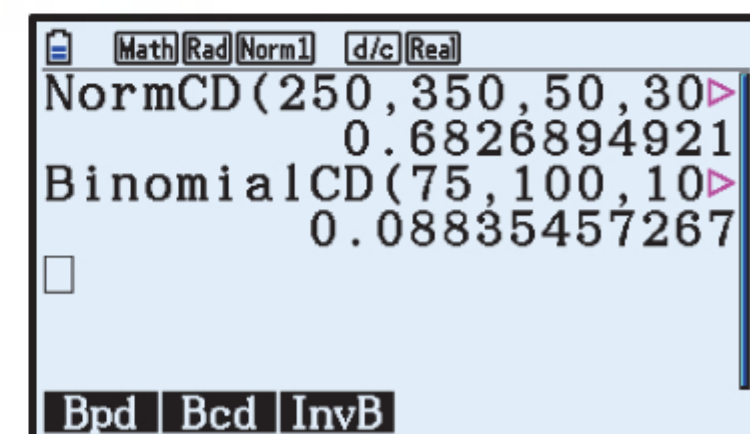
- 11 a Let X grams be the weight of an apple.
 $X \sim N(300, 50^2)$



$$P(250 \leq X \leq 350) \approx 0.682689$$

So, approximately 68.3% of apples are fit for sale.

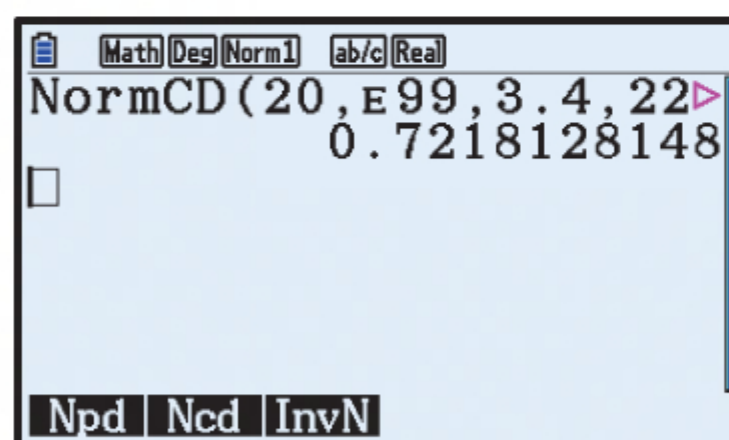
- b Let Y be the number of apples that are fit for sale.
 $Y \sim B(100, 0.682689)$
 $P(Y \geq 75) \approx 0.0884$



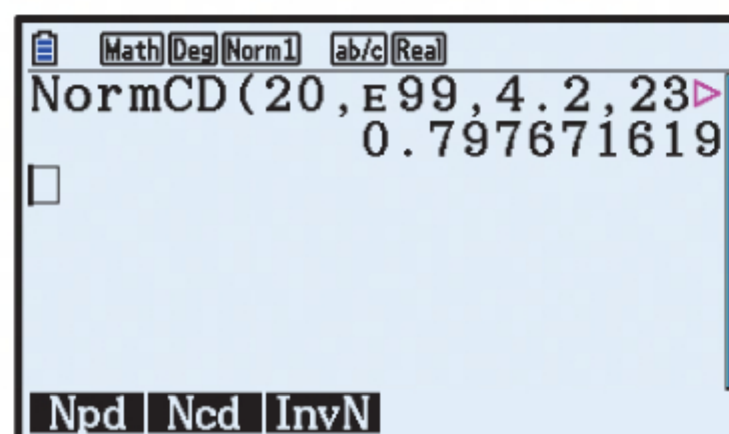
- 12 a** Let X_G be the length in cm of a carrot from Giovanni's farm, and X_B be the length in cm of a carrot from Beppe's farm.

$$X_G \sim N(22, 3.4^2) \text{ and } X_B \sim N(23.5, 4.2^2)$$

i $P(X_G > 20) \approx 0.722$



ii $P(X_B > 20) \approx 0.798$



- b** Probability that neither carrot is longer than 20 cm = $P(X_G < 20) \times P(X_B < 20)$
 $\approx (1 - 0.722) \times (1 - 0.798)$
 ≈ 0.0563

Chapter 16

HYPOTHESIS TESTING

EXERCISE 16A

- 1 **a** $H_0: \mu = 80$ {new globe lasts as long as old globe}
 $H_1: \mu > 80$ {new globe lasts longer than old globe}
- b** $H_0: \mu = 80$ {new globe lasts as long as old globe}
 $H_1: \mu < 80$ {new globe does not last as long as old globe}
- 2 $H_0: \mu = 26.3$ {new top speed is the same as old top speed}
 $H_1: \mu > 26.3$ {new top speed is greater than old top speed}
- 3 $H_0: \mu = 250$ {mean weight of chips per bag is 250 g}
 $H_1: \mu \neq 250$ {mean weight of chips per bag is *not* 250 g}
- 4 $H_0: \mu = 80$ {mean weight of paper is 80 g per m²}
 $H_1: \mu \neq 80$ {mean weight of paper is *not* 80 g per m²}
- 5 $H_0: \mu = 27$ {mean travel time is the same as before}
 $H_1: \mu < 27$ {mean travel time is lower than before}
- 6 $H_0: \mu = 2.7$ {fat content of Brand B's muesli bars is 2.7 g}
 $H_1: \mu > 2.7$ {fat content of Brand B's muesli bars is *greater* than 2.7 g}

INVESTIGATION 1

THE DISTRIBUTION OF THE TEST STATISTIC

- 1 **a** The t -values appear to be approximately normally distributed. Changing the values of the mean and standard deviation does not appear to affect their distribution.
- b** As the value of n increases, the distribution of the t -values becomes less spread out, and the standard deviation gets slightly lower.
- 2 Estimating based on the frequency histogram with $n = 10$:
 - a** about 60% of the t -values appear to lie between -1 and 1
 - b** about 95% of the t -values appear to lie between -2 and 2
 - c** about 99% of the t -values appear to lie between -3 and 3 .

EXERCISE 16B

1 $H_0: \mu = 25$

$H_1: \mu < 25$

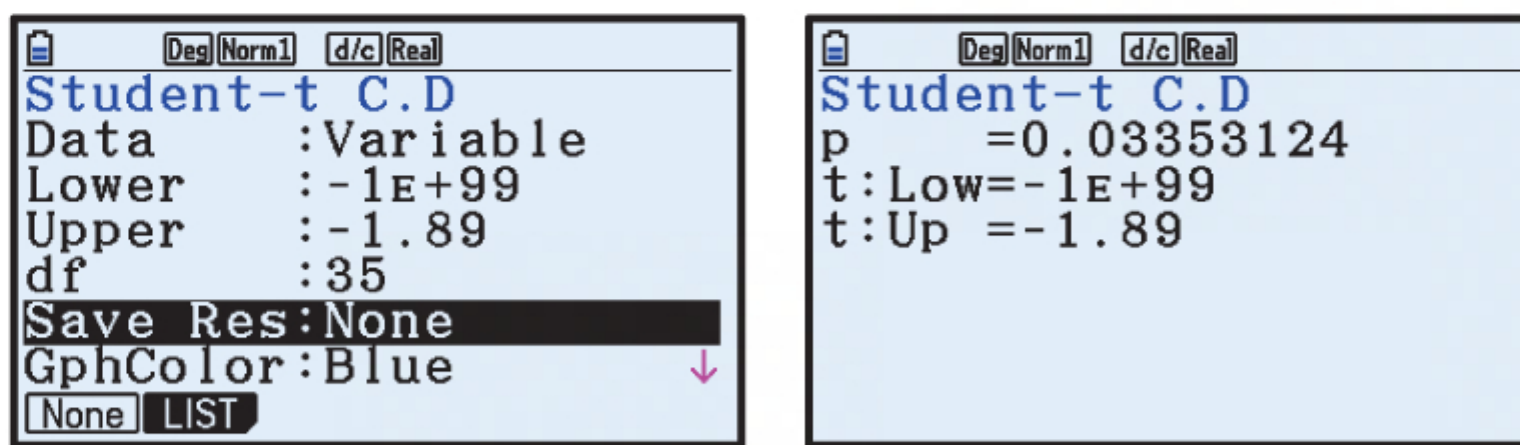
a i $\bar{x} = 23.75, \mu_0 = 25, s = 3.97, n = 36$

The value of the test statistic $t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}}$

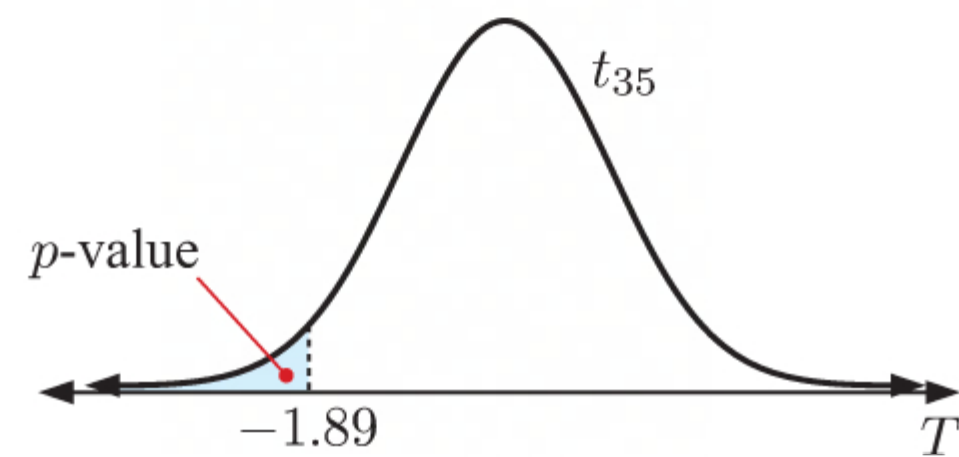
$$= \frac{23.75 - 25}{\frac{3.97}{\sqrt{36}}}$$

$$\approx -1.89$$

ii



Since $H_1: \mu < 25$, $t \approx -1.89$, and $n = 36$,
the $p\text{-value} = P(T \leq t)$ where $T \sim t_{35}$
 $\approx P(T \leq -1.89)$
 ≈ 0.0335



b Since $p\text{-value} < 0.05$, we have enough evidence to reject H_0 in favour of H_1 on a 5% significance level. We therefore accept that the population mean is less than 25.

2 a $H_0: \mu = 80$ {the population mean is 80}

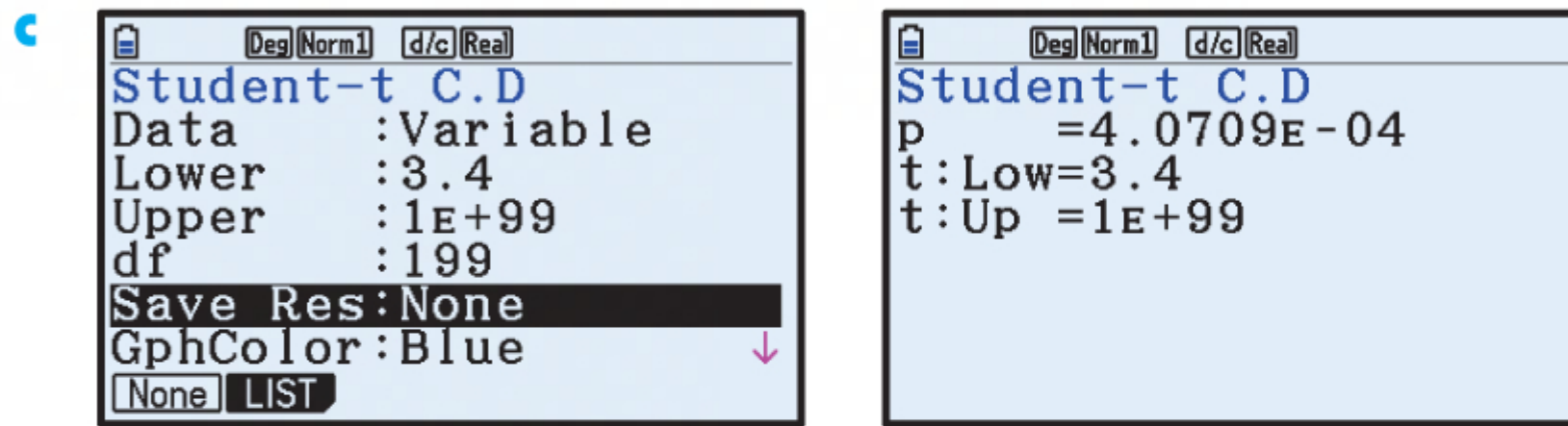
$H_1: \mu > 80$ {the population mean is greater than 80}

b $\bar{x} = 83.1, \mu_0 = 80, s = 12.9, n = 200$

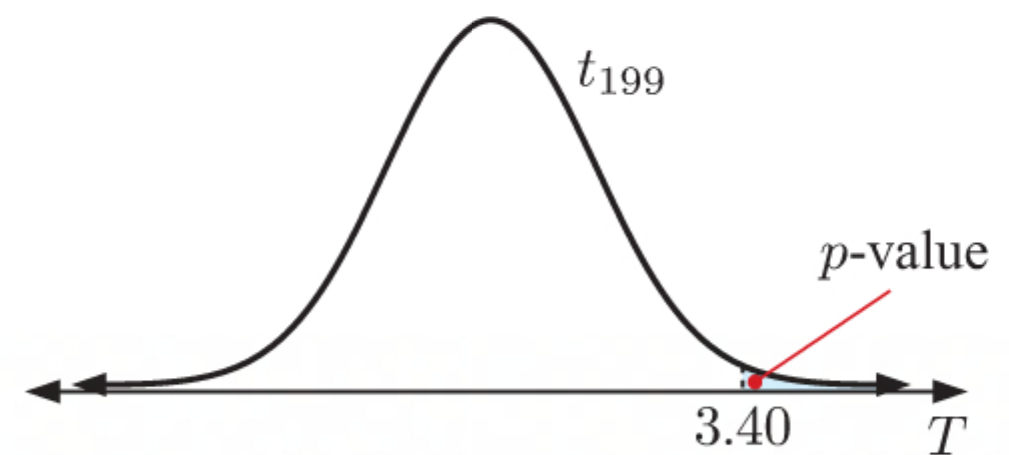
The value of the test statistic $t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}}$

$$= \frac{83.1 - 80}{\frac{12.9}{\sqrt{200}}}$$

$$\approx 3.40$$



Since $H_1: \mu > 80$, $t \approx 3.40$, and $n = 200$,
the p -value $= P(T \geq t)$ where $T \sim t_{199}$
 $\approx P(T \geq 3.40)$
 ≈ 0.000407



- d Since $p\text{-value} < 0.01 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 on a 1% significance level.
- e We conclude that the population mean is greater than 80 at the 1% significance level.

- 3 *Step 1:* Let μ be the population mean weight of the bags. The factory wants to determine whether the weight has *decreased*, so the hypotheses to be considered are:

$$H_0: \mu = 100 \quad \{\text{the mean weight is 100 g per bag}\}$$

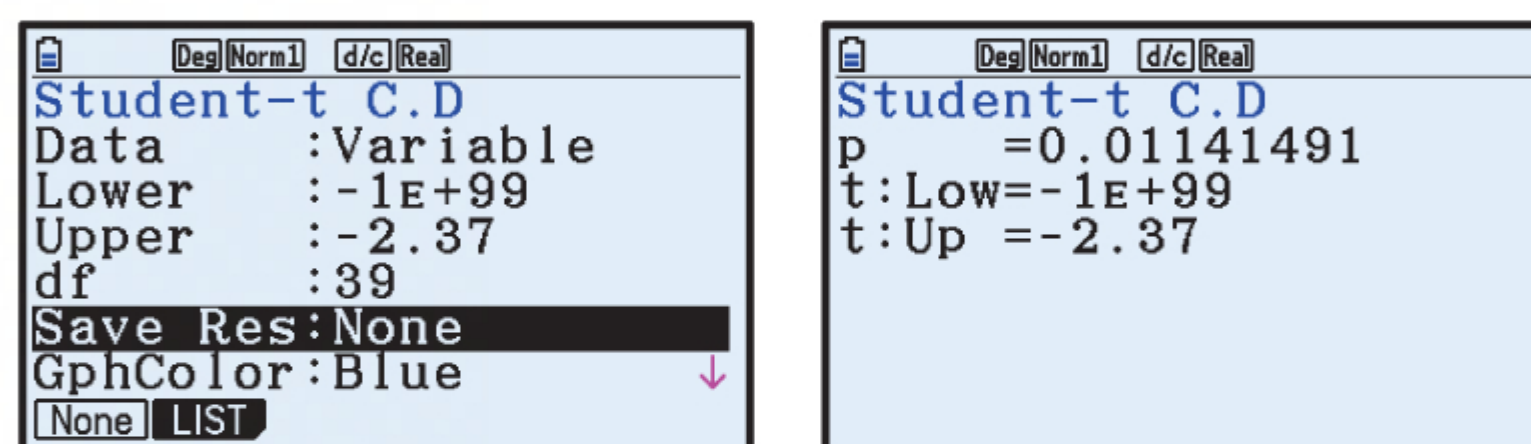
$$H_1: \mu < 100 \quad \{\text{the mean weight is less than 100 g per bag}\}$$

Step 2: The significance level is $\alpha = 0.05$.

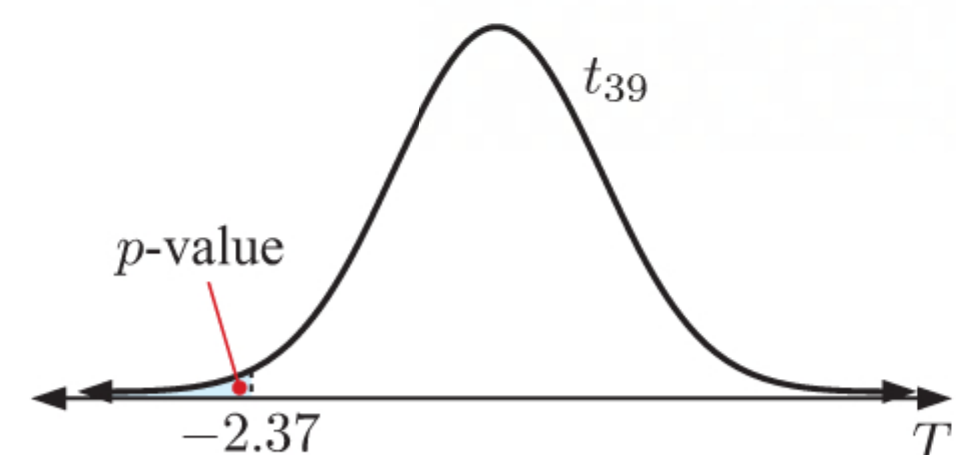
Step 3: $\bar{x} = 99.4$ g, $\mu_0 = 100$ g, $s = 1.6$ g, $n = 40$ bags

$$\begin{aligned} \text{The value of the test statistic } t &= \frac{99.4 - 100}{\frac{1.6}{\sqrt{40}}} \\ &\approx -2.37 \end{aligned}$$

Step 4:



Since $H_1: \mu < 100$ and $n = 40$,
the p -value $= P(T \leq t)$ where $T \sim t_{39}$
 $\approx P(T \leq -2.37)$
 ≈ 0.0114



Step 5: Since $p\text{-value} < 0.05 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 on a 5% significance level.

Step 6: Since we have accepted H_1 , we conclude on a 5% significance level that the mean weight is less than 100 g. The customer's claim is valid.

- 4** *Step 1:* Let μ be the population mean fineness of the herd's fleece. The breeder wants to determine whether the fineness has *changed*, so the hypotheses to be considered are:

$$H_0: \mu = 20.3 \quad \{\text{the herd's mean fineness has not changed}\}$$

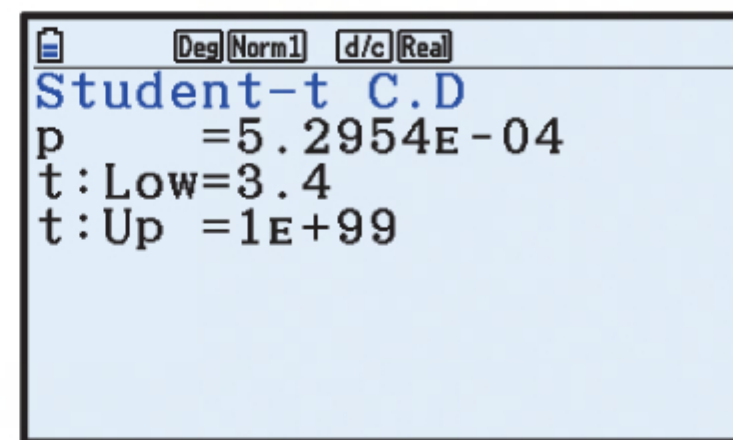
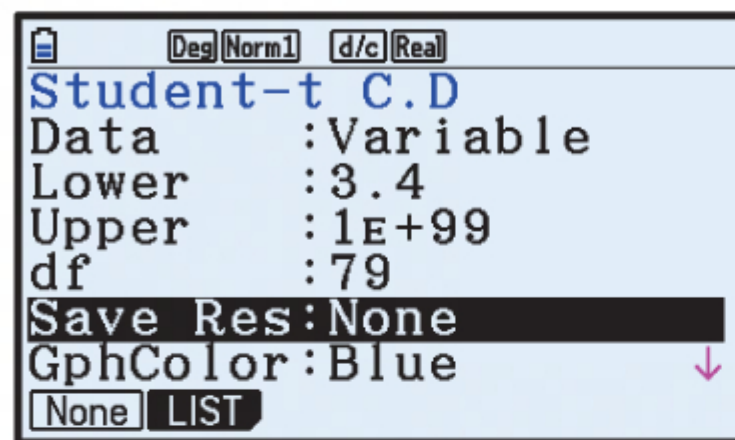
$$H_1: \mu \neq 20.3 \quad \{\text{the herd's mean fineness has changed}\}$$

Step 2: The significance level is $\alpha = 0.05$.

Step 3: $\bar{x} = 19.2$ microns, $\mu_0 = 20.3$ microns, $s = 2.89$ microns, $n = 80$ alpacas

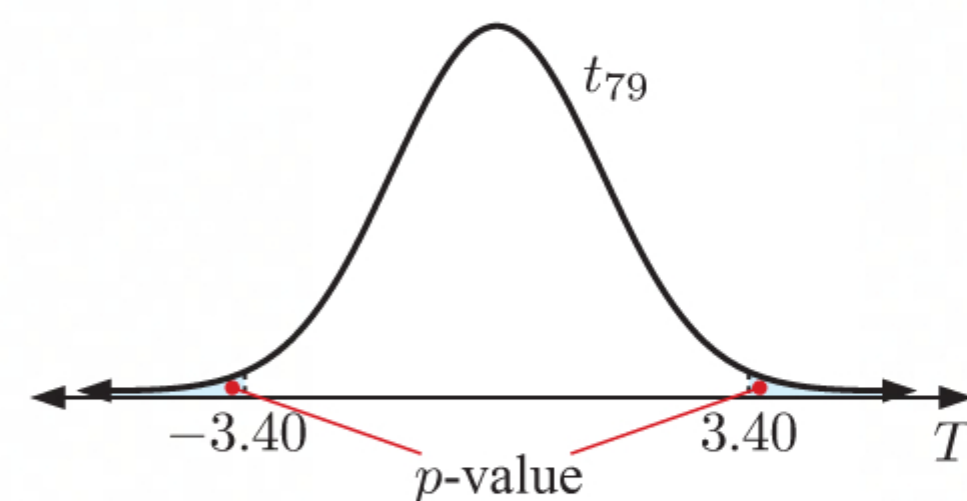
$$\begin{aligned} \text{The value of the test statistic } t &= \frac{19.2 - 20.3}{\frac{2.89}{\sqrt{80}}} \\ &\approx -3.40 \end{aligned}$$

Step 4:



Since $H_1: \mu \neq 20.3$ and $n = 80$,

$$\begin{aligned} \text{the } p\text{-value} &= 2 \times P(T \geq |t|) \quad \text{where } T \sim t_{79} \\ &\approx 2 \times P(T \geq 3.40) \\ &\approx 2 \times 0.000530 \\ &\approx 0.00106 \end{aligned}$$



Step 5: Since $p\text{-value} < 0.05 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 on a 5% significance level.

Step 6: Since we have rejected H_0 , we conclude on a 5% significance level that the fineness of the herd's fleece has changed.

- 5** *Step 1:* Let μ be the population mean screw length. The quality controller wants to determine whether the machine should be adjusted, so the hypotheses to be considered are:

$$H_0: \mu = 2.00 \quad \{\text{the machine is producing screws of the correct length}\}$$

$$H_1: \mu \neq 2.00 \quad \{\text{the machine is not producing screws of the correct length}\}$$

Step 2: The significance level is $\alpha = 0.02$.

Step 3: $\bar{x} = 2.04$ cm, $\mu_0 = 2.00$ cm, $s = 0.09$ cm, $n = 15$ screws

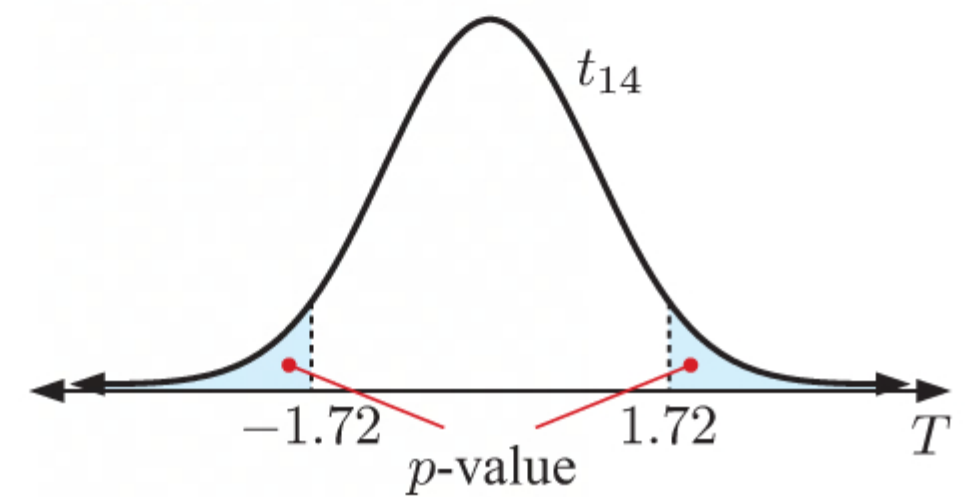
$$\begin{aligned} \text{The value of the test statistic } t &= \frac{2.04 - 2.00}{\frac{0.09}{\sqrt{15}}} \\ &\approx 1.72 \end{aligned}$$

Step 4:

Student-t C.D
Data : Variable
Lower : 1.72
Upper : 1E+99
df : 14
Save Res: None
GphColor: Blue
None LIST

Student-t C.D
p = 0.05372517
t:Low=1.72
t:Up = 1E+99

Since $H_1: \mu \neq 2.00$ and $n = 15$,
the p -value $= 2 \times P(T \geq |t|)$ where $T \sim t_{14}$
 $\approx 2 \times P(T \geq 1.72)$
 $\approx 2 \times 0.0537$
 ≈ 0.107



Step 5: Since $p\text{-value} > 0.02 = \alpha$, we do not have sufficient evidence to reject H_0 in favour of H_1 on a 2% significance level. We therefore accept H_0 .

Step 6: Since we have accepted H_0 , we cannot conclude on a 2% significance level that the mean length is significantly different from 2.00. Adjusting the machine is not justified.

6 Step 1: Let μ be the population mean mass of the bags of sugar. We want to determine whether the machine is *under*-filling the bags, so the hypotheses to be considered are:

$H_0: \mu = 1000$ {the machine is filling the bags correctly}

$H_1: \mu < 1000$ {the machine is not filling the bags enough}

Step 2: The significance level is $\alpha = 0.01$.

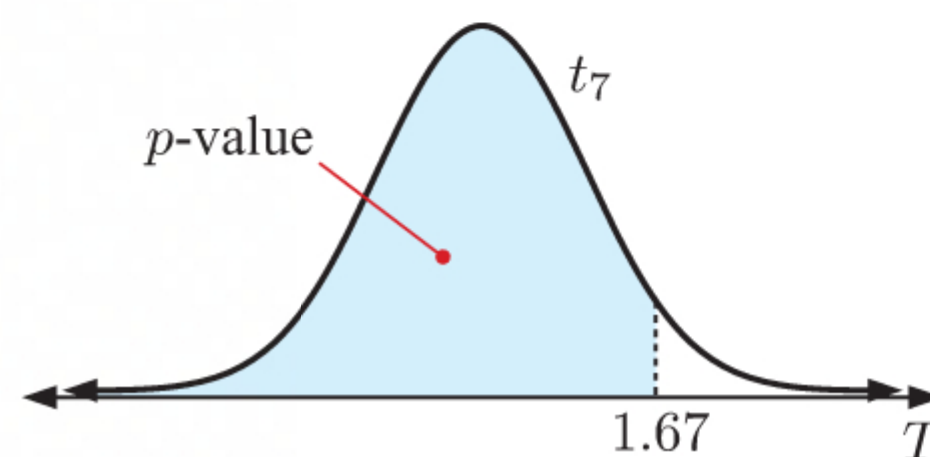
Steps 3 and 4:

	List 1	List 2	List 3	List 4
SUB				
1	1001			
2	998			
3	999			
4	1002			

1-Sample tTest
Data : List
μ : $< \mu_0$
μ_0 : 1000
List : List1
Freq : 1
Save Res: None
1 LIST

1-Sample tTest
μ : < 1000
t = 1.67332005
p = 0.93091002
\bar{x} = 1001
sx = 1.69030851
n = 8

Using technology, the test statistic $t \approx 1.67$ and the p -value ≈ 0.931 .



Step 5: Since $p\text{-value} > 0.01 = \alpha$, we do not have enough evidence to reject H_0 in favour of H_1 on a 1% significance level. We therefore accept H_0 .

Step 6: Since we have accepted H_0 , we cannot conclude on a 1% significance level that the bags have been under-filled.

7 a The distribution of the carrots' weights is likely to be centred around the mean, and will vary randomly due to factors such as amount of sunlight and soil conditions, so it is reasonable to assume that their weights are normally distributed.

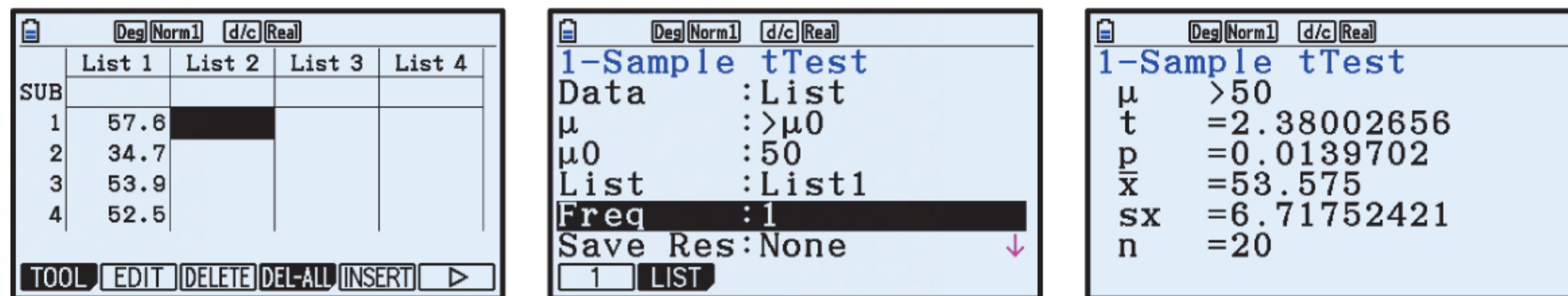
- b** *Step 1:* Let μ be the population mean weight of the carrots. The buyer wants to determine whether the mean is more than 50 g, so the hypotheses to be considered are:

$$H_0: \mu = 50 \quad \{\text{the carrots have a mean weight of 50 g}\}$$

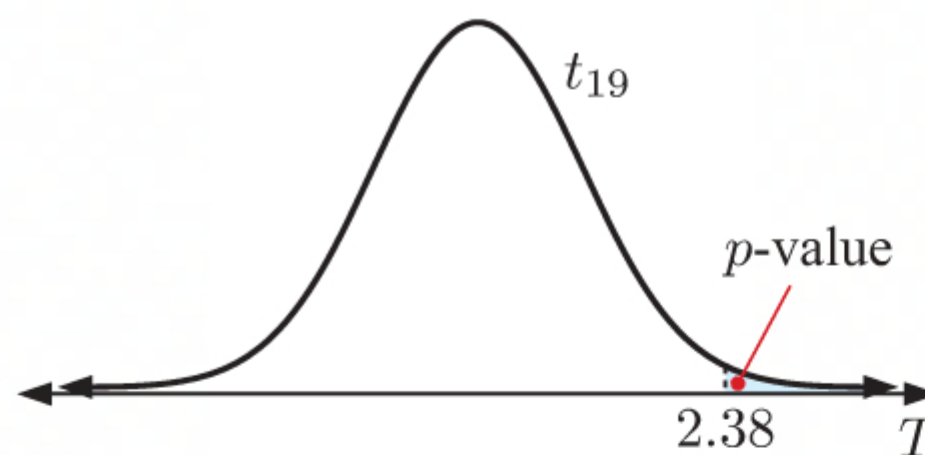
$$H_1: \mu > 50 \quad \{\text{the carrots have a mean weight of more than 50 g}\}$$

Step 2: The significance level is $\alpha = 0.05$.

Steps 3 and 4:



Using technology, the test statistic $t \approx 2.38$ and the p -value ≈ 0.0140 .



- Step 5:* Since $p\text{-value} < 0.05 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 on a 5% significance level. We therefore accept H_1 .
- Step 6:* Since we have accepted H_1 , we conclude on a 5% significance level that the mean weight is greater than 50 g. The buyer will therefore purchase the crop.

INVESTIGATION 2

MULTIPLE TESTING AND STATISTICAL FALLACY

- 1 a** The value of the test statistic $t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}}$
- $$= \frac{\bar{x} - 2}{\frac{s}{\sqrt{10}}}$$

m	Number of times H_0 is rejected	Proportion of samples where H_0 was rejected
20	1	0.05
50	3	0.06
100	4	0.04
500	21	0.042
1000	52	0.052
5000	258	0.0516
10 000	523	0.0523

c i

m	Number of times H_0 is rejected	Proportion of samples where H_0 was rejected
20	3	0.15
50	4	0.08
100	9	0.09
500	63	0.126
1000	101	0.101
5000	489	0.0978
10 000	1013	0.1013

ii

m	Number of times H_0 is rejected	Proportion of samples where H_0 was rejected
20	1	0.05
50	1	0.02
100	4	0.04
500	16	0.032
1000	26	0.026
5000	137	0.0274
10 000	264	0.0264

iii

m	Number of times H_0 is rejected	Proportion of samples where H_0 was rejected
20	0	0
50	1	0.02
100	0	0
500	10	0.02
1000	9	0.009
5000	44	0.0088
10 000	106	0.0106

In each table, including **b**, we see that the proportion of samples where H_0 was rejected is close to the particular value of α used. As the number of samples m increases, the proportion of samples rejected approaches α .

- 2 a** $H_0: \mu = 2$ is true in every sample because each sample is generated from the distribution $N(2, 5^2)$, which has mean $\mu = 2$.
- b** If a statistical hypothesis has significance level α , the probability of incorrectly rejecting H_0 is α .
- c** The number of times H_0 is rejected is a random variable with a fixed number of trials $n = m$, and the same probability of success (rejecting H_0) $p = \alpha$. It therefore has the distribution $B(m, \alpha)$, and so the *expected* number of samples where H_0 is incorrectly rejected is $m\alpha$.
- 3 a** Sabeen is most likely to report on the effects of diet cola on females aged 15 to 19, as this result has the lowest p -value, and is therefore seen as the most significant result.
- b** From **2 c** we know that if we expect to see 1 result for which H_0 is incorrectly rejected, then $m\alpha = 1$.
Sabeen divided her data into 20 different groups, so we would expect to see H_0 rejected once on the significance level $\alpha = \frac{1}{m} = \frac{1}{20} = 0.05$.

This is exactly what we see in Sabeen's data, for the group of females aged 15 to 19, so it appears that this result is simply due to random variation.

Therefore, we would not expect Mysha to replicate Sabeen's finding that diet cola has a significant effect on females aged 15 to 19.

EXERCISE 16C

- 1 a Let μ_1 be the population mean weight of tomatoes from last year's crop, and μ_2 be the population mean weight of tomatoes from this year's crop. Frank wants to determine whether the new fertiliser increased the mean weight of the crop, so the hypotheses to be considered are:

$H_0: \mu_1 = \mu_2$ {the mean weight of this year's crop is the same as last year's crop}

$H_1: \mu_1 < \mu_2$ {the mean weight of this year's crop is more than last year's crop}

- b The significance level $\alpha = 0.01$.

For last year's sample, $\bar{x} = 106.3$, $s = 12.41$, and $n = 50$.

For this year's sample, $\bar{x} = 110.1$, $s = 13.1$, and $n = 65$.

	Deg	Norm1	d/c	Real
2-Sample tTest				
Data	:	Variable		
μ_1	:	< μ_2		
\bar{x}_1	:	106.3		
s_{x1}	:	12.41		
n_1	:	50		
\bar{x}_2	:	110.1		

	Deg	Norm1	d/c	Real
2-Sample tTest				
\bar{x}_2	:	110.1		
s_{x2}	:	13.1		
n_2	:	65		
Pooled	:	On		
Save Res	:	None		
GphColor	:	Blue		
		On	Off	

	Deg	Norm1	d/c	Real
2-Sample tTest				
μ_1	:	< μ_2		
t	:	-1.5775548		
p	:	0.05873167		
df	:	113		
\bar{x}_1	:	106.3		
\bar{x}_2	:	110.1		

Using technology, the test statistic $t \approx -1.58$ and the p -value ≈ 0.587 .

Since the p -value $> 0.01 = \alpha$, there is not enough evidence to reject H_0 in favour of H_1 on a 1% significance level. Frank therefore cannot conclude that the new fertiliser was more effective.

- 2 Step 1: Let μ_1 be the population mean sleeping time for middle school students, and μ_2 be the population mean sleeping time for high school students.

The hypotheses that should be considered are:

$H_0: \mu_1 = \mu_2$ {middle school and high school students sleep the same amount on average}

$H_1: \mu_1 > \mu_2$ {high school students sleep less on average than middle school students}

Step 2: The significance level is $\alpha = 0.05$.

Step 3:

	Deg	Norm1	d/c	Real
2-Sample tTest				
Data	:	Variable		
μ_1	:	> μ_2		
\bar{x}_1	:	8		
s_{x1}	:	0.2		
n_1	:	49		
\bar{x}_2	:	7.9		

	Deg	Norm1	d/c	Real
2-Sample tTest				
\bar{x}_2	:	7.9		
s_{x2}	:	0.5		
n_2	:	55		
Pooled	:	On		
Save Res	:	None		
GphColor	:	Blue		
		On	Off	

	Deg	Norm1	d/c	Real
2-Sample tTest				
μ_1	:	> μ_2		
t	:	1.30924407		
p	:	0.09669668		
df	:	102		
\bar{x}_1	:	8		
\bar{x}_2	:	7.9		

Using technology, the value of the test statistic $t \approx 1.31$.

Step 4: From the screenshots above, the p -value ≈ 0.0967 .

Step 5: Since $p\text{-value} > 0.05 = \alpha$, we do not have sufficient evidence to reject H_0 in favour of H_1 on a 5% significance level. We therefore accept H_0 .

Step 6: Since we have accepted H_0 , we cannot conclude on a 5% significance level that high school students sleep less on average than middle school students. The researcher's claim is therefore invalid.

- 3 Step 1: Let μ_A be the population mean of the height of the seedlings using *Brand A*, and μ_B be the population mean of the height of the seedlings using *Brand B*. The hypotheses that should be considered are:

$H_0: \mu_A = \mu_B$ {the seedlings grow the same amount in both brands of soil}

$H_1: \mu_A < \mu_B$ {the seedlings grow higher in *Brand B*'s soil than *Brand A*'s soil}

Step 2: The significance level is $\alpha = 0.05$.

Step 3:

	List 1	List 2	List 3	List 4
SUB				
1	12.1	12.3		
2	14.6	15.2		
3	10.1	9.9		
4	8.7	9.5		

	List
2-Sample tTest	
Data	:List
μ_1	:< μ_2
List(1)	:List1
List(2)	:List2
Freq(1)	:1
Freq(2)	:1

	List
2-Sample tTest	
μ_1	:< μ_2
t	=-0.1979233
p	=0.42297507
df	=14
\bar{x}_1	=13.1125
\bar{x}_2	=13.375

Using technology, the value of the test statistic $t \approx -0.198$.

Step 4: From the screenshots above, the $p\text{-value} \approx 0.423$.

Step 5: Since $p\text{-value} > 0.05 = \alpha$, we do not have enough evidence to reject H_0 in favour of H_1 on a 5% significance level. We therefore accept H_0 .

Step 6: Since we have accepted H_0 , we cannot conclude on a 5% significance level that *Brand B*'s soil improves the growth of seedlings. *Brand B*'s guarantee is therefore invalid.

- 4 Step 1: Let μ_1 be the population mean test result of the students before receiving tutoring, and μ_2 be the population mean test result of the students after receiving tutoring. The hypotheses that should be considered are:

$H_0: \mu_1 = \mu_2$ {the tutoring made no difference to the students' mean test result}

$H_1: \mu_1 < \mu_2$ {the tutoring increased the students' mean test result}

Step 2: The significance level is $\alpha = 0.05$.

Step 3:

	List 1	List 2	List 3	List 4
SUB				
1	15	20		
2	17	16		
3	25	25		
4	11	18		

	List
2-Sample tTest	
Data	:List
μ_1	:< μ_2
List(1)	:List1
List(2)	:List2
Freq(1)	:1
Freq(2)	:1

	List
2-Sample tTest	
μ_1	:< μ_2
t	=-0.4175849
p	=0.34014893
df	=22
\bar{x}_1	=22.1666667
\bar{x}_2	=23.3333333

Using technology, the value of the test statistic $t \approx -0.418$.

Step 4: From the screenshots above, the $p\text{-value} \approx 0.340$.

Step 5: Since $p\text{-value} > 0.05 = \alpha$, we do not have enough evidence to reject H_0 in favour of H_1 on a 5% significance level. We therefore accept H_0 .

Step 6: Since we have accepted H_0 , we cannot conclude at a 5% significance level that the tutoring increased the students' test results. The tutor's claim is therefore invalid.

- 5 a** *Step 1:* Let μ_1 be the population mean of Jesiah's 100 m sprint times, and μ_2 be the population mean of Billy's 100 m sprint times.

The hypotheses that should be considered are:

$$H_0: \mu_1 = \mu_2 \quad \{\text{there is no difference between the runners' mean 100 m sprint times}\}$$

$$H_1: \mu_1 \neq \mu_2 \quad \{\text{there is a significant difference between the runners' mean 100 m sprint times}\}$$

Step 2: The significance level is $\alpha = 0.05$.

Step 3:

	List 1	List 2	List 3	List 4
SUB				
1	12.3	10.4		
2	13	12		
3	11.3	11.7		
4	11.5	13.2		

2-Sample tTest	
Data	: List
μ_1	$\neq \mu_2$
List(1)	: List1
List(2)	: List2
Freq(1)	: 1
Freq(2)	: 1

2-Sample tTest	
μ_1	$\neq \mu_2$
t	= 2.38843973
p	= 0.02513748
df	= 24
\bar{x}_1	= 12.5785714
\bar{x}_2	= 11.725

Using technology, the value of the test statistic $t \approx 2.39$.

Step 4: From the screenshots above, the p -value ≈ 0.0251 .

Step 5: Since $p\text{-value} < 0.05 = \alpha$, we have sufficient evidence to reject H_0 in favour of H_1 at a 5% significance level. We therefore accept H_1 .

Step 6: Since we have accepted H_1 , we conclude that there is a significant difference between the runners' times on a 5% level of significance.

- b** The mean time for Jesiah was about 12.6 seconds and the mean time for Billy was about 11.7 seconds. So, Billy is faster.

EXERCISE 16D.1

- 1 a** Let p_1 and p_2 be the population proportions of "heads" and "tails" respectively. The hypotheses that should be tested are:

$$H_0: p_1 = 0.5, \quad p_2 = 0.5 \quad \{\text{the coin is unbiased}\}$$

$$H_1: \text{at least one of } p_1 \neq 0.5 \text{ or } p_2 \neq 0.5 \quad \{\text{the coin is biased}\}$$

- b** There were 54 heads in 96 tosses, so there were $96 - 54 = 42$ tails.
- c** If the coin is fair, then $P(\text{head}) = P(\text{tail}) = 0.5$, so the expected frequencies are $96 \times 0.5 = 48$ heads and 48 tails.

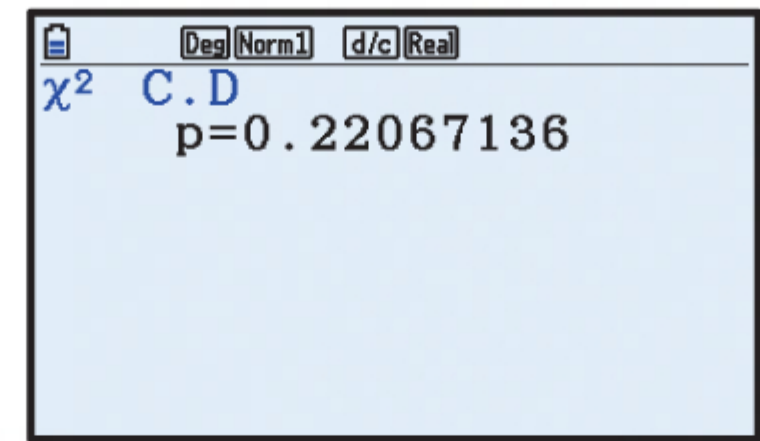
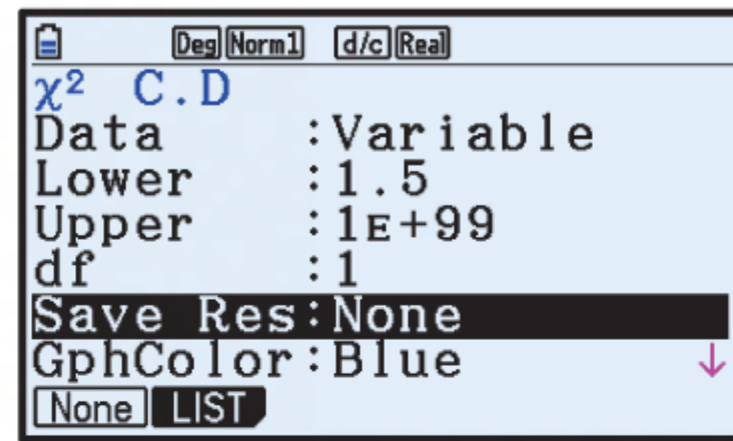
d

Side	f_{obs}	f_{exp}	$\frac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
heads	54	48	0.75
tails	42	48	0.75
Total			1.5

So, $\chi^2_{\text{calc}} = 1.5$

e There are two categories, so $df = 2 - 1 = 1$.

f Using technology,
the p -value ≈ 0.221



g The significance level is $\alpha = 0.05$, and the p -value is > 0.05 , so there is insufficient evidence to reject H_0 at the 5% significance level. There is therefore insufficient evidence to conclude that the coin is biased.

2 **Step 1:** Let p_A , p_B , and p_C be the population proportions of the votes received in the last election by parties A, B, and C respectively.

The hypotheses that should be tested are:

$$H_0: p_A = 0.54, \quad p_B = 0.3, \quad p_C = 1 - 0.54 - 0.3 = 0.16$$

$$H_1: \text{at least one of } p_A \neq 0.54, \quad p_B \neq 0.3, \quad \text{or } p_C \neq 0.16.$$

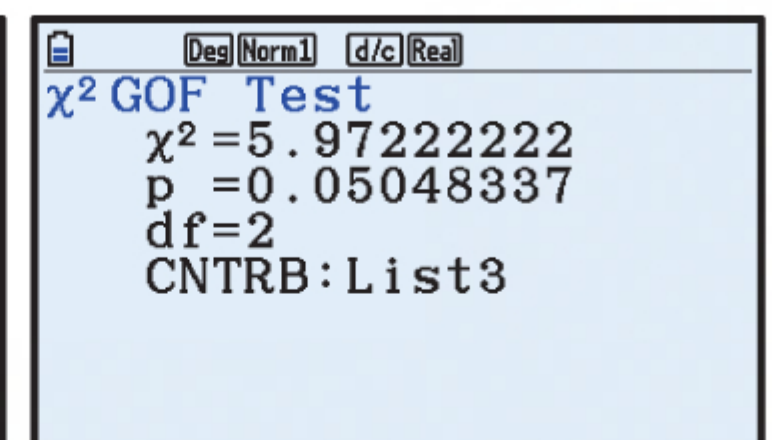
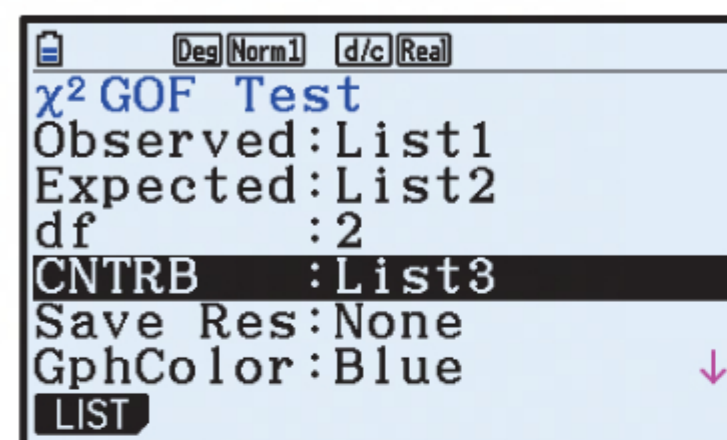
Step 2: The significance level is $\alpha = 0.01$.

Step 3:

Party	f_{obs}	f_{exp}	$\frac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
A	141	$0.54 \times 300 = 162$	≈ 2.7222
B	105	$0.3 \times 300 = 90$	2.5
C	54	$0.16 \times 300 = 48$	0.75
Total			≈ 5.9722

$$\text{So, } \chi^2_{\text{calc}} \approx 5.97$$

Step 4: $df = 3 - 1 = 2$



Using technology, the p -value ≈ 0.0505 .

Step 5: Since $p\text{-value} > 0.01 = \alpha$, there is insufficient evidence to reject H_0 in favour of H_1 on a 1% significance level.

Step 6: Since we have not rejected H_0 , there is insufficient evidence to conclude on a 1% significance level that the proportion of voters supporting each party since the last election has changed.

3 **Step 1:** Let p_1 , p_2 , p_3 , p_4 , and p_5 be the population proportions of the ice cream flavours sold. The hypotheses that should be tested are:

$$H_0: p_1 = \frac{1}{5}, \quad p_2 = \frac{1}{5}, \quad p_3 = \frac{1}{5}, \quad p_4 = \frac{1}{5}, \quad p_5 = \frac{1}{5}$$

$$H_1: \text{at least one of } p_1 \neq \frac{1}{5}, \quad p_2 \neq \frac{1}{5}, \quad p_3 \neq \frac{1}{5}, \quad p_4 \neq \frac{1}{5}, \quad \text{or } p_5 \neq \frac{1}{5}.$$

Step 2: The significance level is $\alpha = 0.1$.

Step 3:

Flavour	f_{obs}	f_{exp}	$\frac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
chocolate	54	$205 \times \frac{1}{5} = 41$	≈ 4.1220
strawberry	48	41	≈ 1.1951
vanilla	35	41	≈ 0.8780
honeycomb	28	41	≈ 4.1220
choc-chip	40	41	≈ 0.0244
		Total	≈ 10.3415

So, $\chi^2_{\text{calc}} \approx 10.3$

Step 4: $df = 5 - 1 = 4$

	List 1	List 2	List 3	List 4
SUB				
1	54	41		
2	48	41		
3	35	41		
4	28	41		

χ^2 GOF Test
Observed: List1
Expected: List2
df: 4
CNTRB: List3
Save Res: None
GphColor: Blue
LIST

χ^2 GOF Test
$\chi^2 = 10.3414634$
p = 0.03505229
df = 4
CNTRB: List3

Using technology, the p -value ≈ 0.0351 .

Step 5: Since $p\text{-value} < 0.1 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 on a 10% significance level.

Step 6: Since we have accepted H_1 , we conclude that the proportions of each ice cream flavour sold are not all the same at a 10% significance level. Brian should therefore change the amounts of each ice cream flavour that he makes.

4 Step 1: Let p_1, p_2, p_3, p_4 , and p_5 be the population proportions of people living in London who identify as being White, Asian/Asian British, Black/Black British, Mixed, and Other respectively.

The hypotheses that should be tested are:

$H_0: p_1 = 0.712, p_2 = 0.121, p_3 = 0.109, p_4 = 0.032, p_5 = 0.026$

$H_1: \text{at least one of } p_1 \neq 0.712, p_2 \neq 0.121, \dots, \text{ or } p_5 \neq 0.026.$

Step 2: Since no significance level is specified, we assume that $\alpha = 0.05$.

Step 3:

Ethnic group	f_{obs}	f_{exp}	$\frac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
White	4 887 435	$8\,173\,941 \times 0.712 \approx 5\,819\,846$	$\approx 149\,384$
Asian/Asian British	1 511 546	$8\,173\,941 \times 0.121 \approx 989\,047$	$\approx 276\,029$
Black/Black British	1 088 640	$8\,173\,941 \times 0.109 \approx 890\,960$	$\approx 43\,860$
Mixed	405 279	$8\,173\,941 \times 0.032 \approx 261\,566$	$\approx 78\,961$
Other	281 041	$8\,173\,941 \times 0.026 \approx 212\,522$	$\approx 22\,091$
		Total	$\approx 570\,325$

So, $\chi^2_{\text{calc}} \approx 570\,000$

Step 4: $df = 5 - 1 = 4$

Using technology, the p -value ≈ 0.000 .

Step 5: Since $p\text{-value} < 0.05 = \alpha$, we have sufficient evidence to reject H_0 in favour of H_1 on a 5% significance level. We therefore accept H_1 .

Step 6: Since we have accepted H_1 , we conclude on a 5% significance level that there was a significant change in London's demographics between 2001 and 2011.

5 a

Band	Expected frequency
10	$150 \times 0.079 = 11.85$
9	$150 \times 0.167 = 25.05$
8	$150 \times 0.298 = 44.7$
7	$150 \times 0.297 = 44.55$
6	$150 \times 0.135 = 20.25$
5 and below	$150 \times 0.024 = 3.6$

b Step 1: Let p_{10} , p_9 , p_8 , p_7 , p_6 , and p_5 be the population proportions of Year 9 students at a particular school who are in NAPLAN bands 10, 9, 8, 7, 6, and 5 and below respectively.

The hypotheses that should be tested are:

$H_0: p_{10} = 0.079, p_9 = 0.167, p_8 = 0.298, p_7 = 0.297, p_6 = 0.135, p_5 = 0.024$

$H_1: \text{at least one of } p_{10} \neq 0.079, p_9 \neq 0.167, \dots, \text{ or } p_5 \neq 0.024.$

Step 2: The significance level is $\alpha = 0.01$.

Step 3:

Band	f_{obs}	f_{exp}	$\frac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
10	5	11.85	≈ 3.9597
9	9	25.05	≈ 10.2835
8	55	44.7	≈ 2.3734
7	53	44.55	≈ 1.6027
6	23	20.25	≈ 0.3735
5 and below	5	3.6	≈ 0.5444
Total			≈ 19.1372

So, $\chi_{\text{calc}}^2 \approx 19.1$

Step 4: $df = 6 - 1 = 5$

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Using technology, the p -value ≈ 0.00181 .

Step 5: Since $p\text{-value} < 0.01 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 on a 1% significance level. We therefore accept H_1 .

Step 6: Since we have accepted H_1 , we conclude on a 1% significance level that the NAPLAN results of Year 9s at this school are significantly different from the national results.

- c Since the expected frequency for “Band 5 and below” is less than 5, the sample size is not large enough for χ^2 to be distributed appropriately. By combining “Band 6” and “Band 5 and below” we can obtain more reliable results.

- d Step 1: Let p_{10} , p_9 , p_8 , p_7 , and p_6 be the population proportions of Year 9 students at a particular school who are in NAPLAN bands 10, 9, 8, 7, and 6 and below respectively.

The hypotheses that should be tested are:

$$H_0: p_{10} = 0.079, p_9 = 0.167, p_8 = 0.298, p_7 = 0.297, \\ p_6 = 0.135 + 0.024 = 0.159$$

$$H_1: \text{at least one of } p_{10} \neq 0.079, p_9 \neq 0.167, \dots, \text{ or } p_6 \neq 0.159.$$

Step 2: The significance level is $\alpha = 0.01$.

Step 3:

Band	f_{obs}	f_{exp}	$\frac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
10	5	11.85	≈ 3.9597
9	9	25.05	≈ 10.2835
8	55	44.7	≈ 2.3734
7	53	44.55	≈ 1.6027
6 and below	$23 + 5 = 28$	$20.25 + 3.6 = 23.85$	≈ 0.7221
Total			≈ 18.9414

$$\text{So, } \chi_{\text{calc}}^2 \approx 18.9$$

Step 4: $df = 5 - 1 = 4$

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Using technology, the p -value ≈ 0.000807 .

Step 5: Since $p\text{-value} < 0.01 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 on a 1% significance level. We therefore accept H_1 .

Step 6: Since we have accepted H_1 , there is still sufficient evidence to support the claim that there is a substantial difference between the school's results and the rest of the nation at a 1% significance level.

Although we have obtained the same result both times, the result is more reliable now that each of the expected frequencies is sufficiently large.

ACTIVITY 1

ASSESSING THE GOODNESS OF FIT OF A PROBABILITY MODEL

- 1 a** Each question has 1 correct choice and 4 possible choices, so the probability of answering the question correctly with a random guess is $\frac{1}{4}$.

- b i**
- Each question can be answered either correctly (success) or incorrectly (failure). So, each trial has only two possible results.
 - The probability of answering correctly (success) is the same for each question.
 - There are 10 questions, and we assume the probability of answering each question correctly does not change based on the results from previous questions. So, there are a fixed number of independent trials.

Therefore, the number of questions the person answers correctly X is a binomial random variable.

ii $X \sim B(10, \frac{1}{4})$

- 2 a, b**

x	$P(X = x)$	Expected frequency
0	$\approx 0.056\ 31$	≈ 8.45
1	$\approx 0.187\ 71$	≈ 28.16
2	$\approx 0.281\ 57$	≈ 42.24
3	$\approx 0.250\ 28$	≈ 37.54
4	$\approx 0.146\ 00$	≈ 21.90
5 or more	$\approx 0.078\ 13$	≈ 11.72

- c** We would expect to see 5 correct answers about $150 \times P(X = 5) \approx 8.76$ times, but expect to see 6 or more correct answers about $11.72 - 8.76 \approx 2.96$ times. This value is less than 5, which means χ^2 may not be distributed appropriately. Therefore outcomes greater than or equal to 5 have been combined into a single “5 or more” category.

- d Step 1:** Let p_0, p_1, p_2, p_3, p_4 , and p_5 be the population proportions of people who correctly guess 0, 1, 2, 3, 4, and 5 or more answers respectively.

The hypotheses to be considered are:

$$H_0: p_0 = P(X = 0), p_1 = P(X = 1), p_2 = P(X = 2), p_3 = P(X = 3), \\ p_4 = P(X = 4), p_5 = P(X \geq 5)$$

$$H_1: \text{at least one of } p_0 \neq P(X = 0), p_1 \neq P(X = 1), \dots, p_4 \neq P(X = 4), \\ \text{or } p_5 \neq P(X \geq 5).$$

Step 2: The significance level is $\alpha = 0.05$.

Step 3: Using the expected frequencies calculated in **b**:

Number of correct answers	f_{obs}	f_{exp}	$\frac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
0	6	≈ 8.45	≈ 0.7089
1	13	≈ 28.16	≈ 8.1589
2	33	≈ 42.24	≈ 2.0194
3	45	≈ 37.54	≈ 1.4814
4	30	≈ 21.90	≈ 2.9962
5 or more	23	≈ 11.72	≈ 10.8593
Total			≈ 26.2241

So, $\chi^2_{\text{calc}} \approx 26.2$

Step 4: $\text{df} = 6 - 1 = 5$

	List 1	List 2	List 3	List 4
SUB				
1	6	8.45		
2	13	28.16		
3	33	42.24		
4	45	37.54		

χ^2 GOF Test
Observed: List1
Expected: List2
df: 5
CNTRB: List3
Save Res: None
GphColor: Blue

χ^2 GOF Test
$\chi^2 = 26.227896$
$p = 8.0597 \times 10^{-5}$
df = 5
CNTRB: List3

Using technology, the p -value ≈ 0.0000806 .

Step 5: Since $p\text{-value} < 0.05 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 on a 5% significance level. We therefore accept H_1 .

Step 6: Since we have accepted H_1 , we conclude on a 5% significance level that the model in which every person randomly guesses every answer is not appropriate.

- 3 a** The model is not likely to be realistic. While it is possible that some people randomly guessed some answers, it is unlikely that every person guessed *every* answer, which is what our model assumes. It may be that the probability of answering correctly is *not* the same for each question. For example, there may be an option which is obviously wrong, and so can be rejected.
- b** From **2 b i**, we have assumed that the probability of answering each question correctly does not change based on the results from previous questions.
- 4 a** If each person can reject exactly one out of the four answers, and will then guess one answer randomly from the remaining 3, then the probability of answering a question correctly becomes $\frac{1}{3}$.

Using the same reasoning as in **1**, we conclude that $Y \sim B(10, \frac{1}{3})$.

The probability distribution of Y is:

y	0	1	2	3	4	5 or more
$P(Y = y)$	≈ 0.01734	≈ 0.08671	≈ 0.19509	≈ 0.26012	≈ 0.22761	≈ 0.21313

- b** *Step 1:* Let p_0, p_1, p_2, p_3, p_4 , and p_5 be the population proportions of people who correctly guess 0, 1, 2, 3, 4, and 5 or more answers respectively.

The hypotheses to be considered are:

$$H_0: p_0 = P(Y = 0), p_1 = P(Y = 1), p_2 = P(Y = 2), p_3 = P(Y = 3), \\ p_4 = P(Y = 4), p_5 = P(Y \geq 5)$$

$$H_1: \text{at least one of } p_0 \neq P(Y = 0), p_1 \neq P(Y = 1), \dots, p_4 \neq P(Y = 4), \\ \text{or } p_5 \neq P(Y \geq 5).$$

Step 2: The significance level is $\alpha = 0.05$.

Step 3: Using the probability distribution from **a**, and $f_{\text{exp}} = 150 \times P(Y = y)$:

Number of correct answers	f_{obs}	f_{exp}	$\frac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
0	6	≈ 2.60	≈ 4.4408
1	13	≈ 13.01	≈ 0.0000
2	33	≈ 29.26	≈ 0.4770
3	45	≈ 39.02	≈ 0.9170
4	30	≈ 34.14	≈ 0.5023
5 or more	23	≈ 31.97	≈ 2.5164
Total			≈ 8.8535

$$\text{So, } \chi_{\text{calc}}^2 \approx 8.85$$

Step 4: $df = 6 - 1 = 5$

	List 1	List 2	List 3	List 4
SUB				
1	6	2.6		
2	13	13.01		
3	33	29.26		
4	45	39.02		

χ^2 GOF Test
Observed: List1
Expected: List2
df: 5
CNTRB: List3
Save Res: None
GphColor: Blue

χ^2 GOF Test
$\chi^2 = 8.85947125$
$p = 0.11480237$
df = 5
CNTRB: List3

Using technology, the p -value ≈ 0.115 .

Step 5: Since $p\text{-value} > 0.05 = \alpha$, we do not have enough evidence to reject H_0 in favour of H_1 on a 5% significance level. We therefore accept H_0 .

Step 6: Since we have accepted H_0 , we conclude on a 5% significance level that the model in which every person rejects one incorrect answer and then randomly guesses from the remaining three answers for each question is appropriate.

- 5** The goodness of fit test could also be used to assess continuous distributions, such as the normal distribution, by considering the class intervals that the data is organised into as the categories.

EXERCISE 16D.2

- 1 a** There are 5 categories so $df = 5 - 1 = 4$, and the significance level is $\alpha = 0.05$.
So, from the table, $\chi_{\text{crit}}^2 = 9.49$.
- b** $\chi_{\text{calc}}^2 = 10.3 > \chi_{\text{crit}}^2$, so we reject H_0 on a 5% significance level.

- 2 a** Let p_1, p_2, p_3 , and p_4 be the population proportions of the lollies made by Chewy Chews, corresponding to the colours red, yellow, green, and blue respectively.

Chewy Chews claims to make the same proportions of each colour, so the hypotheses that should be tested are:

$$H_0: p_1 = \frac{1}{4}, p_2 = \frac{1}{4}, p_3 = \frac{1}{4}, p_4 = \frac{1}{4}$$

$$H_1: \text{at least one of } p_1, p_2, p_3, p_4 \neq \frac{1}{4}.$$

b

Colour	f_{obs}	f_{exp}	$\frac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
red	12	$65 \times 0.25 = 16.25$	≈ 1.112
yellow	17	16.25	≈ 0.035
green	20	16.25	≈ 0.865
blue	16	16.25	≈ 0.004
Total			≈ 2.016

So, $\chi^2_{\text{calc}} \approx 2.02$

- c** There are 4 categories, so $df = 4 - 1 = 3$, and the significance level $\alpha = 0.1$.
So, from the table, $\chi^2_{\text{crit}} = 6.25$.
- d** $\chi^2_{\text{calc}} < \chi^2_{\text{crit}}$, so H_0 should not be rejected at a 10% significance level. Lucy therefore does not have enough evidence to reject the manufacturer's claim.

e

The first screenshot shows the data entry screen for the chi-square GOF test. The 'SUB' menu is open, and the following values are entered: List 1: 12, List 2: 16.25, List 3: 16.25, List 4: 16.25. The second screenshot shows the 'χ² GOF Test' screen with the following settings: Observed: List1, Expected: List2, df: 3, CNTRB: List3, Save Res: None, GphColor: Blue. The third screenshot shows the results of the test: χ² = 2.01538462, p = 0.56921971, df = 3, CNTRB: List3.

Using technology, the p -value ≈ 0.569

$p\text{-value} > 0.1 = \alpha$, so we again conclude that H_0 should not be rejected at a 10% significance level.

- 3 Step 1:** Let p_1, p_2, p_3, p_4 , and p_5 be the proportions of responses from the initial survey, corresponding to “very satisfied”, “satisfied”, “neutral”, “dissatisfied”, and “very satisfied” respectively.

The hypotheses that should be tested are:

$$H_0: p_1 = 0.05, p_2 = 0.25, p_3 = 0.41, p_4 = 0.2, p_5 = 0.09$$

$$H_1: \text{at least one of } p_1 \neq 0.05, p_2 \neq 0.25, \dots, \text{ or } p_5 \neq 0.09.$$

Step 2: The significance level is $\alpha = 0.01$.

Step 3:

<i>Response</i>	f_{obs}	f_{exp}	$\frac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
very satisfied	25	$233 \times 0.05 = 11.65$	≈ 15.298
satisfied	78	$233 \times 0.25 = 58.25$	≈ 6.696
neutral	77	$233 \times 0.41 = 95.53$	≈ 3.594
dissatisfied	36	$233 \times 0.2 = 46.6$	≈ 2.411
very dissatisfied	17	$233 \times 0.09 = 20.97$	≈ 0.752
		<i>Total</i>	≈ 28.751

So, $\chi_{\text{calc}}^2 \approx 28.8$ Step 4: $df = 5 - 1 = 4$, so from the table, $\chi_{\text{crit}}^2 = 13.28$.Step 5: $\chi_{\text{calc}}^2 > \chi_{\text{crit}}^2$, so we have sufficient evidence to reject H_0 in favour of H_1 at the 1% significance level. We therefore accept H_1 .Step 6: Since we have accepted H_1 , we conclude at the 1% significance level that the ISP's changes had a significant impact. The expected numbers of people who were either "dissatisfied" or "very dissatisfied" based on the initial survey were both higher than the results observed in the latter survey, so we conclude that the changes were effective.**ACTIVITY 2****MENDEL'S DATA****1**

<i>Type of pea</i>	<i>Proportion</i>	<i>Expected frequency</i>
Yellow round seeds	$\frac{9}{9+3+3+1} = \frac{9}{16}$	$556 \times \frac{9}{16} = 312.75$
Green round seeds	$\frac{3}{9+3+3+1} = \frac{3}{16}$	$556 \times \frac{3}{16} = 104.25$
Yellow wrinkled seeds	$\frac{3}{9+3+3+1} = \frac{3}{16}$	$556 \times \frac{3}{16} = 104.25$
Green wrinkled seeds	$\frac{1}{9+3+3+1} = \frac{1}{16}$	$556 \times \frac{1}{16} = 34.75$

2 Let p_1, p_2, p_3 , and p_4 be the population proportions of yellow round, green round, yellow wrinkled, and green wrinkled seeds respectively.

The hypotheses to be tested are:

$$H_0: p_1 = \frac{9}{16}, p_2 = \frac{3}{16}, p_3 = \frac{3}{16}, p_4 = \frac{1}{16} \quad \{\text{Mendel's model is true}\}$$

$$H_1: \text{at least one of } p_1 \neq \frac{9}{16}, p_2 \neq \frac{3}{16}, p_3 \neq \frac{3}{16}, \text{ or } p_4 \neq \frac{1}{16}. \quad \{\text{Mendel's model is false}\}$$

Type of pea	f_{obs}	f_{exp}	$\frac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
Yellow round seeds	315	312.75	≈ 0.0162
Green round seeds	108	104.25	≈ 0.1349
Yellow wrinkled seeds	101	104.25	≈ 0.1013
Green wrinkled seeds	32	34.75	≈ 0.2176
Total			≈ 0.4700

	List 1	List 2	List 3	List 4
SUB				
1	315	312.75		
2	108	104.25		
3	101	104.25		
4	32	34.75		

χ^2 GOF Test
Observed: List1
Expected: List2
df: 3
CNTRB: List3
Save Res: None
GphColor: Blue
LIST

χ^2 GOF Test
$\chi^2 = 0.47002398$
p = 0.92542589
df = 3
CNTRB: List3

So, $\chi^2_{\text{calc}} \approx 0.470$, and using technology, the p -value ≈ 0.925 .

- 3 For common significance levels ($\alpha = 0.1, 0.05$, or 0.01 for example), we would retain H_0 since the p -value is quite large.

So, we would conclude that Mendel's model is true.

EXERCISE 16E.1

Note: Expected frequency tables can be found either by hand, or using the χ^2 test functionality on your graphics calculator.

1 a

	Drove to work	Cycled to work	Public transport	Sum
Male	$\frac{44 \times 46}{80} = 25.3$	$\frac{44 \times 14}{80} = 7.7$	$\frac{44 \times 20}{80} = 11$	44
Female	$\frac{36 \times 46}{80} = 20.7$	$\frac{36 \times 14}{80} = 6.3$	$\frac{36 \times 20}{80} = 9$	36
Sum	46	14	20	80

b

	Junior school	Middle school	High school	Sum
Plays sport	$\frac{165 \times 58}{250} = 38.28$	$\frac{165 \times 86}{250} = 56.76$	$\frac{165 \times 106}{250} = 69.96$	165
Does not play sport	$\frac{85 \times 58}{250} = 19.72$	$\frac{85 \times 86}{250} = 29.24$	$\frac{85 \times 106}{250} = 36.04$	85
Sum	58	86	106	250

c The 2×3 contingency table is:

	<i>Wore hat and sunscreen</i>	<i>Wore hat or sunscreen</i>	<i>Wore neither</i>	<i>Sum</i>
<i>Sunburnt</i>	3	5	13	$3 + 5 + 13 = 21$
<i>Not sunburnt</i>	36	17	1	$36 + 17 + 1 = 54$
<i>Sum</i>	$3 + 36 = 39$	$5 + 17 = 22$	$13 + 1 = 14$	$21 + 54 = 75$

The expected frequency table is:

	<i>Wore hat and sunscreen</i>	<i>Wore hat or sunscreen</i>	<i>Wore neither</i>
<i>Sunburnt</i>	$\frac{21 \times 39}{75} = 10.92$	$\frac{21 \times 22}{75} = 6.16$	$\frac{21 \times 14}{75} = 3.92$
<i>Not sunburnt</i>	$\frac{54 \times 39}{75} = 28.08$	$\frac{54 \times 22}{75} = 15.84$	$\frac{54 \times 14}{75} = 10.08$

2 a

	<i>Pass Maths test</i>	<i>Fail Maths test</i>	<i>Sum</i>
<i>Male</i>	$\frac{50 \times 60}{100} = 30$	$\frac{50 \times 40}{100} = 20$	50
<i>Female</i>	$\frac{50 \times 60}{100} = 30$	$\frac{50 \times 40}{100} = 20$	50
<i>Sum</i>	60	40	100

b In a sample of 100 students, we would expect 30 to be male and pass the Maths test.

c

f_{obs}	f_{exp}	$f_{\text{obs}} - f_{\text{exp}}$	$(f_{\text{obs}} - f_{\text{exp}})^2$	$\frac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
24	30	$24 - 30 = -6$	$(-6)^2 = 36$	$\frac{36}{30} = 1.2$
26	20	$26 - 20 = 6$	$6^2 = 36$	$\frac{36}{20} = 1.8$
36	30	$36 - 30 = 6$	$6^2 = 36$	$\frac{36}{30} = 1.2$
14	20	$14 - 20 = -6$	$(-6)^2 = 36$	$\frac{36}{20} = 1.8$
<i>Total</i>				6

So, $\chi^2_{\text{calc}} = 6$.

3 Step 1: H_0 : weight and diabetes are independent

H_1 : weight and diabetes are dependent.

Step 2: The significance level is $\alpha = 0.05$.

Step 3: $df = (2 - 1)(3 - 1) = 2$

Step 4: The 2×3 contingency table is:

	<i>Weight</i>		
	<i>light</i>	<i>medium</i>	<i>heavy</i>
<i>Diabetic</i>	11	19	26
<i>Non-diabetic</i>	79	68	69

<p> Deg Norm1 d/c Real A 1 2 3 1 [11 19 26] 2 [79 68 69] 69 ROW-OP ROW COLUMN EDIT </p>	<p> Deg Norm1 d/c Real χ^2 Test Observed: Mat A Expected: Mat B Save Res: None GphColor: Blue Execute Mat ▶MAT </p>	<p> Deg Norm1 d/c Real χ^2 Test $\chi^2 = 6.60722773$ $p = 0.03675011$ $df = 2$ ▶MAT </p>
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Using technology, $\chi^2_{\text{calc}} \approx 6.61$.

Step 5: From the screenshots above, the p -value ≈ 0.0368 .

Step 6: Since the p -value $< 0.05 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 at the 5% significance level. We therefore accept H_1 .

Step 7: We conclude that *weight* and *diabetes* are dependent.

4 a $df = (2 - 1)(3 - 1) = 2$, and the significance level $\alpha = 0.1$, so from the table, $\chi^2_{\text{crit}} = 4.61$.

b Step 1: H_0 : age of a voter and party they wish to vote for are independent
 H_1 : age of a voter and party they wish to vote for are dependent.

Steps 2 and 3: From a, $\alpha = 0.1$ and $df = 2$.

Step 4: The 2×3 contingency table is:

	Age of voter		
	18 to 35	36 to 59	60+
Party A	85	95	131
Party B	168	197	173

<p> Deg Norm1 d/c Real A 1 2 3 1 [85 95 131] 2 [168 197 173] 173 ROW-OP ROW COLUMN EDIT </p>	<p> Deg Norm1 d/c Real χ^2 Test Observed: Mat A Expected: Mat B Save Res: None GphColor: Blue Execute Mat ▶MAT </p>	<p> Deg Norm1 d/c Real χ^2 Test $\chi^2 = 8.58175739$ $p = 0.01369288$ $df = 2$ ▶MAT </p>
--	--	---

Using technology, $\chi^2_{\text{calc}} \approx 8.58$.

Step 5: From a, $\chi^2_{\text{crit}} = 4.61$, so $\chi^2_{\text{calc}} > \chi^2_{\text{crit}}$.

Step 6: Since $\chi^2_{\text{calc}} > \chi^2_{\text{crit}}$, we have sufficient evidence to reject H_0 in favour of H_1 at a 10% significance level. We therefore accept H_1 .

Step 7: Since we have accepted H_1 , we conclude at a 10% significance level that *age of voter* and *party they wish to vote for* are dependent.

5 a Step 1: H_0 : reason for travelling and rating are independent
 H_1 : reason for travelling and rating are dependent.

Step 2: The significance level is $\alpha = 0.05$.

Step 3: $df = (2 - 1)(4 - 1) = 3$

Step 4: The 2×4 contingency table is:

		Rating			
		Poor	Fair	Good	Excellent
Reason for travelling	Business	27	25	20	8
	Holiday	9	17	24	30

TI-84 Plus calculator screen showing the input of the 2x4 contingency table into matrix A. The matrix is displayed as follows:

A	1	2	3	4
1	27	25	20	8
2	9	17	24	30

The total for the second row is 30.

TI-84 Plus calculator screen showing the chi-square test settings. The settings are:

- χ^2 Test
- Observed: Mat A
- Expected: Mat B
- Save Res: None
- GphColor: Blue
- Execute

TI-84 Plus calculator screen showing the results of the chi-square test. The results are:

- $\chi^2 = 23.624288$
- $p = 2.9923E-05$
- df = 3

Using technology, $\chi^2_{\text{calc}} \approx 23.6$.

Step 5: From the screenshots above, the p -value ≈ 0.0000299 .

Step 6: Since the p -value $< 0.05 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 at a 5% significance level. We therefore accept H_1 .

Step 7: Since we have accepted H_1 , we conclude that, at a 5% significance level, *reason for travelling* and *rating* are dependent.

- b From the contingency table, it appears that guests travelling for a holiday are more likely to give a higher rating than those travelling for business.

- 6 Step 1: H_0 : position and injury type are independent
 H_1 : position and injury type are dependent.

Step 2: The significance level is $\alpha = 0.1$.

Step 3: $df = (3 - 1)(4 - 1) = 6$

Step 4: The 3×4 contingency table is:

		Position			
		Forward	Midfielder	Defender	Goalkeeper
Injury type	No injury	23	18	24	7
	Mild injury	14	34	23	11
	Serious injury	10	16	13	7

TI-84 Plus calculator screen showing the input of the 3x4 contingency table into matrix A. The matrix is displayed as follows:

A	1	2	3	4
1	23	18	24	7
2	14	34	23	11
3	10	16	13	7

The total for the third row is 7.

TI-84 Plus calculator screen showing the chi-square test settings. The settings are:

- χ^2 Test
- Observed: Mat A
- Expected: Mat B
- Save Res: None
- GphColor: Blue
- Execute

TI-84 Plus calculator screen showing the results of the chi-square test. The results are:

- $\chi^2 = 7.9418852$
- $p = 0.24239194$
- df = 6

Using technology, $\chi^2_{\text{calc}} \approx 7.94$.

Step 5: From the screenshots above, the p -value ≈ 0.242 .

Step 6: Since p -value $> 0.1 = \alpha$, we do not have enough evidence to reject H_0 in favour of H_1 at a 10% significance level. We therefore accept H_0 .

Step 7: Since we have accepted H_0 , we conclude at a 10% significance level that *position* and *injury type* are independent.

7 a The 4×2 contingency table is:

		Own a pet?		
		Yes	No	Sum
Age	0 - 19	5	3	8
	20 - 29	32	22	54
	30 - 49	42	58	100
	50+	39	34	73
	Sum	118	117	235

The expected frequency table is:

		Own a pet?	
		Yes	No
Age	0 - 19	$\frac{8 \times 118}{235} \approx 4.02$	$\frac{8 \times 117}{235} \approx 3.98$
	20 - 29	$\frac{54 \times 118}{235} \approx 27.1$	$\frac{54 \times 117}{235} \approx 26.9$
	30 - 49	$\frac{100 \times 118}{235} \approx 50.2$	$\frac{100 \times 117}{235} \approx 49.8$
	50+	$\frac{73 \times 118}{235} \approx 36.7$	$\frac{73 \times 117}{235} \approx 36.3$

b Yes, both the expected frequencies for ages 0 - 19 are less than 5 (4.02 and 3.98).

c We combine ages 0 - 19 with ages 20 - 29 to give:

The new 3×2 contingency table:

		Own a pet?		
		Yes	No	Sum
Age	0 - 29	$5 + 32 = 37$	$3 + 22 = 25$	$37 + 25 = 62$
	30 - 49	42	58	100
	50+	39	34	73
	Sum	118	117	235

The new expected frequency table:

		Own a pet?	
		Yes	No
Age	0 - 29	$\frac{62 \times 118}{235} \approx 31.1$	$\frac{62 \times 117}{235} \approx 30.9$
	30 - 49	50.2	49.8
	50+	36.7	36.3

- d** Step 1: H_0 : age and owning a pet are independent
 H_1 : age and owning a pet are not independent.
- Step 2: The significance level is $\alpha = 0.05$.
- Step 3: $df = (3 - 1)(2 - 1) = 2$
- Step 4:

Using technology and the tables from **c**, $\chi^2_{\text{calc}} \approx 5.22$.

- Step 5: From the screenshots above, the p -value ≈ 0.0735 .
- Step 6: Since the p -value $> 0.05 = \alpha$, we do not have sufficient evidence to reject H_0 in favour of H_1 at a 5% significance level. We therefore accept H_0 .
- Step 7: Since we have accepted H_0 , we conclude that there is not a link between *age* and *owning a pet*.

- 8 a** The 3×4 contingency table is:

		Intelligence level				
		Low	Average	High	Very high	Sum
Smoking habits	Non smoker	279	386	96	2	763
	Medium level smoker	123	201	58	5	387
	Heavy smoker	100	147	64	2	313
	Sum	502	734	218	9	1463

The expected frequency table is:

		Intelligence level			
		Low	Average	High	Very high
Smoking habits	Non smoker	$\frac{763 \times 502}{1463}$ ≈ 262	$\frac{763 \times 734}{1463}$ ≈ 383	$\frac{763 \times 218}{1463}$ ≈ 114	$\frac{763 \times 9}{1463}$ ≈ 4.69
	Medium level smoker	$\frac{387 \times 502}{1463}$ ≈ 133	$\frac{387 \times 734}{1463}$ ≈ 194	$\frac{387 \times 218}{1463}$ ≈ 57.7	$\frac{387 \times 9}{1463}$ ≈ 2.38
	Heavy smoker	$\frac{313 \times 502}{1463}$ ≈ 107	$\frac{313 \times 734}{1463}$ ≈ 157	$\frac{313 \times 218}{1463}$ ≈ 46.6	$\frac{313 \times 9}{1463}$ ≈ 1.93

- b** Step 1: H_0 : *smoking habits and intelligence level are independent*
 H_1 : *smoking habits and intelligence level are not independent.*

Step 2: The significance level is $\alpha = 0.01$.

Step 3: $df = (3 - 1)(4 - 1) = 6$

Step 4:

Using technology and the tables from **a**, $\chi^2_{\text{calc}} \approx 16.9$.

Step 5: From the screenshots above, the p -value ≈ 0.00959 .

Step 6: Since the p -value $< 0.01 = \alpha$, we have sufficient evidence to reject H_0 in favour of H_1 at a 1% significance level. We therefore accept H_1 .

Step 7: Since we have accepted H_1 , we conclude at a 1% significance level that there is a link between *smoking habits and intelligence level*.

- c** Each expected frequency in the *very high intelligence level* is less than 5, so we should combine it with the *high intelligence level*, which gives:

The new 3×3 contingency table:

		Intelligence level			
		Low	Average	High/Very high	Sum
Smoking habits	Non smoker	279	386	$96 + 2 = 98$	763
	Medium level smoker	123	201	$58 + 5 = 63$	387
	Heavy smoker	100	147	$64 + 2 = 66$	313
	Sum	502	734	$98 + 63 + 66 = 227$	1463

The new expected frequency table:

		Intelligence level		
		Low	Average	High/Very high
Smoking habits	Non smoker	261.8	382.8	$\frac{763 \times 227}{1463} \approx 118.4$
	Medium level smoker	132.8	194.2	$\frac{387 \times 227}{1463} \approx 60.0$
	Heavy smoker	107.4	157.0	$\frac{313 \times 227}{1463} \approx 48.6$

- d** Step 1: H_0 : smoking habits and intelligence level are independent
 H_1 : smoking habits and intelligence level are not independent.

Step 2: The significance level is $\alpha = 0.01$.

Step 3: $df = (3 - 1)(3 - 1) = 4$

Step 4:

--	--	--

Using technology and the tables from **c**, $\chi^2_{\text{calc}} \approx 13.2$.

Step 5: From the screenshots above, the p -value ≈ 0.0104 .

Step 6: Since the p -value $> 0.01 = \alpha$, we do not have sufficient evidence to reject H_0 in favour of H_1 at a 1% significance level. We therefore accept H_0 .

Step 7: Since we have accepted H_0 , we conclude that there is not a significant link between *smoking habits* and *intelligence level*, which is different from the conclusion we arrived at in **b**.

EXERCISE 16E.2

- 1 a** The 2×2 contingency table is:

		Result		
		Heads	Tails	Sum
Guess	Heads	54	50	$54 + 50 = 104$
	Tails	41	55	$41 + 55 = 96$
	Sum	$54 + 41 = 95$	$50 + 55 = 105$	$104 + 96 = 200$

The expected frequency table is:

		Result	
		Heads	Tails
Guess	Heads	$\frac{104 \times 95}{200} = 49.4$	$\frac{104 \times 105}{200} = 54.6$
	Tails	$\frac{96 \times 95}{200} = 45.6$	$\frac{96 \times 105}{200} = 50.4$

b We find χ^2_{calc} using Yates' continuity correction:

f_{obs}	f_{exp}	$f_{\text{obs}} - f_{\text{exp}}$	$ f_{\text{obs}} - f_{\text{exp}} - 0.5$	$(f_{\text{obs}} - f_{\text{exp}} - 0.5)^2$	$\frac{(f_{\text{obs}} - f_{\text{exp}} - 0.5)^2}{f_{\text{exp}}}$
54	49.4	4.6	4.1	16.81	≈ 0.3403
50	54.6	-4.6	4.1	16.81	≈ 0.3079
41	45.6	-4.6	4.1	16.81	≈ 0.3686
55	50.4	4.6	4.1	16.81	≈ 0.3335
<i>Total</i>					≈ 1.3503

So, $\chi^2_{\text{calc}} \approx 1.35$

c H_0 : Horace's *guess* and the *result* of the toss are independent

H_1 : Horace's *guess* and the *result* of the toss are not independent.

Since $\alpha = 0.05$, $\chi^2_{\text{crit}} = 3.84$, and $\chi^2_{\text{calc}} \approx 1.35$, then $\chi^2_{\text{calc}} < \chi^2_{\text{crit}}$. This means that we do not have enough evidence to reject H_0 in favour of H_1 at a 5% significance level. We conclude that Horace's *guess* and the *result* of the toss are independent.

d According to the test performed in **c**, we conclude that Horace's claim is not valid.

2 a The 2×2 contingency table is:

		<i>Result</i>		
		<i>Pass</i>	<i>Fail</i>	<i>Sum</i>
<i>Country</i>	<i>France</i>	56	29	$56 + 29 = 85$
	<i>Germany</i>	176	48	$176 + 48 = 224$
	<i>Sum</i>	$56 + 176 = 232$	$29 + 48 = 77$	$85 + 224 = 309$

The expected frequency table is:

		<i>Result</i>	
		<i>Pass</i>	<i>Fail</i>
<i>Country</i>	<i>France</i>	$\frac{85 \times 232}{309} \approx 63.8$	$\frac{85 \times 77}{309} \approx 21.2$
	<i>Germany</i>	$\frac{224 \times 232}{309} \approx 168.2$	$\frac{224 \times 77}{309} \approx 55.8$

b At a 10% significance level with $\text{df} = (2 - 1)(2 - 1) = 1$, $\chi^2_{\text{crit}} = 2.71$.

c

f_{obs}	f_{exp}	$f_{\text{obs}} - f_{\text{exp}}$	$ f_{\text{obs}} - f_{\text{exp}} - 0.5$	$(f_{\text{obs}} - f_{\text{exp}} - 0.5)^2$	$\frac{(f_{\text{obs}} - f_{\text{exp}} - 0.5)^2}{f_{\text{exp}}}$
56	≈ 63.8	≈ -7.8	≈ 7.3	≈ 53.29	≈ 0.835
29	≈ 21.2	≈ 7.8	≈ 7.3	≈ 53.29	≈ 2.514
176	≈ 168.2	≈ 7.8	≈ 7.3	≈ 53.29	≈ 0.317
48	≈ 55.8	≈ -7.8	≈ 7.3	≈ 53.29	≈ 0.955
<i>Total</i>					≈ 4.621

So, $\chi^2_{\text{calc}} \approx 4.62$.

- d** H_0 : the *country* where a motorbike test took place and the *result* are independent
 H_1 : the *country* where a motorbike test took place and the *result* are not independent.

From **b** and **c** we know that $\chi^2_{\text{calc}} > \chi^2_{\text{crit}}$, and so we have sufficient evidence to reject H_0 at the 10% level of significance. We therefore conclude that the *country* in which a motorbike test took place is not independent of the *result* at a 10% level of significance.

REVIEW SET 16A

- Let μ be the population mean number of minutes after 9:45 am that the bus arrives. The hypotheses that should be considered are:
 $H_0: \mu = 0$ {the buses arrive on time}
 $H_1: \mu > 0$ {the buses arrive late}
- There is a $\approx 7.94\%$ chance of observing this result if the null hypothesis is true.
 - For a 10% significance level, we reject H_0 if there is less than a 10% chance of observing this result.
 - As the p -value $< 0.1 = \alpha$, there is enough evidence to reject H_0 in favour of H_1 .
- We are asked to test a hypothesis about a single population mean, so we conduct Student's t -test for a population mean.

Step 1: Let μ be the population mean number of shaves. The hypotheses to be tested are:

$$H_0: \mu = 13 \quad \{\text{the manufacturer's blades last 13 shaves}\}$$

$$H_1: \mu \neq 13 \quad \{\text{the manufacturer's blades do not last 13 shaves}\}$$

Step 2: The significance level is $\alpha = 0.05$.

Step 3: $\bar{x} = 12.8$, $\mu_0 = 13$, $s = 1.6$, $n = 30$

$$\text{The value of the test statistic is } t = \frac{12.8 - 13}{\frac{1.6}{\sqrt{30}}} \approx -0.685.$$

Step 4: Since $H_1: \mu \neq 13$ and $n = 30$,

the p -value

$$= 2 \times P(T \geq |t|)$$

$$\text{where } T \sim t_{29}$$

$$\approx 2 \times P(T \geq 0.685)$$

$$\approx 0.499$$

	Deg Norm1 d/c Real
Student-t C.D	
Data	:Variable
Lower	:0.685
Upper	:1E+99
df	:29
Save Res	:None
GphColor	:Blue
None	LIST

	Deg Norm1 d/c Real
Student-t C.D	
p	=0.24939144
t:Low	=0.685
t:Up	=1E+99

Step 5: Since $p\text{-value} > 0.05 = \alpha$, we do not have enough evidence to reject H_0 in favour of H_1 at a 5% significance level. We therefore accept H_0 .

Step 6: Since we have accepted H_0 , we conclude that the population mean number of shaves is 13. The manufacturer's claim is valid.

- 4 We need to test a hypothesis about a single population mean, so we conduct Student's t -test for a population mean.

- a Let μ be the population mean of Rosario's apricots this year. The hypotheses to be tested are:
 $H_0: \mu = 90$ {the mean weight of the harvest is the same as last year}
 $H_1: \mu < 90$ {the mean weight of the harvest is less than it was last year}
- b The significance level is $\alpha = 0.01$.

	List 1	List 2	List 3	List 4
SUB				
1	88			
2	72			
3	93			
4	71			

	Deg	Norm1	d/c	Real
1-Sample tTest				
Data	:	List		
μ	:	< μ_0		
μ_0	:	90		
List	:	List1		
Freq	:	1		
Save Res	:	None		
		1		

	Deg	Norm1	d/c	Real
1-Sample tTest				
μ	:	<90		
t	:	=-3.5940756		
p	:	=9.6717E-04		
\bar{x}	:	=83.05		
s_x	:	=8.6479386		
n	:	=20		

Using technology, $t \approx -3.59$ and the p -value ≈ 0.000967 .

Since the p -value $< 0.01 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 at the 1% significance level.

We therefore accept at the 1% significance level that the mean weight of Rosario's harvest is less than it was last year, and so his concerns are justified.

- 5 a Let μ_1 and μ_2 be the population mean numbers of fish caught per trip by Joe and Ruben respectively. The hypotheses that Joe should test are:

$$H_0: \mu_1 = \mu_2 \quad \{\text{they catch the same number of fish}\}$$

$$H_1: \mu_1 > \mu_2 \quad \{\text{Joe catches more fish than Ruben}\}$$

- b The significance level is $\alpha = 0.05$.

	Deg	Norm1	d/c	Real
2-Sample tTest				
Data	:	Variable		
μ_1	:	> μ_2		
\bar{x}_1	:	10.9		
s_{x1}	:	3.34		
n_1	:	12		
\bar{x}_2	:	10.25		

	Deg	Norm1	d/c	Real
2-Sample tTest				
n_1	:	12		
\bar{x}_2	:	10.25		
s_{x2}	:	2.26		
n_2	:	12		
Pooled	:	On		
Save Res	:	None		
		None		

	Deg	Norm1	d/c	Real
2-Sample tTest				
μ_1	:	> μ_2		
t	:	=0.55834287		
p	:	=0.29112441		
df	:	=22		
\bar{x}_1	:	=10.9		
\bar{x}_2	:	=10.25		

Using technology, the value of the test statistic $t \approx 0.558$, and the p -value ≈ 0.291 .

Since the p -value $> 0.05 = \alpha$, we do not have enough evidence to reject H_0 in favour of H_1 at the 5% significance level.

We therefore conclude that the two friends catch the same number of fish on average. So Joe's claim is not justified.

- 6 Step 1: Let μ_A and μ_B be the population mean amount of time spent shopping in supermarkets A and B respectively. The hypotheses to be considered are:

$$H_0: \mu_A = \mu_B \quad \{\text{customers spend the same amount of time at each supermarket}\}$$

$$H_1: \mu_A \neq \mu_B \quad \{\text{customers spend different amounts of time at each supermarket}\}$$

Step 2: The significance level is $\alpha = 0.1$.

Step 3:

Sub	List 1	List 2	List 3	List 4
1	12	14		
2	28	35		
3	13	32		
4	7	21		

2-Sample tTest
Data : List
μ_1 : $\neq \mu_2$
List(1) : List1
List(2) : List2
Freq(1) : 1
Freq(2) : 1

2-Sample tTest
μ_1 : $\neq \mu_2$
t = -1.8040534
p = 0.08630669
df = 20
\bar{x}_1 = 13.5
\bar{x}_2 = 21.1666667

Using technology, the value of the test statistic is $t \approx -1.80$.

Step 4: From the screenshots above, the p -value ≈ 0.0863 .

Step 5: Since the p -value $< 0.1 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 at the 10% significance level. We therefore accept H_1 .

Step 6: Since we have accepted H_1 at the 10% significance level, we therefore conclude that there is a significant difference between the mean amount of time spent at supermarket A and supermarket B.

- 7 a i Let p_1, p_2, p_3, p_4 , and p_5 be the population proportions of sizes Small, Medium, Large, X-Large, and XX-Large respectively.

The null hypothesis is therefore:

$$H_0: p_1 = 0.1, p_2 = 0.2, p_3 = 0.35, p_4 = 0.25, p_5 = 0.1$$

- ii There are 5 categories, so $df = 5 - 1 = 4$.

b

Size	f_{obs}	f_{exp}
Small	4	$70 \times 0.1 = 7$
Medium	7	$70 \times 0.2 = 14$
Large	22	$70 \times 0.35 = 24.5$
X-Large	24	$70 \times 0.25 = 17.5$
XX-Large	13	$70 \times 0.1 = 7$

Sub	List 1	List 2	List 3	List 4
1	4	7		
2	7	14		
3	22	24.5		
4	24	17.5		

χ^2 GOF Test
Observed: List1
Expected: List2
df : 4
CNTRB : List3
Save Res: None
GphColor: Blue

χ^2 GOF Test
$\chi^2 = 12.5979592$
p = 0.01341683
df = 4
CNTRB: List3

Using technology, the p -value ≈ 0.0134 .

- c Using significance level $\alpha = 0.05$, the p -value $< \alpha$, and so we reject the null hypothesis at a 5% significance level. We therefore conclude that the proportions of shirts sold in the first week are significantly different from the initial proportions, and so the store should change the distribution of shirt sizes it stocks.

8 a

Item rarity	Expected frequency
super rare	$250 \times 0.05 = 12.5$
rare	$250 \times 0.1 = 25$
uncommon	$250 \times 0.25 = 62.5$
common	$250 \times 0.6 = 150$

b Step 1: Let p_1, p_2, p_3 , and p_4 be the population proportions of super rare, rare, uncommon, and common items respectively. The hypotheses to be tested are:

$$H_0: p_1 = 0.05, p_2 = 0.1, p_3 = 0.25, p_4 = 0.6$$

$$H_1: \text{at least one of } p_1 \neq 0.05, p_2 \neq 0.1, p_3 \neq 0.25, \text{ or } p_4 \neq 0.6.$$

Step 2: The significance level is $\alpha = 0.01$.

Step 3:

Item rarity	f_{obs}	f_{exp}	$\frac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
super rare	5	12.5	4.5
rare	17	25	2.56
uncommon	76	62.5	2.916
common	152	150	≈ 0.0267
Total			≈ 10.0027

$$\text{So, } \chi_{\text{calc}}^2 \approx 10.0.$$

Step 4: $\text{df} = 4 - 1 = 3$, and $\alpha = 0.01$, so, using the table on page 403 of the book, $\chi_{\text{crit}}^2 = 11.34$.

Step 5: Since $\chi_{\text{calc}}^2 < \chi_{\text{crit}}^2$, we do not have sufficient evidence to reject H_0 in favour of H_1 with a 1% level of significance. We therefore accept H_0 .

Step 6: Since we have accepted H_0 , we conclude that Emmanuel's suspicions are not justified at a 1% level of significance.

9 Step 1: H_0 : age of a driver and opinion are independent
 H_1 : age of a driver and opinion are not independent.

Step 2: The significance level is $\alpha = 0.1$.

Step 3: $\text{df} = (2 - 1)(3 - 1) = 2$

Step 4: The 2×3 contingency table is:

Age of driver			
	18 to 30	31 to 54	55+
Increase	234	169	134
No increase	156	191	233

--	--	--

Using technology, $\chi^2_{\text{calc}} \approx 42.1$.

Step 5: From the screenshots above, the p -value $\approx 7.37 \times 10^{-10}$.

Step 6: Since $p\text{-value} < 0.05 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 at a 10% significance level. We therefore accept H_1 .

Step 7: Since we have accepted H_1 , we conclude that there is a significant association between the *age of a driver* and their *opinion* on the speed limit.

REVIEW SET 16B

- 1** Let μ be the population mean minimum weight of Quickchick's chickens.

The hypotheses that should be considered are:

$$H_0: \mu = 1.2 \quad \{\text{the advertised weight is correct}\}$$

$$H_1: \mu < 1.2 \quad \{\text{the chickens are less than the advertised weight}\}$$

- 2 a** Using the table on page 403 of the book, if $df = 6$ and $\alpha = 0.05$, then $\chi^2_{\text{crit}} = 12.59$.
- b** $\chi^2_{\text{calc}} \approx 5.71 < 12.59 = \chi^2_{\text{crit}}$, so there is insufficient evidence to reject the null hypothesis at a 5% significance level.
- 3** We need to test a hypothesis about a single population mean, so we conduct Student's t -test for a population mean.

Step 1: Let μ be the population mean systolic blood pressure of the employees.

The hypotheses to be tested are:

$$H_0: \mu = 140 \quad \{\text{the employees' average blood pressure is not too high}\}$$

$$H_1: \mu > 140 \quad \{\text{the employee's average blood pressure is too high}\}$$

Step 2: The significance level is $\alpha = 0.05$.

Step 3: $\bar{x} = 143.7$ mm Hg, $\mu_0 = 140$ mm Hg, $s = 11.2$ mm Hg, $n = 35$ employees.

$$\text{The value of the test statistic is } t = \frac{143.7 - 140}{\frac{11.2}{\sqrt{35}}} \approx 1.95.$$

Step 4: Since $H_1: \mu > 140$ and $n = 35$,

the p -value

$$= P(T \geq t)$$

$$\text{where } T \sim t_{34}$$

$$\approx P(T \geq 1.95)$$

$$\approx 0.0297$$

Step 5: Since $p\text{-value} < 0.05 = \alpha$, we have sufficient evidence to reject H_0 in favour of H_1 at a 5% significance level. We therefore accept H_1 .

Step 6: Since we have accepted H_1 , we conclude that the company's concerns are justified at a 5% significance level.

- 4** We need to test hypotheses about a single population mean, so we conduct Student's t -test for a population mean.

Step 1: Let μ be the population mean distance that Arthur can hit a golf ball.

The hypotheses to be tested are:

$H_0: \mu = 115$ {the professional did not help Arthur's drive distance}

$H_1: \mu > 115$ {the professional improved Arthur's drive distance}

Step 2: The significance level is $\alpha = 0.05$.

Steps 3 and 4:

Sub	List 1	List 2	List 3	List 4
1	100			
2	126			
3	93			
4	171			

1-Sample tTest
Data : List
μ : $>\mu_0$
μ_0 : 115
List : List1
Freq : 1
Save Res: None
LIST

1-Sample tTest
μ : >115
t : 1.40194855
p : 0.08577201
\bar{x} : 122.566667
s_x : 29.5619552
n : 30

Using technology, $t \approx 1.40$ and the p -value ≈ 0.0858 .

Step 5: Since p -value $> 0.05 = \alpha$, we do not have sufficient evidence to reject H_0 in favour of H_1 at a 5% significance level. We therefore accept H_0 .

Step 6: Since we have accepted H_0 , we conclude that there is insufficient evidence at the 5% significance level to claim that Arthur has improved.

- 5** *Step 1:* Let μ_1 and μ_2 be the population mean points totals of people living in Maple Grove and Berkton respectively. The hypotheses to be tested are:

$H_0: \mu_1 = \mu_2$ {there is no difference in average points between the two suburbs}

$H_1: \mu_1 \neq \mu_2$ {there is a difference in average points between the two suburbs}

Step 2: The significance level is $\alpha = 0.1$.

Steps 3 and 4:

Sub	List 1	List 2	List 3	List 4
1	14	11		
2	11	10		
3	13	12		
4	13	14		

2-Sample tTest
Data : List
μ_1 : $\neq\mu_2$
List(1) : List1
List(2) : List2
Freq(1) : 1
Freq(2) : 1
LIST

2-Sample tTest
μ_1 : $\neq\mu_2$
t : 1.17912849
p : 0.24184195
df : 80
\bar{x}_1 : 12.275
\bar{x}_2 : 11.9047619

Using technology, $t \approx 1.18$ and the p -value ≈ 0.242 .

Step 5: Since p -value $> 0.1 = \alpha$, we do not have enough evidence to reject H_0 in favour of H_1 at the 10% significance level. We therefore accept H_0 .

Step 6: Since we have accepted H_0 , we conclude that there is not a significant difference between the points totals of the two suburbs.

6 a Revision course:

	List 1	List 2	List 3	List 4
SUB				
1	32			
2	39			
3	31			
4	35			

1-Variable	
\bar{x}	=34.6428571
Σx	=485
Σx^2	=16905
σx	=2.7152254
sx	=2.81772256
n	=14

Using technology, the sample mean is $\bar{x}_1 \approx 34.6$, and $s_1 \approx 2.82$.

No revision course:

	List 1	List 2	List 3	List 4
SUB				
1	28			
2	31			
3	30			
4	23			

1-Variable	
\bar{x}	=32.1
Σx	=321
Σx^2	=10485
σx	=4.25323406
sx	=4.48330235
n	=10

Using technology, the sample mean is $\bar{x}_2 \approx 32.1$, and $s_2 \approx 4.48$.

- b Step 1: $H_0: \mu_1 = \mu_2$ {the revision course had no effect}
 $H_1: \mu_1 > \mu_2$ {the revision course improved examination scores}

Step 2: The significance level is $\alpha = 0.1$.

Step 3:

2-Sample tTest	
Data : Variable	
μ_1 : $>\mu_2$	
\bar{x}_1 : 34.6	
sx_1 : 2.82	
n_1 : 14	
\bar{x}_2 : 32.1	

2-Sample tTest	
sx_1 : 2.82	
n_1 : 14	
\bar{x}_2 : 32.1	
sx_2 : 4.48	
n_2 : 10	
Pooled : On	

2-Sample tTest	
μ_1 : $>\mu_2$	
t : 1.68050288	
p : 0.05350242	
df : 22	
\bar{x}_1 : 34.6	
\bar{x}_2 : 32.1	

Using technology, the value of the test statistic is $t \approx 1.68$.

Step 4: From the screenshots above, the p -value ≈ 0.0535 .

Step 5: Since the p -value $< 0.1 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 at the 10% significance level. We therefore accept H_1 .

Step 6: Since we have accepted H_1 , we conclude that there is sufficient evidence at the 10% significance level to say that the revision course was effective.

- 7 Step 1: Let p_1, p_2, p_3 , and p_4 be the population proportions of glass, agate, alabaster, and onyx marbles respectively. The hypotheses to be tested are:

$$H_0: p_1 = \frac{4}{4+2+2+1} = \frac{4}{9}, p_2 = \frac{2}{9}, p_3 = \frac{2}{9}, p_4 = \frac{1}{9}$$

$$H_1: \text{at least one of } p_1 \neq \frac{4}{9}, p_2 \neq \frac{2}{9}, p_3 \neq \frac{2}{9}, \text{ or } p_4 \neq \frac{1}{9}.$$

Step 2: The significance level is $\alpha = 0.05$.

Step 3:

Type	f_{obs}	f_{exp}	$\frac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
glass	19	$50 \times \frac{4}{9} = \frac{200}{9}$	≈ 0.4672
agate	16	$50 \times \frac{2}{9} = \frac{100}{9}$	≈ 2.1511
alabaster	13	$50 \times \frac{2}{9} = \frac{100}{9}$	≈ 0.3211
onyx	2	$50 \times \frac{1}{9} = \frac{50}{9}$	≈ 2.2756
Total			≈ 5.215

So, $\chi^2_{\text{calc}} \approx 5.22$ Step 4: There are 4 categories, so $df = 4 - 1 = 3$

--	--	--

Using technology, the p -value ≈ 0.157 .Step 5: Since $p\text{-value} > 0.05 = \alpha$, we do not have sufficient evidence to reject H_0 in favour of H_1 at a 5% significance level. We therefore accept H_0 .Step 6: Since we have accepted H_0 , we conclude that there is not enough evidence to suggest the company's claim is invalid at the 5% significance level.

- 8 Step 1: H_0 : P and Q are independent.
 H_1 : P and Q are not independent.

Step 2: The significance level in **a** is $\alpha_1 = 0.05$.
The significance level in **b** is $\alpha_2 = 0.01$.Step 3: $df = (3 - 1)(4 - 1) = 6$ Step 4: The 3×4 contingency table is:

	Q_1	Q_2	Q_3	Q_4
P_1	19	23	27	39
P_2	11	20	27	35
P_3	26	39	21	30

--	--	--

Using technology, $\chi^2_{\text{calc}} \approx 13.0$.Step 5: From the screenshots above, the p -value ≈ 0.0433 .

- a Since the $p\text{-value} < 0.05 = \alpha_1$, we have sufficient evidence to reject H_0 in favour of H_1 at the 5% significance level. We therefore conclude that P and Q are not independent.
- b Since the $p\text{-value} > 0.01 = \alpha_2$, we do not have sufficient evidence to reject H_0 in favour of H_1 at the 1% significance level. We therefore conclude that P and Q are independent.
- 9 We want to test whether two categories are independent, so we conduct a χ^2 test for independence.

Step 1: H_0 : Business success and education level are independent.

H_1 : Business success and education level are dependent.

Step 2: The significance level is $\alpha = 0.01$.

Step 3: $df = (4 - 1)(4 - 1) = 9$

Step 4: The 4×4 contingency table is:

		Education level			
		High school	Graduate certificate	Undergraduate degree	Postgraduate
Business success	No success	35	30	41	25
	Low success	28	41	26	29
	Success	35	24	41	56
	High success	52	38	63	72

A	1	2	3	4
1	35	30	41	25
2	28	41	26	29
3	35	24	41	56
4	52	38	63	72

χ^2 Test

Observed: Mat A

Expected: Mat B

Save Res: None

GphColor: Blue

Execute

χ^2 Test

$\chi^2 = 25.5557891$

$p = 2.4141E-03$

$df = 9$

Using technology, $\chi^2_{\text{calc}} \approx 25.6$.

- Step 5: $\chi^2_{\text{calc}} \approx 25.6 > \chi^2_{\text{crit}} = 21.67$, so we have enough evidence to reject H_0 in favour of H_1 at a 1% significance level. We therefore accept H_1 .
- Step 6: Since we have accepted H_1 , we conclude at a 1% significance level that there is a link between *education level* and *business success*.

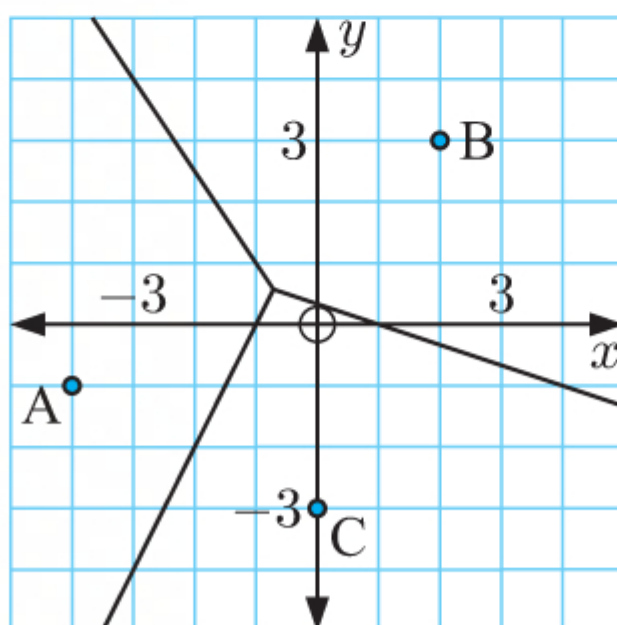
Chapter 17

VORONOI DIAGRAMS

EXERCISE 17A

1 a The diagram contains:

- i 3 cells
- ii 3 edges
- iii 1 vertex.



b i $(-1, 2)$ lies in cell B, so P is closest to site B.

$$\begin{aligned}
 \text{ii A has coordinates } (-4, -1), \text{ so } PA &= \sqrt{(-4 - (-1))^2 + (-1 - 2)^2} \\
 &= \sqrt{(-3)^2 + (-3)^2} \\
 &= \sqrt{18} = 3\sqrt{2} \approx 4.24 \text{ units}
 \end{aligned}$$

$$\begin{aligned}
 \text{B has coordinates } (2, 3), \text{ so } PB &= \sqrt{(2 - (-1))^2 + (3 - 2)^2} \\
 &= \sqrt{3^2 + 1^2} \\
 &= \sqrt{10} \approx 3.16 \text{ units}
 \end{aligned}$$

$$\begin{aligned}
 \text{C has coordinates } (0, -3), \text{ so } PC &= \sqrt{(0 - (-1))^2 + (-3 - 2)^2} \\
 &= \sqrt{1^2 + (-5)^2} \\
 &= \sqrt{26} \approx 5.10 \text{ units}
 \end{aligned}$$

So, $PB < PA$ and $PB < PC$ ✓

c i $(-1, 0)$ lies on the edge adjacent to cells A and C, so Q is equally closest to sites A and C.

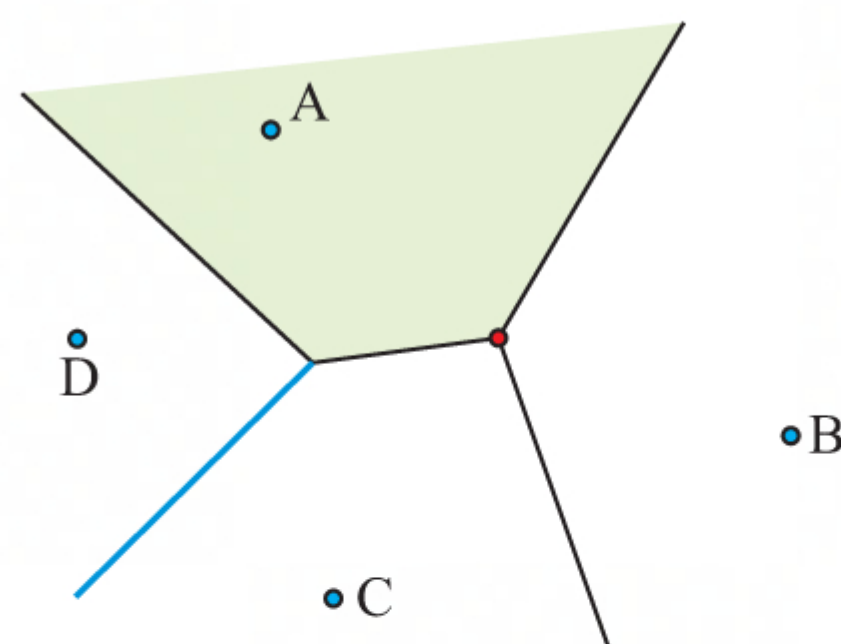
$$\begin{aligned}
 \text{ii } QA &= \sqrt{(-4 - (-1))^2 + (-1 - 0)^2} \\
 &= \sqrt{(-3)^2 + (-1)^2} \\
 &= \sqrt{10} \approx 3.16 \text{ units}
 \end{aligned}$$

$$\begin{aligned}
 QB &= \sqrt{(2 - (-1))^2 + (3 - 0)^2} \\
 &= \sqrt{3^2 + 3^2} \\
 &= \sqrt{18} = 3\sqrt{2} \approx 4.24 \text{ units}
 \end{aligned}$$

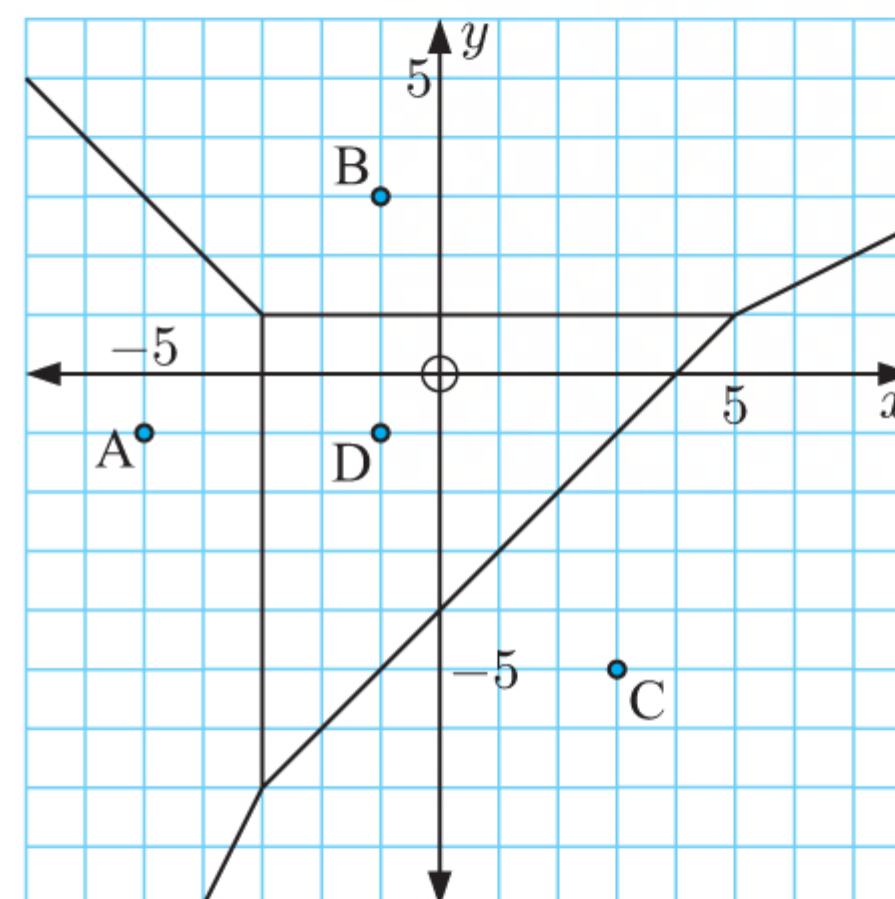
$$\begin{aligned}
 QC &= \sqrt{(0 - (-1))^2 + (-3 - 0)^2} \\
 &= \sqrt{1^2 + (-3)^2} \\
 &= \sqrt{10} \approx 3.16 \text{ units}
 \end{aligned}$$

So, $QA = QC$, $QA < QB$, and $QC < QB$ ✓

- 2**
- a** The green cell is cell A, so points which lie in this cell are closest to site A.
 - b** The blue edge is adjacent to cells C and D, so points which lie on this edge are equally closest to sites C and D.
 - c** The red vertex is where cells A, B, and C meet, so it is equally closest to sites A, B, and C.



- 3**
- a**
 - i** $(2, 3)$ lies in cell B, so it is closest to site B.
 - ii** $(-1, -4)$ lies in cell D, so it is closest to site D.
 - iii** $(6, 0)$ lies in cell C, so it is closest to site C.
 - iv** $(-4, -3)$ lies in cell A, so it is closest to site A.



- b** **i** Let $(-3, 0)$ be the point P.

$$\begin{aligned}
 \text{A has coordinates } (-5, -1), \text{ so } PA &= \sqrt{(-5 - (-3))^2 + (-1 - 0)^2} \\
 &= \sqrt{(-2)^2 + (-1)^2} \\
 &= \sqrt{5} \text{ units}
 \end{aligned}$$

$$\begin{aligned}
 \text{D has coordinates } (-1, -1), \text{ so } PD &= \sqrt{(-1 - (-3))^2 + (-1 - 0)^2} \\
 &= \sqrt{2^2 + (-1)^2} \\
 &= \sqrt{5} \text{ units} \\
 &= PA \quad \checkmark
 \end{aligned}$$

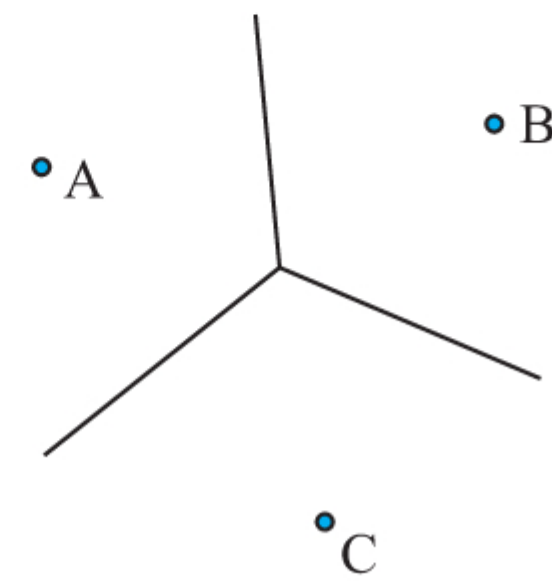
- ii** Let $(-3, 2)$ be the point Q.

$$\begin{aligned}
 QA &= \sqrt{(-5 - (-3))^2 + (-1 - 2)^2} \\
 &= \sqrt{(-2)^2 + (-3)^2} \\
 &= \sqrt{13} \text{ units}
 \end{aligned}$$

$$\begin{aligned}
 QD &= \sqrt{(-1 - (-3))^2 + (-1 - 2)^2} \\
 &= \sqrt{2^2 + (-3)^2} \\
 &= \sqrt{13} \text{ units} \\
 &= QA \quad \checkmark
 \end{aligned}$$

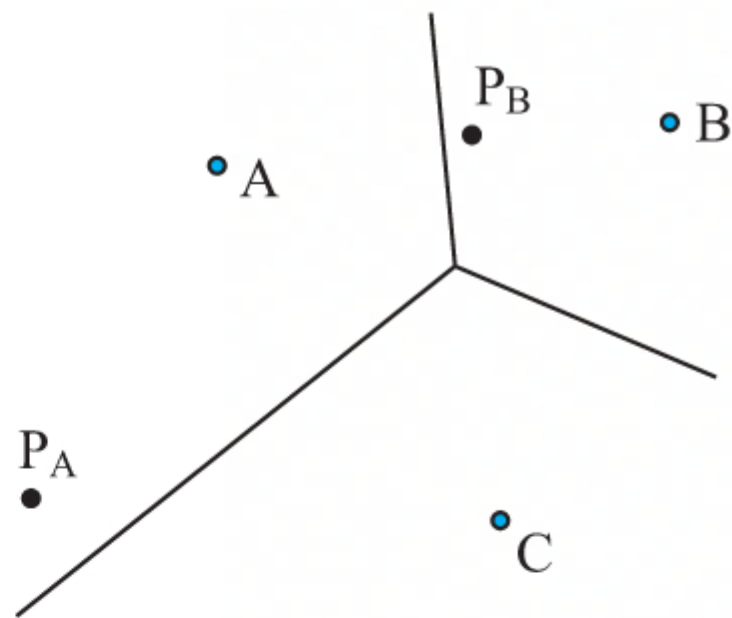
- c** $(-3, 0)$ is equally *closest* to sites A and D, and so lies on the edge adjacent to cells A and D. $(-3, 2)$ is equally *close* to sites A and D, but is *closest* to site B, and so does not lie on an edge.
- d** Cell D is a triangle with base 8 units and height 8 units.
 \therefore area of cell D $= \frac{1}{2} \times 8 \times 8$
 $= 32 \text{ units}^2$

- 4 a** The statement “ P_A is closer to A than to any other site” is true, as P_A lies in cell A.
- b** The statement “ P_B is closer to B than to C” is true, as P_B lies in cell B.



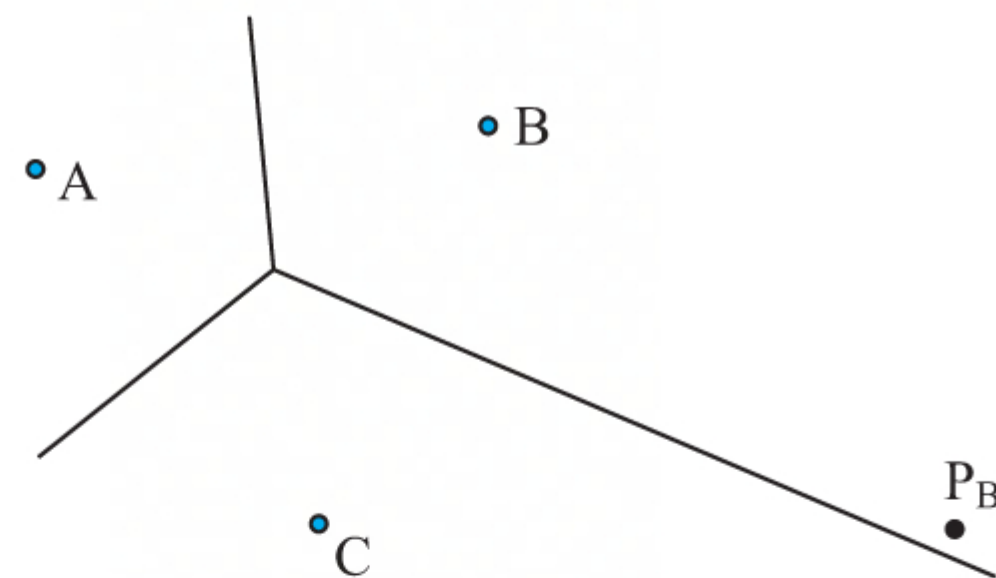
- c** The statement “A is closer to P_A than to P_B ” is not necessarily true.

For example:



- d** The statement “B is closer to P_B than to C” is not necessarily true.

For example:

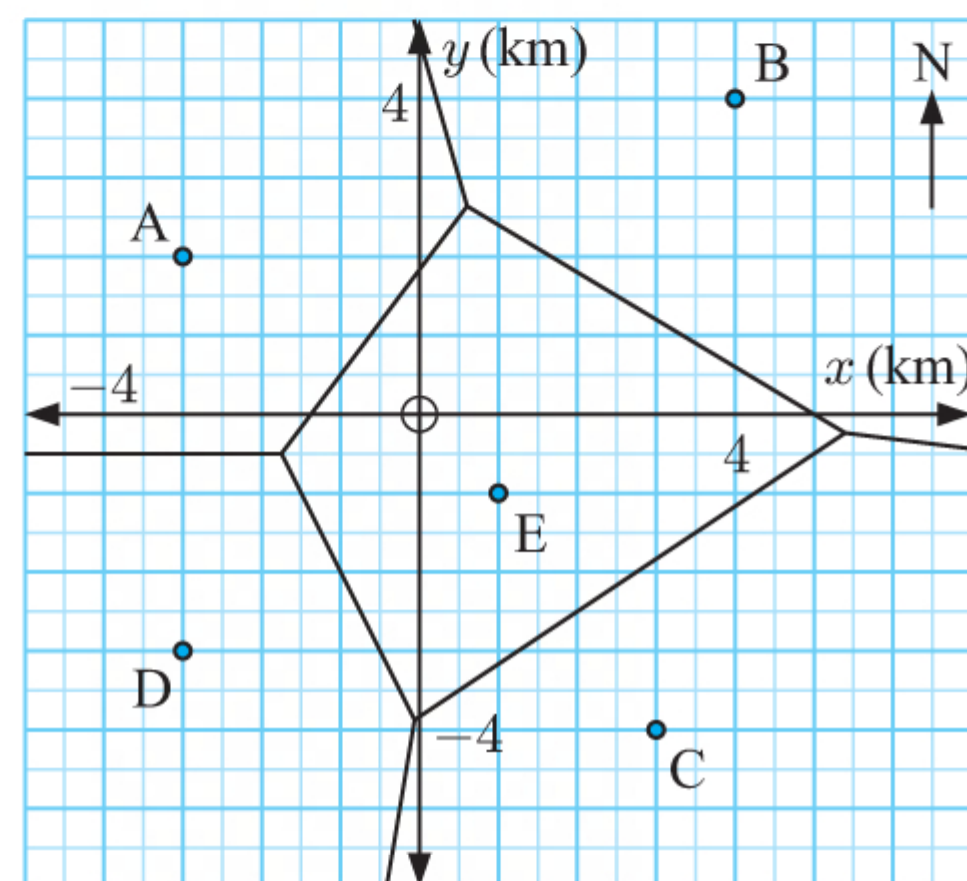


- 5** If the circle passes through another site, then P is equidistant from that site and X, and therefore is not within the interior of cell X.

If another site lies within the circle, then P is closer to that site than to X, and therefore is not within the interior of cell X.

Since P lies within the interior of cell X, this circle cannot contain any other sites.

- 6 a**
- i** $(0, 0)$ lies within cell E, so the holding post office is post office E.
 - ii** $(4, 1)$ lies within cell B, so the holding post office is post office B.
 - iii** $(-2, 0)$ lies within cell A, so the holding post office is post office A.
 - iv** $(3, -2)$ lies within cell C, so the holding post office is post office C.



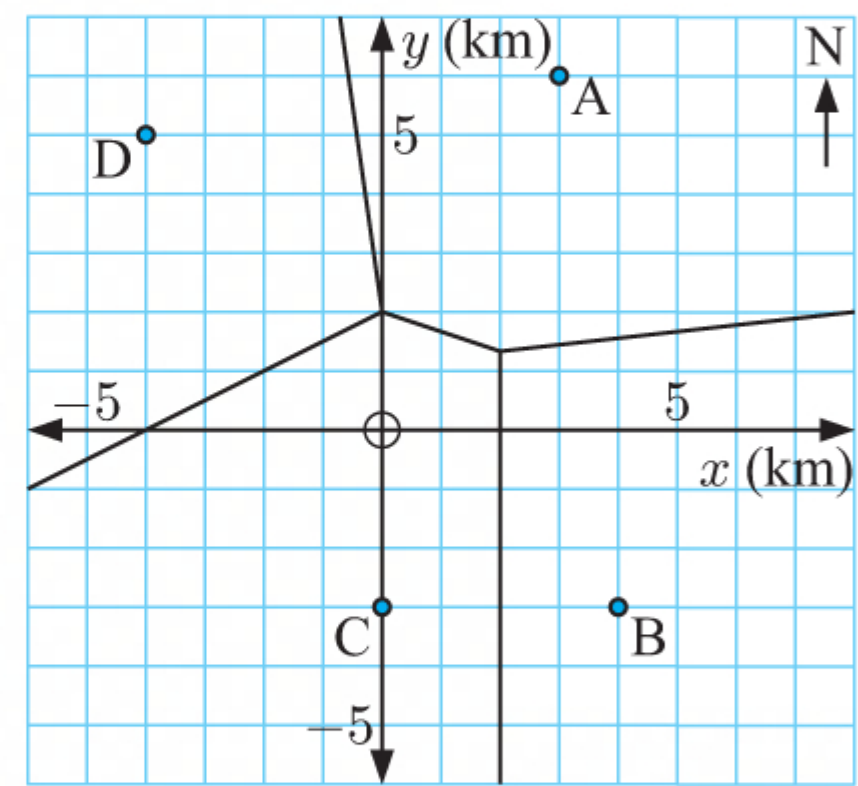
- b** One building lies on the edge adjacent to cells D and E, and the other lies on the edge adjacent to cells A and E.

Consider a building located at $(-1, -2)$ which lies on the edge adjacent to cells D and E. A building 2.5 km north of $(-1, -2)$ is located at $(-1, -2 + 2.5)$, which is $(-1, 0.5)$ or $(-1, \frac{1}{2})$.

Now $(-1, \frac{1}{2})$ lies on the edge adjacent to cells A and E.

So the two buildings are located at $(-1, -2)$ and $(-1, \frac{1}{2})$.

- 7 a**
- i** $(-3, -2)$ lies in cell C, so the nearest public school is school C.
 - ii** $(6, 2)$ lies in cell A, so the nearest public school is school A.
 - iii** $(-2, 4)$ lies in cell D, so the nearest public school is school D.
 - iv** $(5, 1)$ lies in cell B, so the nearest public school is school B.



- b**
- i** $(-1, 4)$ lies in cell D, so Bailey lives in school D's catchment zone.
 - ii** 1 km to the east of $(-1, 4)$ is $(-1 + 1, 4)$ which is $(0, 4)$. This is in cell A rather than cell D, and so Bailey will move into a new catchment zone.
 - iii** The distance from Bailey's home at $(0, 4)$ to school A at $(3, 6)$ is

$$\sqrt{(3-0)^2 + (6-4)^2} = \sqrt{13} \\ \approx 3.61 \text{ km.}$$

The distance from Bailey's home at $(0, 4)$ to school D at $(-4, 5)$ is

$$\sqrt{(-4-0)^2 + (5-4)^2} = \sqrt{17} \\ \approx 4.12 \text{ km.}$$

So, Bailey's new school, school A, is now closer to Bailey than his old school, school D.

- c**
- i** Lizzy lives at a point adjacent to cells A, C, and D, which is $(0, 2)$.
 - ii** The distance from Lizzy's house at $(0, 2)$ to school C at $(0, -3)$ is 5 km.
Since Lizzy's house is equally closest to schools A, C, and D, Lizzy lives 5 km from each of these schools.

- 8** A vertex of a Voronoi diagram is equally closest to at least 3 sites (whose cells meet at that vertex). The circle's edge passes through site X, which is one of the sites V is closest to, and since every point on the edge is equidistant from the centre, the other closest sites must also lie on the edge of the circle.

So, the circle must pass through at least two other sites.

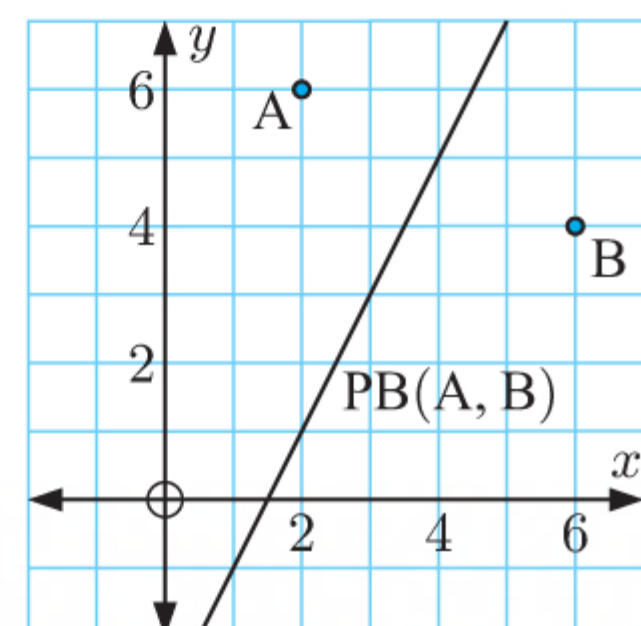
EXERCISE 17B

- 1 a** $A(2, 6), B(6, 4)$

The midpoint of $[AB]$ is $\left(\frac{2+6}{2}, \frac{6+4}{2}\right)$
or $(4, 5)$.

The gradient of $[AB]$ is $\frac{4-6}{6-2} = \frac{-2}{4} = -\frac{1}{2}$.

So, $PB(A, B)$ has gradient 2 and passes through $(4, 5)$.

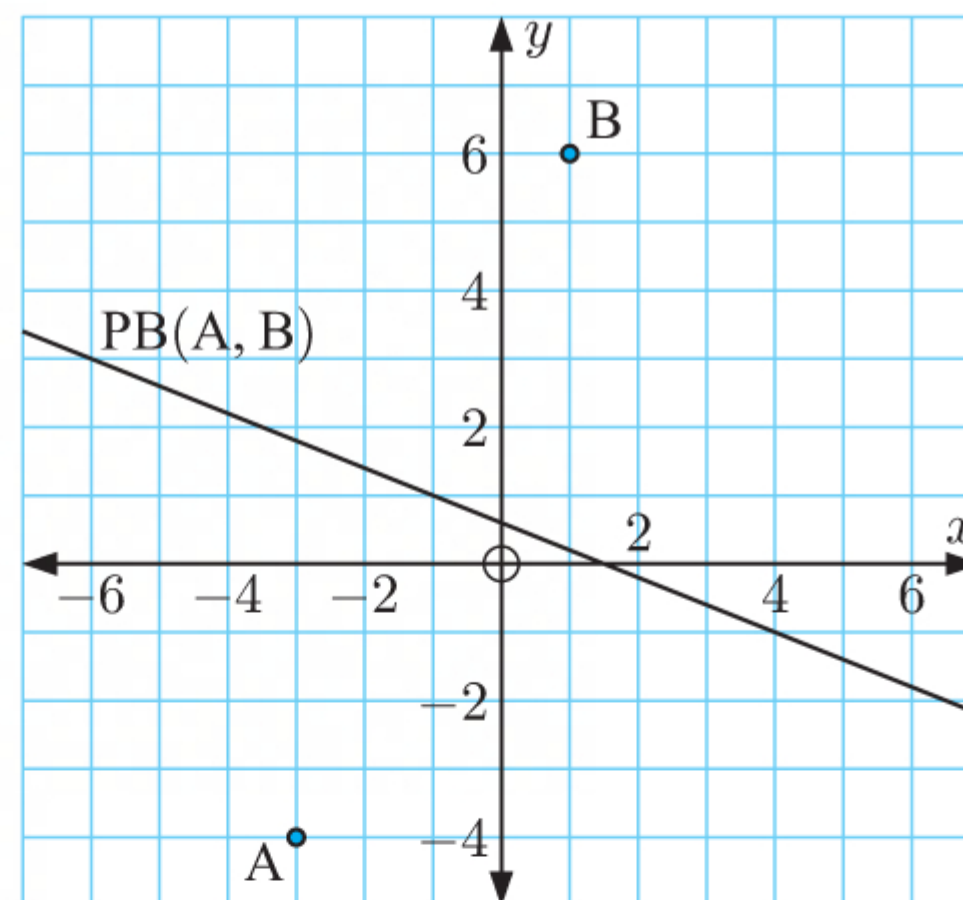


b $A(-3, -4), B(1, 6)$

The midpoint of $[AB]$ is $\left(\frac{-3+1}{2}, \frac{-4+6}{2}\right)$
or $(-1, 1)$.

The gradient of $[AB]$ is $\frac{6-(-4)}{1-(-3)} = \frac{10}{4} = \frac{5}{2}$.

So, $PB(A, B)$ has gradient $-\frac{2}{5}$ and passes through $(-1, 1)$.



2 a $A(-2, 5), B(4, 3)$

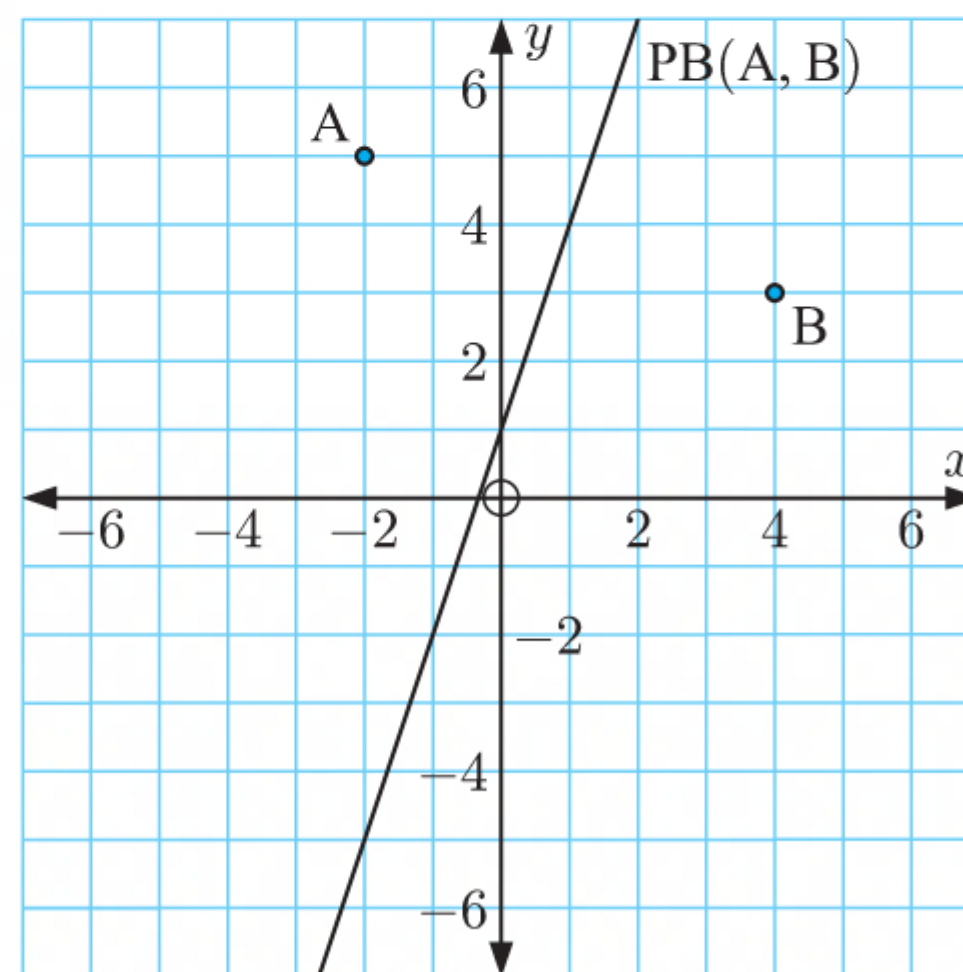
The midpoint of $[AB]$ is $\left(\frac{-2+4}{2}, \frac{5+3}{2}\right)$
or $(1, 4)$.

The gradient of $[AB]$ is $\frac{3-5}{4-(-2)} = \frac{-2}{6} = -\frac{1}{3}$.

So, $PB(A, B)$ has gradient 3 and passes through $(1, 4)$.

b The Voronoi edge has gradient 3 and passes through $(1, 4)$.

\therefore its equation is $3x - y = 3(1) - 1(4)$
or $y = 3x + 1$

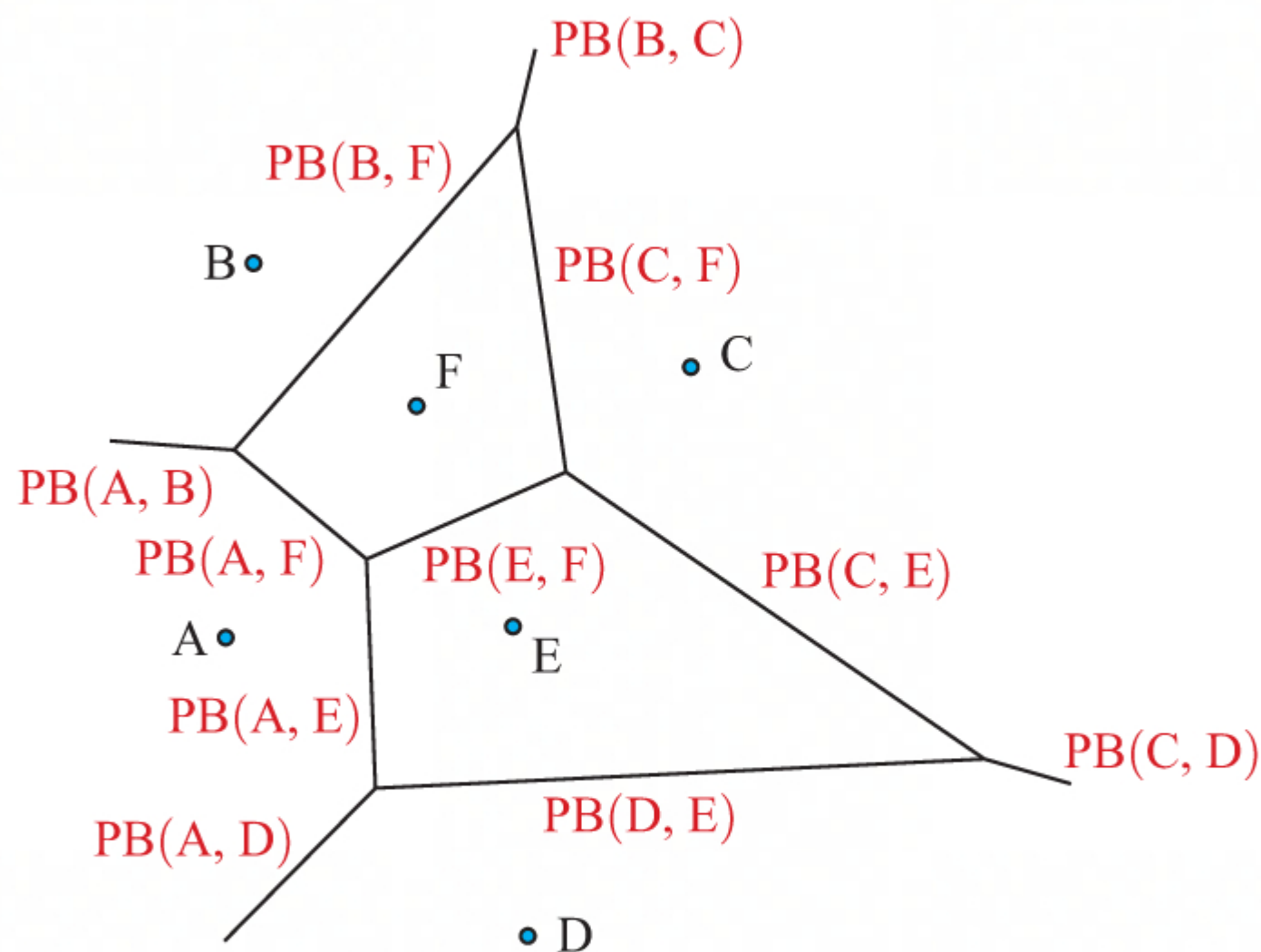


- c i** When $x = -2$, $y = 3(-2) + 1 = -5$
 $\therefore (-2, -5)$ is a point on the line $y = 3x + 1$.
 $\therefore (-2, -5)$ lies on the Voronoi edge.

- ii** The distance from $A(-2, 5)$ to $(-2, -5)$ is $\sqrt{(-2 - (-2))^2 + (-5 - 5)^2} = 10$ units
and the distance from $B(4, 3)$ to $(-2, -5)$ is $\sqrt{(-2 - 4)^2 + (-5 - 3)^2} = \sqrt{36 + 64} = 10$ units ✓

- d i** $(0, 0)$ lies in cell B, so it is closest to site B.
ii $(3, 6)$ lies in cell B, so it is closest to site B.
iii $(-4, -5)$ lies in cell A, so it is closest to site A.

3

4 a $A(4, 7), B(8, 3), C(0, -5)$

The midpoint of $[AB]$ is $\left(\frac{4+8}{2}, \frac{7+3}{2}\right)$ or $(6, 5)$.

The gradient of $[AB]$ is $\frac{3-7}{8-4} = \frac{-4}{4} = -1$.

So, $PB(A, B)$ has gradient 1 and passes through $(6, 5)$.

The midpoint of $[AC]$ is $\left(\frac{4+0}{2}, \frac{7+(-5)}{2}\right)$ or $(2, 1)$.

The gradient of $[AC]$ is $\frac{-5-7}{0-4} = \frac{-12}{-4} = 3$.

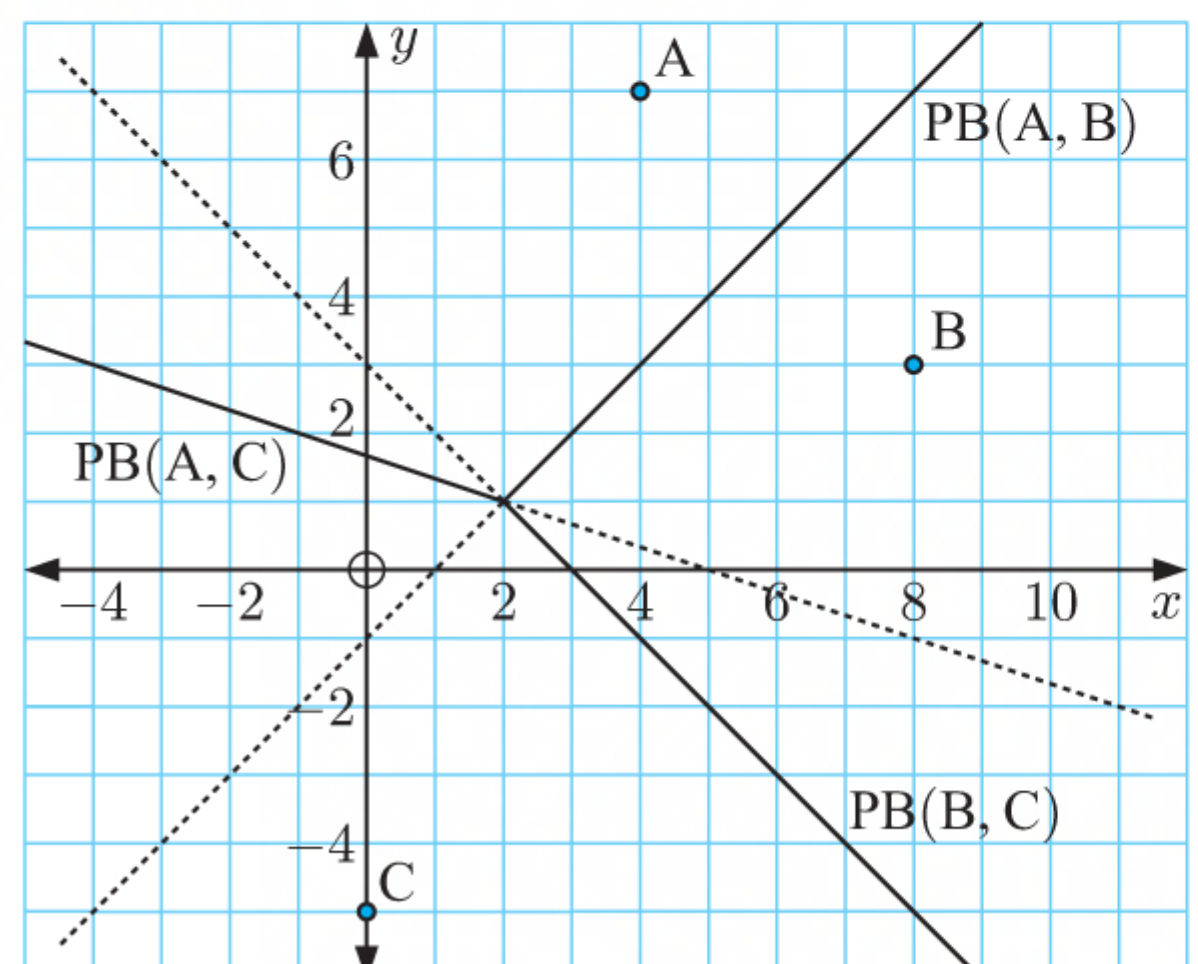
So, $PB(A, C)$ has gradient $-\frac{1}{3}$ and passes through $(2, 1)$.

The midpoint of $[BC]$ is $\left(\frac{8+0}{2}, \frac{3+(-5)}{2}\right)$ or $(4, -1)$.

The gradient of $[BC]$ is $\frac{-5-3}{0-8} = \frac{-8}{-8} = 1$.

So, $PB(B, C)$ has gradient -1 and passes through $(4, -1)$.

We plot sites A, B , and C on a set of axes. We draw $PB(A, B)$, $PB(A, C)$, and $PB(B, C)$ as dashed lines, then make solid only the parts which form the Voronoi edges.



b $A(-1, 4)$, $B(5, 2)$, $C(-5, -4)$

The midpoint of $[AB]$ is $\left(\frac{-1+5}{2}, \frac{4+2}{2}\right)$ or $(2, 3)$.

The gradient of $[AB]$ is $\frac{2-4}{5-(-1)} = \frac{-2}{6} = -\frac{1}{3}$.

So, $PB(A, B)$ has gradient 3 and passes through $(2, 3)$.

The midpoint of $[AC]$ is $\left(\frac{-1+(-5)}{2}, \frac{4+(-4)}{2}\right)$ or $(-3, 0)$.

The gradient of $[AC]$ is $\frac{-4-4}{-5-(-1)} = \frac{-8}{-4} = 2$.

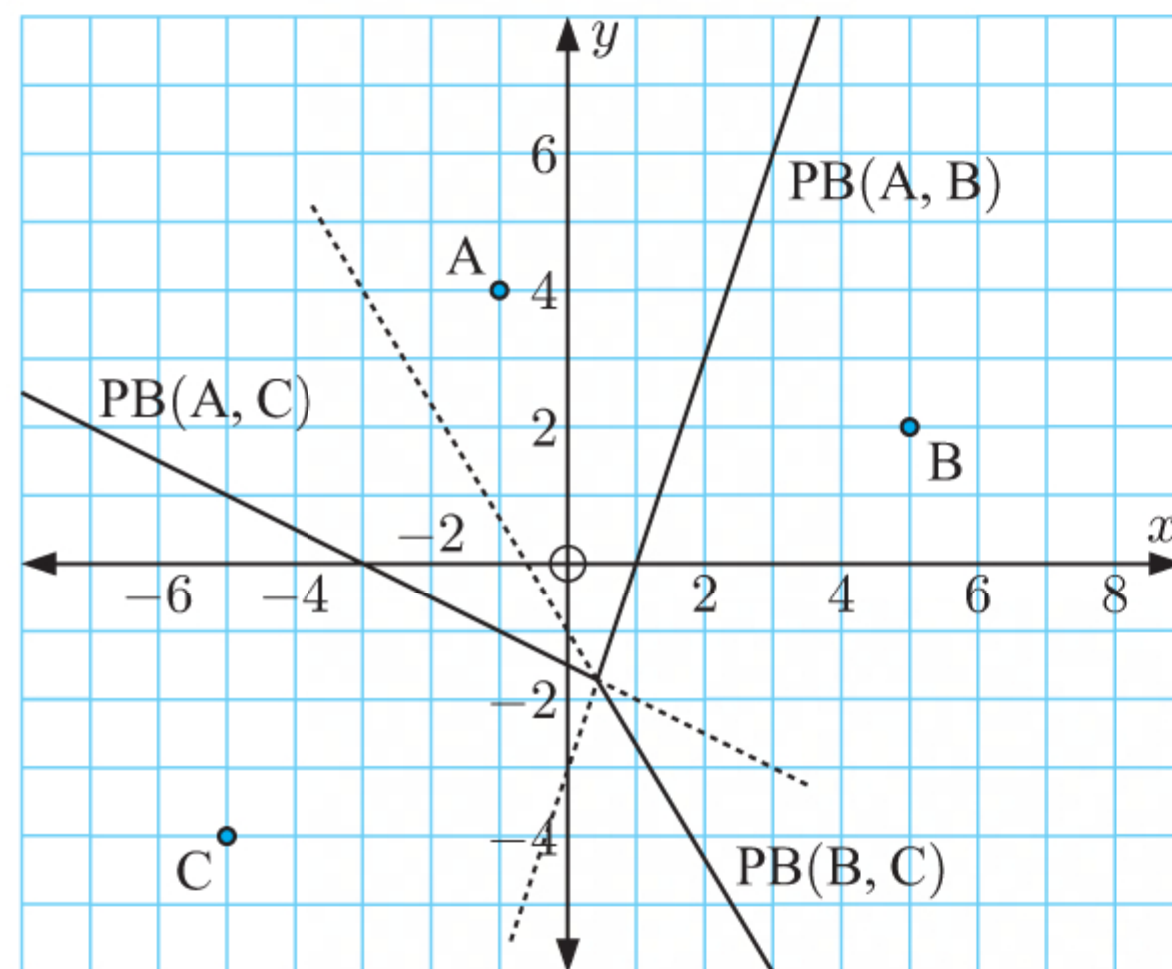
So, $PB(A, C)$ has gradient $-\frac{1}{2}$ and passes through $(-3, 0)$.

The midpoint of $[BC]$ is $\left(\frac{5+(-5)}{2}, \frac{2+(-4)}{2}\right)$ or $(0, -1)$.

The gradient of $[BC]$ is $\frac{-4-2}{-5-5} = \frac{-6}{-10} = \frac{3}{5}$.

So, $PB(B, C)$ has gradient $-\frac{5}{3}$ and passes through $(0, -1)$.

We plot sites A , B , and C on a set of axes. We draw $PB(A, B)$, $PB(A, C)$, and $PB(B, C)$ as dashed lines, then make solid only the parts which form the Voronoi edges.



5 a $A(-10, 9)$, $B(10, 13)$, $C(-2, -7)$

The midpoint of $[AB]$ is $\left(\frac{-10+10}{2}, \frac{9+13}{2}\right)$ or $(0, 11)$.

The gradient of $[AB]$ is $\frac{13-9}{10-(-10)} = \frac{4}{20} = \frac{1}{5}$.

So, $PB(A, B)$ has gradient -5 and passes through $(0, 11)$.

The midpoint of $[AC]$ is $\left(\frac{-10+(-2)}{2}, \frac{9+(-7)}{2}\right)$ or $(-6, 1)$.

The gradient of $[AC]$ is $\frac{-7-9}{-2-(-10)} = \frac{-16}{8} = -2$.

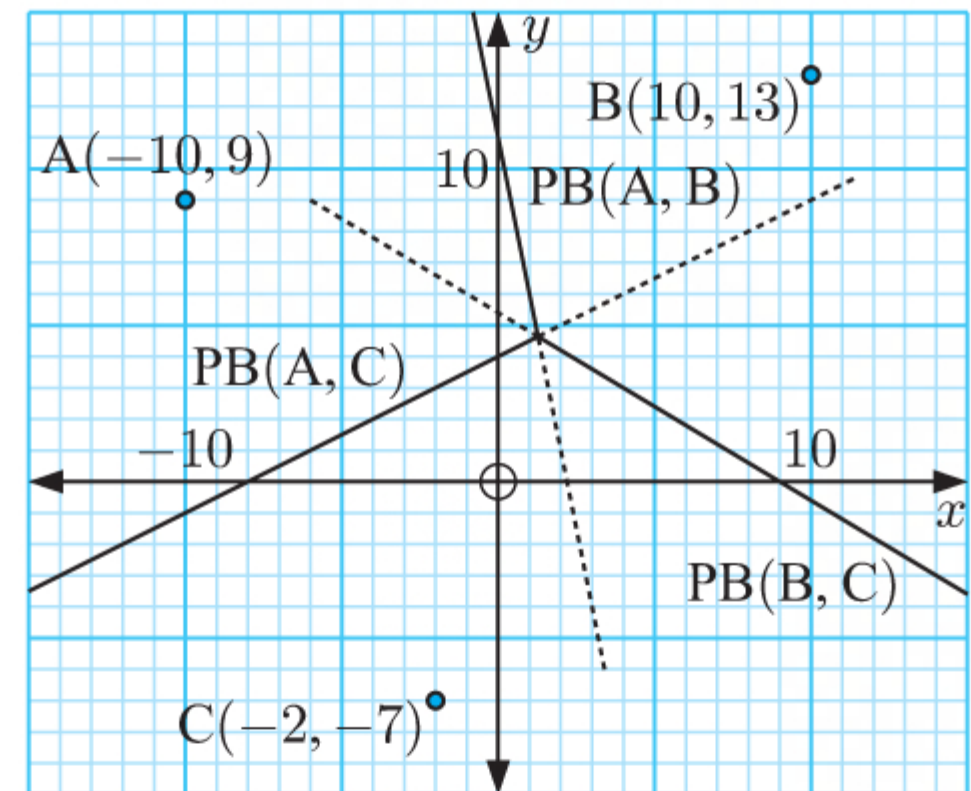
So, $PB(A, C)$ has gradient $\frac{1}{2}$ and passes through $(-6, 1)$.

The midpoint of $[BC]$ is $\left(\frac{10+(-2)}{2}, \frac{13+(-7)}{2}\right)$ or $(4, 3)$.

The gradient of [BC] is $\frac{-7-13}{-2-10} = \frac{-20}{-12} = \frac{5}{3}$.

So, PB(B, C) has gradient $-\frac{3}{5}$ and passes through (4, 3).

We plot sites A, B, and C on a set of axes. We draw PB(A, B), PB(A, C), and PB(B, C) as dashed lines, then make solid only the parts which form the Voronoi edges.



- b** PB(A, B) has gradient -5 and passes through (0, 11).

$$\therefore \text{its equation is } 5x + y = 5(0) + 1(11) \\ \text{or } y = -5x + 11$$

PB(A, C) has gradient $\frac{1}{2}$ and passes through $(-6, 1)$.

$$\therefore \text{its equation is } x - 2y = 1(-6) - 2(1) \\ \text{which is } 2y = x + 8 \\ \text{or } y = \frac{1}{2}x + 4$$

PB(B, C) has gradient $-\frac{3}{5}$ and passes through (4, 3).

$$\therefore \text{its equation is } 3x + 5y = 3(4) + 5(3) \\ \text{which is } 5y = -3x + 27 \\ \text{or } y = -\frac{3}{5}x + \frac{27}{5}$$

- c** The vertex is the point at which all 3 edges intersect, so we equate the equations found in **b**.

PB(A, B) and PB(A, C) intersect where $-5x + 11 = \frac{1}{2}x + 4$

$$\therefore 10x - 22 = -x - 8$$

$$\therefore 11x = 14$$

$$\therefore x = \frac{14}{11}$$

$$\text{When } x = \frac{14}{11}, \quad y = -5\left(\frac{14}{11}\right) + 11 \\ = \frac{51}{11}$$

So, the vertex is $V\left(\frac{14}{11}, \frac{51}{11}\right)$.

Check: Using the equation of PB(B, C), when $x = \frac{14}{11}$, $y = -\frac{3}{5}\left(\frac{14}{11}\right) + \frac{27}{5} = \frac{51}{11}$ ✓

$$\begin{aligned} VA &= \sqrt{\left(-10 - \frac{14}{11}\right)^2 + \left(9 - \frac{51}{11}\right)^2} \\ &= \sqrt{\left(-\frac{124}{11}\right)^2 + \left(\frac{48}{11}\right)^2} \\ &= \sqrt{\frac{17\,680}{121}} \approx 12.1 \text{ units} \end{aligned}$$

$$\begin{aligned} VB &= \sqrt{\left(10 - \frac{14}{11}\right)^2 + \left(13 - \frac{51}{11}\right)^2} \\ &= \sqrt{\left(\frac{96}{11}\right)^2 + \left(\frac{92}{11}\right)^2} \\ &= \sqrt{\frac{17\,680}{121}} \approx 12.1 \text{ units} \end{aligned}$$

$$\begin{aligned}
 VC &= \sqrt{\left(-2 - \frac{14}{11}\right)^2 + \left(-7 - \frac{51}{11}\right)^2} \\
 &= \sqrt{\left(-\frac{36}{11}\right)^2 + \left(-\frac{128}{11}\right)^2} \\
 &= \sqrt{\frac{17\,680}{121}} \approx 12.1 \text{ units}
 \end{aligned}$$

So, the vertex V is equidistant from A, B, and C.

- d**
- i** $(-2, 8)$ lies in cell A, so it is closest to site A.
 - ii** $(5, 5)$ lies in cell B, so it is closest to site B.
 - iii** $(2, -3)$ lies in cell C, so it is closest to site C.

- 6 a** $A(5, 3), B(4, -2), C(-4, -6)$

The midpoint of [AB] is $\left(\frac{5+4}{2}, \frac{3+(-2)}{2}\right)$ or $\left(\frac{9}{2}, \frac{1}{2}\right)$.

The gradient of [AB] is $\frac{-2-3}{4-5} = \frac{-5}{-1} = 5$.

So, PB(A, B) has gradient $-\frac{1}{5}$ and passes through $\left(\frac{9}{2}, \frac{1}{2}\right)$.

The midpoint of [AC] is $\left(\frac{5+(-4)}{2}, \frac{3+(-6)}{2}\right)$ or $\left(\frac{1}{2}, -\frac{3}{2}\right)$.

The gradient of [AC] is $\frac{-6-3}{-4-5} = \frac{-9}{-9} = 1$.

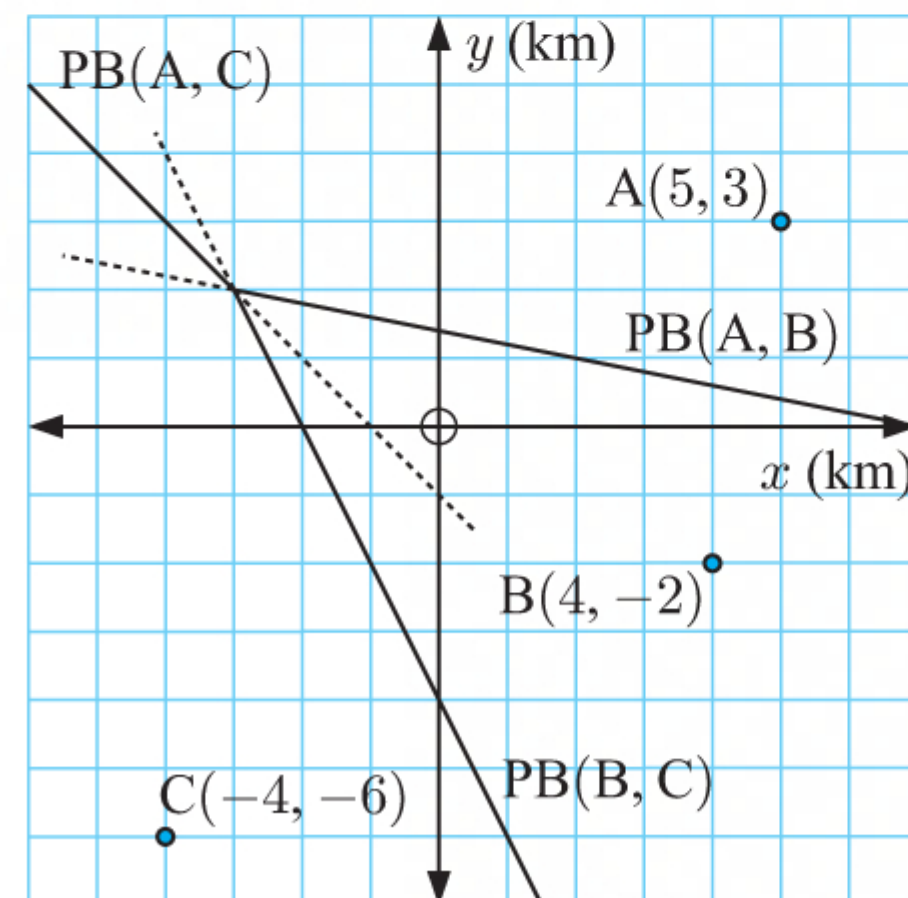
So, PB(A, C) has gradient -1 and passes through $\left(\frac{1}{2}, -\frac{3}{2}\right)$.

The midpoint of [BC] is $\left(\frac{4+(-4)}{2}, \frac{-2+(-6)}{2}\right)$ or $(0, -4)$.

The gradient of [BC] is $\frac{-6-(-2)}{-4-4} = \frac{-4}{-8} = \frac{1}{2}$.

So, PB(B, C) has gradient -2 and passes through $(0, -4)$.

We plot sites A, B, and C on a set of axes. We draw PB(A, B), PB(A, C), and PB(B, C) as dashed lines, then make solid only the parts which form the Voronoi edges.



- b**
- i** $(3, 1)$ lies in cell A, so they are closest to store A.
 - ii** $(-2, 2)$ lies in cell A, so they are closest to store A.
 - iii** $(-5, -1)$ lies in cell C, so they are closest to store C.

- c i** Amanda must live at the vertex of the Voronoi diagram.
Using the information found in **a**:

$$\text{PB(A, B) has equation } x + 5y = 1\left(\frac{9}{2}\right) + 5\left(\frac{1}{2}\right)$$

$$\text{which is } 5y = -x + 7$$

$$\text{or } y = -\frac{1}{5}x + \frac{7}{5}$$

$$\text{PB(A, C) has equation } x + y = \frac{1}{2} - \frac{3}{2}$$

$$\text{or } y = -x - 1$$

$$\text{PB(B, C) has equation } 2x + y = 2(0) - 4$$

$$\text{or } y = -2x - 4$$

$$\begin{aligned} \text{PB(A, C) and PB(B, C) intersect where } -x - 1 &= -2x - 4 \\ \therefore x &= -3 \end{aligned}$$

$$\text{When } x = -3, y = -(-3) - 1 = 2$$

So, Amanda lives at $(-3, 2)$.

Check: Using the equation of PB(A, B), when $x = -3$, $y = -\frac{1}{5}(-3) + \frac{7}{5} = 2$ ✓

- ii** The distance from Amanda's house at $(-3, 2)$ to $A(5, 3)$ is

$$\sqrt{(5 - (-3))^2 + (3 - 2)^2}$$

$$= \sqrt{8^2 + 1^2}$$

$$= \sqrt{65} \approx 8.06 \text{ km}$$

Amanda's house is equidistant from all 3 stores, so she is about 8.06 km from each store.

- 7 a** The Voronoi diagram must have an edge missing because sites A and D are in the same cell.
b The missing edge is the perpendicular bisector of $A(-5, 3)$ and $D(-3, -3)$.

$$\text{The midpoint of [AD] is } \left(\frac{-5 + -3}{2}, \frac{3 + -3}{2}\right) \text{ or } (-4, 0).$$

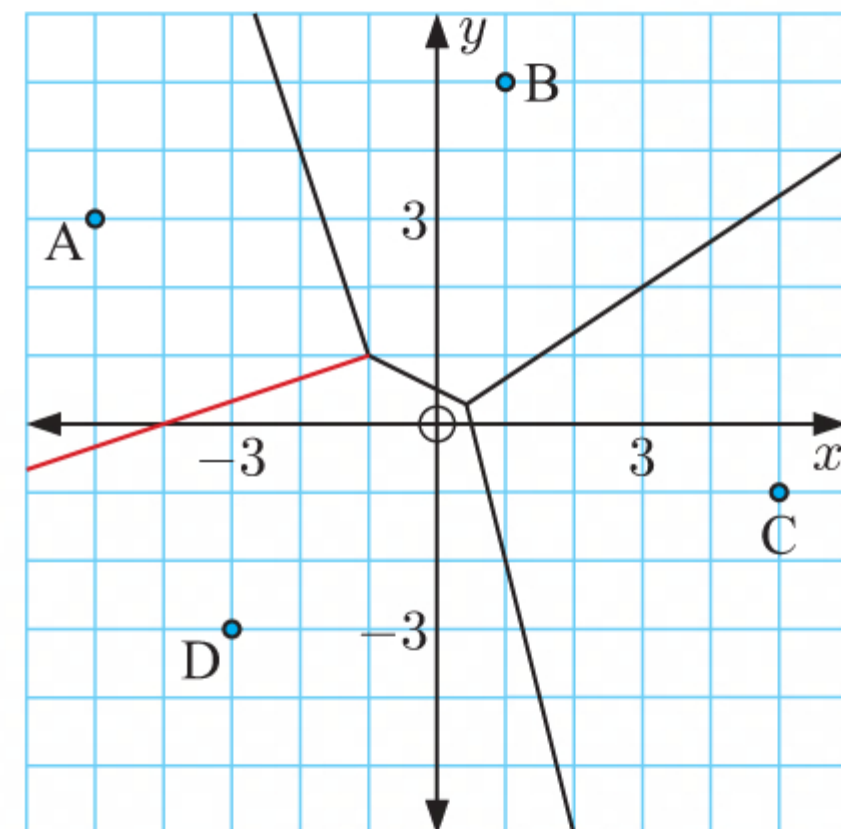
$$\text{The gradient of [AD] is } \frac{-3 - 3}{-3 - -5} = \frac{-6}{2} = -3.$$

So, PB(A, D) has gradient $\frac{1}{3}$ and passes through $(-4, 0)$.

$$\therefore \text{ its equation is } x - 3y = 1(-4) - 3(0)$$

$$\text{which is } 3y = -x + 4$$

$$\text{or } y = \frac{1}{3}x + \frac{4}{3}$$



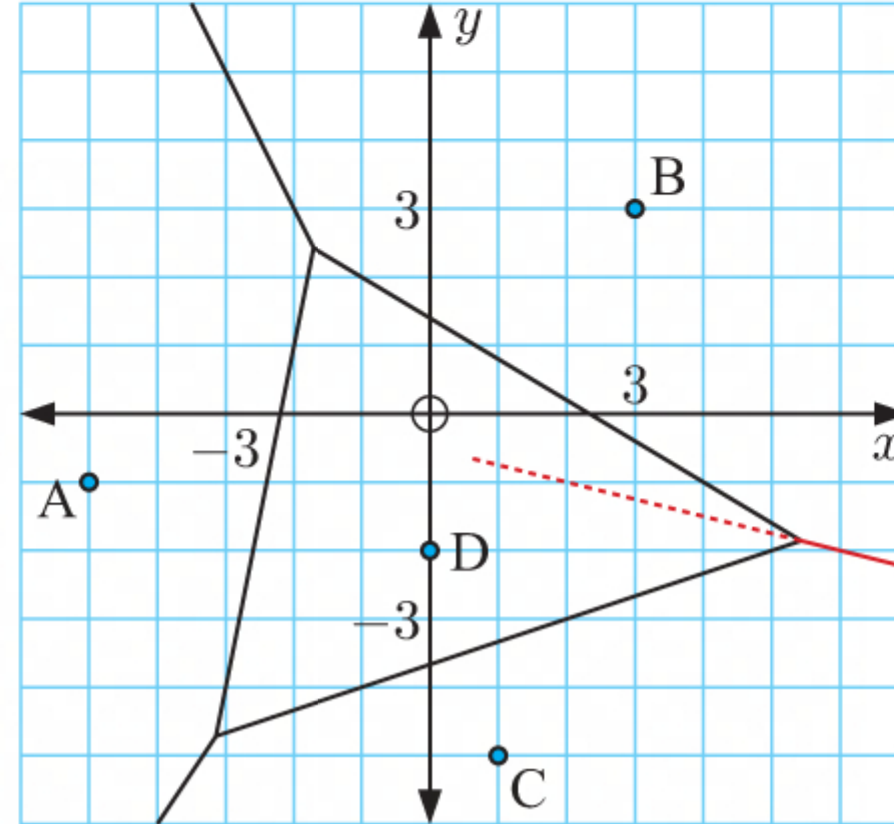
- 8 a** Sites $B(3, 3)$ and $C(1, -5)$ are currently in the same cell, so the missing edge must be the perpendicular bisector of $[BC]$.

The midpoint of $[BC]$ is $\left(\frac{3+1}{2}, \frac{3+(-5)}{2}\right)$ or $(2, -1)$.

The gradient of $[BC]$ is $\frac{-5-3}{1-3} = \frac{-8}{-2} = 4$.

So, $PB(B, C)$ has gradient $-\frac{1}{4}$ and passes through $(2, -1)$.

\therefore its equation is $x + 4y = 1(2) + (-1)$
or $x + 4y + 2 = 0$



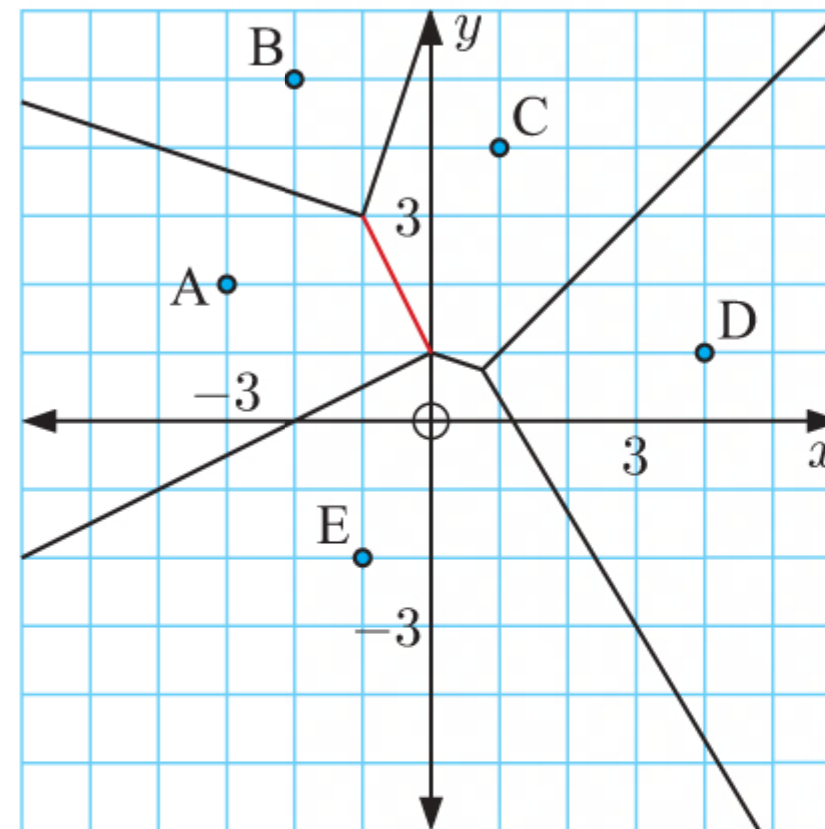
- b** Sites $A(-3, 2)$ and $C(1, 4)$ are currently in the same cell, so the missing edge must be the perpendicular bisector of $[AC]$.

The midpoint of $[AC]$ is $\left(\frac{-3+1}{2}, \frac{2+4}{2}\right)$ or $(-1, 3)$.

The gradient of $[AC]$ is $\frac{4-2}{1-(-3)} = \frac{2}{4} = \frac{1}{2}$.

So, $PB(A, C)$ has gradient -2 and passes through $(-1, 3)$.

\therefore its equation is $2x + y = 2(-1) + 3$
or $2x + y - 1 = 0$



- 9 a** Sites A and D are in the same cell, as are sites C and E.
So, there are 2 missing edges.
- b** The missing edges are the perpendicular bisector of $A(-3, 4)$ and $D(-1, -2)$, and the perpendicular bisector of $C(5, -4)$ and $E(1, 0)$.

The midpoint of $[AD]$ is $\left(\frac{-3+(-1)}{2}, \frac{4+(-2)}{2}\right)$ or $(-2, 1)$.

The gradient of $[AD]$ is $\frac{-2-4}{-1-(-3)} = \frac{-6}{2} = -3$.

So, $PB(A, D)$ has gradient $\frac{1}{3}$ and passes through $(-2, 1)$.

\therefore its equation is $x - 3y = 1(-2) - 3(1)$

which is $3y = x + 5$

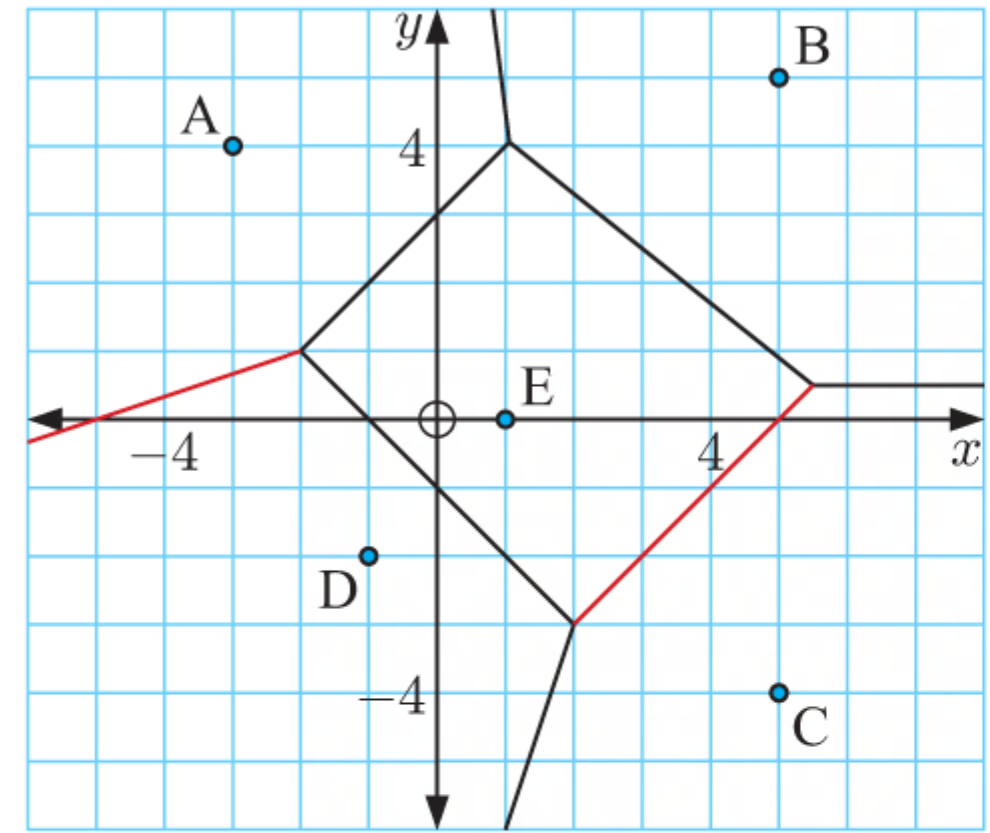
or $y = \frac{1}{3}x + \frac{5}{3}$

The midpoint of [CE] is $\left(\frac{5+1}{2}, \frac{-4+0}{2}\right)$ or $(3, -2)$.

The gradient of [CE] is $\frac{0 - -4}{1 - 5} = \frac{4}{-4} = -1$.

So, PB(C, E) has gradient 1 and passes through $(3, -2)$.

\therefore its equation is $x - y = 1(3) - (-2)$
or $y = x - 5$



- c**
 - i** $(-4, 1)$ lies in cell A, so it is closest to site A.
 - ii** $(2, 3)$ lies in cell E, so it is closest to site E.
- d**
 - i** $(4, -1)$ lies on the edge adjacent to cells C and E, so it is equally closest to sites C and E.
 - ii** $(-2, 1)$ lies on the vertex adjacent to cells A, D, and E, so it is equally closest to sites A, D, and E.

- 10 a** The blue edge has gradient 3 and passes through the point $(-1, 0)$.

\therefore the equation of the blue edge is $3x - y = 3(-1) - 0$
or $y = 3x + 3$

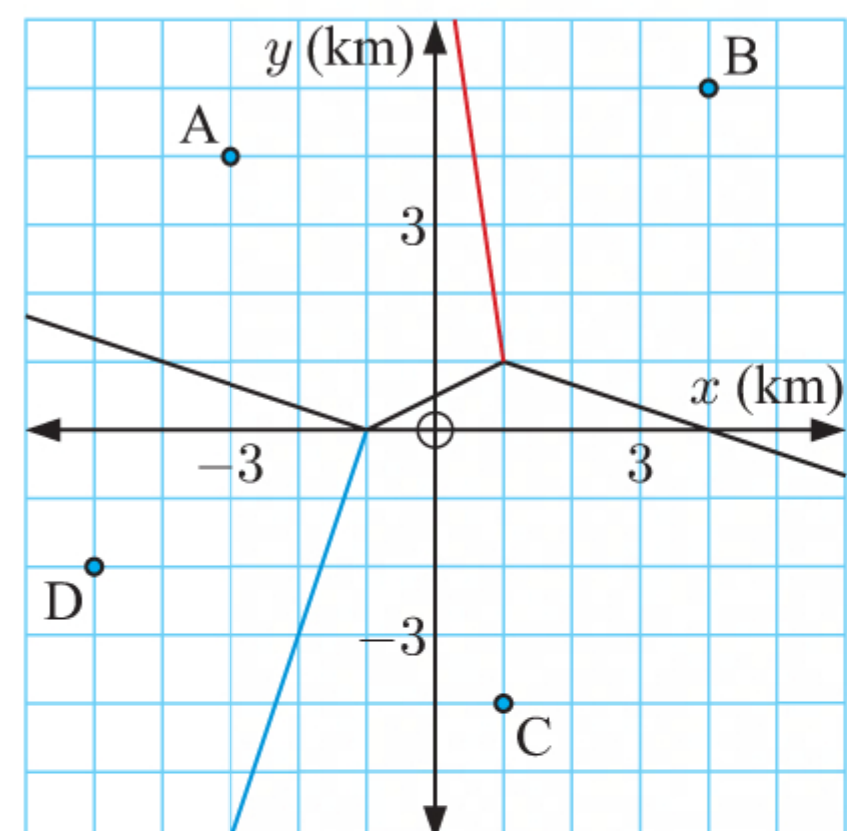
- b** Sites A $(-3, 4)$ and B $(4, 5)$ are currently in the same cell, so the missing edge must be the perpendicular bisector of [AB].

The midpoint of [AB] is $\left(\frac{-3+4}{2}, \frac{4+5}{2}\right)$ or $\left(\frac{1}{2}, \frac{9}{2}\right)$.

The gradient of [AB] is $\frac{5-4}{4-(-3)} = \frac{1}{7}$.

So, PB(A, B) has gradient -7 and passes through $\left(\frac{1}{2}, \frac{9}{2}\right)$.

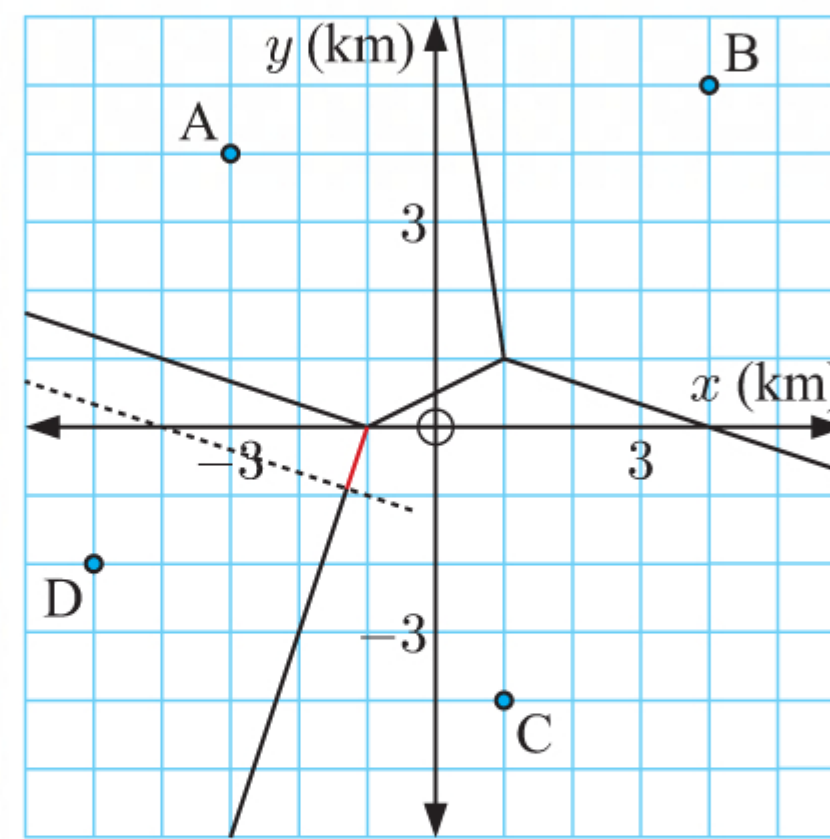
\therefore its equation is $7x + y = 7\left(\frac{1}{2}\right) + \frac{9}{2}$
or $y = -7x + 8$



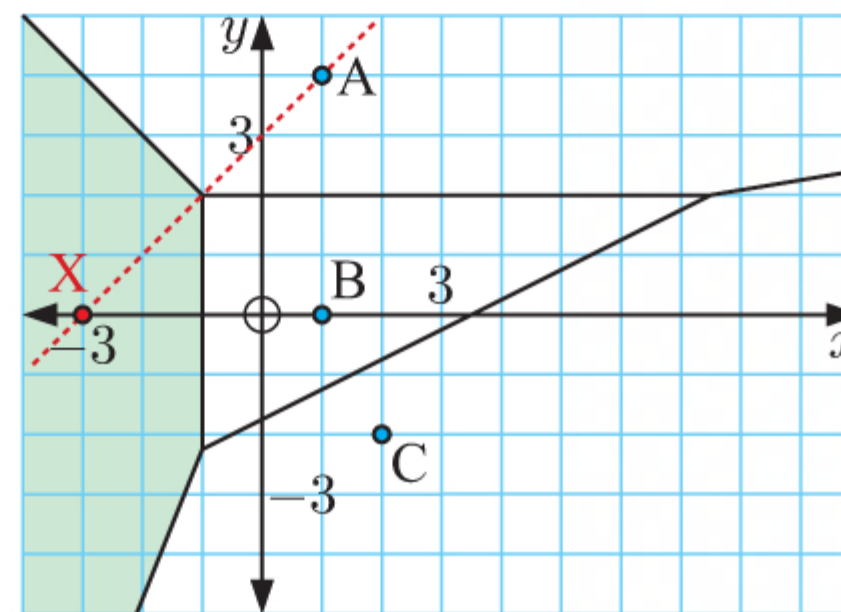
- c**
 - i** $(1, 3)$ lies in cell B, so Julie is nearest to campsite B.
 - ii** The distance from $(1, 3)$ to B $(4, 5)$ is $\sqrt{(4-1)^2 + (5-3)^2}$
 $= \sqrt{3^2 + 2^2}$
 $= \sqrt{13} \approx 3.61$ km

So, Julie is about 3.61 km from campsite B.

- d** Simon is equally closest to campsites C and D, so he is on the edge adjacent to cells C and D. He is less than 1 km south of cell A, so we construct a line which is 1 km south of the edge adjacent to cells A and D. Simon is therefore somewhere on the line segment shown alongside in red.

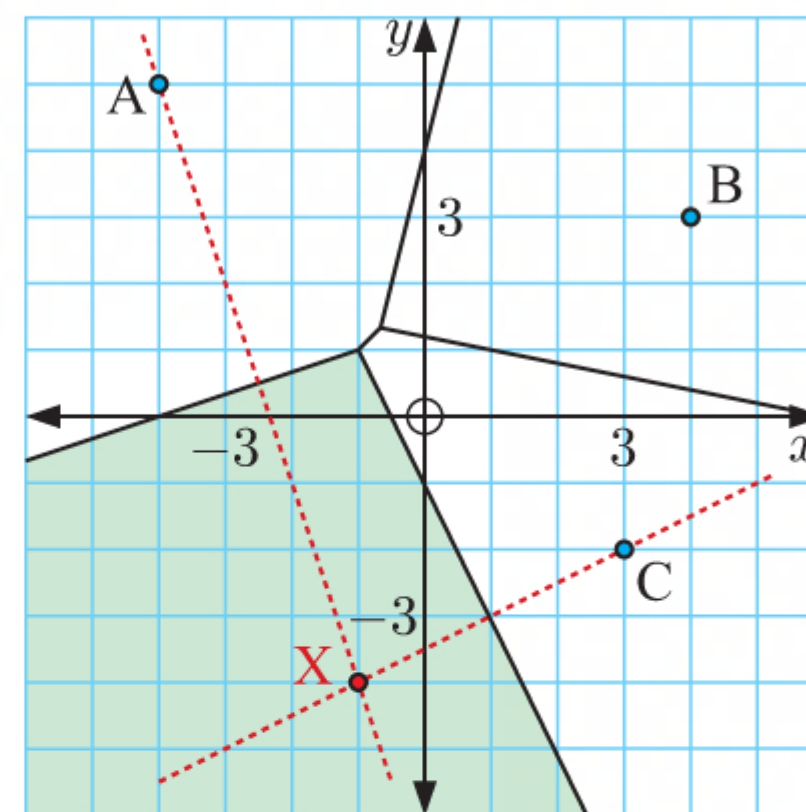


- 11 a** The Voronoi diagram must have a site missing as the shaded cell in the diagram alongside currently has no site.
- b** Let the missing site be X. X must lie on the x -axis because site B lies on the x -axis, and the edge adjacent to sites B and X is perpendicular to the x -axis.

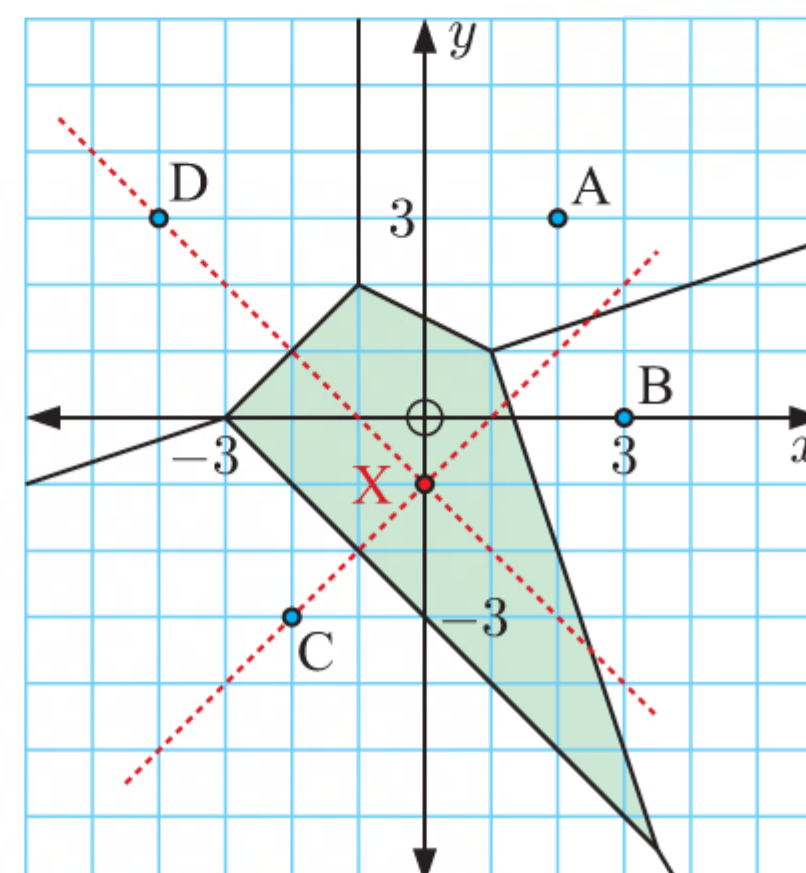


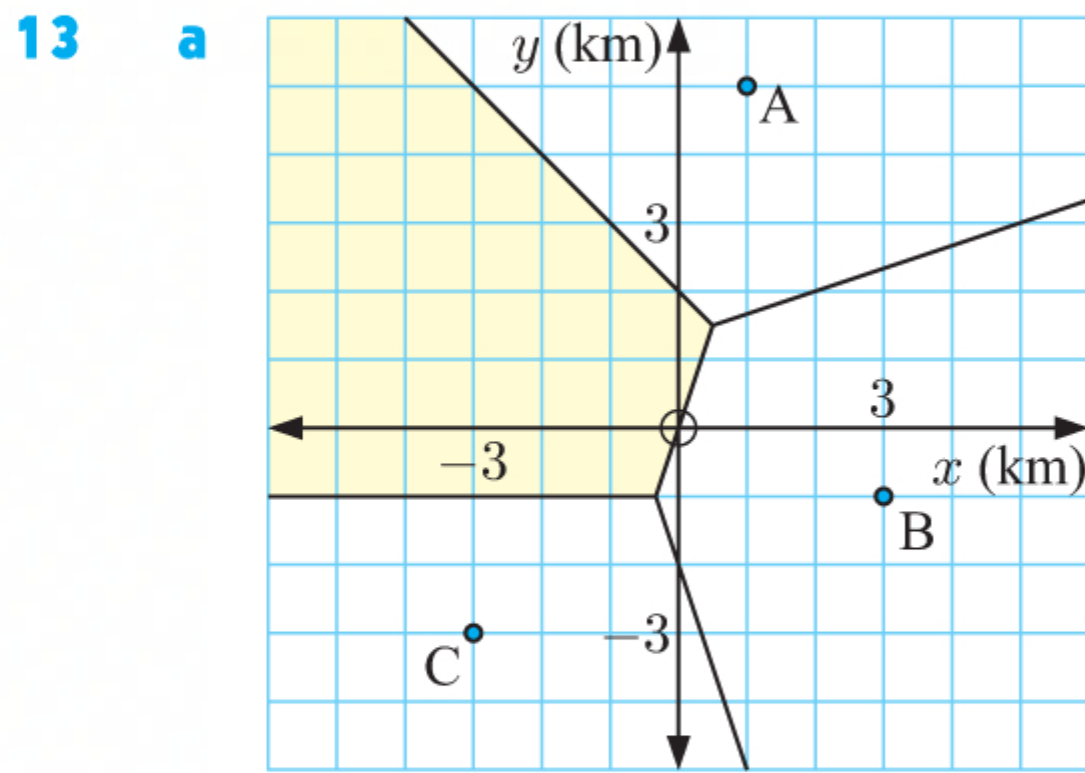
- c** $PB(A, X)$ has gradient -1 , so $[AX]$ has gradient 1 . If we draw the line (AX) , X must be the point where the line cuts the x -axis (from **b**). We observe that X has coordinates $(-3, 0)$.

- 12 a** The missing site X must lie in the shaded cell, as this cell currently has no site. Now $PB(A, X)$ has gradient $\frac{1}{3}$, and $PB(C, X)$ has gradient -2 . $\therefore [AX]$ has gradient -3 , and $[CX]$ has gradient $\frac{1}{2}$. If we draw lines (AX) and (CX) through A and D respectively, their intersection point must be site X. We observe that X has coordinates $(-1, -4)$.



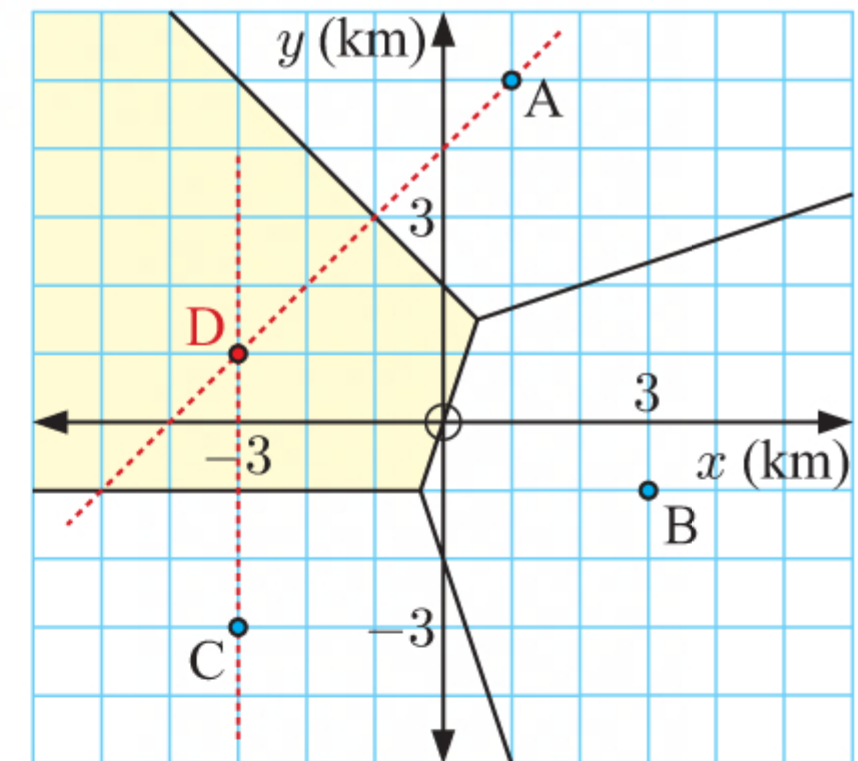
- b** The missing site X must lie in the shaded cell, as this cell currently has no site. Now $PB(C, X)$ has gradient -1 , and $PB(D, X)$ has gradient 1 . $\therefore [CX]$ has gradient 1 , and $[DX]$ has gradient -1 . If we draw lines (CX) and (DX) through C and D respectively, their intersection point must be site X. We observe that X has coordinates $(0, -1)$.





- b** The distance from $(0, -1)$ to $(-2, 1)$ is $\sqrt{(-2-0)^2 + (1-(-1))^2} = \sqrt{8} \approx 2.83$ km.
 The distance from $(0, -1)$ to $B(3, -1)$ is $\sqrt{(3-0)^2 + (-1-(-1))^2} = 3$ km.
 So, if the missing vet clinic was at $(-2, 1)$, then its cell would include $(0, -1)$.
 But $(0, -1)$ is in cell B.
 \therefore the missing vet clinic is not at $(-2, 1)$.

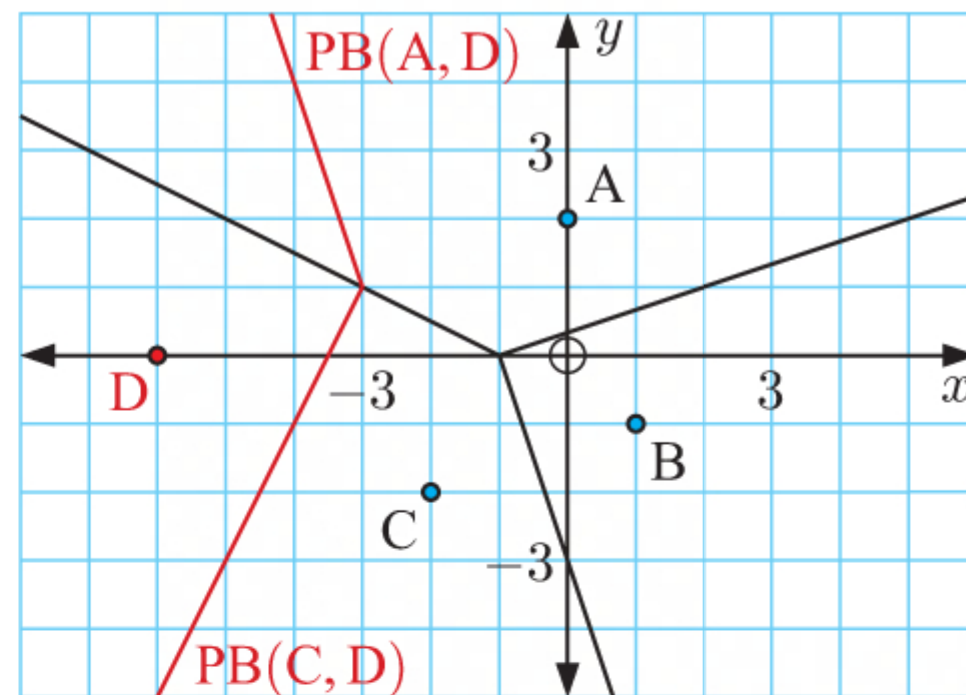
- c** $PB(A, D)$ has gradient -1 , and $PB(C, D)$ is horizontal.
 \therefore $[AD]$ has gradient 1 , and $[CD]$ is vertical.
 If we draw lines (AD) and (CD) through A and D respectively, their intersection must be site D .
 We observe that the missing vet clinic D is at $(-3, 1)$.



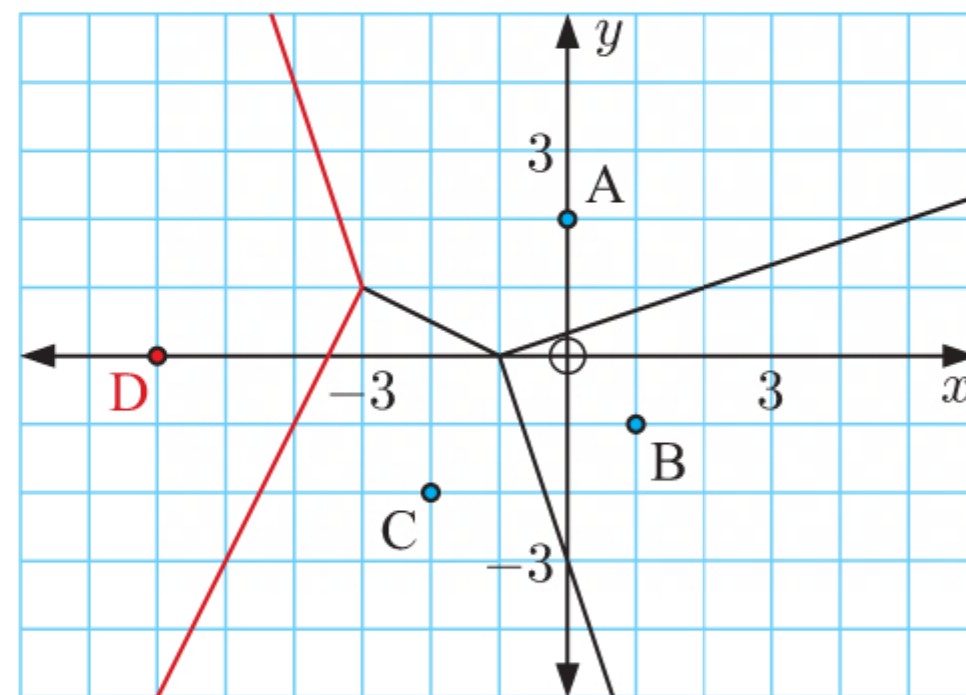
- d** **i** $(-1, 4)$ lies in cell A, so it is closest to vet clinic A.
ii $(1, -5)$ lies on the edge adjacent to cells B and C, so it is equally closest to vet clinics B and C.
- e** Larissa lives equally closest to vet clinics A and D, so Larissa's house must lie on the edge adjacent to cells A and D.
 The shortest possible distance to either vet clinic is the point at which $PB(A, D)$ and $[AD]$ meet.
 From the construction lines drawn in **c**, we observe that this point is $(-1, 3)$.
 The distance from $(-1, 3)$ to $A(1, 5)$ is $\sqrt{(1-(-1))^2 + (5-3)^2} = \sqrt{8} \approx 2.83$ km.
 So, the shortest distance Larissa's house could be from vet clinics A and D is $2\sqrt{2} \approx 2.83$ km.

EXERCISE 17C

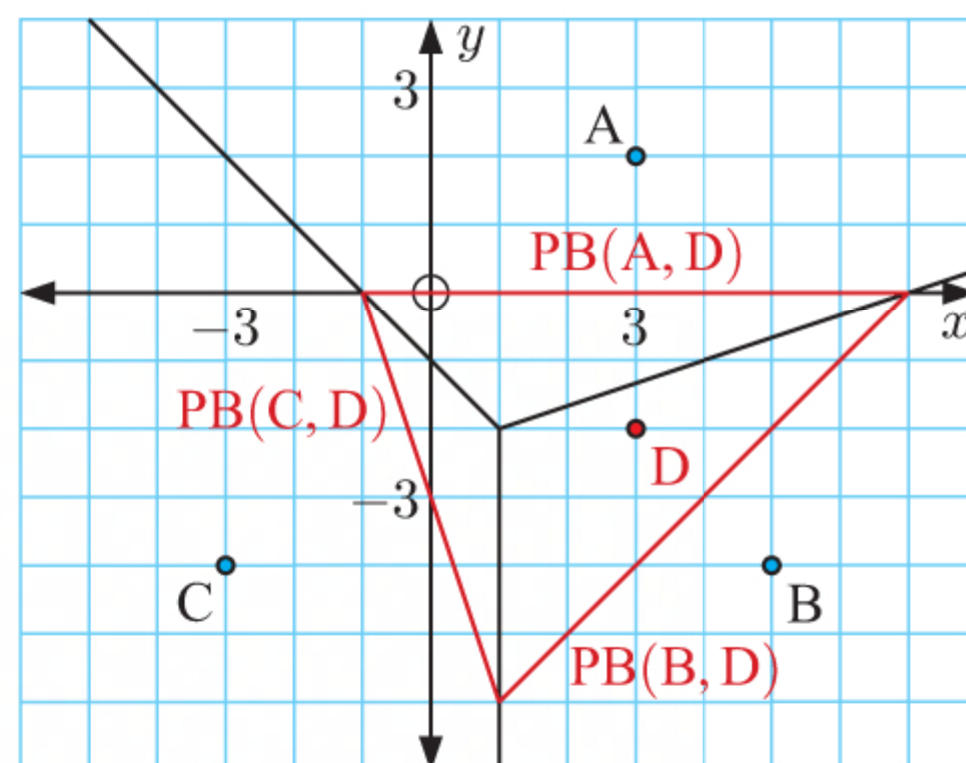
- 1
 - a Site D lies in the existing cell C.
 - b Cells A and C will be affected, as they both contain points which are closest to D.
 - c We construct $PB(C, D)$ within cell C, which creates a new vertex at $(-3, 1)$.
Cell A is adjacent to $(-3, 1)$, so we construct $PB(A, D)$ from $(-3, 1)$ through cell A.



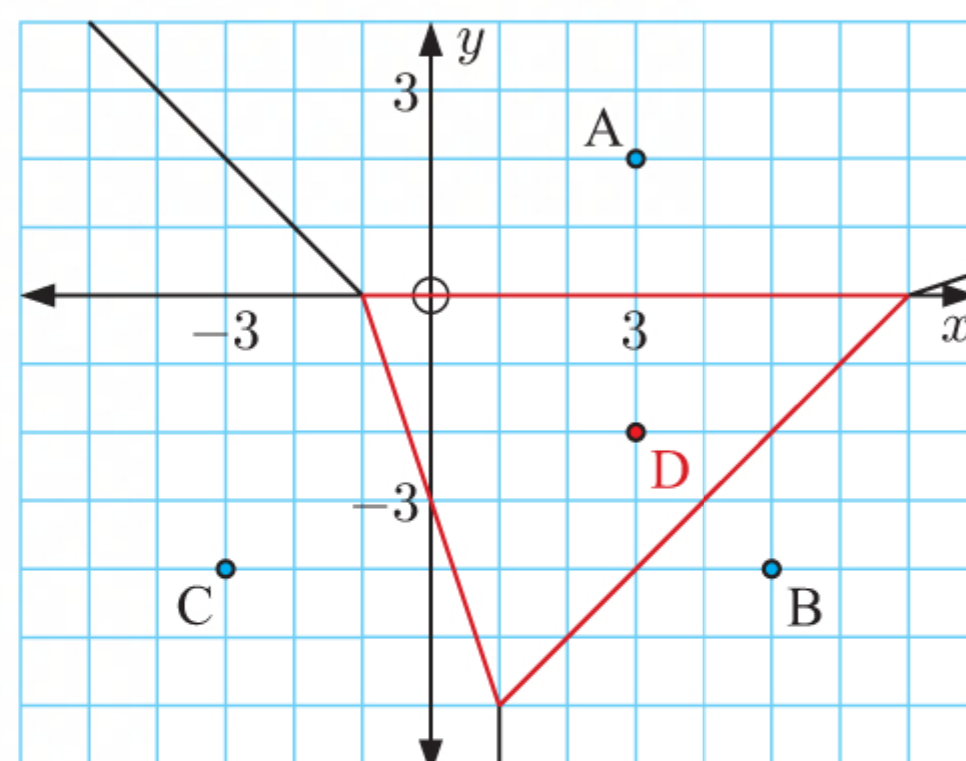
We then remove the edge segment from the original Voronoi diagram which now lies within cell D, giving us the Voronoi diagram which includes site D.



- 2
 - a Adding site D will affect all cells as cells A, B, and C each contain points which are closer to site D than they are to sites A, B, and C respectively.
 - b We construct $PB(A, D)$ within cell A, which creates new vertices at $(-1, 0)$ and $(7, 0)$.
Cell C is adjacent to $(-1, 0)$, so we construct $PB(C, D)$ from $(-1, 0)$ through cell C.
This creates a new vertex at $(1, -6)$.
Cell B is adjacent to $(1, -6)$, so we construct $PB(B, D)$ from $(1, -6)$ through cell B.
This connects us back to $(7, 0)$.

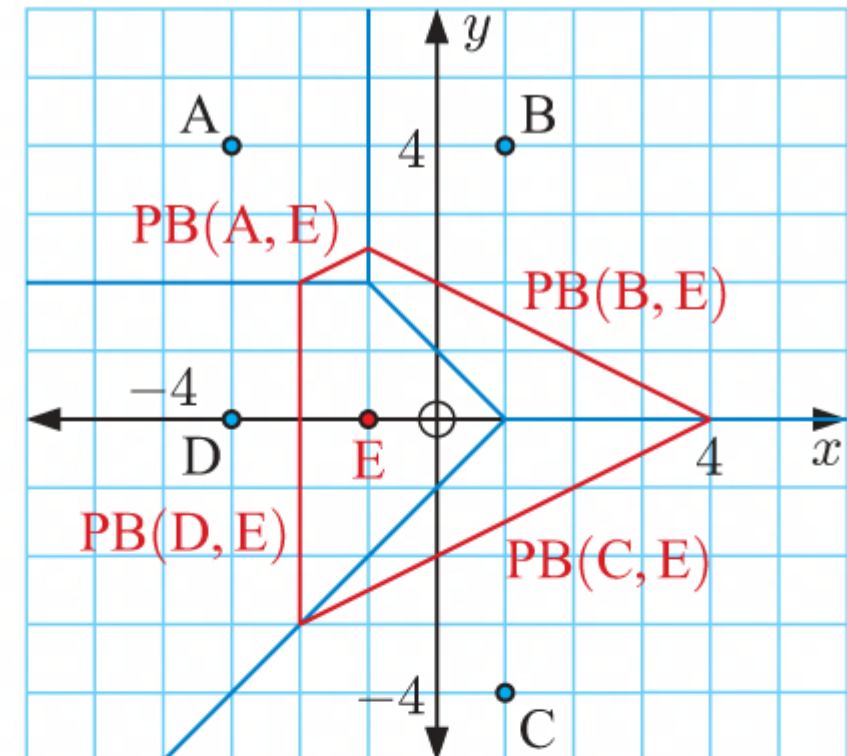


We then remove the segments of edges from the original Voronoi diagram which now lie within cell D, giving us the Voronoi diagram which includes site D.

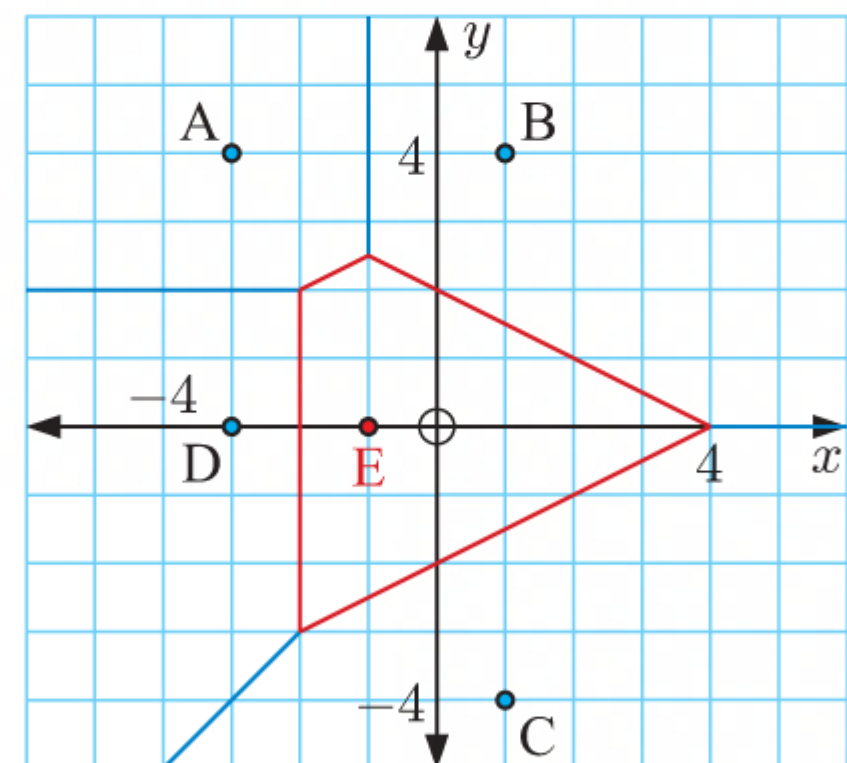


- c From the diagram in b, cell D is a triangle with base 8 units and height 6 units.
 \therefore area of cell D = $\frac{1}{2} \times 8 \times 6 = 24 \text{ units}^2$.

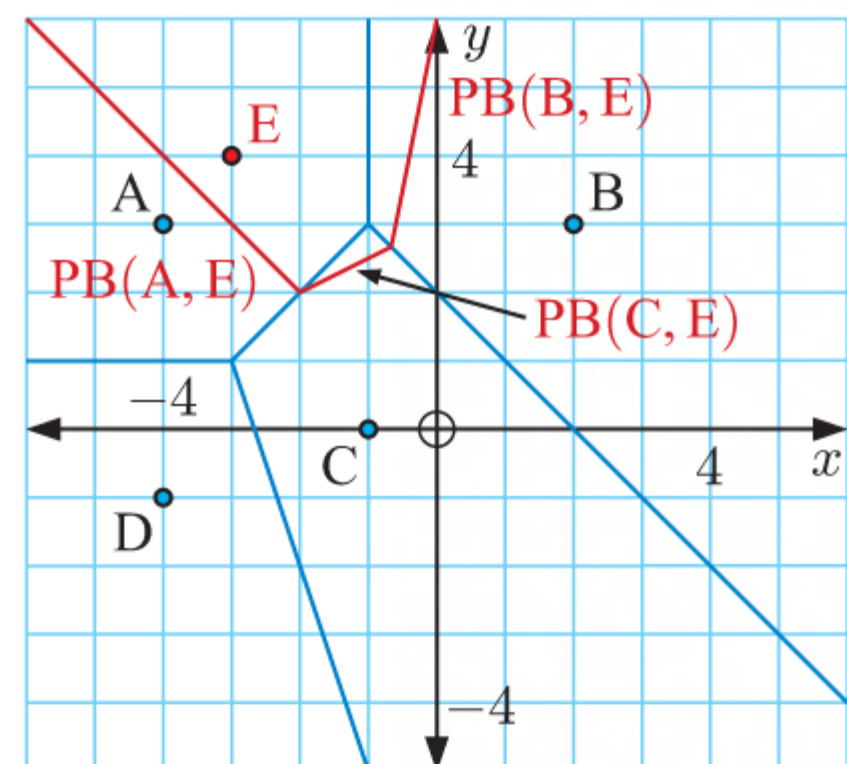
- 3 a We construct $PB(A, E)$, $PB(B, E)$, $PB(C, E)$, and $PB(D, E)$ within the original cells A, B, C, and D respectively.



We then remove the segments of edges from the original Voronoi diagram which now lie within cell E, giving us the Voronoi diagram which includes site E.

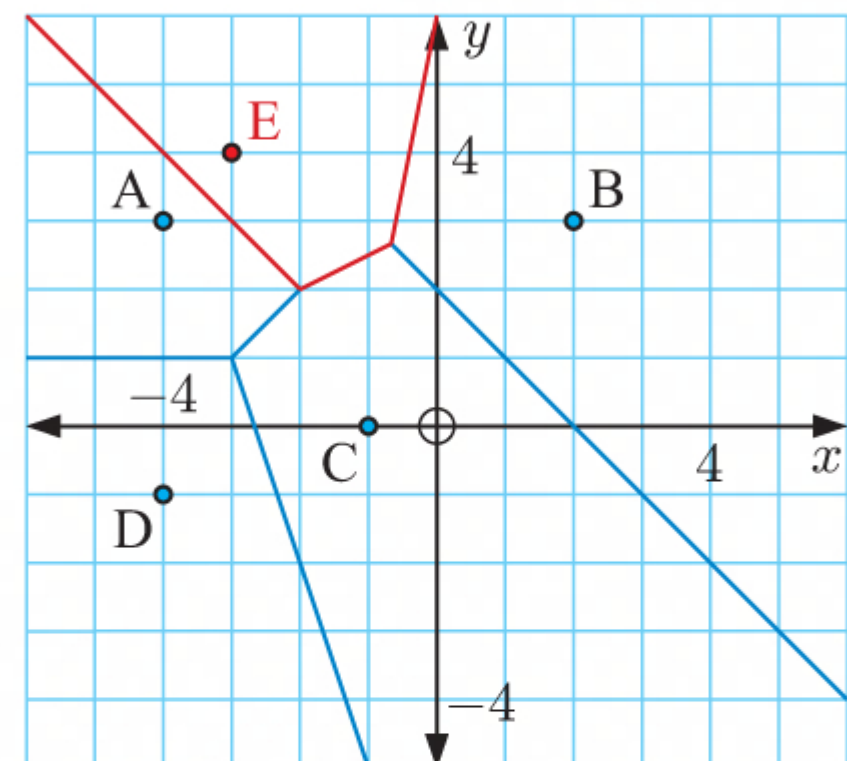


- b We construct $PB(A, E)$, $PB(B, E)$, and $PB(C, E)$ within the original cells A, B, and C respectively.



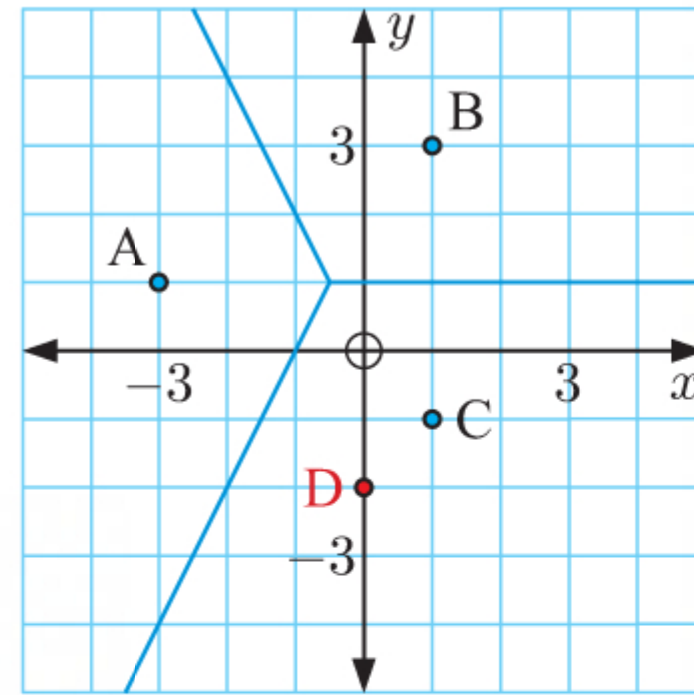
We then remove the segments of edges from the original Voronoi diagram which now lie within cell E, giving us the Voronoi diagram which includes site E.

Note: We do not need to construct $PB(D, E)$ as there are no points in cell D which are now closest to site E.



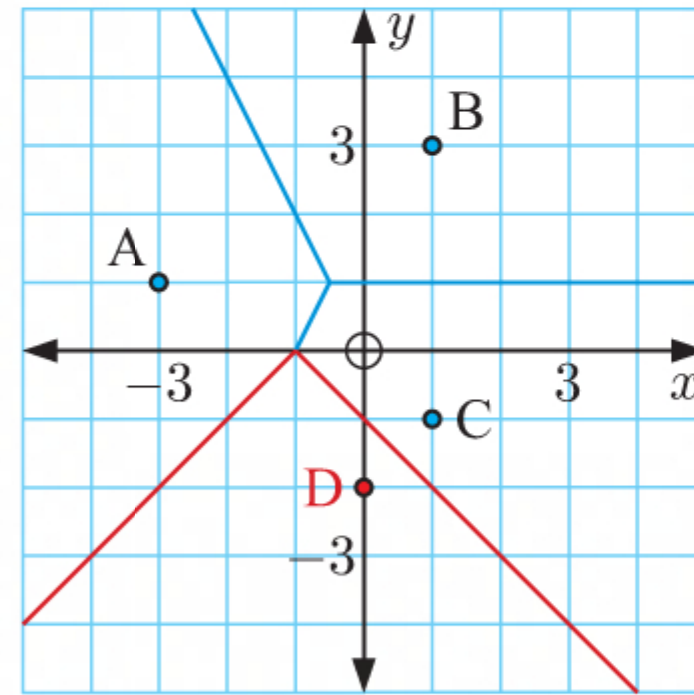
- 4 a** $A(-3, 1)$, $B(1, 3)$, $C(1, -1)$, $D(0, -2)$

We draw $PB(A, B)$, $PB(A, C)$, and $PB(B, C)$, then remove segments which do not form part of the Voronoi diagram.



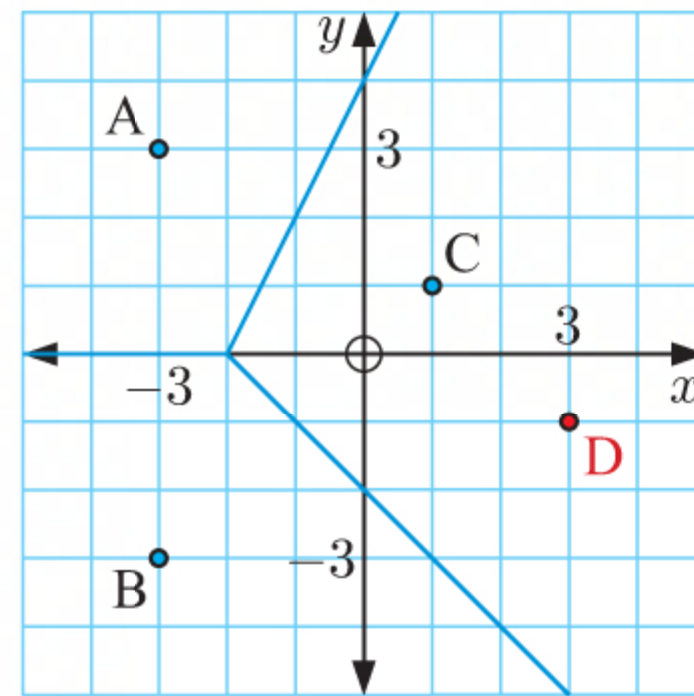
Given this Voronoi diagram, we now construct $PB(A, D)$ and $PB(C, D)$ within the original cells A and C respectively.

Then we remove the segment of the edge from the original Voronoi diagram which now lies within cell D, giving us the Voronoi diagram for A, B, C, and D.



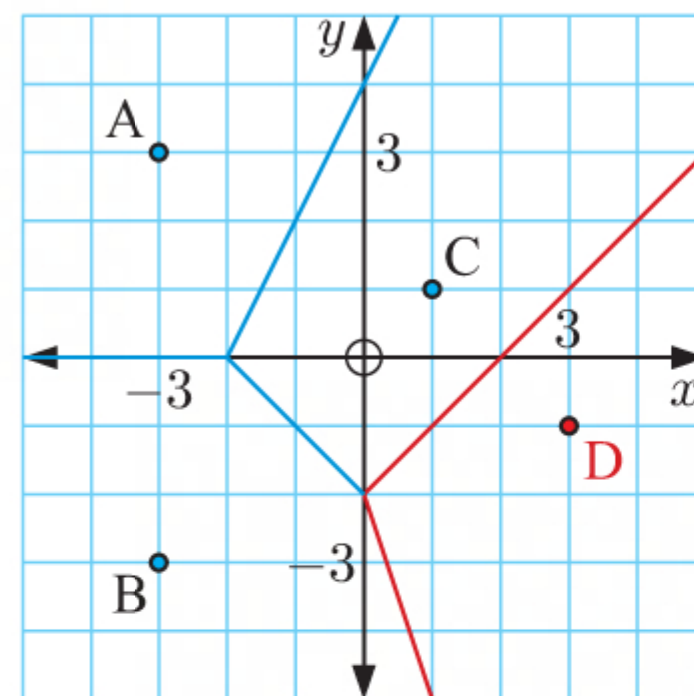
- b** $A(-3, 3)$, $B(-3, -3)$, $C(1, 1)$, $D(3, -1)$

We draw $PB(A, B)$, $PB(A, C)$, and $PB(B, C)$, then remove segments which do not form part of the Voronoi diagram.

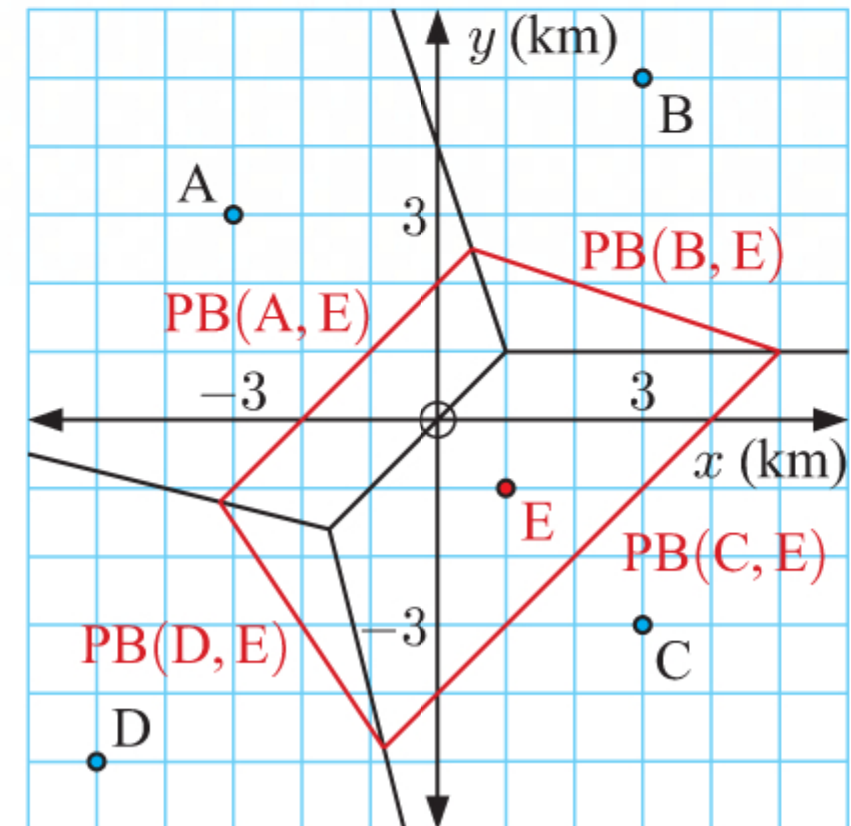


Given this Voronoi diagram, we now construct $PB(B, D)$ and $PB(C, D)$ within the original cells B and C respectively.

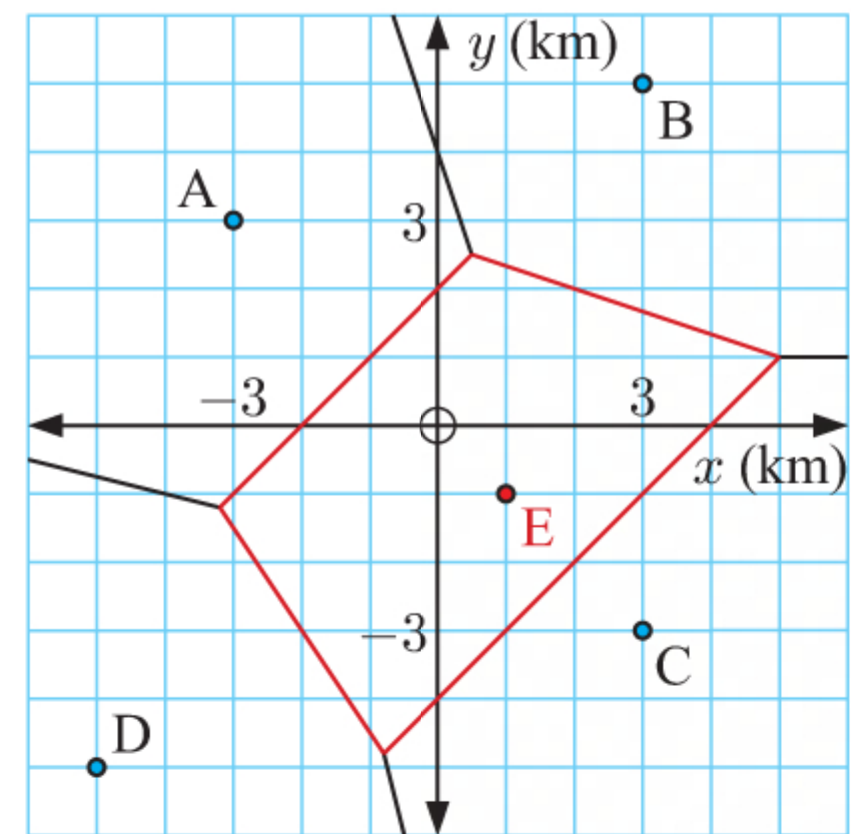
Then we remove the segment of the edge from the original Voronoi diagram which now lies within cell D, giving us the Voronoi diagram for A, B, C, and D.



- 5 a**
- i** $(-2, -1)$ lies in cell A, so it is closest to ATM A.
 - ii** $(5, 2)$ lies in cell B, so it is closest to ATM B.
- b i** We construct $PB(A, E)$, $PB(B, E)$, $PB(C, E)$, and $PB(D, E)$ within the original cells A, B, C, and D respectively.

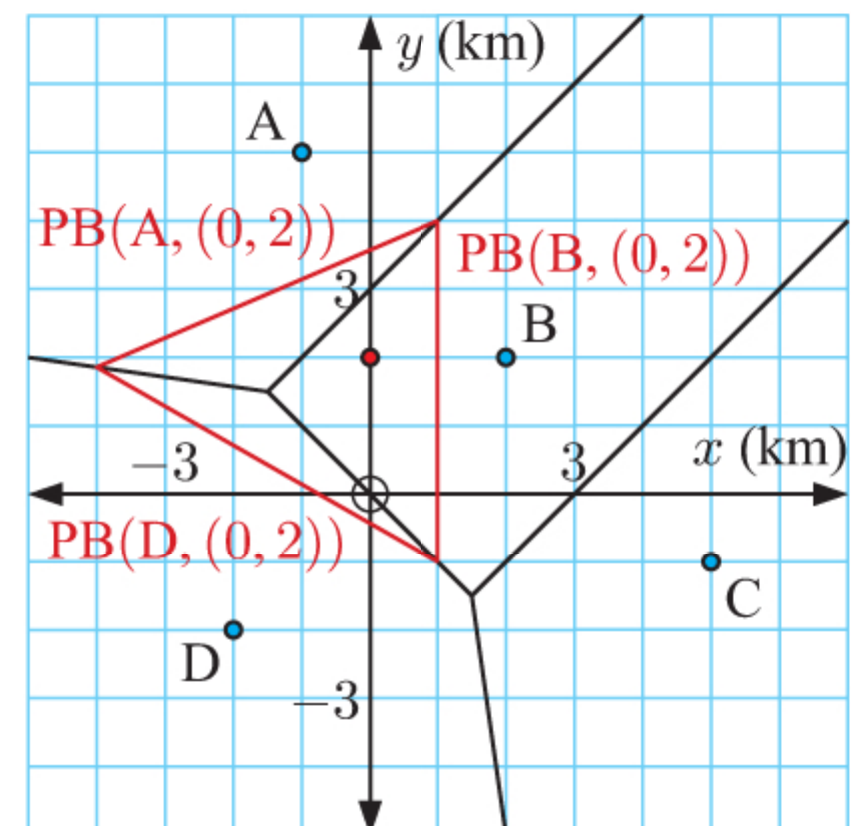


We then remove the segments of edges which now lie within cell E, giving us the Voronoi diagram which includes site E.

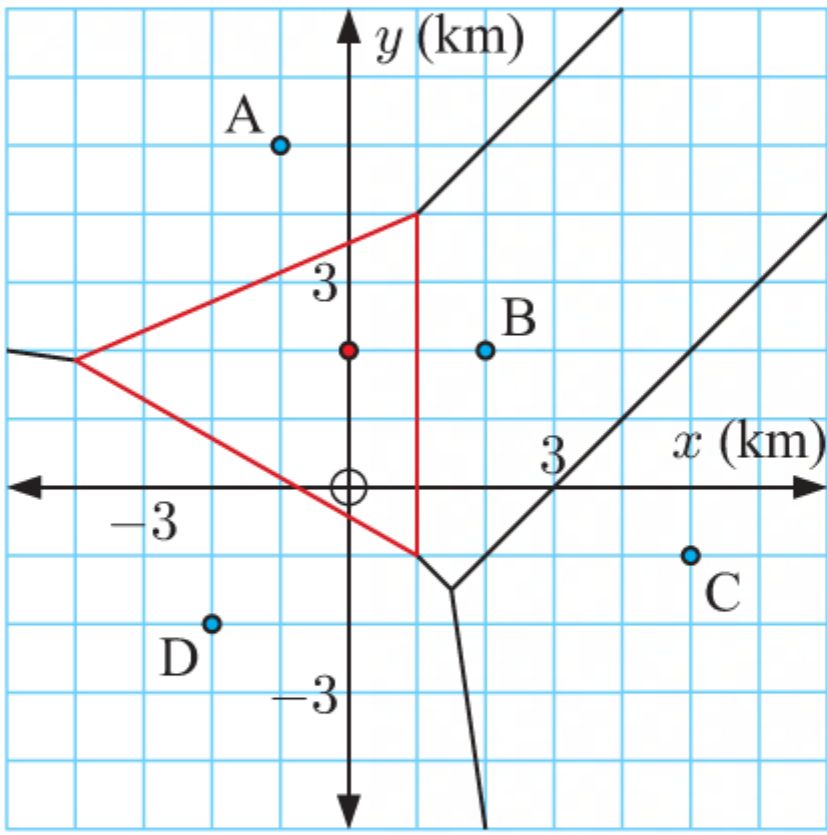


- ii** Yes, there are residents whose nearest ATM has changed from D to E, as part of the original cell D now lies in cell E.
- iii** If Morris is equally close to ATMs B, C, and E, then he is at the vertex adjacent to cells B, C, and E.
We observe from the Voronoi diagram in **i** that this is the point $(5, 1)$.
Morris is therefore located at $(5, 1)$.

- 6 a**
- i** $(-3, 2)$ lies in cell A, so polling booth A is the closest.
 - ii** $(1, -4)$ lies in cell D, so polling booth D is the closest.
- b i** We construct $PB(A, (0, 2))$, $PB(B, (0, 2))$, and $PB(D, (0, 2))$ within the original cells A, B, and D respectively.



We then remove the segments of edges which now lie within the new cell, giving us the Voronoi diagram which includes the polling booth at $(0, 2)$.



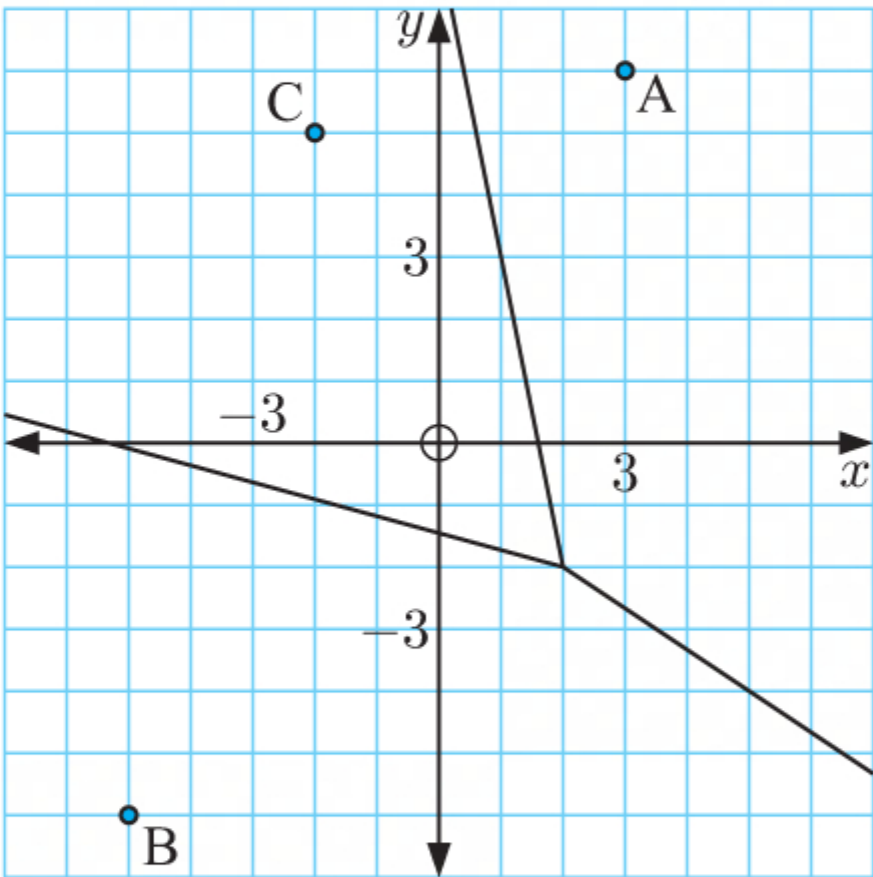
- ii We see from the Voronoi diagram in i that cell C will be unaffected, as it does not contain any points which are closer to site $(0, 2)$ than to site C.
- iii From the diagram in i, the new cell is a triangle with base 5 units and height 5 units.
 \therefore area of new cell $= \frac{1}{2} \times 5 \times 5 = 12.5 \text{ km}^2$.

EXERCISE 17D

1

Location	Temperature ($^{\circ}\text{C}$)
A	28.4
B	25.6
C	27.3

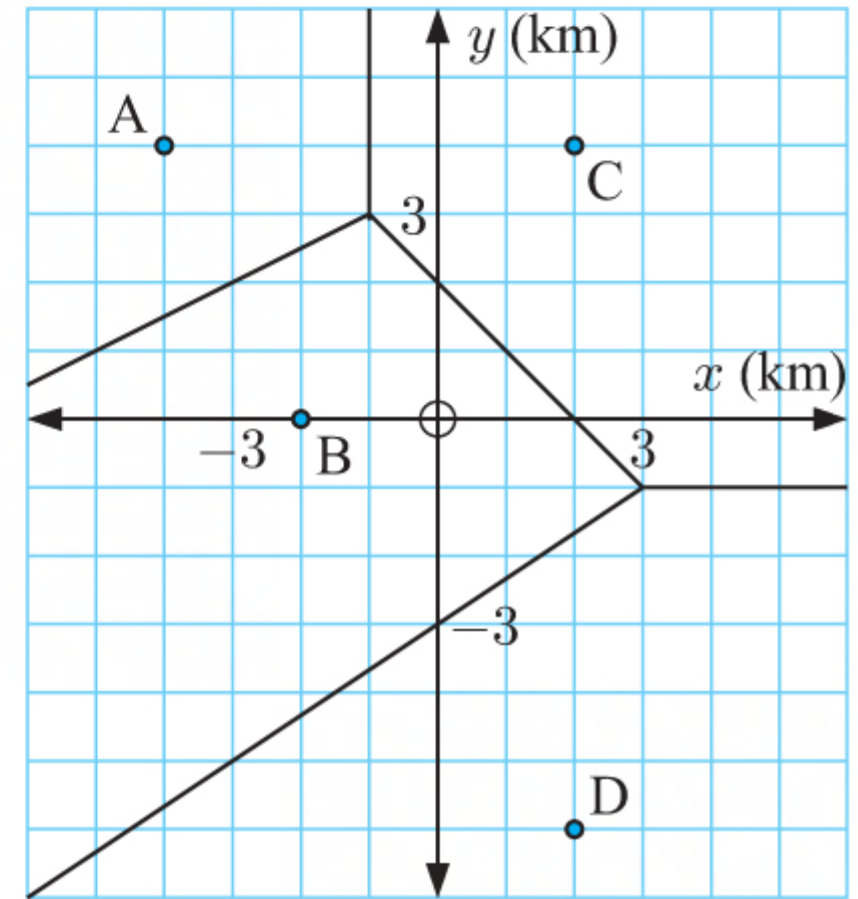
- a $(1, 0)$ is closest to C, so we estimate a temperature of 27.3°C at 3 pm at $(1, 0)$.
- b $(-3, -1)$ is closest to B, so we estimate a temperature of 25.6°C at 3 pm at $(-3, -1)$.
- c $(5, -2)$ is closest to A, so we estimate a temperature of 28.4°C at 3 pm at $(5, -2)$.



2

Location	Elevation (m)
A(-4, 4)	57
B(-2, 0)	48
C(2, 4)	55
D(2, -6)	36

- a** We draw $PB(A, B)$, $PB(A, C)$, $PB(B, C)$, $PB(B, D)$, and $PB(C, D)$, then remove segments which do not form part of the Voronoi diagram.

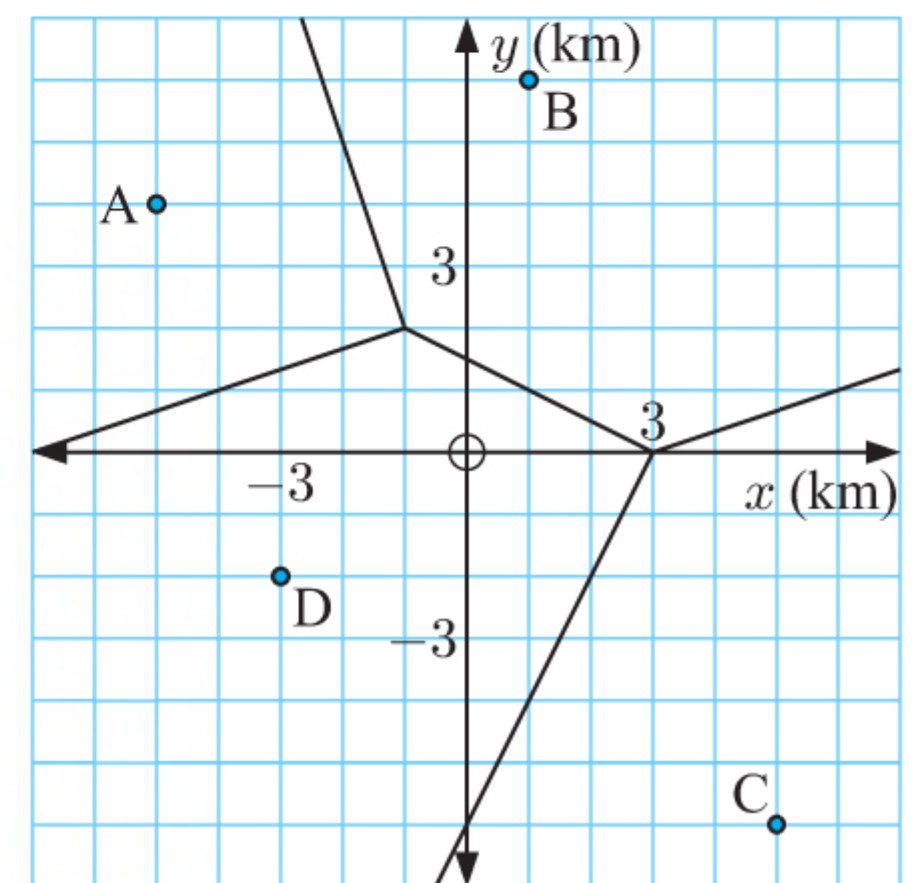


- b**
- i** $(0, 1)$ is closest to B, so we estimate an elevation of 48 m at $(0, 1)$.
 - ii** $(-4, 2)$ is closest to A, so we estimate an elevation of 57 m at $(-4, 2)$.
 - iii** $(3, -4)$ is closest to D, so we estimate an elevation of 36 m at $(3, -4)$.

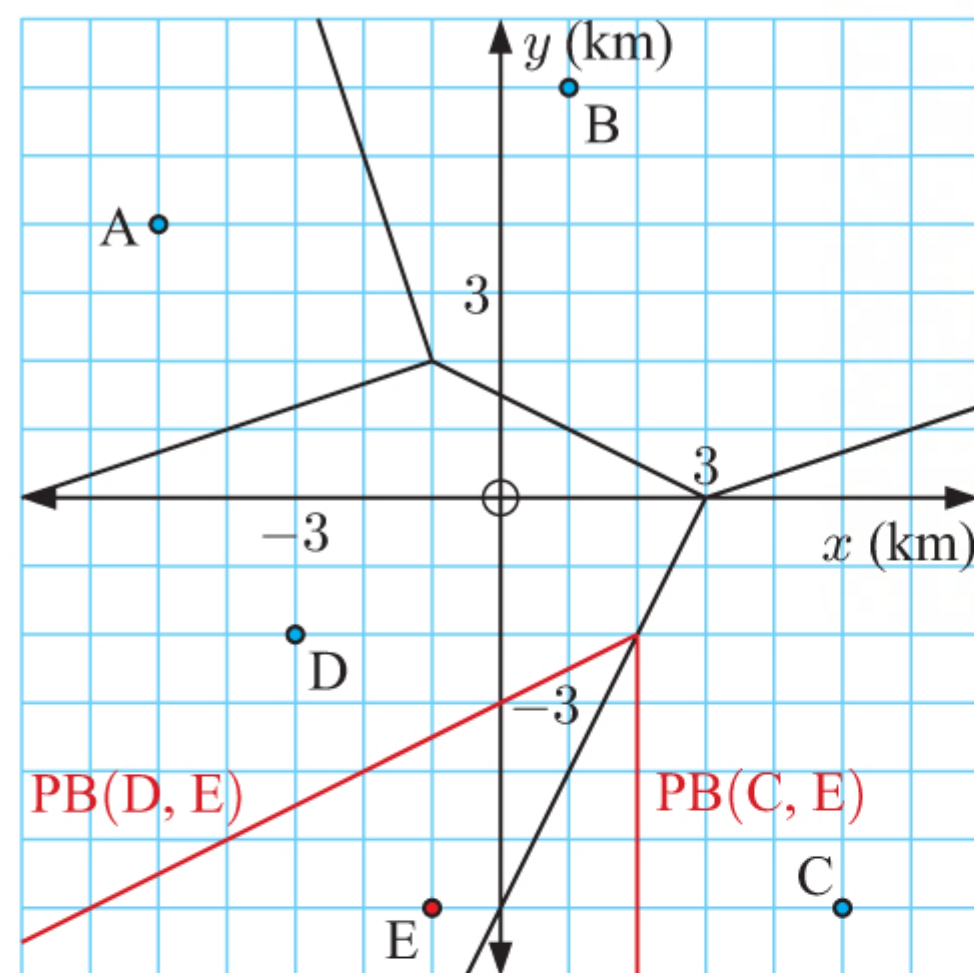
3

Location	Snowfall (inches)
A	7
B	5.5
C	12.2
D	9.3

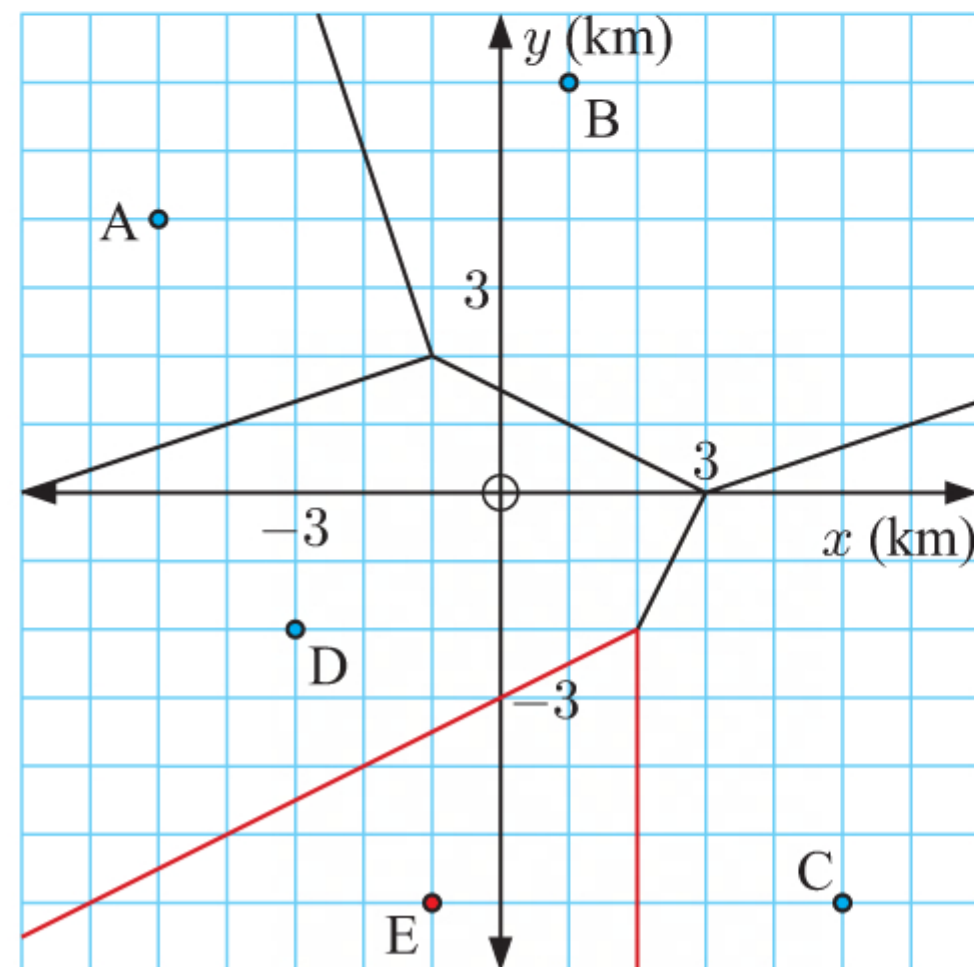
- a**
- i** $(-1, -4)$ is closest to D, so we estimate that 9.3 inches of snow were received at $(-1, -4)$.
 - ii** $(2, 1)$ is closest to B, so we estimate that 5.5 inches of snow were received at $(2, 1)$.
 - iii** $(1, -4)$ is equally closest to C and D, so we estimate that $\frac{12.2 + 9.3}{2} = 10.75$ inches of snow were received at $(1, -4)$.



- b i** We construct $PB(C, E)$ and $PB(D, E)$ within the original cells C and D respectively.



We then remove the edge segment from the original Voronoi diagram which now lies within cell E , giving us the Voronoi diagram which includes location E .



- ii** Both $(-1, -4)$ and $(1, -4)$ from **a i** and **a iii** respectively are now closest to location E .

So, we would now estimate 10.6 inches of snow at both locations.

EXERCISE 17E

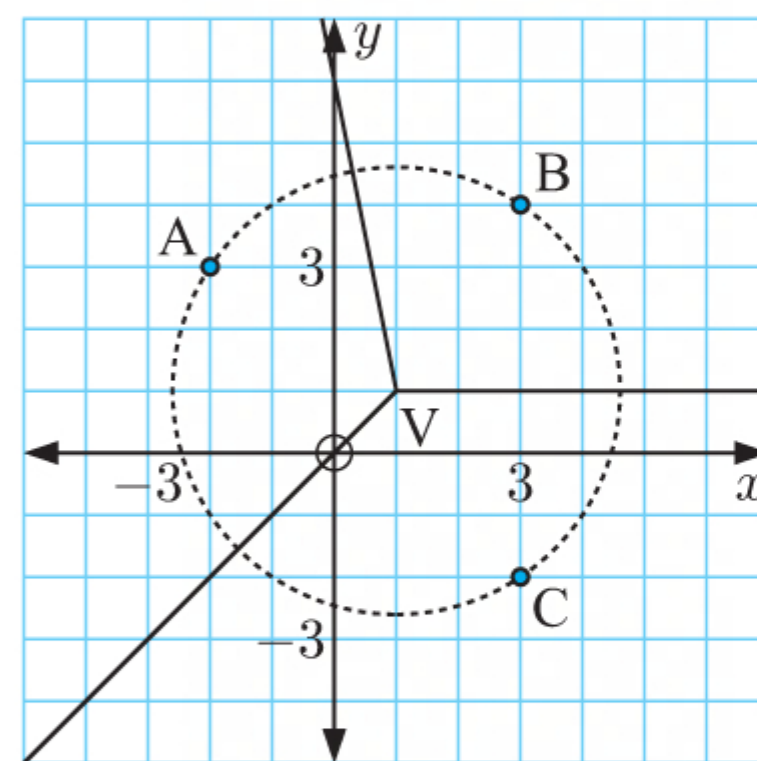
- 1 a** The Voronoi diagram has vertex $V(1, 1)$.

V is equidistant from A , B , and C .

A has coordinates $(-2, 3)$, so

$$\begin{aligned} VA &= \sqrt{(-2-1)^2 + (3-1)^2} \\ &= \sqrt{(-3)^2 + 2^2} \\ &= \sqrt{13} \text{ units} \end{aligned}$$

So the largest empty circle has centre $V(1, 1)$ and radius $\sqrt{13}$ units.



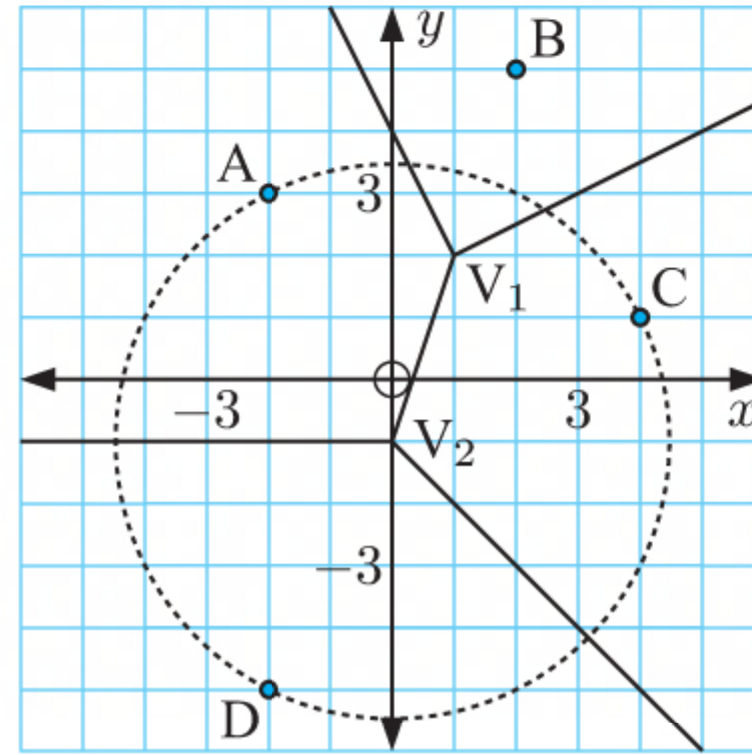
- b** The Voronoi diagram has vertices $V_1(1, 2)$ and $V_2(0, -1)$.

V_1 is equidistant from A, B, and C, and V_2 is equidistant from A, C, and D.

A has coordinates $(-2, 3)$.

$$\begin{aligned} V_1A &= \sqrt{(-2-1)^2 + (3-2)^2} \\ &= \sqrt{(-3)^2 + 1^2} \\ &= \sqrt{10} \text{ units} \end{aligned}$$

$$\begin{aligned} V_2A &= \sqrt{(-2-0)^2 + (3-(-1))^2} \\ &= \sqrt{(-2)^2 + 4^2} \\ &= \sqrt{20} = 2\sqrt{5} \text{ units} \end{aligned}$$



So, the largest empty circle has centre $V_2(0, -1)$ and radius $2\sqrt{5}$ units.

- 2 a** The Voronoi diagram has vertices $V_1(20, 30)$, $V_2(20, 10)$, and $V_3(8, 2)$.

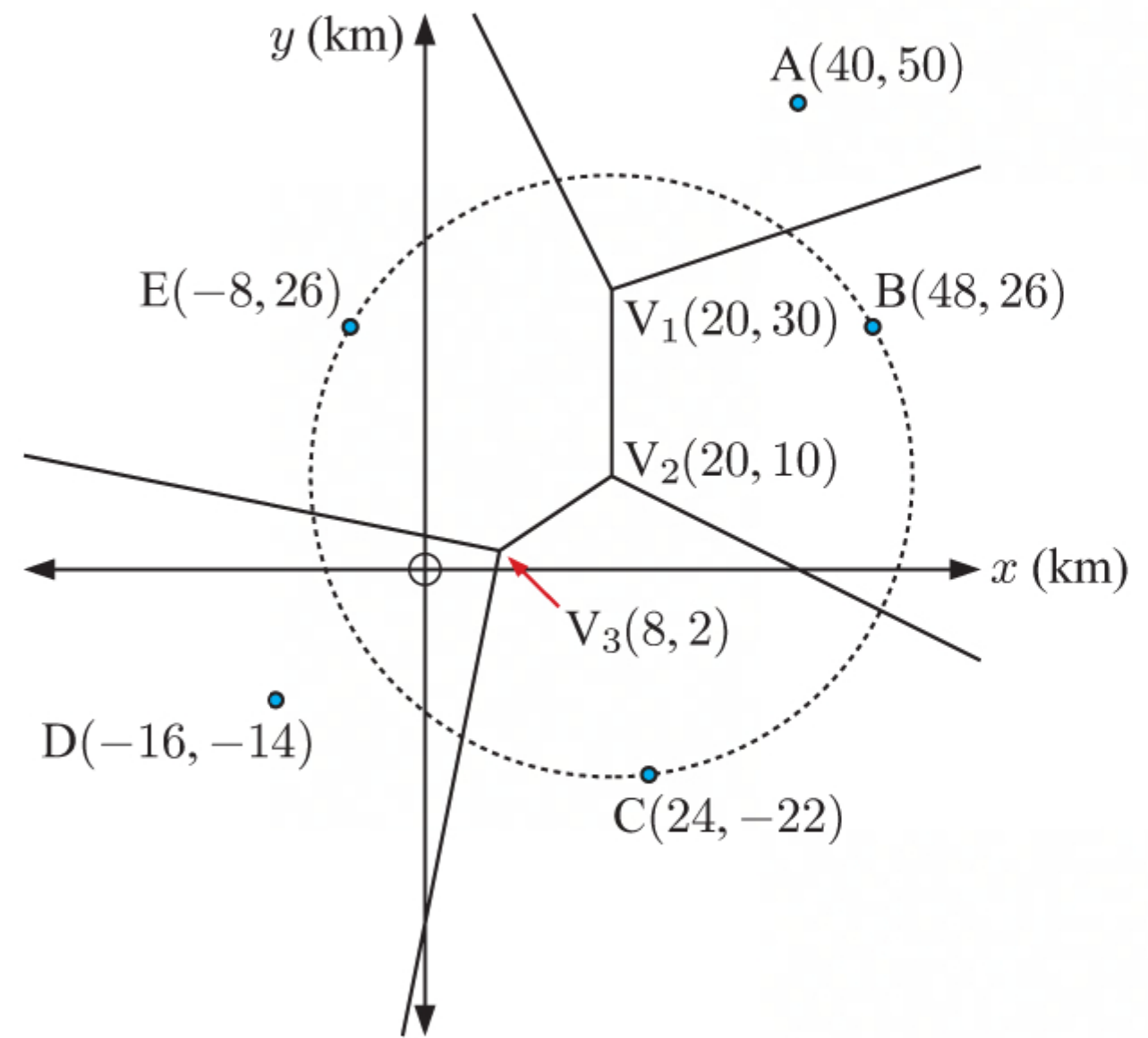
V_1 is equidistant from A, B, and E, V_2 is equidistant from B, C, and E, and V_3 is equidistant from C, D, and E.

E has coordinates $(-8, 26)$.

$$\begin{aligned} V_1E &= \sqrt{(-8-20)^2 + (26-30)^2} \\ &= \sqrt{(-28)^2 + (-4)^2} \\ &= \sqrt{800} = 20\sqrt{2} \text{ km} \end{aligned}$$

$$\begin{aligned} V_2E &= \sqrt{(-8-20)^2 + (26-10)^2} \\ &= \sqrt{(-28)^2 + 16^2} \\ &= \sqrt{1040} = 4\sqrt{65} \text{ km} \end{aligned}$$

$$\begin{aligned} V_3E &= \sqrt{(-8-8)^2 + (26-2)^2} \\ &= \sqrt{(-16)^2 + 24^2} \\ &= \sqrt{832} = 8\sqrt{13} \text{ km} \end{aligned}$$



So, the largest empty circle has centre $V_2(20, 10)$, and radius $4\sqrt{65}$ km.

\therefore the rubbish dump should be established at $(20, 10)$.

- b** From **a**, the largest empty circle has radius $4\sqrt{65}$ km.

\therefore the dump is $4\sqrt{65} \approx 32.2$ km from the nearest town.

- c** $V_2(20, 10)$ is adjacent to cells B, C, and E.

So towns B, C, and E are closest to the rubbish dump.

- d** To show that the location found in **a** is preferable, we must show that there is at least one town which is less than ≈ 32.2 km from $(25, 15)$.

$$\begin{aligned} \text{The distance from } (25, 15) \text{ to } B(48, 26) &\text{ is } \sqrt{(48-25)^2 + (26-15)^2} \\ &= \sqrt{23^2 + 11^2} \\ &= \sqrt{650} \approx 25.5 \text{ km} \end{aligned}$$

So, B is closer to $(25, 15)$ than the answer found in **a** is to any other town.

- 3** Let P lie in cell X . The largest empty circle centred at P would touch X . As P lies within cell X , the largest empty circle does not touch any other site, meaning that a larger empty circle could be created with centre closer to the edge of the cell.

\therefore the circle centred at P is not the largest empty circle.

\therefore the largest empty circle cannot lie within a cell.

- 4 a i** $A(-15, 10)$, $D(-3, -18)$

The midpoint of $[AD]$ is $\left(\frac{-15 + -3}{2}, \frac{10 + -18}{2}\right)$ or $(-9, -4)$.

The gradient of $[AD]$ is $\frac{-18 - 10}{-3 - -15} = \frac{-28}{12} = -\frac{7}{3}$.

So, $PB(A, D)$ has gradient $\frac{3}{7}$ and passes through $(-9, -4)$.

\therefore its equation is $3x - 7y = 3(-9) - 7(-4)$

which is $7y = 3x - 1$

or $y = \frac{3}{7}x - \frac{1}{7}$

- ii** $C(15, 0)$, $D(-3, -18)$

The midpoint of $[CD]$ is $\left(\frac{15 + -3}{2}, \frac{0 + -18}{2}\right)$ or $(6, -9)$.

The gradient of $[CD]$ is $\frac{-18 - 0}{-3 - 15} = \frac{-18}{-18} = 1$.

So, $PB(C, D)$ has gradient -1 and passes through $(6, -9)$.

\therefore its equation is $x + y = 6 + (-9)$

or $y = -x - 3$

- b** $PB(A, D)$ and $PB(C, D)$ intersect where $\frac{3}{7}x - \frac{1}{7} = -x - 3$

$$\therefore 3x - 1 = -7x - 21$$

$$\therefore 10x = -20$$

$$\therefore x = -2$$

When $x = -2$, $y = -(-2) - 3 = -1$

So, V_2 has coordinates $(-2, -1)$.

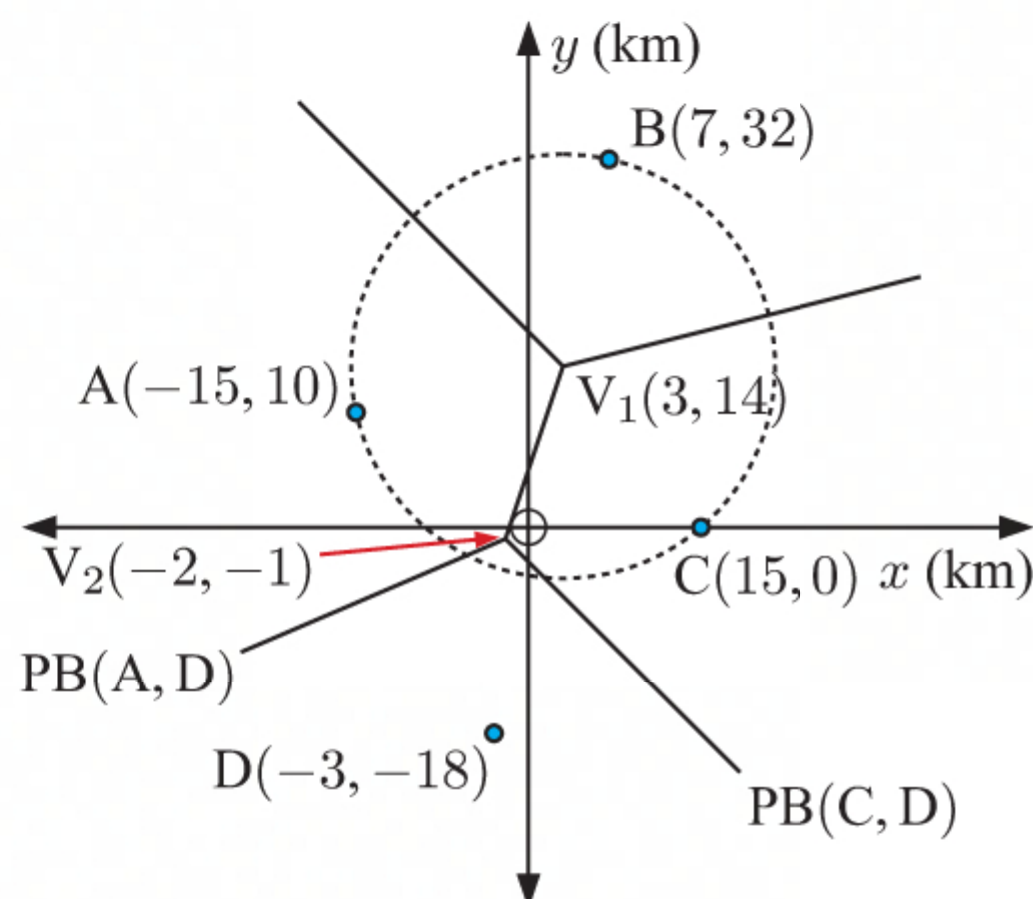
- c** From **b**, the Voronoi diagram has vertices $V_1(3, 14)$ and $V_2(-2, -1)$.

V_1 is adjacent to cells A , B , and C , and V_2 is adjacent to vertices A , C , and D .

$$\begin{aligned} V_1C &= \sqrt{(15 - 3)^2 + (0 - 14)^2} \\ &= \sqrt{12^2 + (-14)^2} \\ &= \sqrt{340} \text{ km} \end{aligned}$$

$$\begin{aligned} V_2C &= \sqrt{(15 - (-2))^2 + (0 - (-1))^2} \\ &= \sqrt{17^2 + 1^2} \\ &= \sqrt{290} \text{ km} \end{aligned}$$

So, the largest empty circle has centre $V_1(3, 14)$, and radius $\sqrt{340}$ km.



\therefore the optimal position for a toxic waste dump is $(3, 14)$.

5 A(-5, -10), B(11, 18), C(5, -12)

a The midpoint of [AB] is $\left(\frac{-5+11}{2}, \frac{-10+18}{2}\right)$ or (3, 4).

The gradient of [AB] is $\frac{18-(-10)}{11-(-5)} = \frac{28}{16} = \frac{7}{4}$.

So, PB(A, B) has gradient $-\frac{4}{7}$ and passes through (3, 4).

\therefore its equation is $4x + 7y = 4(3) + 7(4)$

which is $7y = -4x + 40$

or $y = -\frac{4}{7}x + \frac{40}{7}$

The midpoint of [AC] is $\left(\frac{-5+5}{2}, \frac{-10+(-12)}{2}\right)$ or (0, -11).

The gradient of [AC] is $\frac{-12-(-10)}{5-(-5)} = \frac{-2}{10} = -\frac{1}{5}$.

So, PB(A, C) has gradient 5 and passes through (0, -11).

\therefore its equation is $5x - y = 5(0) - (-11)$

or $y = 5x - 11$

The midpoint of [BC] is $\left(\frac{11+5}{2}, \frac{18+(-12)}{2}\right)$ or (8, 3).

The gradient of [BC] is $\frac{-12-18}{5-11} = \frac{-30}{-6} = 5$.

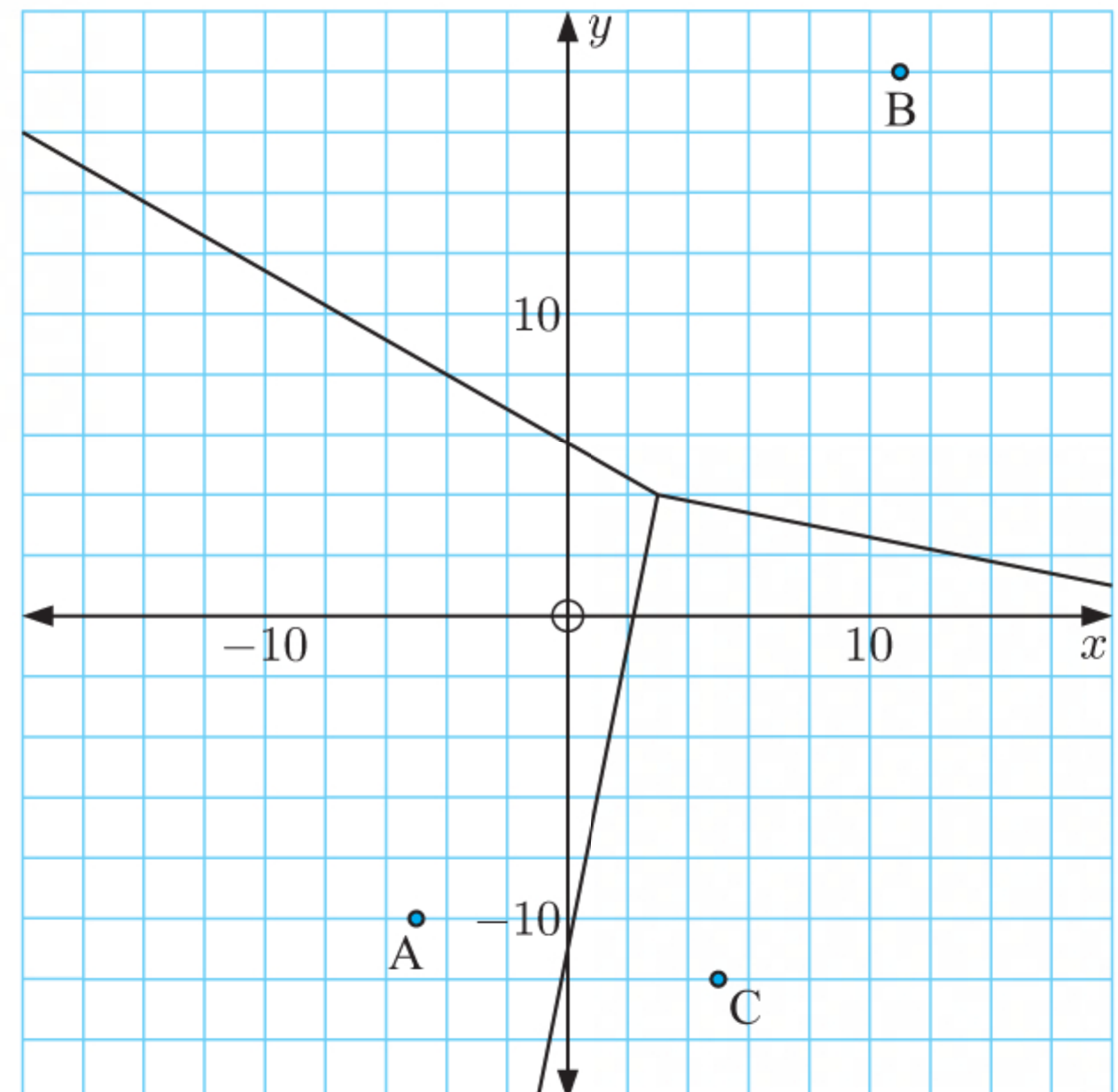
So, PB(B, C) has gradient $-\frac{1}{5}$ and passes through (8, 3).

\therefore its equation is $x + 5y = 1(8) + 5(3)$

which is $5y = -x + 23$

or $y = -\frac{1}{5}x + \frac{23}{5}$

We plot sites A, B, and C on a set of axes. We draw PB(A, B), PB(A, C), and PB(B, C), then remove segments which do not form part of the Voronoi diagram.



- b** $PB(A, C)$ and $PB(B, C)$ intersect where $5x - 11 = -\frac{1}{5}x + \frac{23}{5}$
 $\therefore 25x - 55 = -x + 23$
 $\therefore 26x = 78$
 $\therefore x = 3$

When $x = 3$, $y = 5(3) - 11 = 4$, so the point of intersection is $(3, 4)$.

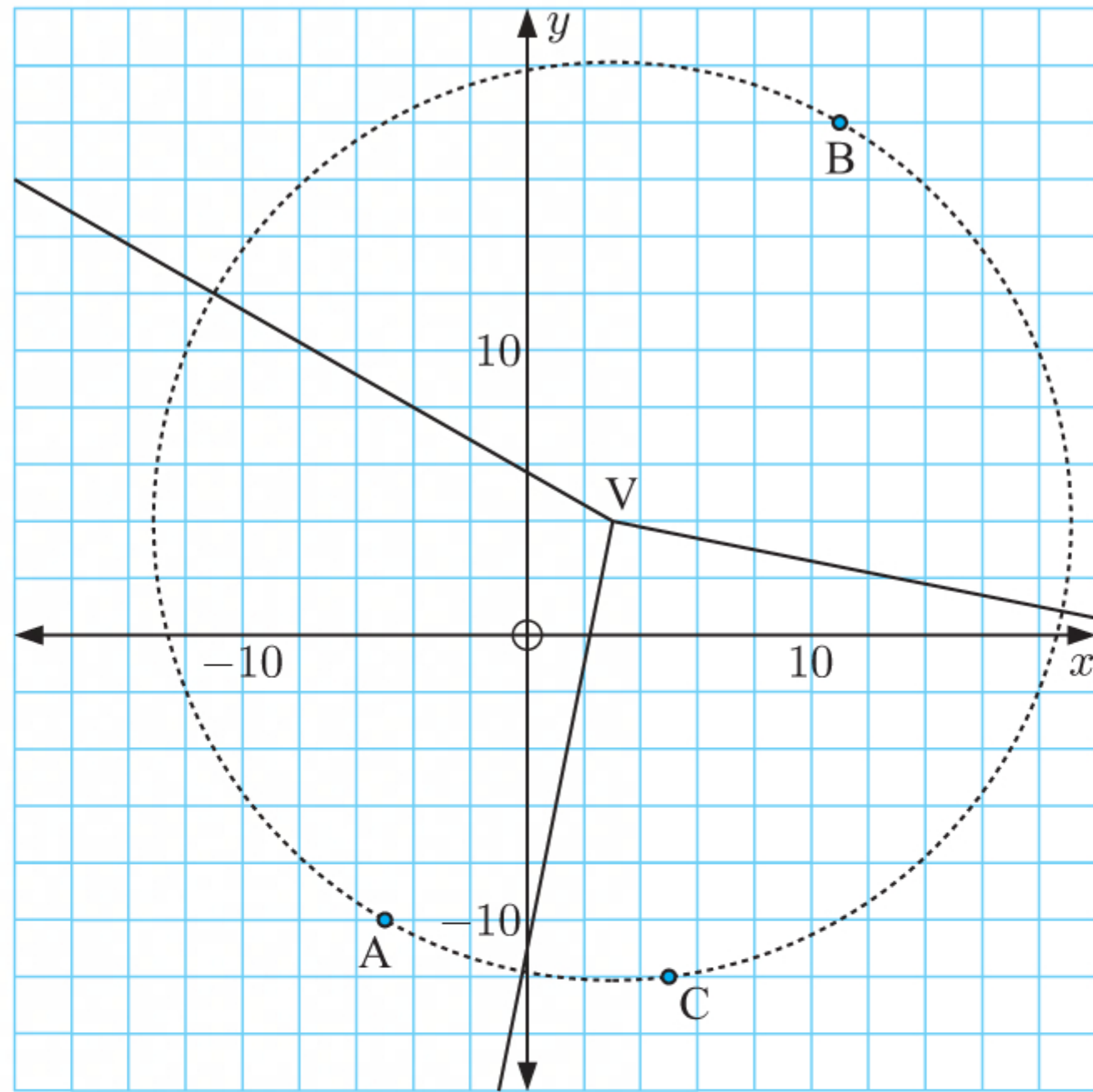
Check: Using the equation of $PB(A, B)$, when $x = 3$, $y = -\frac{4}{7}(3) + \frac{40}{7} = 4$ ✓

So, $(3, 4)$ is the vertex of the Voronoi diagram.

- c** The Voronoi diagram has vertex $V(3, 4)$.
 V is equidistant from A , B , and C .

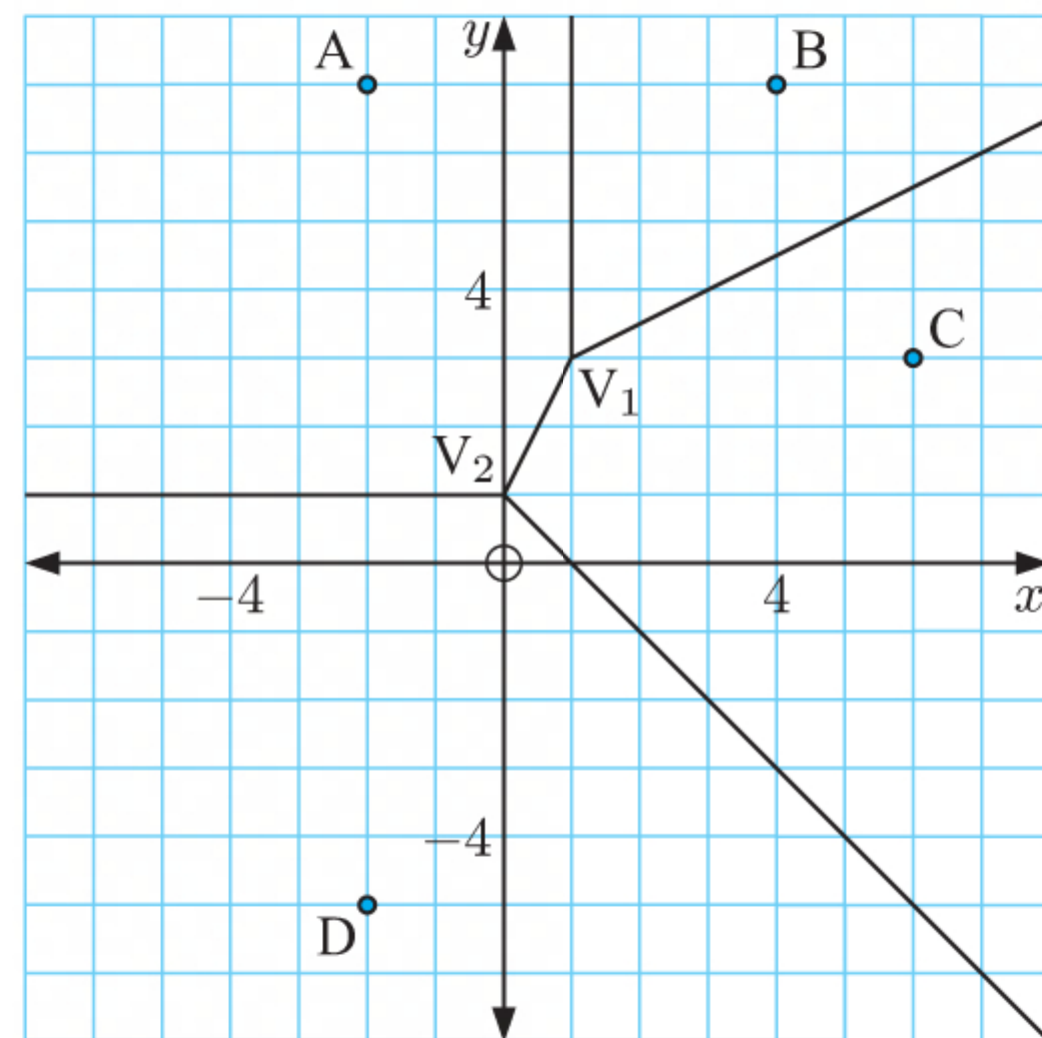
$$\begin{aligned} VA &= \sqrt{(-5-3)^2 + (-10-4)^2} \\ &= \sqrt{(-8)^2 + (-14)^2} \\ &= \sqrt{260} = 2\sqrt{65} \text{ units} \end{aligned}$$

So, the largest empty circle has centre $V(3, 4)$ and radius $2\sqrt{65}$ units.



- 6** $A(-2, 7)$, $B(4, 7)$, $C(6, 3)$, $D(-2, -5)$

- a** We draw $PB(A, B)$, $PB(A, C)$, $PB(A, D)$, $PB(B, C)$, and $PB(C, D)$, then remove segments which do not form part of the Voronoi diagram.



- b** From the Voronoi diagram in **a**, we observe that the vertices are $V_1(1, 3)$ and $V_2(0, 1)$.

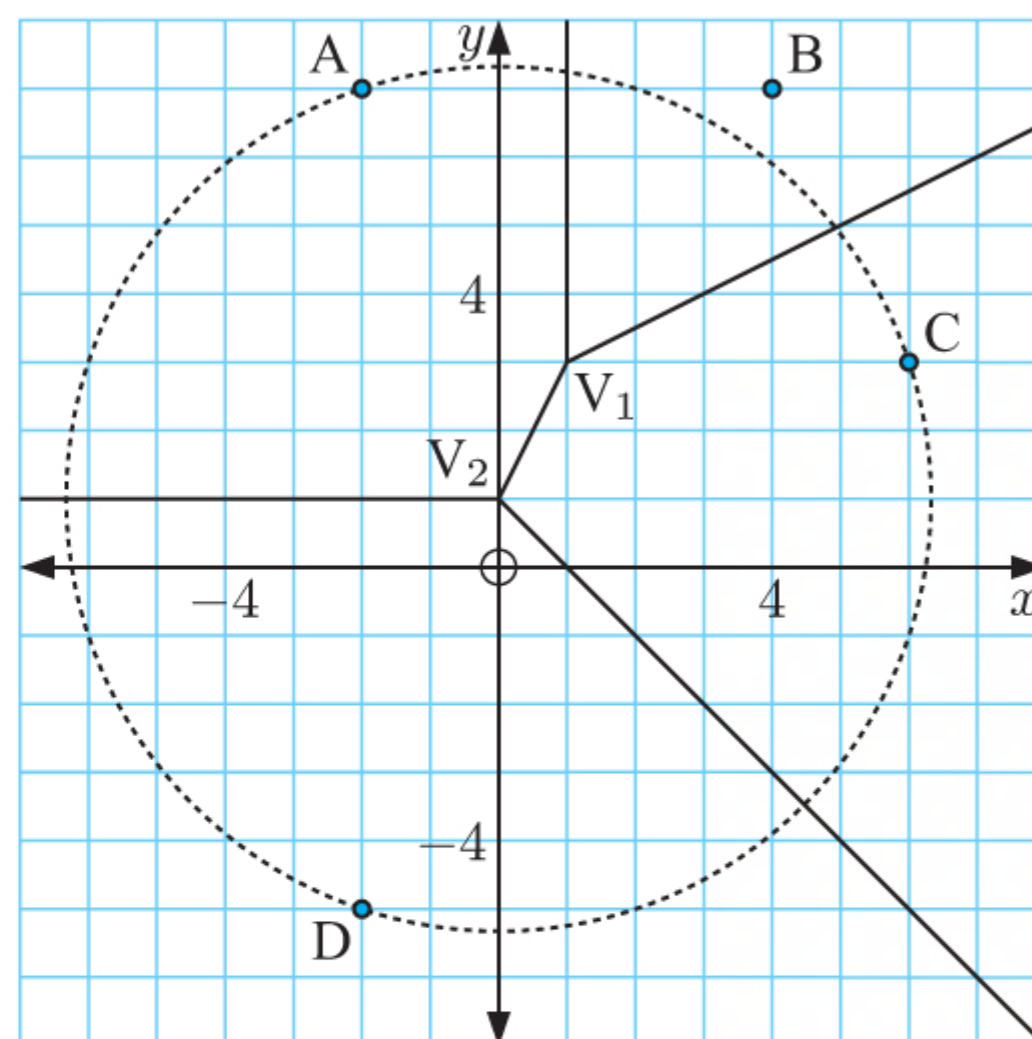
- c** The Voronoi diagram has vertices $V_1(1, 3)$ and $V_2(0, 1)$.

V_1 is equidistant from A, B, and C, and V_2 is equidistant from A, C, and D.

$$\begin{aligned} V_1A &= \sqrt{(-2-1)^2 + (7-3)^2} \\ &= \sqrt{(-3)^2 + 4^2} \\ &= \sqrt{25} = 5 \text{ units} \end{aligned}$$

$$\begin{aligned} V_2A &= \sqrt{(-2-0)^2 + (7-1)^2} \\ &= \sqrt{(-2)^2 + 6^2} \\ &= \sqrt{40} = 2\sqrt{10} \text{ units} \end{aligned}$$

So, the largest empty circle has centre $V_2(0, 1)$ and radius $2\sqrt{10} \approx 6.32$ units.



- 7 a i** $(3, 4)$ lies in cell B, so station B is the closest to $(3, 4)$.
ii $(-3, -3)$ lies in cell E, so station E is the closest to $(-3, -3)$.

- b** $PB(C, E)$ and $PB(D, E)$ intersect at V_3 .

$$C(3, -1), \quad D(-3, -9), \quad E(-5, -1)$$

$$\text{The midpoint of } [CE] \text{ is } \left(\frac{3+(-5)}{2}, \frac{-1+(-1)}{2} \right) \text{ or } (-1, -1).$$

Now $[CE]$ is horizontal.

So, $PB(C, E)$ is vertical and passes through $(-1, -1)$.

\therefore its equation is $x = -1$.

$$\text{The midpoint of } [DE] \text{ is } \left(\frac{-3+(-5)}{2}, \frac{-9+(-1)}{2} \right) \text{ or } (-4, -5).$$

$$\text{The gradient of } [DE] \text{ is } \frac{-1-(-9)}{-5-(-3)} = \frac{8}{-2} = -4.$$

So, $PB(D, E)$ has gradient $\frac{1}{4}$ and passes through $(-4, -5)$.

$$\therefore \text{ its equation is } x - 4y = 1(-4) - 4(-5)$$

$$\text{which is } 4y = x - 16$$

$$\text{or } y = \frac{1}{4}x - 4$$

So, $PB(C, E)$ and $PB(D, E)$ intersect where $x = -1$.

$$\text{When } x = -1, \quad y = \frac{1}{4}(-1) - 4 = -\frac{17}{4}.$$

$$\therefore V_3 \text{ has coordinates } \left(-1, -\frac{17}{4}\right).$$

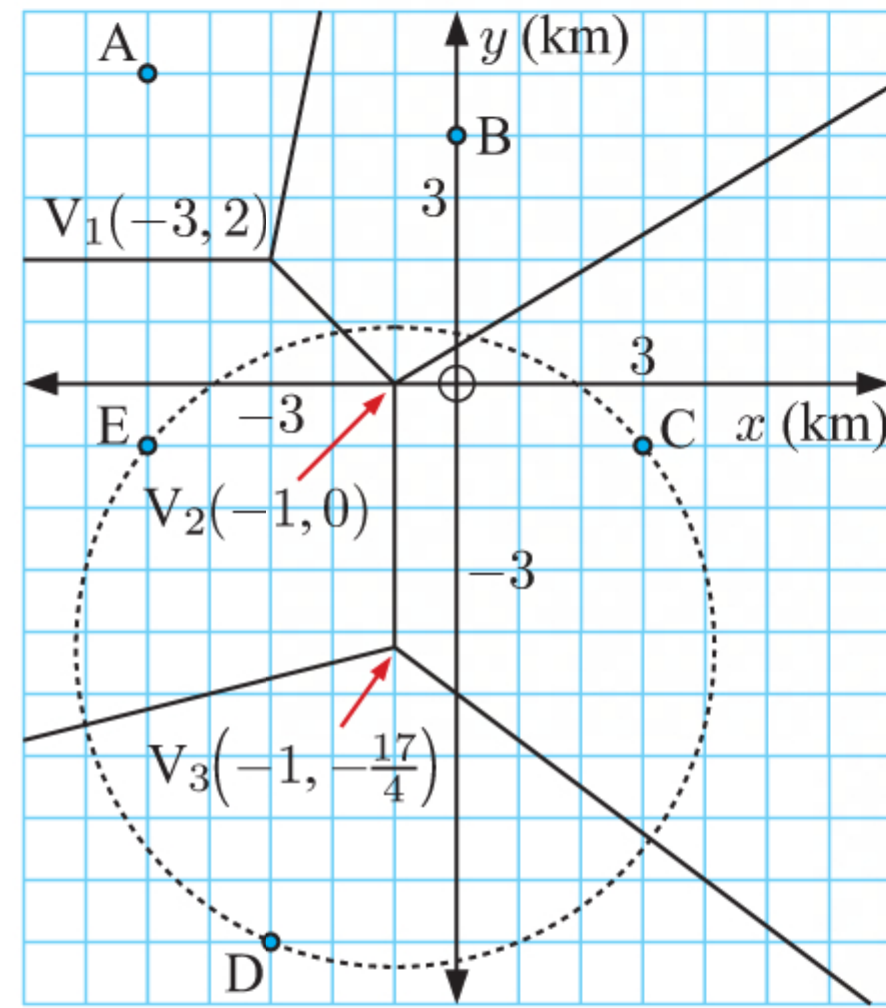
- i** The Voronoi diagram has vertices $V_1(-3, 2)$, $V_2(-1, 0)$, and $V_3(-1, -\frac{17}{4})$.

V_1 is equidistant from stations A, B, and E, V_2 is equidistant from stations B, C, and E, and V_3 is equidistant from stations C, D, and E.

$$\begin{aligned} V_1E &= \sqrt{(-5 - (-3))^2 + (-1 - 2)^2} \\ &= \sqrt{(-2)^2 + (-3)^2} \\ &= \sqrt{13} \approx 3.61 \text{ km} \end{aligned}$$

$$\begin{aligned} V_2E &= \sqrt{(-5 - (-1))^2 + (-1 - 0)^2} \\ &= \sqrt{(-4)^2 + (-1)^2} \\ &= \sqrt{17} \approx 4.12 \text{ km} \end{aligned}$$

$$\begin{aligned} V_3E &= \sqrt{(-5 - (-1))^2 + (-1 - (-\frac{17}{4}))^2} \\ &= \sqrt{(-4)^2 + (\frac{13}{4})^2} \\ &= \sqrt{\frac{425}{16}} = \frac{5\sqrt{17}}{4} \approx 5.15 \text{ km} \end{aligned}$$



So, the largest empty circle has centre $V_3(-1, -\frac{17}{4})$ and radius $\frac{5\sqrt{17}}{4} \approx 5.15$ km.

\therefore the new police station should be built at $(-1, -\frac{17}{4})$.

- ii** Before the new police station is built, $(-3, -3)$ lies in cell E, so its closest station is E.

$$\begin{aligned} \text{The distance from } (-3, -3) \text{ to } E(-5, -1) &= \sqrt{(-5 - (-3))^2 + (-1 - (-3))^2} \\ &= \sqrt{(-2)^2 + 2^2} \\ &= \sqrt{8} \approx 2.82 \text{ km} \end{aligned}$$

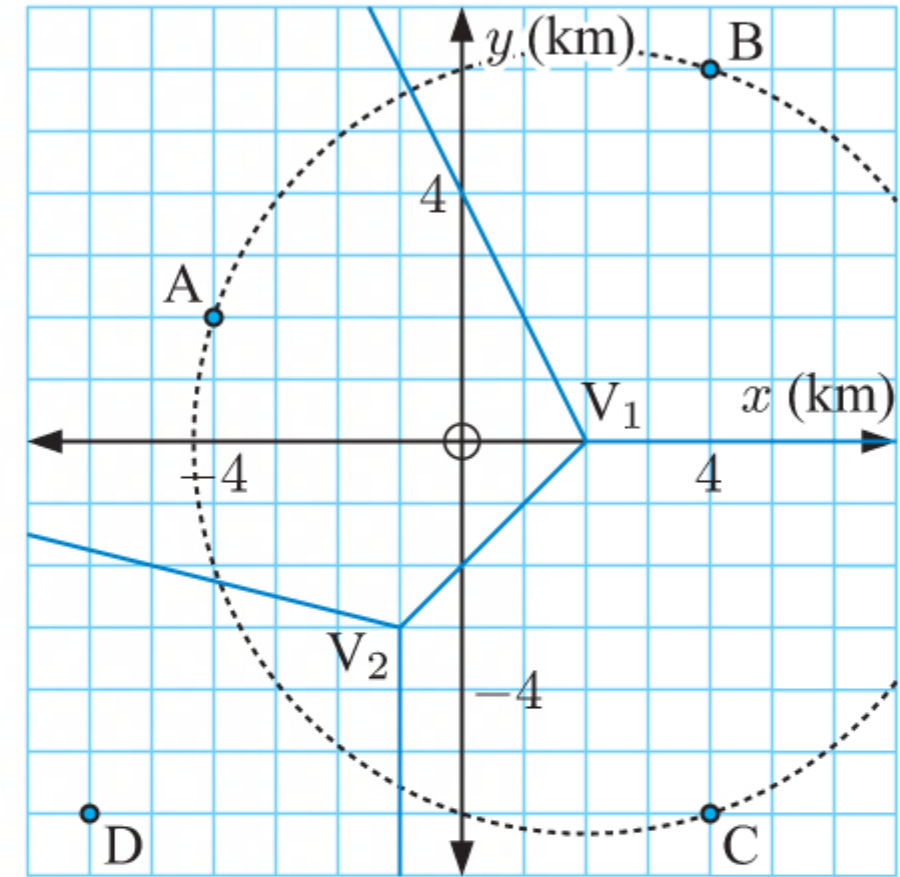
$$\begin{aligned} \text{The distance from } (-3, -3) \text{ to } V_3(-1, -\frac{17}{4}) &= \sqrt{(-1 - (-3))^2 + (-\frac{17}{4} - (-3))^2} \\ &= \sqrt{2^2 + (-\frac{5}{4})^2} \\ &= \sqrt{\frac{89}{16}} \approx 2.36 \text{ km} \end{aligned}$$

So, the new station will now be the closest station to $(-3, -3)$.

- 8 a** The Voronoi diagram has vertices $V_1(2, 0)$ and $V_2(-1, -3)$.
 V_1 is adjacent to cells A, B, and C, and V_2 is adjacent to cells A, C, and D.
 A has coordinates $(-4, 2)$.

$$\begin{aligned} V_1A &= \sqrt{(-4-2)^2 + (2-0)^2} \\ &= \sqrt{(-6)^2 + 2^2} \\ &= \sqrt{40} = 2\sqrt{10} \text{ km} \end{aligned}$$

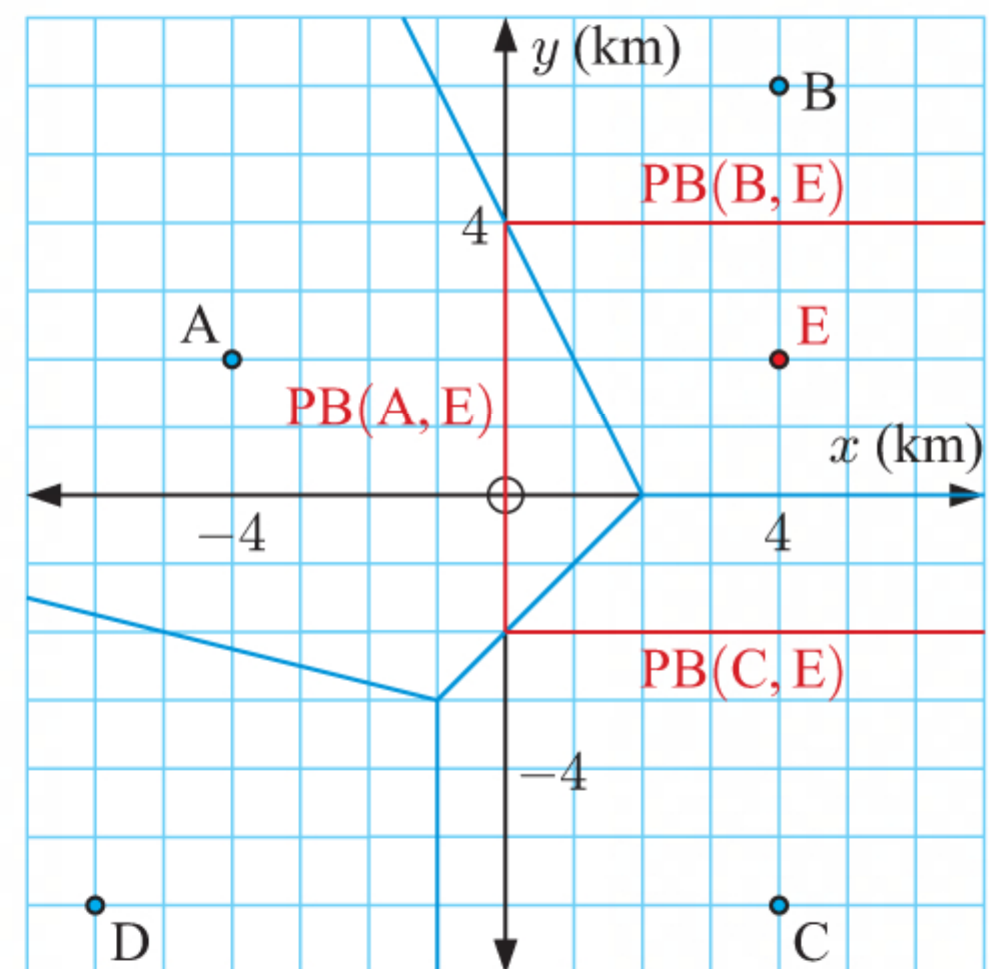
$$\begin{aligned} V_2A &= \sqrt{(-4-(-1))^2 + (2-(-3))^2} \\ &= \sqrt{(-3)^2 + 5^2} \\ &= \sqrt{34} \text{ km} \end{aligned}$$



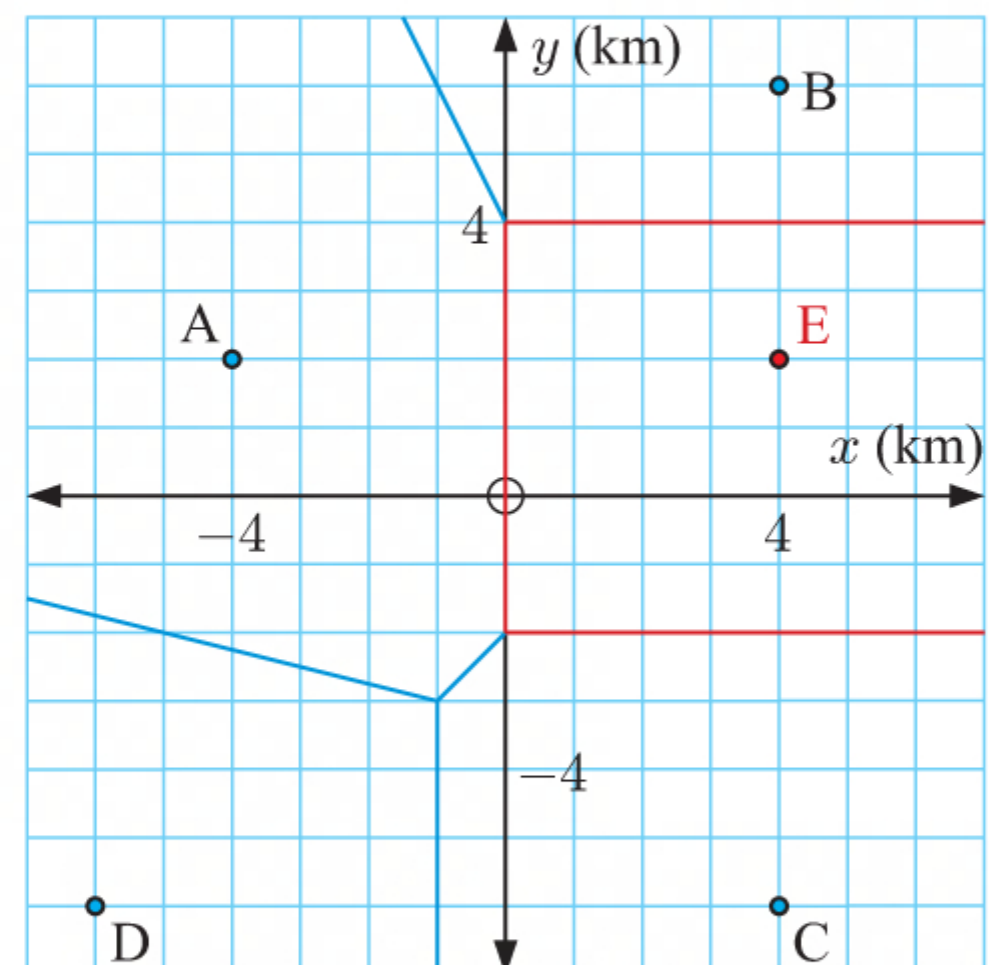
So, the largest empty circle has centre $V_1(2, 0)$ and radius $2\sqrt{10}$ km.

\therefore the optimal position for Brigitte's store is $(2, 0)$, which is $2\sqrt{10} \approx 6.32$ km from the nearest competitor.

- b i** Let E be the site with coordinates $(4, 2)$.
 We construct $PB(A, E)$, $PB(B, E)$, and $PB(C, E)$ within the original cells A, B, and C respectively.



We then remove the segments of edges from the original Voronoi diagram which now lie within cell E, giving us the Voronoi diagram with the new store at $(4, 2)$.



- ii One vertex of the original Voronoi diagram remains, $V_2(-1, -3)$, and so is still $\sqrt{34}$ km from its nearest stores.

The new Voronoi diagram also has vertices $V_3(0, 4)$ and $V_4(0, -2)$.

V_3 is adjacent to A, B, and E, and V_4 is adjacent to A, C, and E.

$$\begin{aligned} V_3A &= \sqrt{(-4-0)^2 + (2-4)^2} \\ &= \sqrt{(-4)^2 + (-2)^2} \\ &= \sqrt{20} = 2\sqrt{5} \text{ km} \end{aligned}$$

$$\begin{aligned} V_4A &= \sqrt{(-4-0)^2 + (2-(-2))^2} \\ &= \sqrt{(-4)^2 + 4^2} \\ &= \sqrt{32} = 4\sqrt{2} \text{ km} \end{aligned}$$

So, the largest empty circle has centre $V_2(-1, -3)$ and radius $\sqrt{34}$ km.

\therefore the new optimal position for Brigitte's store is $(-1, -3)$.

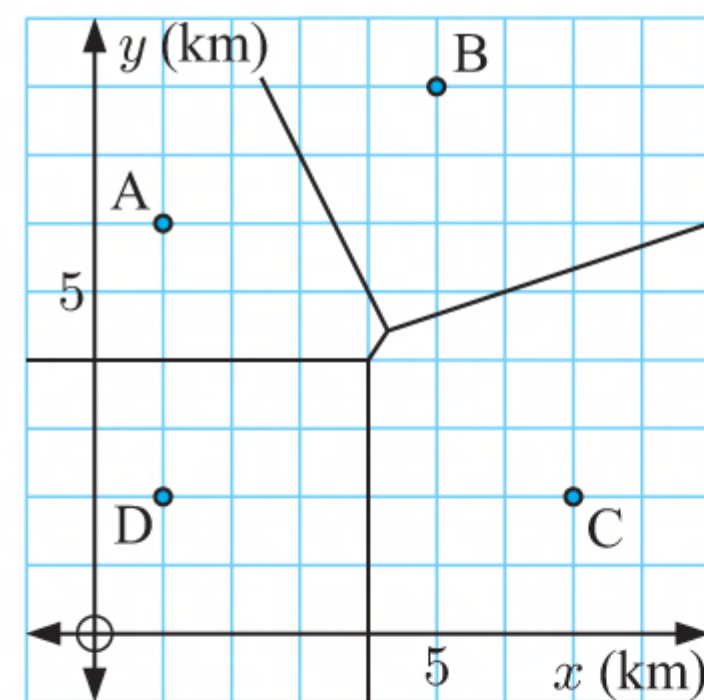
- 9 a To determine which fire station is closest to a particular point, we first draw a Voronoi diagram.

$A(1, 6)$, $B(5, 8)$, $C(7, 2)$, $D(1, 2)$

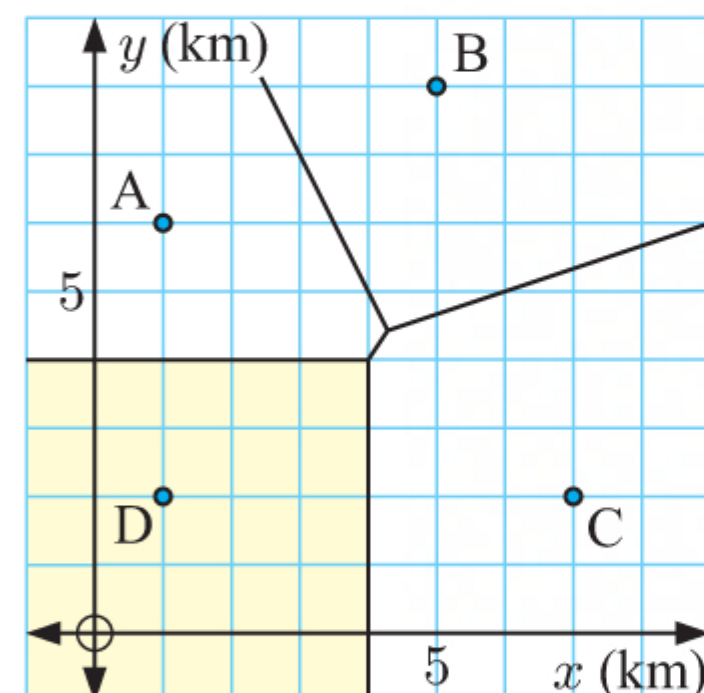
We draw $PB(A, B)$, $PB(A, C)$, $PB(A, D)$, $PB(B, C)$, and $PB(C, D)$, then remove segments which do not form part of the Voronoi diagram.

We observe from the Voronoi diagram that:

- i $(6, 3)$ lies in cell C, so fire station C is closest to $(6, 3)$.
- ii $(4, 6)$ lies in cell B, so fire station B is closest to $(4, 6)$.



- b The region closest to fire station D is shaded in the Voronoi diagram alongside.



- c We could create a Voronoi diagram like in a.
- d The vertex adjacent to cells A, C, and D is $V_1(4, 4)$.
The vertex adjacent to cells A, B, and C is where $PB(A, B)$ and $PB(B, C)$ intersect.
 $PB(A, B)$ has gradient -2 and passes through $(4, 5)$.
 \therefore its equation is $2x + y = 2(4) + 5$
or $y = -2x + 13$

PB(B, C) has gradient $\frac{1}{3}$ and passes through (6, 5).

\therefore its equation is $x - 3y = 1(6) - 3(5)$

which is $3y = x + 9$

or $y = \frac{1}{3}x + 3$

PB(A, B) intersects PB(B, C) where $-2x + 13 = \frac{1}{3}x + 3$

$\therefore -\frac{7}{3}x = -10$

$\therefore x = \frac{30}{7}$

When $x = \frac{30}{7}$, $y = \frac{1}{3}\left(\frac{30}{7}\right) + 3 = \frac{31}{7}$.

So, V_2 has coordinates $\left(\frac{30}{7}, \frac{31}{7}\right)$.

$$V_1A = \sqrt{(1-4)^2 + (6-4)^2}$$

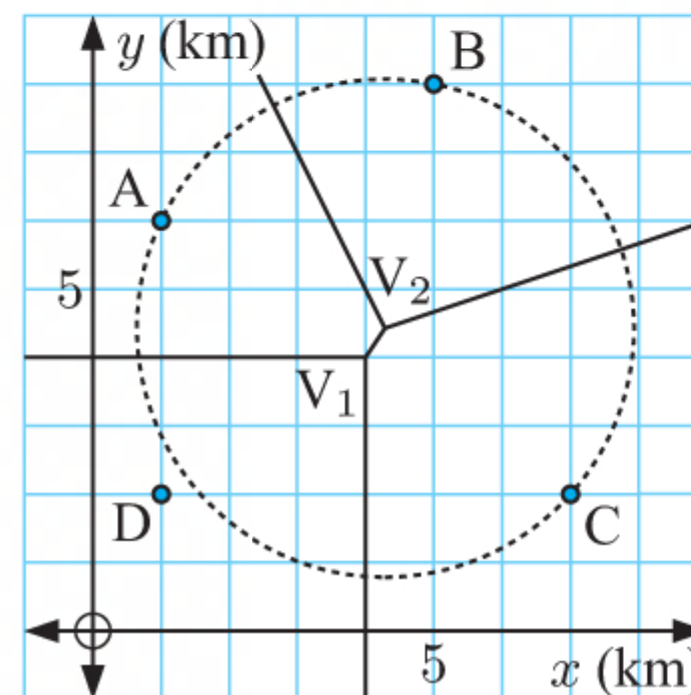
$$= \sqrt{(-3)^2 + 2^2}$$

$$= \sqrt{13} \approx 3.61 \text{ km}$$

$$V_2A = \sqrt{\left(1 - \frac{30}{7}\right)^2 + \left(6 - \frac{31}{7}\right)^2}$$

$$= \sqrt{\left(-\frac{23}{7}\right)^2 + \left(\frac{11}{7}\right)^2}$$

$$= \sqrt{\frac{650}{49}} \approx 3.64 \text{ km}$$

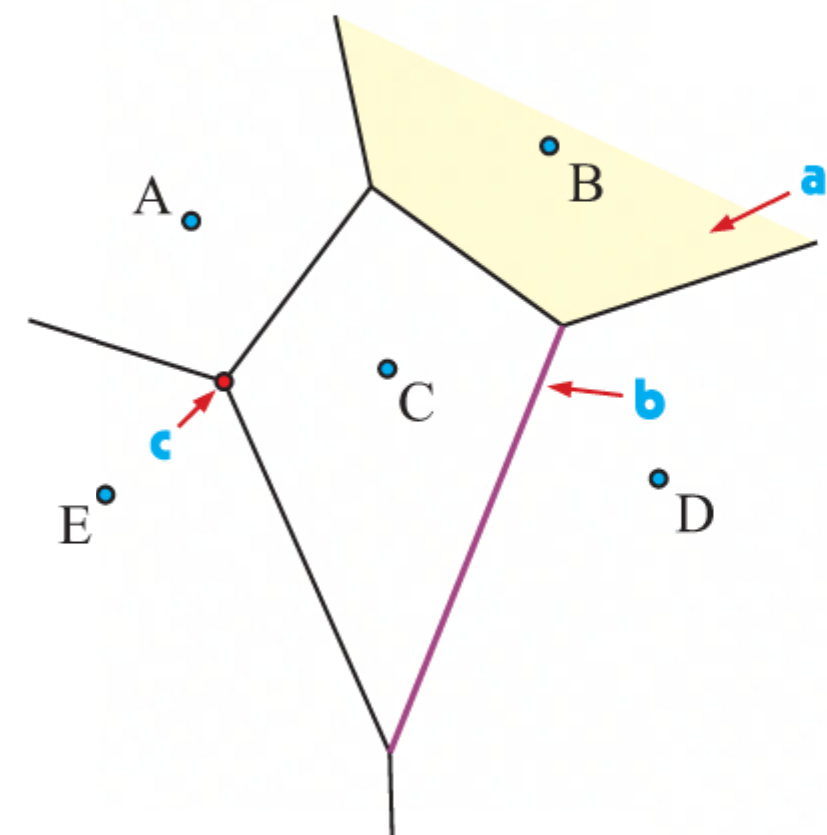


So, the largest empty circle has centre $V_2\left(\frac{30}{7}, \frac{31}{7}\right)$ and radius $\sqrt{13}$ km.

\therefore the new station should be built at $\left(\frac{30}{7}, \frac{31}{7}\right)$.

REVIEW SET 17A

- 1
 - a The parts closest to site B are in the yellow area, which is cell B.
 - b The parts equally closest to C and D are on the purple line, which is the edge adjacent to cells C and D.
 - c The part equally closest to A, C, and E is the red point, which is the vertex adjacent to cells A, C, and E.



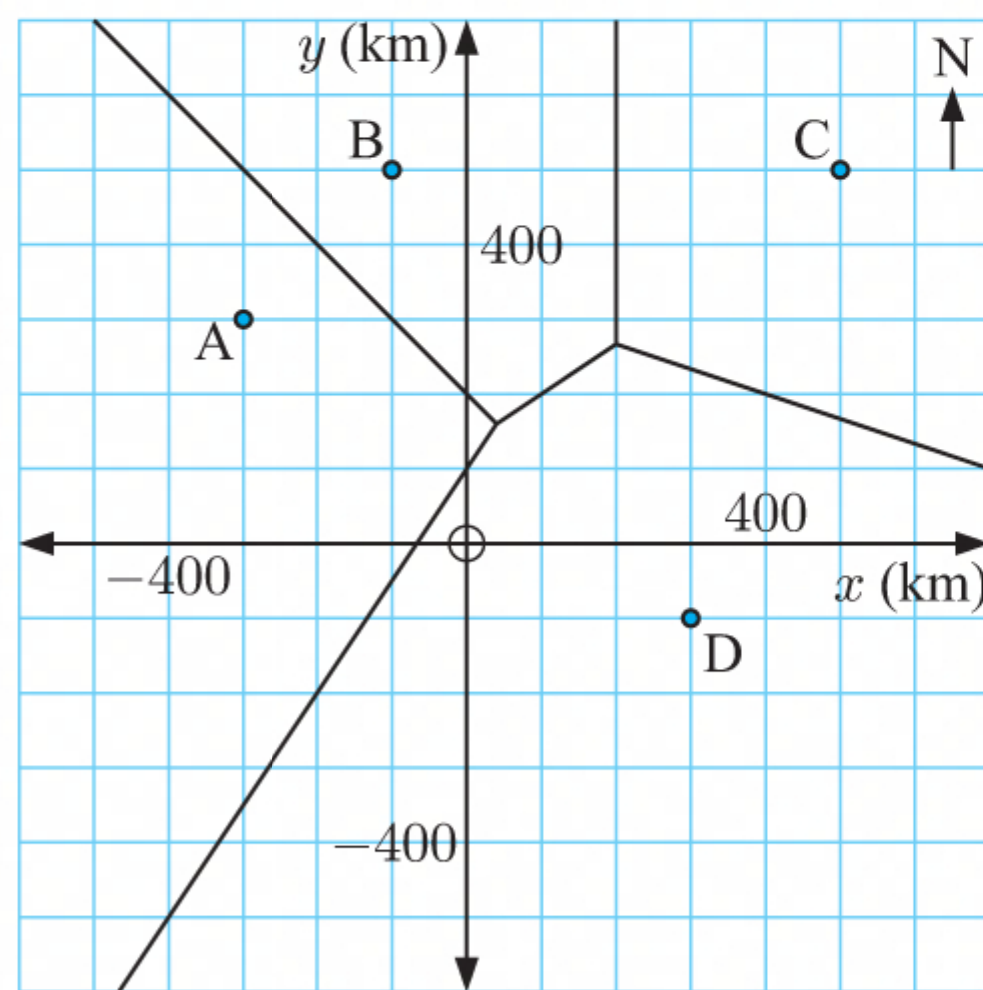
- 2 a**
- i** (300, 400) lies in cell C, so the nearest airport is Airport C.
 - ii** (−100, 0) lies in cell A, so the nearest airport is Airport A.
- b**
- i** (100, 200) lies on the edge adjacent to cells B and D, so it is equally closest to Airports B and D.
 - ii** The distance from (100, 200) to D(300, −100) is

$$\begin{aligned}
 & \sqrt{(300 - 100)^2 + (-100 - 200)^2} \\
 &= \sqrt{200^2 + (-300)^2} \\
 &= \sqrt{130\,000} \approx 361 \text{ km}
 \end{aligned}$$

So, the aeroplane is about 361 km from Airports B and D.

- iii** (400, 200) lies on the edge adjacent to cells C and D, and is $400 - 100 = 300$ km east of (100, 200).

So, the aeroplane must travel more than 300 km east before it is closest to Airport C.



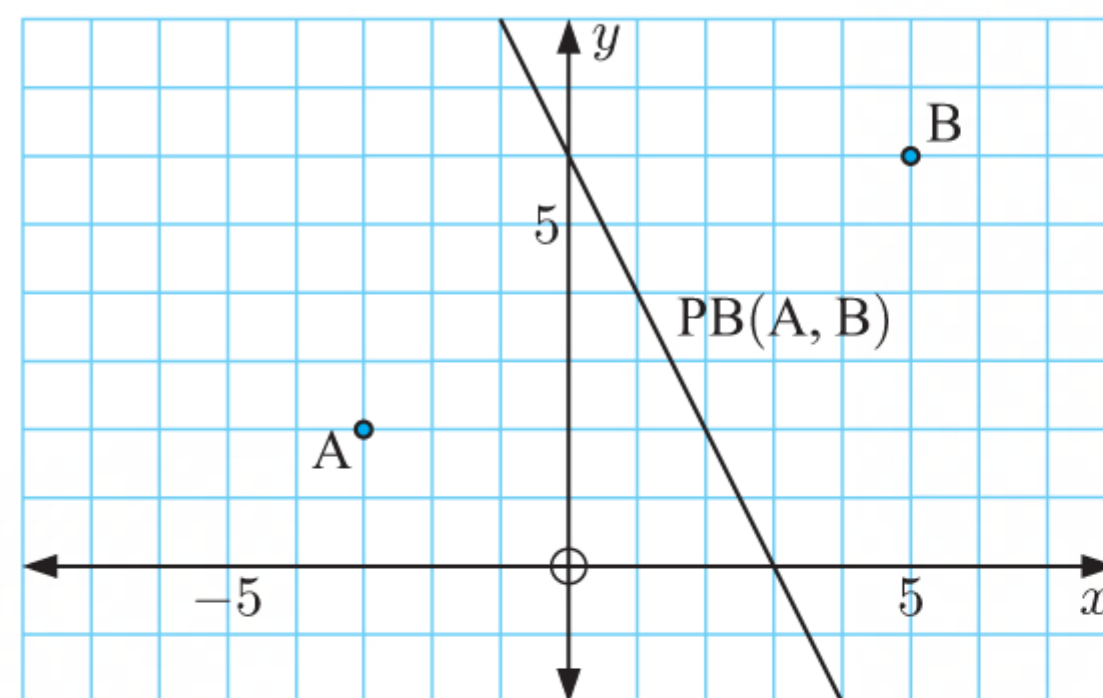
- 3 a** A(−3, 2), B(5, 6)

The midpoint of [AB] is $\left(\frac{-3+5}{2}, \frac{2+6}{2}\right)$
or (1, 4).

The gradient of [AB] is $\frac{6-2}{5-(-3)} = \frac{4}{8} = \frac{1}{2}$.

So, PB(A, B) has gradient −2 and passes through (1, 4).

We plot sites A and B, then draw PB(A, B) to give us the Voronoi diagram.



- b** PB(A, B) has gradient −2 and passes through (1, 4).

∴ its equation is $2x + y = 2(1) + 4$
or $y = -2x + 6$

- c i** When $x = 3$, $y = -2(3) + 6 = 0$

∴ (3, 0) is a point on the line $y = -2x + 6$.

∴ (3, 0) lies on the edge.

- ii** The distance from (3, 0) to A(−3, 2) is $\sqrt{(-3-3)^2 + (2-0)^2}$
 $= \sqrt{(-6)^2 + 2^2}$
 $= \sqrt{40} = 2\sqrt{10}$ units

The distance from (3, 0) to B(5, 6) is $\sqrt{(5-3)^2 + (6-0)^2}$
 $= \sqrt{2^2 + 6^2}$
 $= \sqrt{40} = 2\sqrt{10}$ units ✓

- d**
- i** $(-1, 7)$ lies in cell A, so it is closest to site A.
 - ii** $(2, -5)$ lies in cell A, so it is closest to site A.

4 $A(0, 20)$, $B(20, 0)$, $C(50, 10)$

a The midpoint of $[AB]$ is $\left(\frac{0+20}{2}, \frac{20+0}{2}\right)$ or $(10, 10)$.

The gradient of $[AB]$ is $\frac{0-20}{20-0} = \frac{-20}{20} = -1$.

So, $PB(A, B)$ has gradient 1 and passes through $(10, 10)$.

The midpoint of $[AC]$ is $\left(\frac{0+50}{2}, \frac{20+10}{2}\right)$ or $(25, 15)$.

The gradient of $[AC]$ is $\frac{10-20}{50-0} = \frac{-10}{50} = -\frac{1}{5}$.

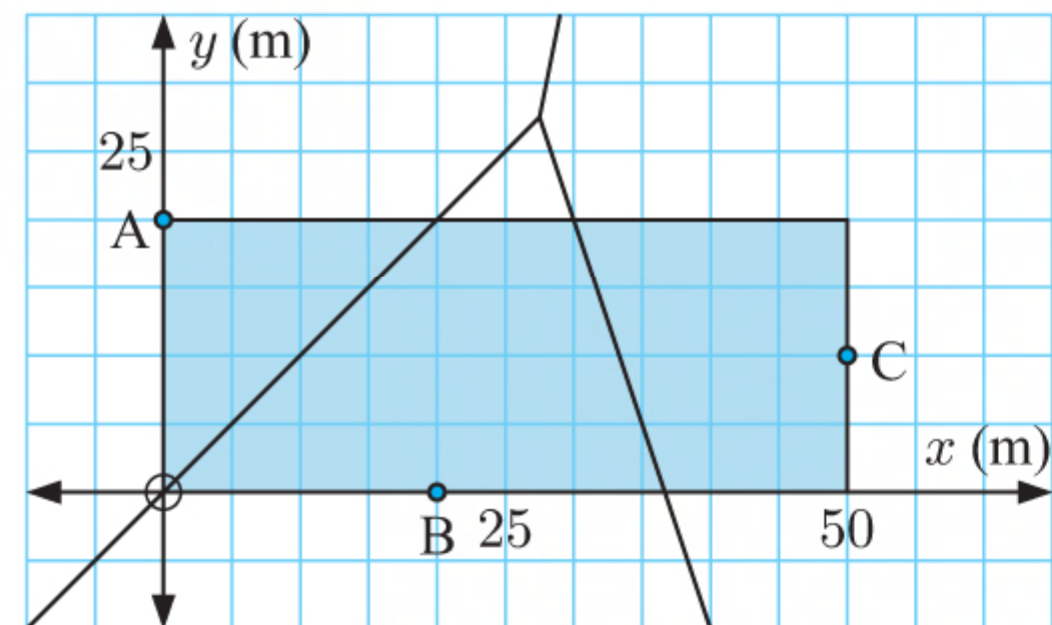
So, $PB(A, C)$ has gradient 5 and passes through $(25, 15)$.

The midpoint of $[BC]$ is $\left(\frac{20+50}{2}, \frac{0+10}{2}\right)$ or $(35, 5)$.

The gradient of $[BC]$ is $\frac{10-0}{50-20} = \frac{10}{30} = \frac{1}{3}$.

So, $PB(B, C)$ has gradient -3 and passes through $(35, 5)$.

We draw $PB(A, B)$, $PB(A, C)$, and $PB(B, C)$, then remove segments which do not form part of the Voronoi diagram.



- b** There is no point in the pool equidistant from all three exits, as the vertex of the Voronoi diagram lies outside the pool.
- c**
- i** $(35, 10)$ lies in cell C, so Jenny is closest to exit C.
 - ii** The distance from $(35, 10)$ to $C(50, 10)$ is $\sqrt{(50-35)^2 + (10-10)^2} = \sqrt{15^2} = 15$ m

So, Jenny is 15 m away from exit C.

d Area of the pool $= 50 \times 20 = 1000 \text{ m}^2$

The region of the pool which is within cell:

- i** A is a triangle with base 20 m and height 20 m.

So the area is $\frac{1}{2} \times 20 \times 20 = 200 \text{ m}^2$.

$\therefore \frac{200}{1000} = \frac{1}{5}$ of the pool is closest to exit A.

- ii** B is a trapezium which is 20 m high, and has parallel sides 10 m and

$$35 + 5\left(\frac{1}{3}\right) = \frac{110}{3} \text{ m long.}$$

$$\text{So the area is } \left(\frac{10 + \frac{110}{3}}{2} \right) \times 20 = \frac{1400}{3} \text{ m}^2.$$

$$\therefore \frac{\frac{1400}{3}}{1000} = \frac{7}{15} \text{ of the pool is closest to exit B.}$$

- iii** Using **i** and **ii**, the proportion of the pool that is closest to exit C is

$$1 - \frac{1}{5} - \frac{7}{15} = \frac{5}{15} = \frac{1}{3}.$$

- 5 a** The Voronoi diagram must have an edge missing because sites C and D currently lie within the same cell.

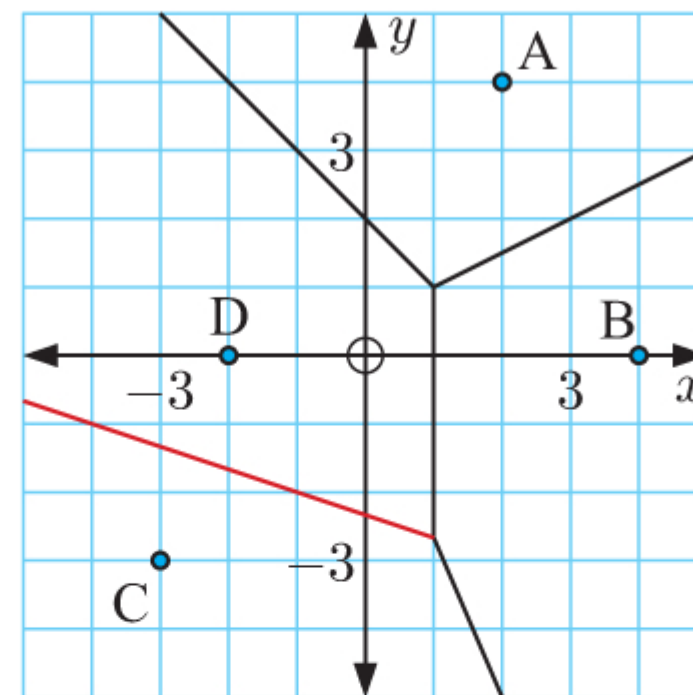
- b** $C(-3, -3)$, $D(-2, 0)$

$$\text{The midpoint of [CD] is } \left(\frac{-3 + -2}{2}, \frac{-3 + 0}{2} \right) \text{ or } \left(-\frac{5}{2}, -\frac{3}{2} \right).$$

$$\text{The gradient of [CD] is } \frac{0 - -3}{-2 - -3} = \frac{3}{1} = 3.$$

So, $PB(C, D)$ has gradient $-\frac{1}{3}$ and passes through $\left(-\frac{5}{2}, -\frac{3}{2}\right)$.

$$\therefore \text{ its equation is } x + 3y = 1\left(-\frac{5}{2}\right) + 3\left(-\frac{3}{2}\right) \\ \text{or } x + 3y + 7 = 0$$



- 6 a** The blue edge has gradient $\frac{1}{3}$ and passes through $(4, 0)$.

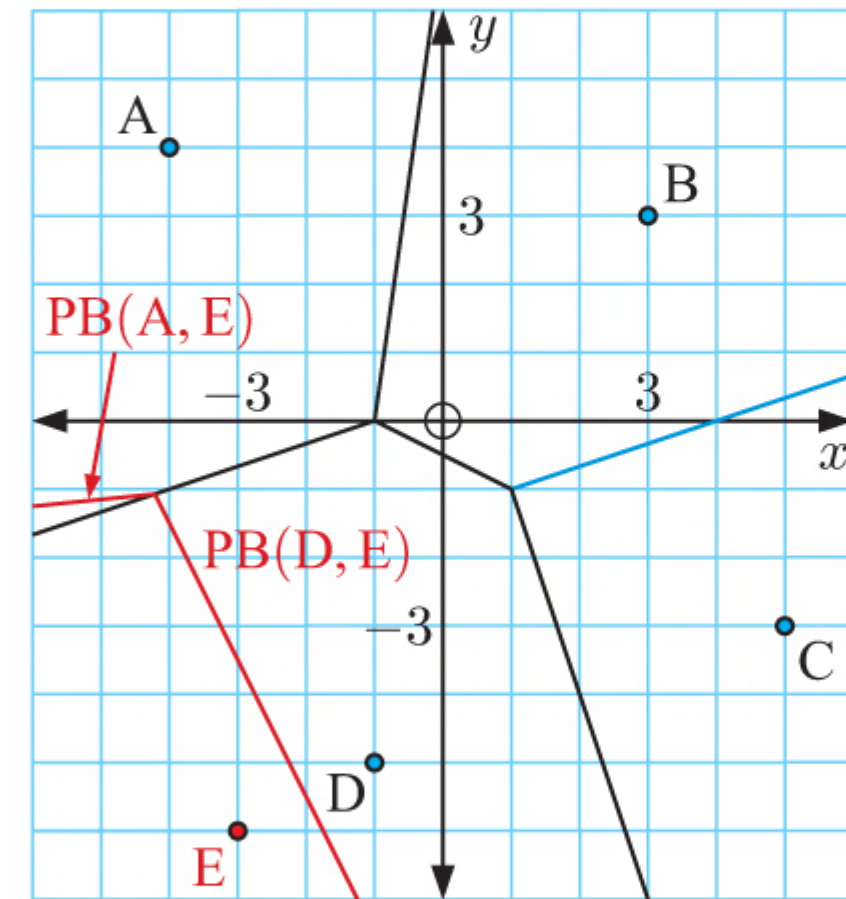
$$\therefore \text{ its equation is } x - 3y = 1(4) - 3(0)$$

$$\text{which is } 3y = x - 4$$

$$\text{or } y = \frac{1}{3}x - \frac{4}{3}$$

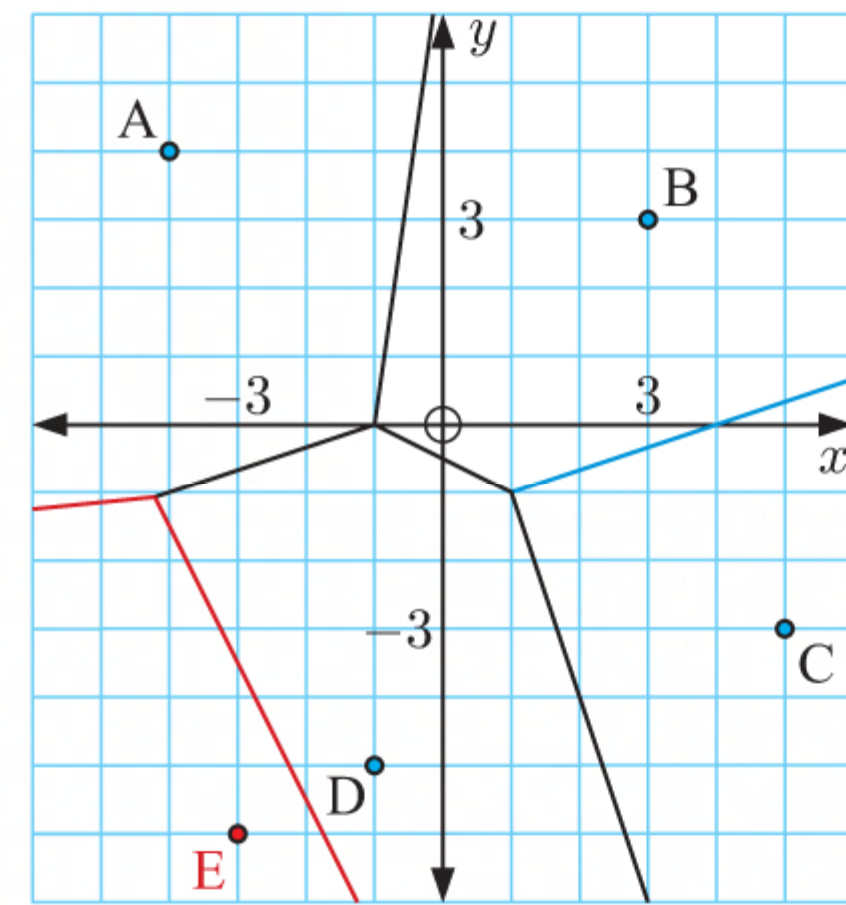
- b i** $(2, -1)$ lies in cell C, so it is closest to site C.
ii $(-5, -2)$ lies in cell D, so it is closest to site D.

- c** We construct $PB(A, E)$ and $PB(D, E)$ within the original cells A and D respectively.



We then remove the segments of edges from the original Voronoi diagram which now lie within cell E, giving us the Voronoi diagram which includes site E.

Note: We do not need to construct $PB(B, E)$ or $PB(C, E)$, as there are no points in cells B or C which are now closest to site E.

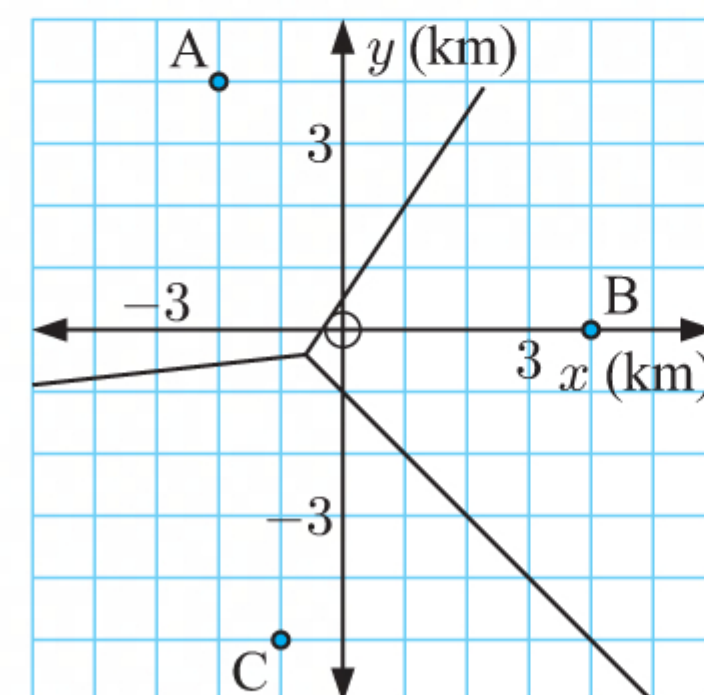


- d** The addition of site E affects our answer to **b ii**, as $(-5, -2)$ now lies within cell E, and so is now closest to site E.

7

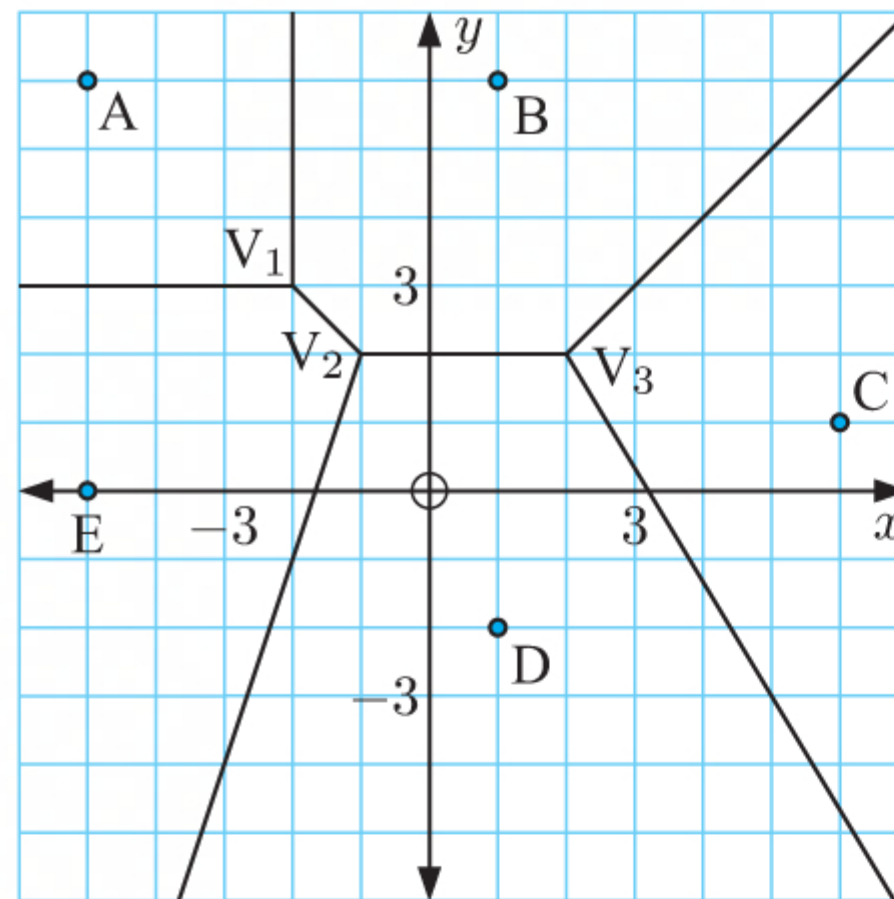
Location	Wind speed (km h^{-1})
A	14
B	11
C	19

- a** $(0, 0)$ is closest to B, so we estimate a wind speed of 11 km h^{-1} at $(0, 0)$.
- b** $(-1, 3)$ is closest to A, so we estimate a wind speed of 14 km h^{-1} at $(-1, 3)$.
- c** $(-4, -2)$ is closest to C, so we estimate a wind speed of 19 km h^{-1} at $(-4, -2)$.



- 8 a $A(-5, 6)$, $B(1, 6)$, $C(6, 1)$, $D(1, -2)$, $E(-5, 0)$

We draw $PB(B, C)$, $PB(B, D)$, $PB(C, D)$, and $PB(D, E)$, then remove segments which do not form part of the Voronoi diagram.



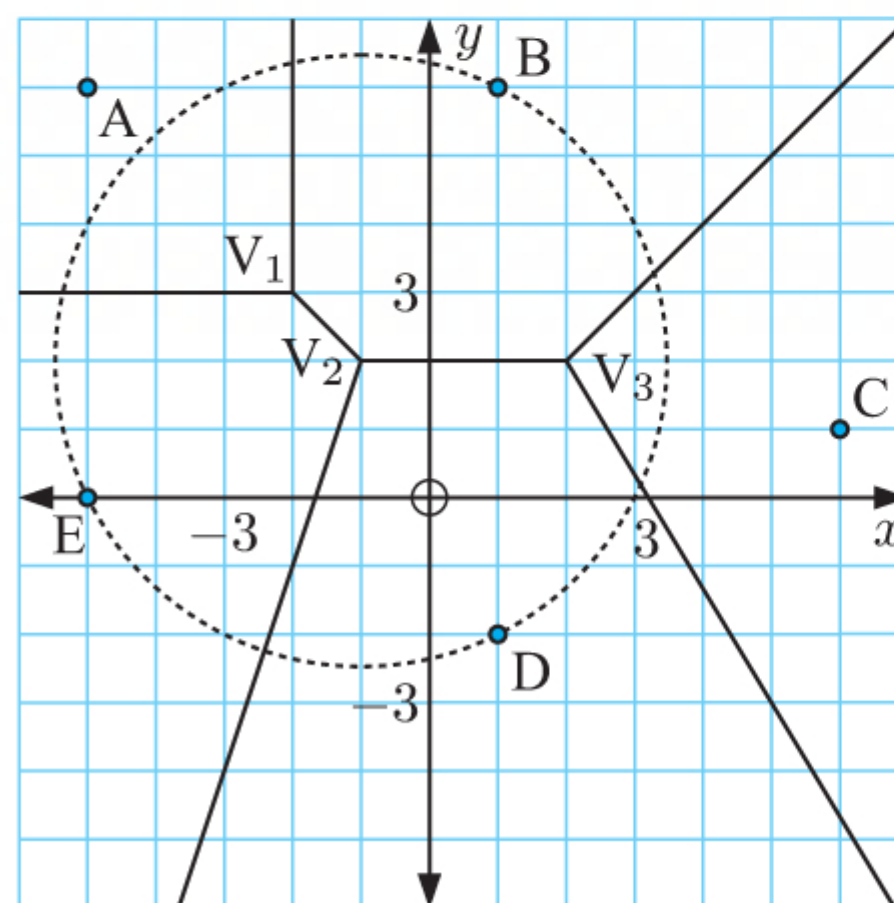
- b We observe from the Voronoi diagram in a that:

- the vertex adjacent to A, B, and E is $V_1(-2, 3)$
- the vertex adjacent to B, D, and E is $V_2(-1, 2)$
- the vertex adjacent to B, C, and D is $V_3(2, 2)$.

$$\begin{aligned} V_1B &= \sqrt{(1 - (-2))^2 + (6 - 3)^2} \\ &= \sqrt{3^2 + 3^2} \\ &= \sqrt{18} = 3\sqrt{2} \text{ units} \end{aligned}$$

$$\begin{aligned} V_2B &= \sqrt{(1 - (-1))^2 + (6 - 2)^2} \\ &= \sqrt{2^2 + 4^2} \\ &= \sqrt{20} = 2\sqrt{5} \text{ units} \end{aligned}$$

$$\begin{aligned} V_3B &= \sqrt{(1 - 2)^2 + (6 - 2)^2} \\ &= \sqrt{(-1)^2 + 4^2} \\ &= \sqrt{17} \text{ units} \end{aligned}$$



So, the largest empty circle has centre $V_2(-1, 2)$ and radius $2\sqrt{5}$ units.

- 9 a i The midpoint of $[BC]$ is $\left(\frac{14+22}{2}, \frac{7+(-9)}{2}\right)$
or $(18, -1)$.

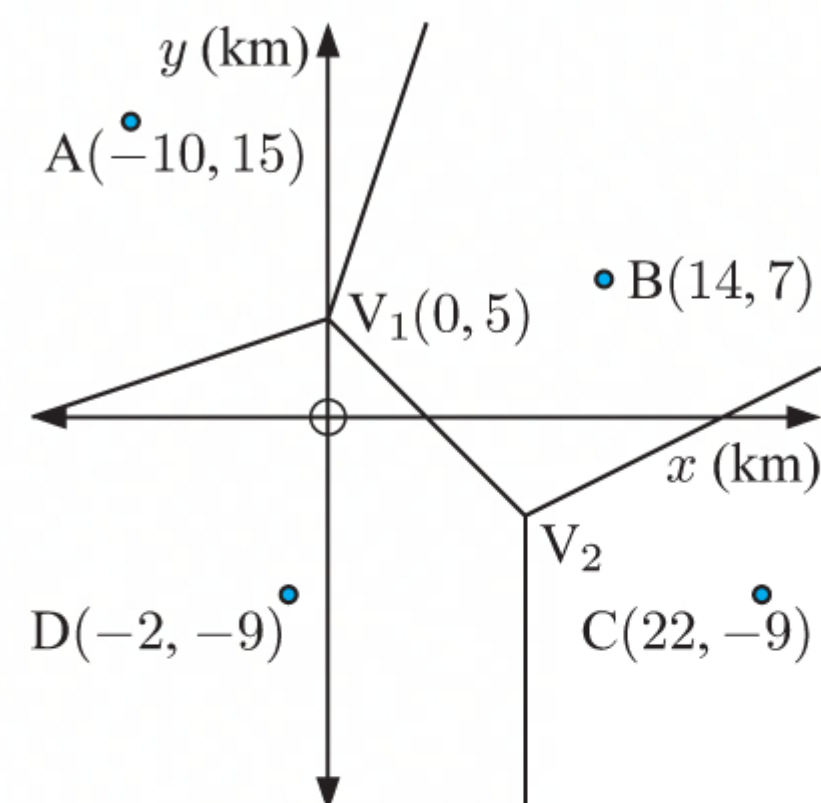
$$\text{The gradient of } [BC] \text{ is } \frac{-9-7}{22-14} = \frac{-16}{8} = -2.$$

So, $PB(B, C)$ has gradient $\frac{1}{2}$ and passes through $(18, -1)$.

$$\therefore \text{ its equation is } x - 2y = 1(18) - 2(-1)$$

$$\text{which is } 2y = x - 20$$

$$\text{or } y = \frac{1}{2}x - 10$$



- ii The midpoint of $[CD]$ is $\left(\frac{22 + (-2)}{2}, \frac{-9 + (-9)}{2}\right)$ or $(10, -9)$.

The gradient of $[CD]$ is horizontal.

So, $PB(C, D)$ is vertical and passes through $(10, -9)$.

\therefore its equation is $x = 10$.

- b $PB(B, C)$ and $PB(C, D)$ intersect where $x = 10$

When $x = 10$, $y = \frac{1}{2}(10) - 10 = -5$

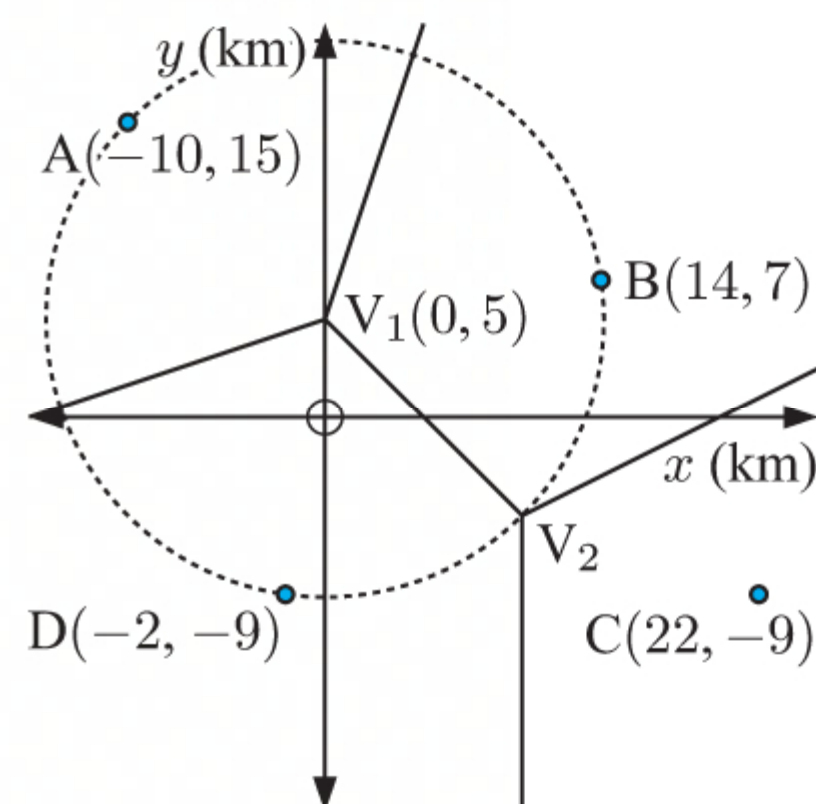
So, the vertex V_2 has coordinates $(10, -5)$.

- c i The Voronoi diagram has vertices $V_1(0, 5)$ and $V_2(10, -5)$.

V_1 is adjacent to cells A, B, and D, and V_2 is adjacent to cells B, C, and D.

$$\begin{aligned} V_1D &= \sqrt{(-2 - 0)^2 + (-9 - 5)^2} \\ &= \sqrt{(-2)^2 + (-14)^2} \\ &= \sqrt{200} = 10\sqrt{2} \text{ km} \end{aligned}$$

$$\begin{aligned} V_2D &= \sqrt{(-2 - 10)^2 + (-9 - (-5))^2} \\ &= \sqrt{(-12)^2 + (-4)^2} \\ &= \sqrt{160} = 4\sqrt{10} \text{ km} \end{aligned}$$



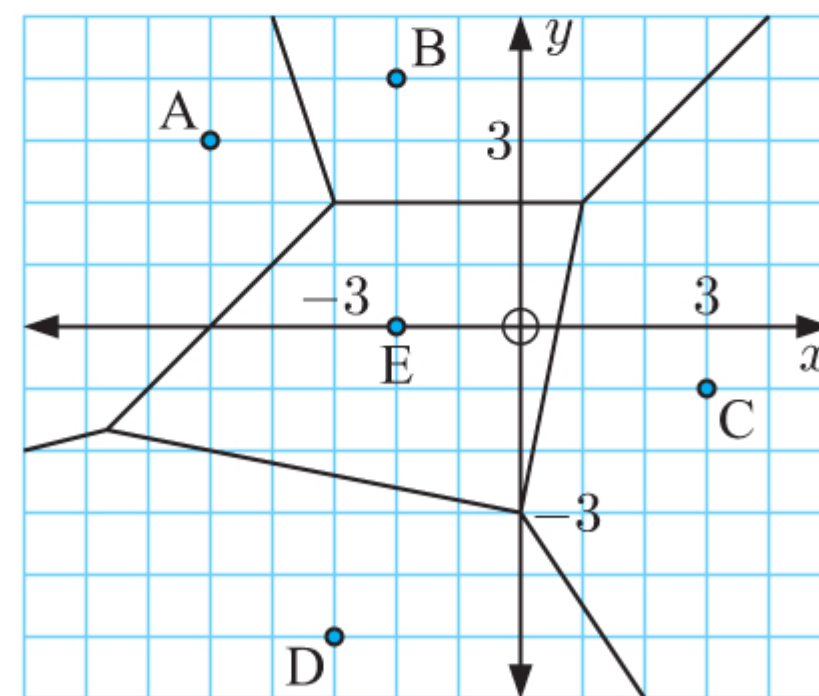
So, the largest empty circle has centre $V_1(0, 5)$ and radius $10\sqrt{2}$ km.

\therefore the observatory should be built at $V_1(0, 5)$.

- ii From i, the observatory will be $10\sqrt{2} \approx 14.1$ km from the nearest towns.
iii From i, the observatory will be equally closest to towns A, B, and D.

REVIEW SET 17B

- 1 a i $(1, 0)$ lies in cell C, so it is closest to site C.
ii $(-4, -2)$ lies in cell E, so it is closest to site E.
iii $(3, -5)$ lies in cell C, so it is closest to site C.
iv $(-3, 2)$ lies on the vertex adjacent to cells A, B, and E, so it is equally closest to sites A, B, and E.



- b There are no points equally closest to sites B and D, as there is no edge or vertex which is adjacent to both cell B and cell D.
c The vertex which is adjacent to cells B, C, and E is $(1, 2)$.
So, $(1, 2)$ is equally closest to sites B, C, and E.

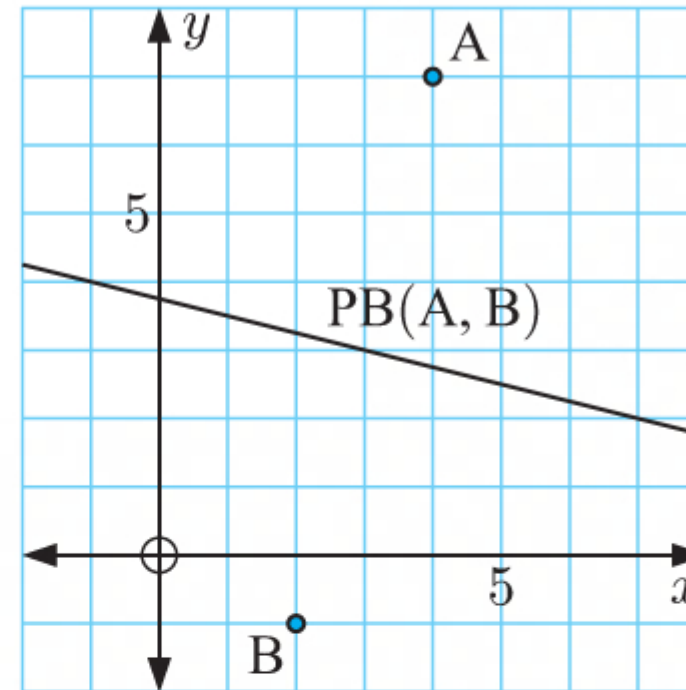
- 2** A point on an edge is equally closest to two sites. If the circle passes through a site, then the radius of the circle is the distance to that closest site.
 \therefore the radius would also be the distance to the other site.
 \therefore the circle passes through two sites.

- 3 a** $A(4, 7), B(2, -1)$

The midpoint of $[AB]$ is $\left(\frac{4+2}{2}, \frac{7+(-1)}{2}\right)$
 or $(3, 3)$.

The gradient of $[AB]$ is $\frac{-1-7}{2-4} = \frac{-8}{-2} = 4$.

So, $PB(A, B)$ has gradient $-\frac{1}{4}$, and passes through $(3, 3)$.



- b** $A(-5, 0), B(1, 6), C(7, 4)$

The midpoint of $[AB]$ is $\left(\frac{-5+1}{2}, \frac{0+6}{2}\right)$ or $(-2, 3)$.

The gradient of $[AB]$ is $\frac{6-0}{1-(-5)} = \frac{6}{6} = 1$.

So, $PB(A, B)$ has gradient -1 and passes through $(-2, 3)$.

The midpoint of $[AC]$ is $\left(\frac{-5+7}{2}, \frac{0+4}{2}\right)$ or $(1, 2)$.

The gradient of $[AC]$ is $\frac{4-0}{7-(-5)} = \frac{4}{12} = \frac{1}{3}$.

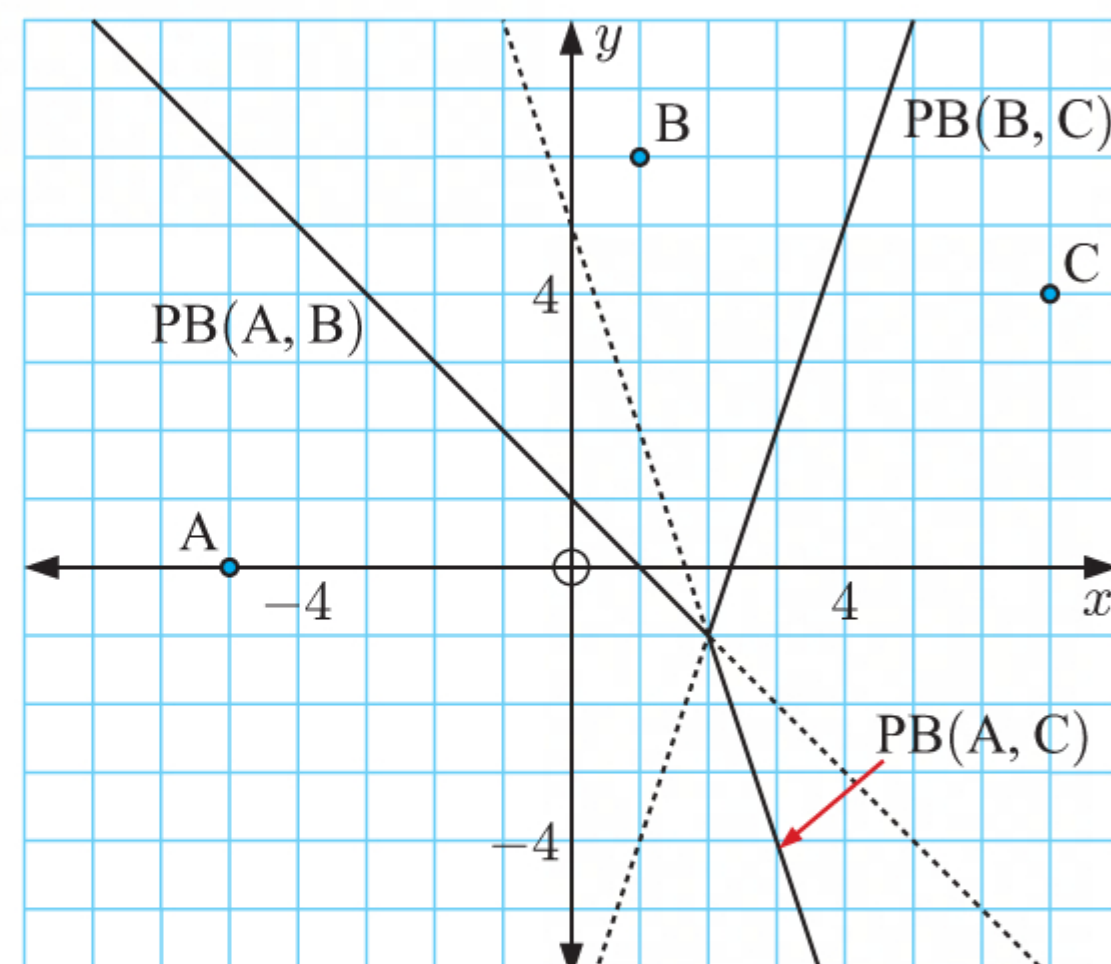
So, $PB(A, C)$ has gradient -3 and passes through $(1, 2)$.

The midpoint of $[BC]$ is $\left(\frac{1+7}{2}, \frac{6+4}{2}\right)$ or $(4, 5)$.

The gradient of $[BC]$ is $\frac{4-6}{7-1} = \frac{-2}{6} = -\frac{1}{3}$.

So, $PB(B, C)$ has gradient 3 and passes through $(4, 5)$.

We plot sites A, B , and C on a set of axes. We draw $PB(A, B)$, $PB(A, C)$, and $PB(B, C)$ as dashed lines, then make solid only the parts which form the Voronoi edges.



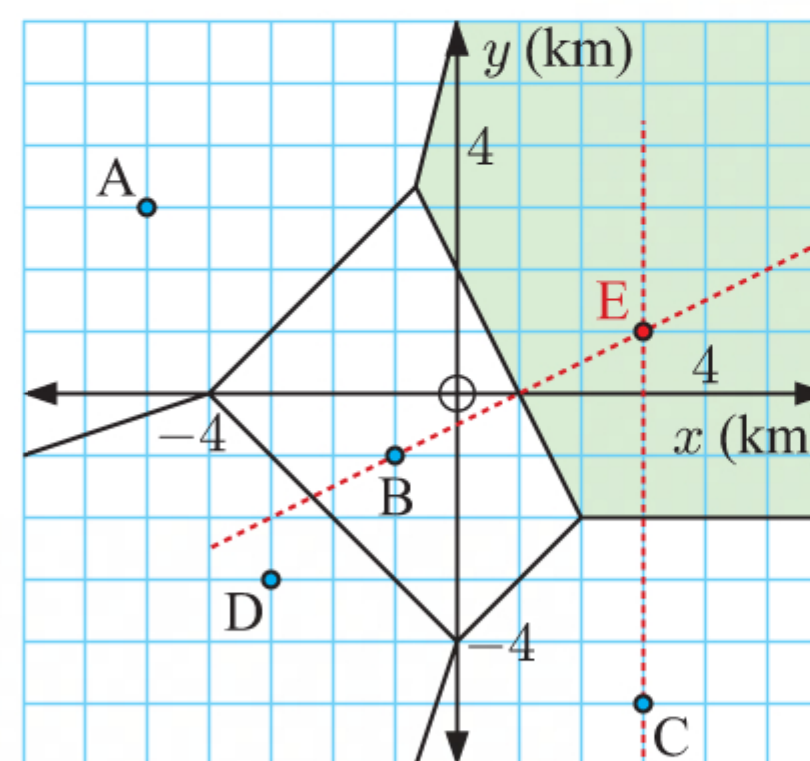
- 4 a** The Voronoi diagram must have a site missing as there is a cell without a site.

- b** The missing site E must lie in the shaded cell.
Now $PB(B, E)$ has gradient -2 , and $PB(C, E)$ is horizontal.

$\therefore [BE]$ has gradient $\frac{1}{2}$, and $[CE]$ is vertical.

If we draw lines (BE) and (CE) through B and C respectively, their intersection point must be E.

We observe that E has coordinates $(3, 1)$.



- c i** $(-3, 2)$ lies in cell A, so Elizabeth is closest to taxi rank A.

ii The distance from $(-3, 2)$ to $A(-5, 3)$ is
$$\begin{aligned} & \sqrt{(-5 - (-3))^2 + (3 - 2)^2} \\ &= \sqrt{(-2)^2 + 1^2} \\ &= \sqrt{5} \text{ km} \end{aligned}$$

Elizabeth can walk at 5 km h^{-1} , so it will take $\frac{\sqrt{5}}{5} \approx 0.44721$ hours, or about 26 minutes 50 seconds, for her to walk to taxi rank A.

We assume that she can walk in a straight line to A, and is walking at a constant speed of 5 km h^{-1} .

- d i** $(-2, -2)$ lies on the edge adjacent to cells B and D, so Albert is equally closest to taxi ranks B and D.
- ii** Albert might consider:
- whether he can walk there in a straight line
 - the price
 - his direction of travel once he gets into the taxi.

- 5 a** $A(-6, -2)$, $B(-4, 2)$, $C(4, 4)$

The midpoint of $[AB]$ is $\left(\frac{-6 + -4}{2}, \frac{-2 + 2}{2}\right)$ or $(-5, 0)$.

The gradient of $[AB]$ is $\frac{2 - -2}{-4 - (-6)} = \frac{4}{2} = 2$.

So, $PB(A, B)$ has gradient $-\frac{1}{2}$ and passes through $(-5, 0)$.

The midpoint of $[AC]$ is $\left(\frac{-6 + 4}{2}, \frac{-2 + 4}{2}\right)$ or $(-1, 1)$.

The gradient of $[AC]$ is $\frac{4 - -2}{4 - -6} = \frac{6}{10} = \frac{3}{5}$.

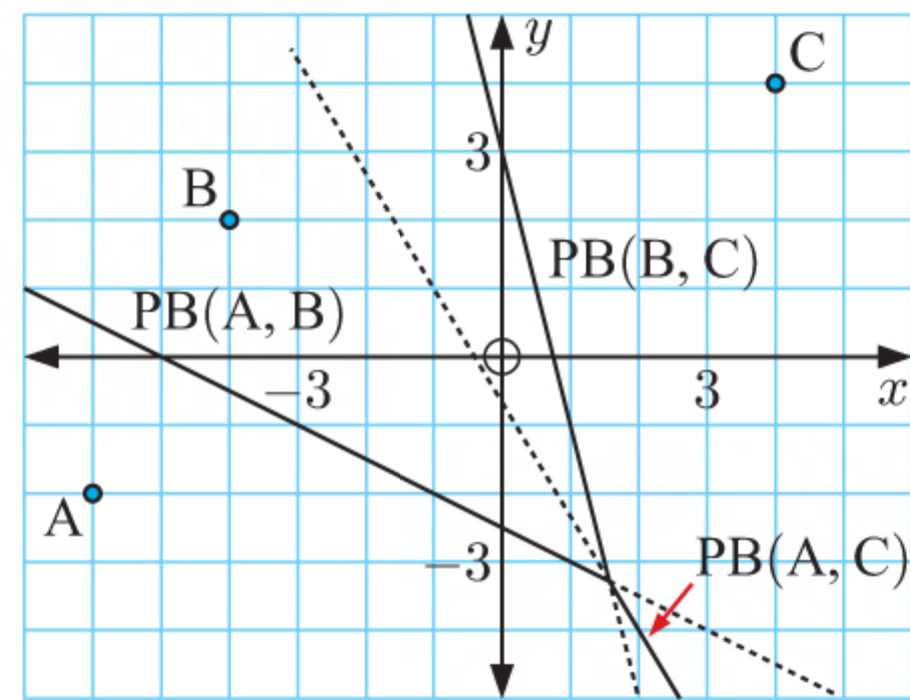
So, $PB(A, C)$ has gradient $-\frac{5}{3}$ and passes through $(-1, 1)$.

The midpoint of $[BC]$ is $\left(\frac{-4 + 4}{2}, \frac{2 + 4}{2}\right)$ or $(0, 3)$.

The gradient of $[BC]$ is $\frac{4 - 2}{4 - -4} = \frac{2}{8} = \frac{1}{4}$.

So, $PB(B, C)$ has gradient -4 and passes through $(0, 3)$.

We draw $PB(A, B)$, $PB(A, C)$, and $PB(B, C)$ as dashed lines, then make solid only the parts which form the Voronoi edges.



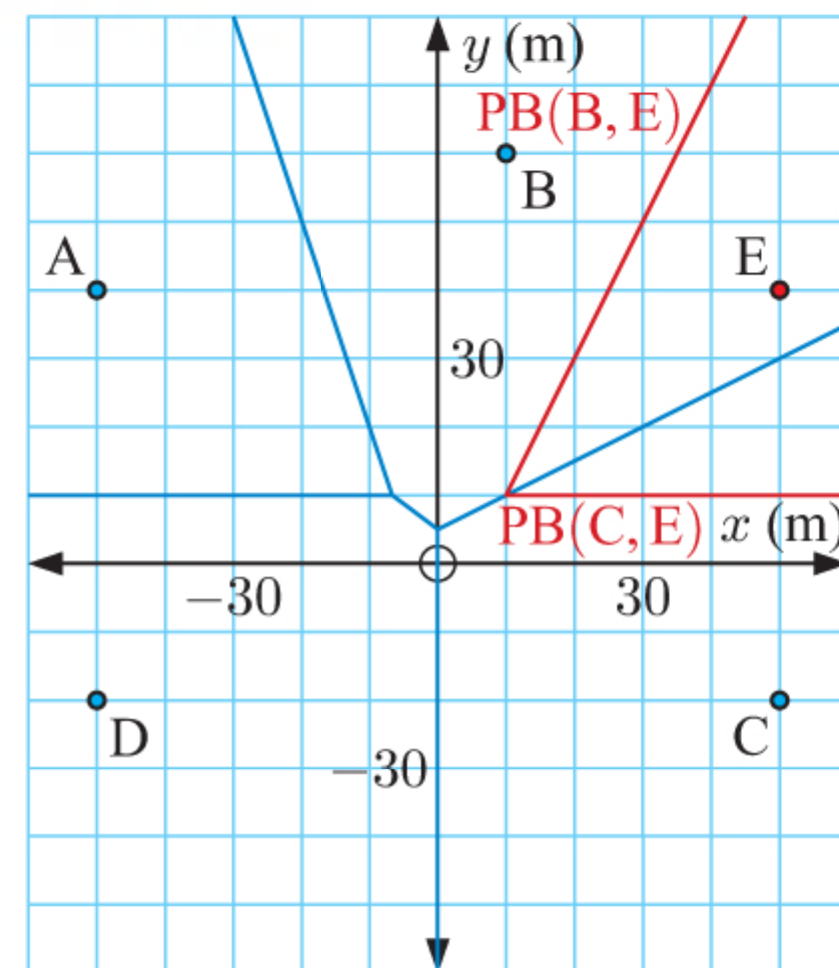
- b**
 - i** $(1, 2)$ lies in cell C, so it is closest to site C.
 - ii** $(-1, -5)$ lies in cell A, so it is closest to site A.
 - iii** $(-3, -1)$ lies on the edge adjacent to cells A and B, so it is equally closest to sites A and B.

- 6 a** $(20, 20)$ lies in cell B, so Boris is closest to Bin B.

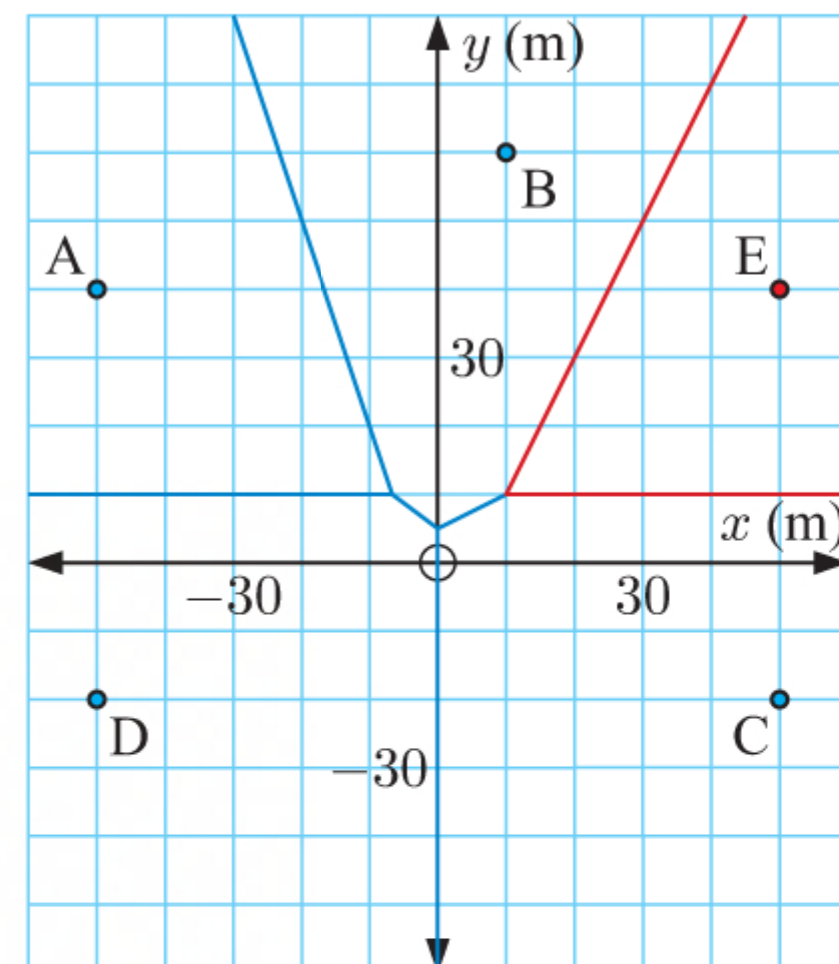
$$\begin{aligned}
 \text{The distance from } (20, 20) \text{ to } B(10, 60) \text{ is } & \sqrt{(10 - 20)^2 + (60 - 20)^2} \\
 &= \sqrt{(-10)^2 + 40^2} \\
 &= \sqrt{1700} = 10\sqrt{17} \approx 41.2 \text{ m}
 \end{aligned}$$

So, Boris is about 41.2 m from the closest bin.

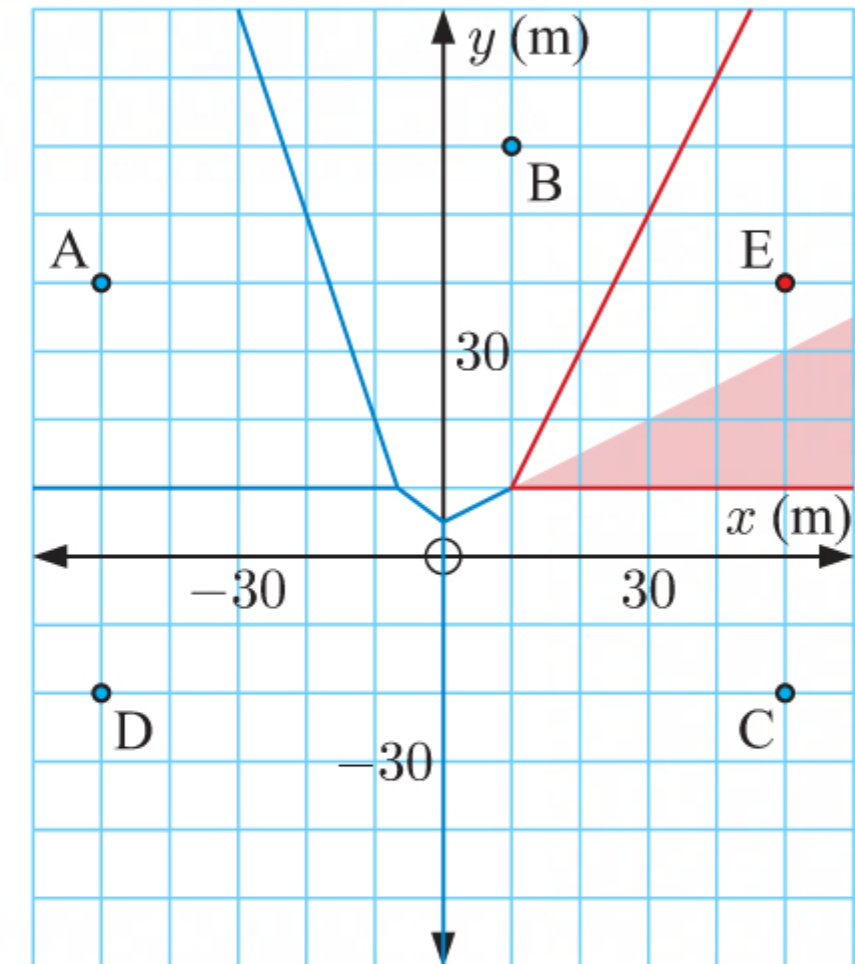
- b i** We construct $PB(B, E)$ and $PB(C, E)$ within the original cells B and E respectively.



We then remove the segments of edges which now lie within cell E, giving us the Voronoi diagram which includes site E.



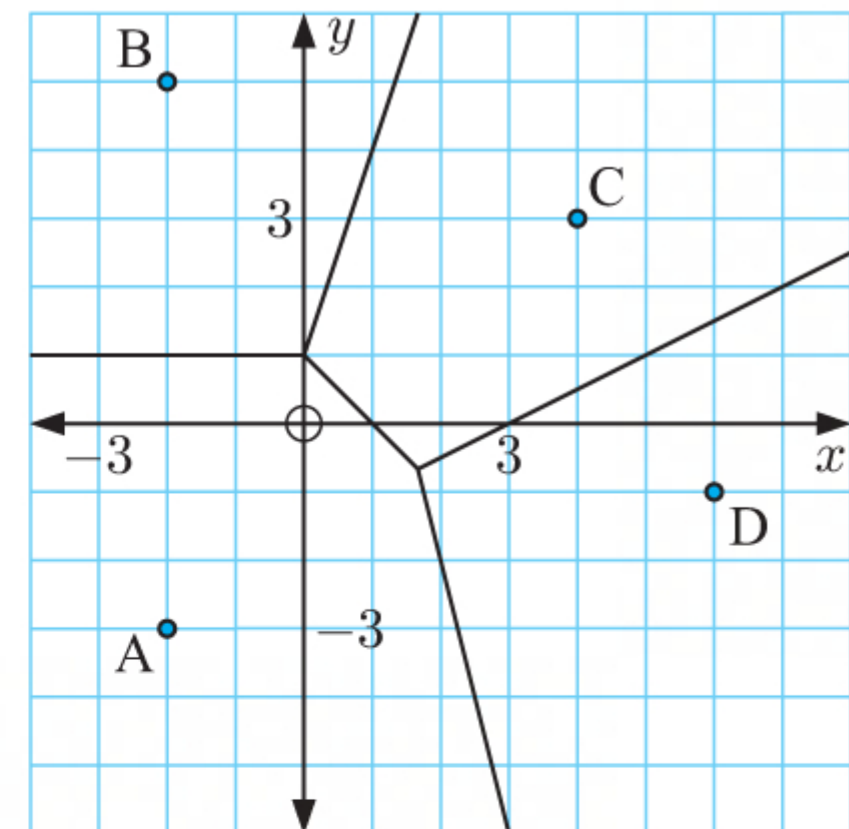
- ii Cells B and C have been affected by the introduction of the new bin at E.
- iii The intersection between the original cell C and the current cell E is shaded in red, which is the area of park which used to be closest to Bin C and is now closest to Bin E.



7

Location	Depth (m)
A(-2, -3)	7.5
B(-2, 5)	9.2
C(4, 3)	6.1
D(6, -1)	6.9

- a We plot sites A, B, C, and D on a set of axes. We draw $PB(A, B)$, $PB(A, C)$, $PB(A, D)$, $PB(B, C)$, and $PB(C, D)$, then remove segments which do not form part of the Voronoi diagram.



- b
 - i $(-1, 3)$ is closest to B, so we estimate the depth of the lake to be 9.2 m at $(-1, 3)$.
 - ii $(4, 0)$ is closest to D, so we estimate the depth of the lake to be 6.9 m at $(4, 0)$.

8

- a
- i $(-100, 200)$ lies in cell E, so it is closest to parking lot E.
 - ii $(0, -400)$ lies in cell D, so it is closest to parking lot D.
- b
- i The Voronoi diagram has vertices $V_1(-100, 300)$, $V_2(0, 200)$, and $V_3(-100, -100)$. V_1 is adjacent to cells A, B, and E, V_2 is adjacent to cells B, C, and E, and V_3 is adjacent to cells C, D, and E.
 - E has coordinates $(-400, 0)$.

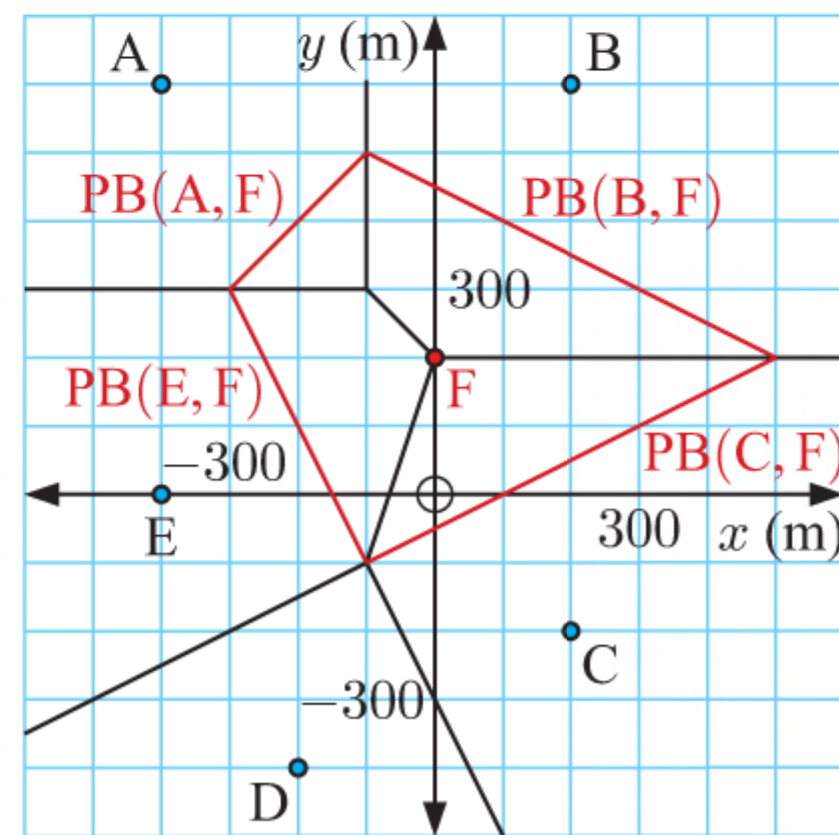
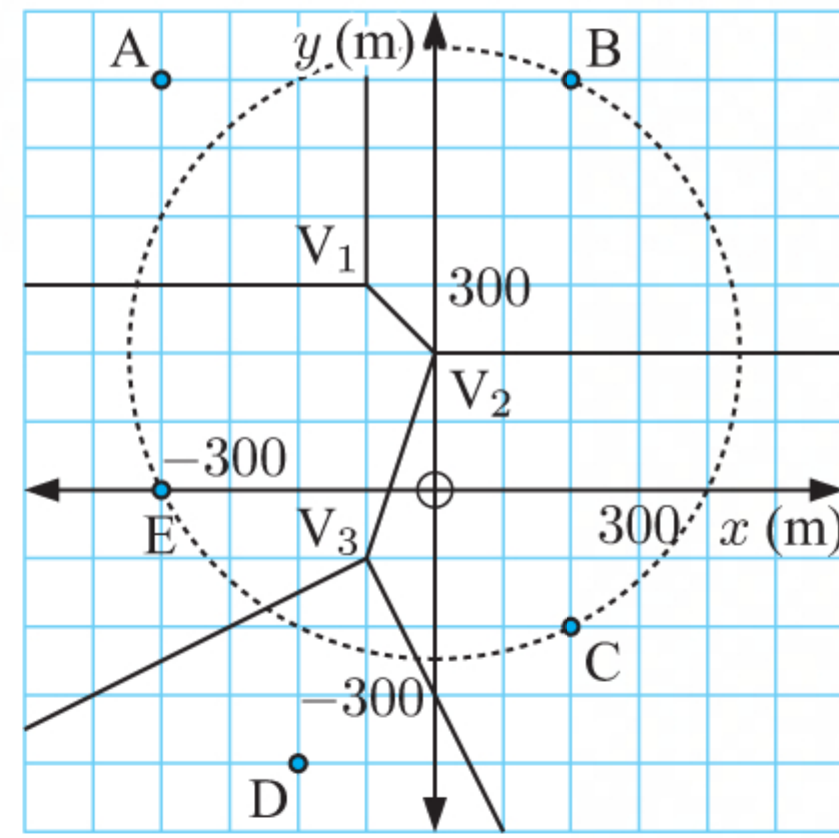
$$\begin{aligned}
 V_1E &= \sqrt{(-400 - (-100))^2 + (0 - 300)^2} \\
 &= \sqrt{(-300)^2 + (-300)^2} \\
 &= \sqrt{180\,000} = 300\sqrt{2} \text{ m}
 \end{aligned}$$

$$\begin{aligned}
 V_2E &= \sqrt{(-400 - 0)^2 + (0 - 200)^2} \\
 &= \sqrt{(-400)^2 + (-200)^2} \\
 &= \sqrt{200\,000} = 200\sqrt{5} \text{ m}
 \end{aligned}$$

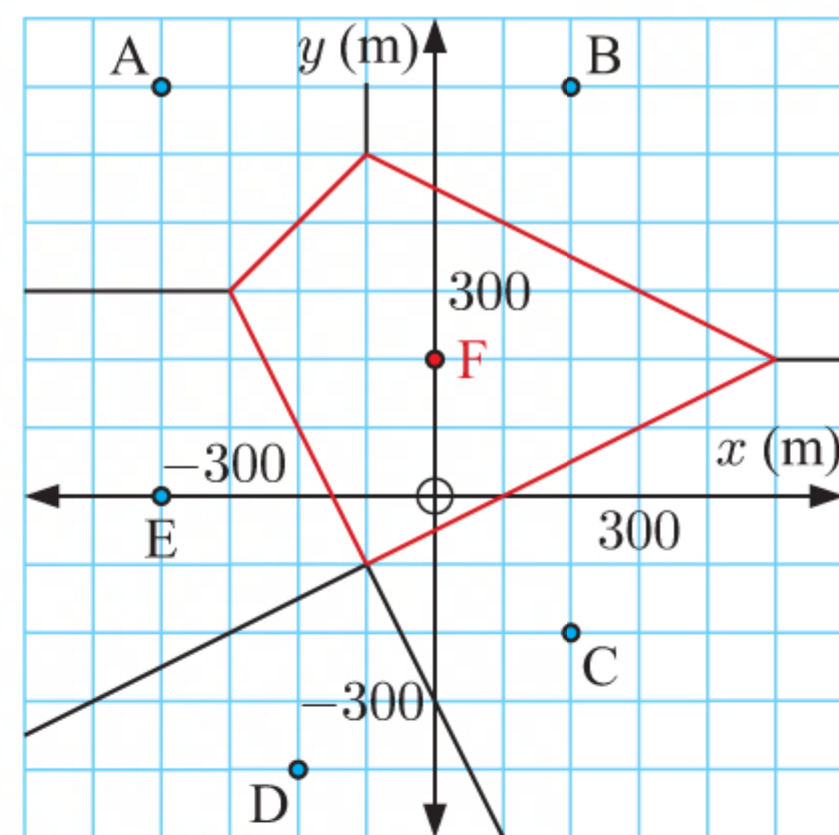
$$\begin{aligned}
 V_3E &= \sqrt{(-400 - (-100))^2 + (0 - (-100))^2} \\
 &= \sqrt{(-300)^2 + 100^2} \\
 &= \sqrt{100\,000} = 100\sqrt{10} \text{ m}
 \end{aligned}$$

So, the largest empty circle has centre $V_2(0, 200)$ and radius $200\sqrt{5}$ m.
 \therefore the optimal position for the new parking lot F is $(0, 200)$.

- ii From i, the new parking lot at $F(0, 200)$ is $200\sqrt{5} \approx 447$ m from the closest existing parking lots.
- iii We construct $PB(A, F)$, $PB(B, F)$, $PB(C, F)$, and $PB(E, F)$ within the original cells A, B, C, and E respectively.

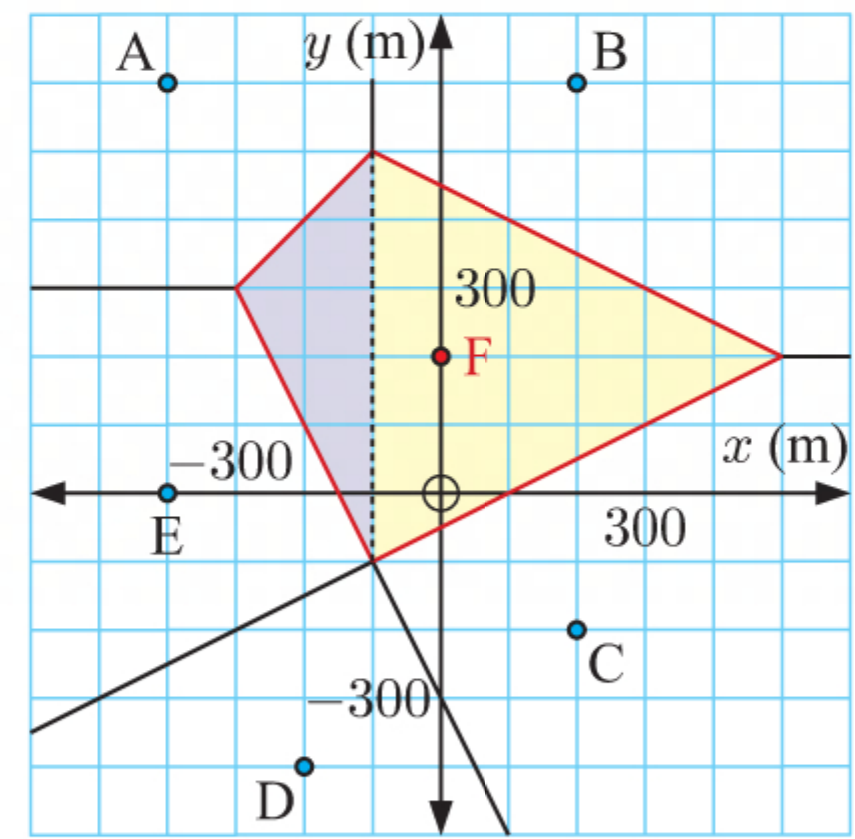


We then remove the segments of edges from the original Voronoi diagram which now lie within cell F, giving us the Voronoi diagram with the new parking lot added at F.



- iv** The region closest to the new parking lot is cell F, which can be divided into two triangles. One has base length 600 m and height 200 m, and the other has base length 600 m and height 600 m.

$$\begin{aligned}\therefore \text{ area of cell F} \\ &= \frac{1}{2} \times 600 \times 200 + \frac{1}{2} \times 600 \times 600 \\ &= 240\,000 \text{ m}^2\end{aligned}$$



- c i** From the diagram in **b iii**, we observe that $(200, 200)$ is in cell F.
 \therefore Joseph is closest to the new parking lot F.
- ii** The distance between $(200, 200)$ and $F(0, 200)$ is

$$\begin{aligned}&\sqrt{(0 - 200)^2 + (200 - 200)^2} \\ &= \sqrt{(-200)^2 + 0^2} \\ &= 200 \text{ m}\end{aligned}$$

We observe that $(200, 200)$ was previously on the edge adjacent to cells B and C.

The distance between $(200, 200)$ and $B(200, 600)$ is

$$\begin{aligned}&\sqrt{(200 - 200)^2 + (600 - 200)^2} \\ &= \sqrt{0^2 + 400^2} \\ &= 400 \text{ m}\end{aligned}$$

So, Joseph is now $400 - 200 = 200 \text{ m}$ closer to a parking lot.